IMA Journal of Numerical Analysis (1983) **3**, 81–85

A Fourier-series Solution of the Crank–Gupta Equation

H. O. DAHMARDAH AND D. F. MAYERS
Oxford University Computing Laboratory,
19 Parks Road, Oxford OX1 3PL

[Received 27 January 1981 and in revised form 16 August 1982]

The Crank–Gupta equation is one of the simplest examples of a moving boundary problem in a partial differential equation; it has been used as a test problem by a number of different authors. We show how an accurate solution may be obtained by expansion in a Fourier series.

1. Introduction

THE PROBLEM of oxygen diffusion in one dimension is one of the simplest examples of a parabolic equation involving a moving boundary; it has been used as a test case in the development of several new methods for such problems. A recent summary of methods and results is given by Furzeland (1977); the original problem was introduced by Crank & Gupta (1972). In this paper we show how accurate results may be obtained by expanding the solution in a Fourier series.

We wish to determine $C(x, t)$, the solution of

$$\frac{\partial C}{\partial t} = \frac{\partial^2 C}{\partial x^2} - 1 \quad \text{in } 0 \leqslant x \leqslant s(t), \quad t \geqslant 0, \tag{1}$$

with the initial conditions

$$C(x, 0) = \tfrac{1}{2}(1-x)^2, \quad 0 < x < 1, \tag{2}$$

$$s(0) = 1 \tag{3}$$

and the boundary conditions

$$\frac{\partial C}{\partial x} = 0 \quad \text{on } x = 0, \tag{4}$$

$$C = 0 \quad \text{and} \quad \frac{\partial C}{\partial x} = 0 \quad \text{on } x = s(t). \tag{5}$$

2. Solution in Series

If the moving boundary conditions were replaced by a fixed boundary at $x = 1$, with the simple condition $\partial C/\partial x = 0$ on $x = 1$, we could immediately write the classical solution of the problem as the Fourier series in the x variable:

$$C(x, t) = -t + \tfrac{1}{2}A_0 + \sum_{k=1}^{\infty} A_k \cos k\pi x \, e^{-k^2\pi^2 t} \tag{6}$$

0272-4979/83/010081+05 $03.00/0

where the coefficients A_k are the Fourier coefficients of the initial value $C(x, 0)$:

$$C(x,0) = \tfrac{1}{2}(1-x)^2 = \tfrac{1}{2}A_0 + \sum_1^{\infty} A_k \cos k\pi x, \quad 0 < x < 1. \tag{7}$$

A simple calculation shows that

$$A_k = \frac{2}{k^2\pi^2} \quad \text{if } k > 0, \tag{8}$$

$$A_0 = \tfrac{1}{3}.$$

This form of the solution suggests that we represent the solution of the Crank–Gupta problem in the form

$$C(x,t) = -t + \tfrac{1}{2}v_0(t) + \sum_1^{\infty} v_k(t) \cos \frac{k\pi x}{s(t)} \tag{9}$$

where $s(t)$ is the position of the moving boundary at time t, and the functions $v_k(t)$ are to be determined. This function clearly satisfies the boundary condition $\partial C/\partial x = 0$ on $x = 0$ and $x = s(t)$. It will also satisfy the initial condition on $t = 0$ provided that

$$v_k(0) = A_k, \tag{10}$$

and it will satisfy the second condition on the moving boundary, $C(t, s(t)) = 0$, provided that

$$\tfrac{1}{2}V_0 + \sum_1^{\infty} (-1)^k v_k = t \quad \text{for all } t > 0. \tag{11}$$

Finally we must also require that $C(x, t)$ satisfies the differential equation. By direct substitution we find that this requires

$$-1 + \tfrac{1}{2}v_0' + \sum_1^{\infty} v_k' \cos\frac{k\pi x}{s} + \sum_1^{\infty} v_k \frac{k\pi x}{s^2} s' \sin\frac{k\pi x}{s} = -\sum_1^{\infty} v_k \frac{k^2\pi^2}{s^2} \cos\frac{k\pi x}{s} - 1 \tag{12}$$

where v_k' and s' denote derivatives with respect to t.

In this equation we may multiply by $\cos(r\pi x/s)$ and integrate over x, from 0 to s. Using the simple orthogonal properties of the functions $\sin x$ and $\cos x$, we easily obtain the system of ordinary differential equations

$$v_r' = \left(-\frac{r^2\pi^2}{s^2} + \frac{s'}{2s}\right)v_r - \sum_{\substack{k=1\\k\neq r}}^{\infty} (-1)^{k-r+1} \frac{2k^2}{k^2-r^2}\left(\frac{s'}{s}\right)v_k \quad (r > 0) \tag{13}$$

$$v_0' = \sum_{k=1}^{\infty} (-1)^k \frac{2s'}{s} V_k \tag{14}$$

By differentiating Equation (11), which was obtained from the second condition on the moving boundary, we obtain

$$\tfrac{1}{2}v_0' + \sum_1^{\infty} (-1)^k v_k' = 1 \tag{15}$$

which together with (14) gives an equation for s'.

Using the notation

$$D_r = -\frac{r^2\pi^2}{s^2}, \qquad B_{r,k} = (-1)^{k-r}\frac{2k^2}{k^2-r^2} \quad (r \neq k),$$
$$P = \frac{1-\sum_{r=1}^{\infty}(-1)^r D_r v_r}{\sum_{k=1}^{\infty}\left\{v_k\left[\tfrac{3}{2}(-1)^k+\sum_{\substack{r=1\\ r\neq k}}^{\infty}(-1)^r B_{r,k}\right]\right\}}, \tag{16}$$

we thus obtain a system of differential equations:

$$\begin{aligned} s' &= Ps,\\ v_0' &= 2P\sum_{1}^{\infty}(-1)^k v_k,\\ v_r' &= (\tfrac{1}{2}P+D_r)v_r+P\sum_{\substack{k=1\\ k\neq r}}^{\infty} B_{r,k}v_k, \end{aligned} \tag{17}$$

which are to be solved with the initial conditions

$$\begin{aligned} v_0(0) &= \tfrac{1}{3},\\ v_k(0) &= 2/k^2\pi^2, \quad k>0,\\ s(0) &= 1. \end{aligned} \tag{18}$$

This constitutes an infinite system of equations, but if the Fourier series converges rapidly we can solve a finite segment of the equations, by assuming that v_k is negligible for $k > N$. We have obtained useful accuracy with N about 20. Notice that for values of r of this size the coefficient of v_r in the equation for v_r' is negative and fairly large. For example, when $s = 1$, we find that D_{10} is about -1000; moreover the value of s decreases to zero, and so each of the coefficients D_r remains negative and increases in magnitude. The equations are therefore stiff; we have used a readily available version of Gear's algorithm which has given an accurate solution without difficulty, and it appears that the equations are not so stiff as to cause any serious problem.

For small values of t the series converges rather slowly, as can be seen from the values of the coefficients when $t = 0$, given in (18). We therefore begin the numerical integration from a positive t_0, using as a starting condition the approximation given by Crank & Gupta (1972)

$$s(t_0) = 1, \qquad v_0(t_0) = \tfrac{1}{3}, \qquad v_r(t_0) = v_r(t_0)\,e^{-r^2\pi^2 t_0}.$$

This gives very good accuracy for t_0 about 0·01, where the Fourier series now converges rapidly.

3. Critical Value of t

As t increases, the boundary moves to the left, so that s decreases. The boundary reaches $s = 0$ at a time t^* shortly before $t = 0{\cdot}2$, and at this point the differential equations become singular. As there is some interest in investigating the behaviour of the solution near this point, we must make a change of variables. We first write

$$y = -\log s \tag{19}$$

so that as we approach the singular point, y tends to infinity. It is then convenient to rewrite the equations in terms of y as independent variable; at the same time we write

$$w_r = v_r/s^2, \qquad \phi = Ps^2 \tag{20}$$

and obtain the new system of equations

$$\begin{aligned} \frac{dt}{dy} &= -e^{-2y}/\phi, \\ \frac{dw_r}{dy} &= \left\{\frac{3}{2} - \frac{D_r s^2}{\phi}\right\} w_r - \sum B_{r,k} w_k. \end{aligned} \tag{21}$$

We can then continue the use of Gear's algorithm to solve the original Equations (17) up to a time about $t = 0{\cdot}19$. We then compute the values of y and w_r at this point, and use them as starting condition for the solution of the transformed system (21), again by Gear's algorithm. As y increases we find that ϕ and w_r tend quite rapidly to constant limits, and t tends to the critical value t^* at which the solution becomes singular.

TABLE 1

t	$C(0, t)$	$s(t)$	
		This work	Hansen & Hougaard (1974)
0·01	0·387162	1·000000	1·00000
0·02	0·340423	0·999999	1·00000
0·03	0·304559	0·999911	
0·04	0·274324	0·999180	0·99918
0·05	0·247687	0·996793	0·99679
0·10	0·143177	0·935018	0·93501
0.12	0·109129	0·879171	0·87916
0·14	0·077850	0·798944	0·79891
0·16	0·048823	0·683449	0·68337
0·18	0·021781	0·501329	0·50109
0·19	0·009021	0·346000	0·34537
0·195	0·002883	0·208453	0·20652
0·196	0·001685	0·163125	0·16266
0·197	0·000491	0·093054	0·09175
0·1972	0·000274	0·069437	0·06708
0·1974	0·000040	0·027818	
0·19743	0·000005	0·010789	
0·197434	0·000001	0·004838	

4. Numerical Results

We have performed a number of calculations, varying the parameters involved in the numerical algorithm, and the number of terms, N, in the Fourier series. The results show very good consistency, and the values given in the table are believed to be correct to the number of figures quoted.

The table gives the values of C at $x = 0$, and the position, s, of the moving boundary, for various values of t. Our results are in generally good agreement with those of Hansen & Hougaard (1974). The values of $C(0, t)$ agree up to $t = 0{\cdot}195$, which is as far as the latter tabulation extends. As shown in the table, there is a small discrepancy in the position of the boundary, which increases as the critical point is reached. Our calculations give the critical value as $t^* = 0{\cdot}197434$.

5. Conclusion

The method of expanding in Fourier series has given very accurate results in this particular problem. This is a rather favourable case, owing to the simple conditions specified on the moving boundary. It is not yet clear how satisfactory the technique may be when applied to more general moving-boundary problems.

References

CRANK, J. & GUPTA, R. S. 1972 A moving boundary problem arising from the diffusion of oxygen in absorbing tissue. *J. Inst. Maths Applics* **10,** 19–33.

FURZELAND, R. M. 1977 A survey of the formulation and solution of free and moving boundary (Stefan) problems. Brunel University, TR/76.

HANSEN, E. & HOUGAARD, P. 1974 On a moving boundary problem from biomechanics. *J. Inst. Maths Applics* **13,** 385–398.