Homework 1

## Exercises 1.1

### 2(f)

| $P$ | $Q$ | $\sim Q$ | $Q \vee \sim Q$ | $P \wedge (Q \vee \sim Q)$ |
|---|---|---|---|---|
| T | T | F | T | T |
| T | F | T | T | T |
| F | T | F | T | F |
| F | F | T | T | F |

### 5(d)

The proposition "Horses have four legs but three quarters do not equal one dollar" is of the form $A \wedge \sim C$. Since $A$ is true and $C$ is false (so $\sim C$ is true), the compound proposition is true.

### 10(b)

A useful denial of the statement "We will win the first game or the second one" is "We will lose the first two games."

### 11(a)

Restoring parentheses to the propositional form $\sim\sim P \vee \sim Q \wedge \sim S$, we first apply each $\sim$ to the smallest proposition following it, which yields $\sim(\sim P) \vee (\sim Q) \wedge (\sim S)$. Next, the symbol $\wedge$ connects the smallest propositions surrounding it, so we have

$$\sim(\sim P) \vee ((\sim Q) \wedge (\sim S)) .$$

If desired, we may omit the inner parentheses without losing clarity: $\sim(\sim P) \vee (\sim Q \wedge \sim S)$ .

## Exercises 1.2

### 4

Use part (a) of Theorem 1.2, which says if $P$ and $Q$ are propositions, then the conditional $P \Rightarrow Q$ is equivalent to $(\sim P) \vee Q$. That is, a conditional sentence is true if either the antecedent is false or the consequent is true.

### (b)

The antecedent "a hexagon has six sides" is true, while the consequent "the moon is made of cheese" is false. So the conditional "If a hexagon has six sides, then the moon is made of cheese" is **false**.

### (d)

The antecedent "$5 < 2$" is false. This truth value alone is sufficient to conclude the conditional "If $5 < 2$, then $10 < 7$" is **true**.

### (f)

Since the consequent "rectangles have four sides" is true, we need not ascertain the truth value of the antecedent "Euclid's birthday was April 2." We can immediately conclude the given conditional is **true**.

**5(d)**

By part (b) Theorem 1.2, the biconditional "$m$ is odd iff $m^2$ is odd" is equivalent to the compound proposition "if $m$ is odd, then $m^2$ is odd, and if $m^2$ is odd, then $m$ is odd." Both of the conditionals in this compound are true (as we will show), so the compound proposition is true. It follows that the biconditional "$m$ is odd iff $m^2$ is odd" is **true**.

Let us prove the two conditional statements.

**Claim 1** *Let $m$ be a positive integer. If $m$ is odd, then $m^2$ is odd.*

**Proof.** Since $m$ is odd, $m$ is equal to $2j+1$ for some nonnegative integer $j$. Thus,

$$\begin{aligned} m^2 &= (2j+1)^2 \\ &= 4j^2+4j+1 \\ &= 2\left(2j^2+2j\right)+1. \end{aligned}$$

The last expression $2\left(2j^2+2j\right)+1$ is of the form $2r+1$ where $r$ is the positive integer $r=2j^2+2j$. In other words, $m^2$ is odd, as claimed. ■

**Claim 2** *Let $m$ be a positive integer. If $m^2$ is odd, then $m$ is odd.*

**Proof.** Since $m^2$ is odd, for some nonnegative integer $k$, we have

$$m^2 = 2k+1,$$

so

$$\begin{aligned} 2k &= m^2-1 \\ &= (m-1)(m+1). \end{aligned}$$

Since 2 divides the left-hand side, it must divide one of the factors on the right-hand side.

- Suppose 2 divides $m-1$. Then $m-1$ is even; hence, $m$ is odd, as claimed.
- If 2 divides $m+1$, then $m+1$ is even, and the same reasoning shows that $m$ is again odd, completing the proof.

■

**8(b)**

"If $n$ is prime, then $n=2$ or $n$ is odd": ($n$ is prime)$\Rightarrow$($n=2$) $\vee$ ($n$ is odd)

**(h)**

"$6 \geq n-3$ only if $n>4$ or $n>10$": $(6 \geq n-3) \Rightarrow (n>4) \vee (n>10)$

## Exercises 1.3

**4(a)**

Completing the proof of Theorem 1.3(b): $\sim(\exists x)\, A(x)$ is equivalent to $(\forall x) \sim A(x)$.
**Proof.** Let $U$ be any universe.

The sentence $\sim(\exists x)\, A(x)$ is true in $U$
iff $(\exists x)\, A(x)$ is false in $U$
iff the truth set for $A(x)$ is empty
iff the truth set for $\sim A(x)$ is $U$
iff $(\forall x) \sim A(x)$ is true in $U$.

■

**(b)**

Let $A(x)$ be an open sentence with variable $x$. Then $\sim A(x)$ is an open sentence with variable $x$, so we may apply part (a) of Theorem 1.3. Thus $\sim (\forall x) \sim A(x)$ is equivalent to $(\exists x) \sim\sim A(x)$, which is equivalent to $(\exists x) A(x)$. Therefore $\sim (\exists x) A(x)$ is equivalent to $\sim\sim (\forall x) \sim A(x)$, which is equivalent to $(\forall x) \sim A(x)$.

**6(a)**

"Every natural number is greater than or equal to 1."

**8(d)**

**Claim 3** *Let $A(x)$ be an open sentence with variable $x$. The proposition $(\exists! x) A(x)$ is equivalent to*

$$(\exists x)[A(x) \wedge (\forall y)(A(y) \Rightarrow x = y)].$$

**Proof.** Let $U$ be any universe. Suppose $(\exists! x) A(x)$ is true in $U$. Then the truth set for $A(x)$ contains exactly one element, $x_0$. Then for every $y$ in $U$, if $A(y)$ then $x_0 = y$. Thus $x_0$ is in the truth set of $A(x) \wedge (\forall y)(A(y) \Rightarrow x = y)$, so $(\exists x)[A(x) \wedge (\forall y)(A(y) \Rightarrow x = y)]$ is true in $U$.

Conversely, suppose $(\exists x)[A(x) \wedge (\forall y)(A(y) \Rightarrow x = y)]$ is true in $U$. Let $x_0$ be an element in the truth set of $A(x) \wedge (\forall y)(A(y) \Rightarrow x = y)$. Then $x_0$ is the only element in the truth set of $A(x)$. Thus $(\exists! x) A(x)$ is true in $U$. ∎

## Exercises 1.4

**5(b)**

Let $x$ and $y$ be integers.

**Claim 4** *If $x$ and $y$ are even, then $xy$ is divisible by 4.*

**Proof.** We use the definition given in the preface: "An integer $x$ is even if and only if there is an integer $k$ such that $x = 2k$."

Since $x$ is even, there is an integer $k_1$ such that $x = 2k_1$. Since $y$ is even, there exists an integer $k_2$ such that $y = 2k_2$. Then

$$\begin{aligned} xy &= (2k_1)(2k_2) \\ &= 4(k_1 k_2). \end{aligned}$$

Now we see that $xy$ is the product of 4 and an integer (namely, the integer $k_1 k_2$). So $xy$ is divisible by 4. ∎

**6(a)**

Let $a$ and $b$ be real numbers.

**Claim 5** $|ab| = |a||b|$.

**Proof.** If $a = 0$ or $b = 0$, then $|ab| = 0 = |a||b|$. Otherwise, there are four cases.

1. If $a > 0$ and $b > 0$, then $|a| = a$ and $|b| = b$. Also, $ab > 0$, so $|ab| = ab = |a||b|$.
2. If $a > 0$ and $b < 0$, then $|a| = a$ and $|b| = -b$. Also, $ab < 0$, so $|ab| = -ab = a(-b) = |a||b|$.
3. If $a < 0$ and $b > 0$, then $|a| = -a$ and $|b| = b$. Also, $ab < 0$, so $|ab| = -ab = (-a)b = |a||b|$.
4. If $a < 0$ and $b < 0$, then $|a| = -a$ and $|b| = -b$. Also, $ab > 0$, so $|ab| = ab = (-a)(-b) = |a||b|$.

In all cases, $|ab| = |a||b|$. ∎

**9(a)**

If $x$ and $y$ are positive real numbers, then

$$\frac{x+y}{2} \geq \sqrt{xy}.$$

First, multiply both sides of the inequality by 2, then square both sides:

$$\begin{aligned} x+y &\geq 2\sqrt{xy} \\ x^2+2xy+y^2 &\geq 4xy. \end{aligned}$$

Subtract $4xy$ from both sides, and factor:

$$\begin{aligned} x^2-2xy+y^2 &\geq 0 \\ (x-y)^2 &\geq 0. \end{aligned}$$

For any real numbers $x$ and $y$, the square of $x-y$ is nonnegative, as we know from algebra. Now we have the steps to write the proof in the forward direction.

**Claim 6** *If $x$ and $y$ are positive real numbers, then*

$$\frac{x+y}{2} \geq \sqrt{xy}.$$

**Proof.** Since $x-y$ is a real number, we have $(x-y)^2 = x^2-2xy+y^2 \geq 0$. Add $4xy$ to both sides: $x^2+2xy+y^2 = (x+y)^2 \geq 4xy$. Next, take square roots on both sides. (Here we use the assumption that $x$ and $y$ are positive. It implies $4xy$ is nonnegative, so $\sqrt{4xy}$ is a real number, and also implies $\sqrt{(x+y)^2} = x+y$.) This gives $x+y \geq 2\sqrt{xy}$. Dividing both sides of the inequality by 2, we obtain $\frac{x+y}{2} \geq \sqrt{xy}$, as claimed. ■

**11(b)**

Grade: **C**.

The claim is correct and the proof strategy is largely correct, but the author mistakenly assumes both $b$ and $c$ are equal to $aq$ *for the same* $q$. This implies $b=c$, which is not assumed in the claim.

An example of a correct proof might read, "For some integer $q_1$, $b=aq_1$, and for some integer $q_2$, $c=aq_2$. Then $b+c = aq_1+aq_2 = a(q_1+q_2)$, so $a$ divides $b+c$."

## Exercises 1.5

**3(c)**

**Claim 7** *If $x^2$ is not divisible by 4, then $x$ is odd.*

**Proof.** We show the contrapositive is true: If $x$ is even, then $x^2$ is divisible by 4.

If $x$ is even, then for some integer $k$, $x=2k$. Then $x^2=(2k)^2=4k^2$, and $k^2$ is an integer, so $x^2$ is divisible by 4. Thus, if $x^2$ is not divisible by 4, then $x$ is odd. ■

**5(a)**

**Claim 8** *If a circle has center $(2,4)$, then the points $(-1,5)$ and $(5,1)$ are not both on the circle.*

**Proof.** Suppose $P(-1,5)$ and $Q(5,1)$ are both on the circle. Then $P$ and $Q$ are equidistant from the center. Using the distance formula, the distance from $P$ to $(2,4)$ is $\sqrt{(2-(-1))^2+(4-5)^2} = \sqrt{9+1} = \sqrt{10}$. On the other hand, the distance from $Q$ to $(2,4)$ is $\sqrt{(2-5)^2+(4-1)^2} = \sqrt{9+9} = \sqrt{18}$. But $\sqrt{10} \neq \sqrt{18}$, so $P$ and $Q$ are not equidistant from $(2,4)$. We conclude that a circle with center $(2,4)$ cannot contain both $(-1,5)$ and $(5,1)$. ■

## 6(a)

**Claim 9** *Let $a$ and $b$ be positive integers. If $a$ divides $b$, then $a \leq b$.*

**Proof.** Suppose $a$ divides $b$ and $a > b$. Then there is a natural number $k$ such that $b = ak$. Since $k$ is a natural number, $k \geq 1$. Thus, $b = ak \geq a \cdot 1 = a$. Thus, $b \geq a$. This contradicts the assumption that $a > b$. Therefore, if $a$ divides $b$, then $a \leq b$. ■

## 11

**Claim 10** *Let $x, y$, and $z$ be real numbers with $0 < x < y < z < 1$. Then at least two of the numbers $x, y$, and $z$ are within $\frac{1}{2}$ unit from one another.*

**Proof.** Assume that the distances from $x$ to $y$ and from $y$ to $z$ are at least $\frac{1}{2}$; that is, assume $|x - y| = y - x \geq \frac{1}{2}$ and $|y - z| = z - y \geq \frac{1}{2}$. The total distance from 0 to 1 is 1. We may express this as

$$(x - 0) + (y - x) + (z - y) + (1 - z) = 1. \tag{1}$$

(Simplify the left-hand side to see this is correct.) But $x - 0 > 0$ (since $x > 0$) and $1 - z > 0$ (since $z < 1$). So

$$\begin{aligned} (x - 0) + (y - x) + (z - y) + (1 - z) &> 0 + \frac{1}{2} + \frac{1}{2} + 0 \\ &= 1. \end{aligned}$$

This contradicts line (1) above. Therefore at least two of the numbers $x, y$ and $z$ are within $\frac{1}{2}$ unit from one another. ■