Homework 2

Exercises 1.6

5(a)

A counterexample to the statement "For all positive integers x, x^2+x+41 is a prime" is provided by $x = 41$, since

$$41^2 + 41 + 41 = 41\,(41 + 1 + 1) = 41 \cdot 43,$$

a composite integer.

5(b)

Claim 1 *For every real number x, there exists a real number y such that $x + y = 0$.*

Proof. Let $x \in \mathbb{R}$. Then $y = -x$ satisfies

$$\begin{aligned} x + y &= x + (-x) \\ &= x - x \\ &= 0. \end{aligned}$$

Therefore, for every $x \in \mathbb{R}$, there exists $y \in \mathbb{R}$ such that $x + y = 0$. ■

5(d)

Since 12 divides $24 = 6 \cdot 4$, but 12 divides neither 6 nor 4, the statement "For integers a, b, c, if a divides bc, then either a divides b or a divides c" is false. (Let $a = 12$, $b = 6$, and $c = 4$.)

8(c)

Grade: C.

The "proof" misstates the definition of *divides* twice ("assume a divides b. Then $a = kb$ for some integer k," etc.). This could be corrected by interchanging a and b in the two equations.

8(e)

Grade: F.

The claim "Every real function is continuous at $x = 0$" is clearly false since, for example, the greatest integer function $f(x) = \lfloor x \rfloor$ (sometimes denoted $[[x]]$) is discontinuous at $x = 0$, since $\lim_{x \to 0} f(x)$ does not exist. The proof's error is to deduce from the true statement "every real function either is continuous at $x = 0$ or is not continuous at $x = 0$" the false statement "every real function is continuous at $x = 0$ or every real function is not continuous at $x = 0$." The correct disjunction is "every real function is continuous at $x = 0$ or **there exists a real function** which is not continuous at $x = 0$."

Exercises 1.7

1(c)

Claim 2 *The sum of five consecutive integers is always divisible by* 5.

Proof. Let n be the least of the five consecutive integers. Then the sum is

$$\begin{aligned} n + (n+1) + (n+2) + (n+3) + (n+4) &= 5n + (1+4) + (2+3) \\ &= 5n + 10 \\ &= 5\,(n+2)\,. \end{aligned}$$

Since $n+2$ is an integer, the left-hand side is divisible by 5. Therefore, the sum of five consecutive integers is always divisible by 5. ■

12(d)

Grade: F.

The "proof" is given only for one real number ($x = \pi$). Therefore, it does not prove the claim for all real numbers.

12(e)

Grade: A.

Exercises 2.2

1(b)

The set of integers whose square is less than 17 is denoted $\{x \mid x \in \mathbb{Z} \text{ and } x^2 < 17\}$.

4(b)

True. By Theorem 2.1, for any set A, we have $\emptyset \subseteq A$.

5(b)

An example of set A, B, and C satsifying $A \subseteq B$, $B \subseteq C$, and $C \subseteq A$ is given by $A = B = C = \{1\}$.

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Claim 3 *If $x \notin B$ and $A \subseteq B$, then $x \notin A$.*

Proof. Assume $x \notin B$ and $A \subseteq B$. Suppose $x \in A$. Then, since $A \subseteq B$, from the definition of a subset, we deduce $x \in B$, contradicting our assumption. Thus, $x \notin A$. We conclude that if $x \notin B$ and $A \subseteq B$, then $x \notin A$. ■

Exercises 2.2

10(b)

Claim 4 *If $A \subseteq B \cup C$ and $A \cap B = \emptyset$, then $A \subseteq C$.*

Proof. Assume $A \subseteq B \cup C$ and $A \cap B = \emptyset$, and suppose $x \in A$. Then $x \in B$ or $x \in C$, but x can't be in B since $A \cap B = \emptyset$. Thus, $x \in C$, from which we conclude $A \subseteq C$. Therefore, if $A \subseteq B \cup C$ and $A \cap B = \emptyset$, then $A \subseteq C$. ■

12(d)

An example of nonempty sets A, B, and C such that $A \nsubseteq B \cup C$, $B \nsubseteq A \cup C$, and $C \subseteq A \cup B$ is given by $A = \{x, z\}$, $B = \{y, z\}$, and $c = \{z\}$.

14(b)

A counterexample to the claim "If $A \cap C \subseteq B \cap C$, then $A \subseteq B$" is provided by $A = \{1\}$ and $B = \{2\} = C$. Then $A \cap C = \emptyset \subseteq B \cap C$, but $A \nsubseteq B$.

Exercises 2.3

1(h)

$\bigcup_{r\in\mathbb{R}} A_r = [-\pi, \infty)$; $\bigcup_{r\in\mathbb{R}} A_r = [-\pi, 0]$

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Claim 5 *Let $\mathcal{A}$ be a family of sets, and suppose $\emptyset \in \mathcal{A}$. Then* $\bigcap_{A\in\mathcal{A}} A = \emptyset$.

Proof. Since $\emptyset \in \mathcal{A}$, $\bigcap_{A\in\mathcal{A}} A \subseteq \emptyset$, by Theorem 2.9(a). Therefore, $\bigcap_{A\in\mathcal{A}} A = \emptyset$. Thus, if the empty set is a member of a family of sets $\mathcal{A}$, then $\bigcap_{A\in\mathcal{A}} A = \emptyset$. ■

Exercises 2.4

8(e)

Claim 6 *For all natural numbers n, we have the identity*

$$1^3 + 2^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2}\right]^2.$$

Proof. Let $S = \left\{ n \in \mathbb{N} : 1^3 + 2^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2}\right]^2 \right\}$.

(i) Observe that $1^3 = 1 = \left[\frac{1(1+1)}{2}\right]^2$, so $1 \in S$.

(ii) Assume $n \in S$, for some natural number n. Then

$$\begin{aligned} 1^3 + 2^3 + \cdots + n^3 + (n+1)^3 &= \left[1^3 + 2^3 + \cdots + n^3\right] + (n+1)^3 \\ &= \left[\frac{n(n+1)}{2}\right]^2 + (n+1)^3 \\ &= (n+1)^2 \left[\frac{n^2}{4} + n + 1\right] \\ &= (n+1)^2 \left[\frac{n^2 + 4n + 4}{4}\right] \\ &= \left[\frac{(n+1)(n+2)}{2}\right]^2. \end{aligned}$$

Thus, $n + 1 \in S$.

(iii) By the PMI, $S = \mathbb{N}$.

Therefore, for all $n \in \mathbb{N}$, $1^3 + 2^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$. ■