The role of the Crank–Gupta model in the theory of free and moving boundary problems

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Dedicated to Professor J. Crank on the occasion of his 80th birthday

A very brief review is given of the striking way in which the Crank–Gupta model has enhanced our understanding of the well-posedness of free and moving boundary problems.

1. Introduction

In [4, 5], Crank and Gupta introduced what is now known as the Crank–Gupta, or oxygen-consumption problem for the concentration c of oxygen in a tissue in which it is absorbed volumetrically at a prescribed constant rate. In one dimension, c satisfies the dimensionless equation

$$\frac{\partial c}{\partial t} = \frac{\partial^2 c}{\partial x^2} - 1 \tag{1}$$

in those parts of the tissue where oxygen is present ($c > 0$), together with suitable initial and boundary conditions. This represents a mass balance for the oxygen but, crucially, in the medical context that motivated [4], c falls to zero in some parts of the tissue that are to be determined as part of the problem. It is the condition at the free boundary between regions where $c > 0$ and $c = 0$ that lies at the heart of the Crank–Gupta model and a careful mass balance shows that if $x = s(t)$ is such a free boundary,

$$c = \frac{\partial c}{\partial x} = 0 \quad \text{at } x = s(t). \tag{2}$$

We will not describe any mathematical details of the particular Crank–Gupta problem as posed in 1972 here, except to say that, as explained in [7] and to be discussed shortly, it has a perfectly well-behaved solution for which analytical bounds can be obtained and for which accurate numerical algorithms are available. Rather than going into such technical details, this review will concentrate on the implications of the combination of the field equation (1) and the free boundary condition (2) for the whole subject of free and moving boundary problems, which has burgeoned over the past three decades.

When (1)–(2) are being solved without any specification of the sign of c, we will say we are solving a (CG) problem. However, in order to obtain a physically acceptable solution to a (CG) problem, we need to exclude regions where $c < 0$ and we cannot over-emphasise that such physically acceptable solutions cannot be obtained by chopping off, or "truncating" those regions where $c < 0$. Such truncations were used in the early days of mathematical biology but they can only be justified when they are carefully incorporated into certain numerical schemes; we will return to this point in the conclusion.

Since the positivity or otherwise of the solutions of (CG) is so crucial to the theme of this article, we will henceforth make a clear distinction between situations where

(i) $c \geqslant 0$, the constrained (CG) problem (CCG), which can be shown to be equivalent to solving

$$\frac{\partial c}{\partial t} = \frac{\partial^2 c}{\partial x^2} - H(c), \tag{3}$$

where H is the Heaviside function;

(ii) c can have either sign, the unconstrained (CG) problem (UCG).[1] Clearly (CCG) is likely to have more components in its free boundary than (UCG).

The motivation for the all-important distinction between (CCG) and (UCG) lies in the relationship between (CG) and the grandfather of all free boundary problems, the Stefan problem[2] for the temperature u in a stationary heat-conducting medium, which satisfies the field equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}; \tag{4}$$

if there is a change of phase say from solid to liquid at zero temperature at $x = s(t)$, and this phase change is associated with a unit latent heat, then

$$u = 0, \qquad \left[\frac{\partial u}{\partial x}\right]_{\text{solid}}^{\text{liquid}} = -\dot{s} \quad \text{at } x = s(t), \tag{5}$$

where $[\cdot]$ denotes the jump across the phase boundary and we denote (4)–(5) by (S). If there is no superheating or supercooling, so that $u > 0$ in the liquid and $u < 0$ in the solid, then this jump is

$$\left(\lim_{u\downarrow 0} - \lim_{u\uparrow 0}\right)\left(\frac{\partial u}{\partial x}\right).$$

[1] This terminology was introduced in [8].

[2] A version of the (CG) problem is also the fundamental model for the Black–Scholes theory of American option pricing, and its relationship to the Stefan problem is mentioned in [27].

At one level there seems to be a trivial relationship between (CG) and (S). As noticed by Schatz [24],[3] if we formally write

$$u = \frac{\partial c}{\partial t} \tag{6}$$

and differentiate (1)–(2) we get (4) and

$$u = 0, \qquad \frac{\partial u}{\partial x} = \frac{\partial^2 c}{\partial x \partial t} = -\dot{s} \quad \text{at } x = s(t) \tag{7}$$

respectively, since

$$\frac{\partial^2 c}{\partial x^2}\dot{s} + \frac{\partial^2 c}{\partial x \partial t} = 0 \quad \text{and} \quad \frac{\partial^2 c}{\partial x^2} = 1 \quad \text{at } x = s(t).$$

Hence u can be identified as the temperature in a *one-phase* Stefan problem (1PS) in which $u_{\text{solid}} \equiv 0$. However, we note that whenever $\partial c/\partial t$ is negative in (CG), as is the case in [4], this identification implies that the liquid is supercooled.

Schatz' motivation for relating (1)–(2) to (4)–(5) was entirely mathematical. Because the free boundary velocity appears explicitly in (S), it is much easier to prove results about the classical solution than it is for (CG) where s only appears implicitly. Indeed, the whole existence and uniqueness theory for (S) given in [21] depends on this fact. However, there is an irony here because, as we shall see, (CG) is really much nicer than (S) from many mathematical points of view.

Although the transformation (6) looks innocuous enough, care needs to be exercised in many situations, for example concerning invertibility when $\dot{s}$ is not of one sign. Also consider, for example, the initial/boundary value problem of [4], namely (1)–(2) in $0 < x < s(t)$ with

$$\frac{\partial c}{\partial x}(0,t) = 0, \qquad c(x,0) = c_0(x) = \frac{1}{2}(1-x)^2, \quad 0 < x < 1. \tag{8}$$

If we formally write (6), we obtain (4)–(5) with

$$\frac{\partial u}{\partial x}(0,t) = 0, \qquad u(x,0) = \frac{\mathrm{d}^2 c_0}{\mathrm{d}x^2} - 1 = 0 \quad \text{for } 0 < x < 1, \tag{9}$$

for which $u \equiv 0$ would be the obvious solution were it not for an impulsive thermal source at $x = t = 0$. Other interesting possibilities arise when c_0 in (8) is such that

$$\left.\frac{\mathrm{d}c_0}{\mathrm{d}x}\right|_{x=1} = 0.$$

[3] In fact the (CG) problem might well be called the Schatz problem if it not were not for the idiosyncratic spelling of Stefan in [24]. Schatz' work was much more in the context of optimal stopping problems than in physical applied mathematics.

However, these pitfalls are minor compared to those that are encountered when well-posedness questions are studied, and we now devote some paragraphs to this aspect.

Concerning (CCG), numerical and analytical evidence points to the fact that the solution usually exists, and is unique and well-behaved for all time, with c vanishing everywhere for all sufficiently large time. This is a manifestation of the fact that (3) can be written as a parabolic variational inequality [7], from which many mathematical properties can be inferred and discretisations suggested. It is not in the spirit of this review to go into technicalities here (see [8] for details and examples of exceptional behaviour) but the above remark applies to general (CCG) problems in which the unit absorption is replaced by an arbitrary smooth function of x and t, positive or negative.

In complete contrast, the global behaviour of solutions of (UCG) and (S) is a very delicate matter. Whereas the methods of [21] can be used to demonstrate the well-posedness of a large class of Stefan problems when neither superheating nor supercooling occurs, in all other cases it is common to find "blow-up" in which $\dot{s}$ tends to infinity in finite time. In particular, this always happens for the data

$$\frac{\partial u}{\partial x}(0,t) = 0, \qquad u(x,0) = u_0(x), \quad 0 < x < 1, \tag{10a}$$

as long as

$$\int_0^1 \big(1 + u_0(x)\big)\,\mathrm{d}x < 0, \tag{10b}$$

but this is a sufficient, not a necessary condition for blow-up. By integrating the Stefan problem with respect to x, this inequality can be seen to have a simple physical interpretation as saying that the initial thermal energy in the material is too large and negative to be absorbed by the total latent heat that is available. The blow-up phenomenon was first pointed out in [25] and has since been intensively investigated [9, 10]; it appears paradoxical because the solution of (CCG) exists for all $t > 0$, whether or not (10b) holds, i.e., whether or not

$$\int_0^1 \frac{\mathrm{d}^2 c_0}{\mathrm{d}x^2}\,\mathrm{d}x < 0.$$

We will return to this in section 2.

There is one other intriguing connection between (CG) and (S) and this concerns the infamous "mushy region". As pointed out in [1], the "volumetrically heated" *two phase* Stefan problem in which (4) is replaced by

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + 1 \tag{11}$$

with (5) and, say,

$$\frac{\partial u}{\partial x}(0,t) = 0, \qquad u = u_0(x) < 0 \quad \text{at } t = 0 \tag{12}$$

can lead to the existence of a region of superheated solid when we try to solve a "classical problem" in which (11) holds on either side of a free boundary where (5) holds. However, if (11), (5) and (12) are solved by the enthalpy method [7] in which (11) and (5) are replaced by

$$\frac{\partial h}{\partial t} = \frac{\partial^2}{\partial x^2}\{u(h)\} + 1 \tag{13}$$

with

$$u = \begin{cases} h-1, & h > 1, \\ 0, & 1 > h > 0, \\ h, & 0 > h, \end{cases} \tag{14}$$

taken in the sense of distributions, we find that (5), which would have been the anticipated "Rankine–Hugoniot" condition, is lost. Instead, the solid domain abuts not a liquid one but a mushy region in which $u \equiv 0$ and $0 < h < 1$ and it can be proved that at the interface between the two

$$u = \frac{\partial u}{\partial x} = 0.$$

Hence, identifying h with $(-c)$, the behaviour in the solid in the vicinity of the solid/mush interface is precisely that of a (CG) problem with oxygen being removed volumetrically. Thus there is no tendency to blow-up as there might have been in the absence of mush.

A helpful way of thinking about good and bad Stefan problems is to observe that it is only when $h(u)$ *cannot* be defined as a monotone increasing function that the weak formulation (13) will possibly lead to a backwards heat equation, and $h(u)$ can only be defined to be monotone as in (14) in cases where both superheating and supercooling are absent.

An even more subtle and dramatic link between (1PS) and (CG) is revealed when we consider their multidimensional generalisations, replacing $\partial^2/\partial x^2$ by ∇^2, $\partial/\partial x$ by the normal derivative $\partial/\partial n$ and $\dot{s}$ by V_n, the normal velocity of the free boundary. Whereas (CG) is, at least in its weak form as a parabolic variational inequality, still as well-behaved as in one space dimension except in very rare cases, the superheated and/or supercooled Stefan problem exhibits not just blow-up possibilities for which results like (10) are available, but also genuine ill-posedness. This is presaged by the famous "Mullins–Sekerka" linear stability analysis for a nearly planar interface; any corrugation in such an interface will grow at a rate that increases without bound as the wavelength decreases. The stability analysis leading to this result is quite complicated but fortunately it can be illustrated with a simple non-trivial limit of the one-phase two-space-dimensional Stefan problem, called the Hele–Shaw problem [22]. In this problem, to which we will refer as (HS), the pressure p in a liquid in a Hele–Shaw cell satisfies

$$\nabla^2 p = 0 \tag{15}$$

with

$$p = 0, \qquad \frac{\partial p}{\partial n} = -V_n \tag{16}$$

at the free boundary, conveniently written $t = \omega(x, y)$; hence (HS) corresponds to (1PS) in a liquid with zero specific heat (i.e., the coefficient of $\partial u/\partial t$ in (4) vanishes) with negative pressures relative to ambient, corresponding to suction, being interpreted as "supercooling". The stability analysis referred to above shows that a fluid region

$$x < Vt + \varepsilon e^{\sigma t} \cos ny, \quad n > 0, \tag{17}$$

evolves so that $\sigma = -nV$ which demonstrates ill-posedness when $V < 0$. However, a much more important attribute of (HS) compared to (1PS) is that it is susceptible to complex analysis and many very helpful explicit solutions can be written down. These demonstrate that blow-up is much more likely to occur in two dimensions via a singularity in the geometry of the boundary such as a cusp, than via the infinite-speed "Sherman" blow-up in which a tangent plane always exists.

We can now ask about the connection between (HS) and the "quasi-steady" two-dimensional (CG) problem in which $\partial c/\partial t$ is dropped, viz

$$\nabla^2 c = 1, \qquad c = \frac{\partial c}{\partial n} = 0 \quad \text{on } t = \omega(x, y), \tag{18}$$

in which t only appears as a parameter. It is tempting to conjecture that if c satisfies (18) then $\partial c/\partial t$ can be interpreted as the pressure in a Hele–Shaw cell, but the relationship between the generic behaviour of solutions of (CG) and (HS) is far less well-understood than that between (CG) and (SP) in one dimension (the one-dimensional Hele–Shaw problem is, of course, trivial and no blow-up can occur). However, (18) has recently been shown to be a very effective tool for revealing some dramatic and unexpected regularity properties of the free boundary of solutions of (HS) in the unstable (suction) case. This will be discussed further in section 3 and this leads, in section 4, to another important consequence of the relation between the full two-dimensional (CG) and (1PS); this is that comparison results can be made between Stefan problems *with different specific heats*. Hence some of the welter of information that is available in the zero-specific heat case can be exploited to analyse the hitherto unexplored area of supercooled two-dimensional Stefan problems.

It is amusing, and also sometimes helpful mathematically to note that there is a free boundary problem intermediate between (18) and (HS), namely

$$\nabla^2 p = f(\boldsymbol{x}, t), \qquad p = 0, \quad \frac{\partial p}{\partial n} = -V_n \quad \text{on } t = \omega, \tag{19}$$

which models Hele–Shaw "squeeze film" flows in a cell whose geometry is characterised by $f(\boldsymbol{x}, t)$. This problem is such that if we set $u = p + fc$, where c satisfies (18), then u, rather than p, satisfies (HS). Hence (18) can be used as a stepping stone to relate standard (HS) problems to those with a distributed driving mechanism f.

In the conclusion we will make very brief mention of some possible generalisations of these very fruitful analogies. They are all based on the central theme that (CG) is, perhaps, the nicest free boundary problem ever posed from the mathematical point of view. Its apparently close relationship to some not-so-nice problems has meant that it has played and continues to play a pivotal role in the theories of well-posedness of free boundary problems.

2. One-dimensional problems: (CG) and (1PS) models

The identification between (CG) and (1PS) began by assuming the existence of a "classical" solution $c(x,t)$ of (CG). As proposed in [8], the formal approach (6) can be made much more rigorous if we begin with the solution u of (4)–(5), say with initial and boundary data (12). The existence of u is assured from the results of [21], so we can now define

$$c(x,t) = \int_{s(t)}^{x} \mathrm{d}\xi \int_{s(t)}^{\xi} \big(u(\eta,t)+1\big)\,\mathrm{d}\eta. \tag{20}$$

Then, known smoothness properties allow us to manipulate this expression to show that c satisfies (UCG) with data

$$\frac{\partial c}{\partial x}(0,t) = 0, \qquad c(x,0) = c_0(x) \quad \text{where } \frac{\mathrm{d}^2 c_0}{\mathrm{d}x^2} - 1 = u_0(x) \tag{21}$$

and similar manipulations are possible for more general initial and boundary data. Hence there is a more-or-less one-to-one correspondence between (UCG) and (1PS): the blow-up behaviour that is known to occur in the latter when supercooling is present will also occur in the former, and thus the elaborate blow-up categorisation of [10] applies to (UCG). We will not describe all the possibilities here but we remark that

(i) (10b) is only a very crude sufficiency condition for blow-up. The precise distribution of $u_0(x)$ is important, as has been discussed in [18] by the introduction of a "weighting function" into (10b). Even though (10b) can be violated, it is vital that $u_0 < -1$ at some points of the initial interval;

(ii) the behaviour near blow-up is far from obvious and published asymptotic results are only just available [11];

(iii) in most cases, blow-up is such that no temporal continuation is possible thereafter, but exceptional cases of so-called "non-essential blow-up" can occur [8].

For our purposes the most important attribute of blow-up is its close association with the existence of a negativity set of c as defined by (6) or (20). The approach of the free boundary $x = s(t)$ to this negativity set can be seen to engender blow-up because, conversely, if blow-up does not occur, then $\partial u/\partial t$ and hence c must ultimately be non-negative [8].

None of this blow-up theory is relevant to (CCG); far from it. The constraint of forbidding negative values of c, and thereby introducing new free boundaries, exerts a remarkable tranquilising effect on the model and endows it with the smoothness properties mentioned in the introduction that enable it in turn to be turned into a parabolic variational inequality.

This state of affairs, and in particular the relationship between blow-up and the negativity set of c, suggests the intriguing possibility of using (CCG) to "regularise" the badly-behaved (1PS). For, suppose we have a supercooled situation, with the potential for blow-up, and suppose also that the corresponding c defined by (20) is positive for some short initial time interval before it develops an interior zero. We could now hypothesise a "mathematical nucleation" event in which we introduce a new region of zero-temperature that spreads out from this zero, thereby introducing two new components into the free boundary. Moreover, we could continue to do this whenever c threatened to vanish and, because we would be associating (1PS) with (CCG), (1PS) would have become equivalent to a well-posed parabolic variational inequality.

As described in [8], such a mathematical nucleation and growth procedure is vague because it neither encompasses all the ways in which a negativity set of c could be born, nor is it in any sense a unique regularisation. For example, we could "pin" c to be zero at a point rather than nucleate new solid as soon as a zero occurs, and all these possibilities are enumerated in [8]. Also that paper shows how to formulate a "least nucleation" principle in the hope that one day some physical relevance might be ascribed to the regularisation.

The main point to emerge from this discussion is that (CCG) is the starting point for a method for smoothing what would otherwise be a commonly occurring ill-posed mathematical model. As such it is to be compared with the idea of introducing a mushy region (as in (13)–(14)) or of introducing a smoothing term, such as might model chemical kinetics, into the free boundary conditions. The mathematical nucleation and growth philosophy has been applied in more space dimensions, but only with radial symmetry; without this last assumption, things become much more complicated as we will discover in the next section.

3. Two-dimensional problems: (CG) and (HS) models

Perhaps the best understood, and certainly one of the commonest two-dimensional free boundary problems is (HS) (15)–(16). As can be seen by explicit calculations, this problem is innocuous enough in one dimension, but its extreme sensitivity to two-dimensional disturbances has been mentioned in the introduction.

We have already pointed out that the links between (HS) and the two-dimensional "zero specific heat" (CG) (henceforth referred to as (ZSCG)) can, like (6), be related to the ideas of Schatz. However, more fundamentally, they are applications of the famous "Baiocchi transform" introduced for "dam-type" free boundary problems. Such problems are themselves generalisations of (HS) to account for the presence of a body force and steady states of dam problems correspond to travelling wave solutions of (HS).

For the "well-posed" (HS) where p is positive and the free boundary is being blown outwards, the Baiocchi transform was used in [6] to achieve a reduction to a one-parameter family of elliptic variational inequalities and hence to deduce many well-posedness results. Roughly speaking, the idea is to relate c in (ZSCG) to p in (HS) by

$$c = \int_0^t p(x, y, \tau)\,\mathrm{d}\tau \tag{22}$$

in Ω_0, the region initially occupied by fluid, and by

$$c = \int_{\omega(x,y)}^{t} p(x,y,\tau)\,\mathrm{d}\tau \tag{23}$$

in Ω/Ω_0, where $\partial\Omega$ is the free boundary for $t > 0$, given by $t = \omega(x,y)$; this gives the now-fashionable variational formulation for

$$\nabla^2 c = \chi(x)H(c) \tag{24a}$$

which is equivalent to the linear complementarity problem

$$c \geqslant 0, \qquad \chi - \nabla^2 c \geqslant 0, \quad (\chi - \nabla^2 c)c = 0. \tag{24b}$$

Here χ is the characterstic function of the fluid region, and it is inherent in the distributional interpretation of (24) that $c = \partial c/\partial n = 0$ on $\partial\Omega$ as in (18).

As pointed out in [18], (22) is unnecessary when we try formally to use this type of transformation in the ill-posed "suction" (HS) problem. There, when we write (23), we retrieve (24) but, in contrast to the blow-up problem, we cannot even determine $c(x,y,0)$ without solving the ill-posed Cauchy problem

$$\nabla^2 c = 1 \quad \text{in } \Omega_0 \quad \text{with} \quad c = \frac{\partial c}{\partial n} = 0 \quad \text{on } \partial\Omega_0. \tag{25}$$

Another obvious hazard is that (24) only has a chance of being true at points that are crossed by the free boundary, which, in view of the propensity for blow-up, may not form a very large set. Nonetheless, when it does make sense, (23) has many consequences; for example it can be shown that if we put $z = x + \mathrm{i}y$ and consider the function

$$g(z,t) = \bar{z} - 2\left(\frac{\partial c}{\partial x} - \mathrm{i}\frac{\partial c}{\partial y}\right) \quad (\bar{z} = x - \mathrm{i}y), \tag{26}$$

we find that it is analytic ($c - (1/4)z\bar{z}$ is harmonic) and is the Schwarz function of $\partial\Omega$, i.e., the function such that

$$\bar{z} = g(z,t) \tag{27}$$

is $t = \omega(x,y)$. Information about blow-up can now be inferred from a knowledge of the singularities of g. In fact, since

$$c(x,y,t) = c(x,y,0) + \int_0^t p(x,y,\tau)\,\mathrm{d}\tau,$$

it is easy to see from (26) that

$$g(z,t) = g(z,0) - 2\int_0^t \frac{\mathrm{d}w}{\mathrm{d}z}(z,\tau)\,\mathrm{d}\tau, \tag{28}$$

where the fluid velocity $-\nabla p$ is the conjugate of the analytic function $\mathrm{d}w/\mathrm{d}z$. Hence the singularities of g in the fluid region are fixed in time since $\mathrm{d}w/\mathrm{d}z$ is analytic there, and they provide an immovable obstacle to the inward motion of the boundary. Also, as in the one-dimensional cases of the previous section, it is easy to see that any negativity set of c must increase in time (since $p = \partial c/\partial t < 0$) and trigger a singularity when it meets the contracting boundary $\partial\Omega$. Another insight is gained from the fact that sequences of one-parameter variational inequalities of the form (24) can have singularities in the form of cusps that are locally $y \propto x^{(4n+1)/2}$, $n = 1, 2, 3 \ldots$, [23], and hence, if a contracting free boundary is lucky enough to encounter such a singularity, it can be analytically continued thereafter. However, a more powerful application of this idea is in connection with the likelihood that blow-up in (HS) occurs in accordance with the two-dimensional generalisation of the "energy" criterion (10b). In [18] it was found that blow-up could be proved to occur even when we violate the condition that $c_0 < -1$, which, we recall, was a vital necessary condition for one-dimensional blow-up of (UCG). The key idea was to consider initial data for which c_0 has certain judiciously chosen singularities, and it can even be generalised to three-dimensional versions of (HS).

The fact that genuine two-dimensional "blow-up" can occur has been known for many years because of the multitude of explicit solutions that can be written down for (HS) by conformally mapping the flow domain onto a half plane or unit circle and guessing the right mapping function to satisfy the free boundary conditions [12, 19, 20].

All the above evidence points to the fact that (HS) with suction and hence (ZSCG) are all ill-posed and we might reasonably think that the worse the initial data, the less likely would be the existence of any solutions at all. Indeed the appearance of the ill-posed Cauchy problem (25) suggests that $\partial\Omega_0$ should be analytic if we are to even get started and, in fact, it can be proved by generalised Cauchy–Kowalevskaya methods that the analyticity of $\partial\Omega_0$ does indeed permit the existence of a local-in-time solution. It is all the more surprising therefore that there is one wildly nonanalytic initial value problem where the initial values for (ZSCG) do exist, and that is when there is a corner in $\partial\Omega_0$.[4] We cannot catalogue here the astonishing variety of flows that are possible near such a corner but [15] have shown that waiting times and discontinuities in the corner angle as a function of time can both occur. The proof of these results is greatly facilitated by the convenience of the formulation (24) over (15)–(16) for deriving the existence and uniqueness of weak solutions and also for comparison theorems.

Given the ubiquity and practical importance of (HS) in science and industry, it is clear that the indicators provided by (ZSCG) will continue to provide more and more insight into the myriad free boundary problems modelled by (HS).

[4] In fact many other nonanalytic Cauchy problems appear to have local solutions in which the free boundary is nonanalytic [J. R. King, private communication].

4. Two-dimensional problems: (CG) and (S)

When u satisfies a supercooled (1SP) we can, as usual define

$$c = \int_{\omega}^{t} u(x, y, \tau)\,\mathrm{d}\tau$$

to obtain the (UCG)

$$\nabla^2 c = \frac{\partial c}{\partial t} + 1, \qquad c = \frac{\partial c}{\partial n} = 0 \quad \text{on } \partial\Omega \tag{29}$$

with initial data that must again be found by solving an ill-posed Cauchy problem. Exactly as in sections 2 and 3 we can see that, whenever c has a negativity set, blow-up eventually occurs. However, the question of the morphology of $\partial\Omega$ at blow-up has long been a thorn in the side of theoreticians: in [17] it was conjectured that the cusps that occur in (HS) might still exist in cases when specific heat is non-zero but this was the only statement in the literature until very recently.

Although the panoply of complex variable theory is no longer available to help us study (29), the idea of comparison theorems does carry over from (HS). In fact it is relatively easy to construct comparison theorems for (29) when $\partial c/\partial t$ is replaced by $\alpha \partial c/\partial t$, α being a positive specific heat, but only inasmuch as (29) applies to well-posed, unsupercooled problems. Then the guaranteed positivity of c allows one to show that the expanding free boundary for a larger α lies inside that for a smaller α and, moreover, the dependence of $\partial\Omega$ on α is continuous [16]. However, as also shown in [16], this argument can be applied locally in the supercooled case because, even then, $c > 0$ sufficiently close to $\partial\Omega$ while the solution continues to exist. Hence we can assert that, as $\alpha \to 0$, $\partial\Omega$ approaches that of the Hele–Shaw problem. Thus the free boundary of the Stefan problem can approach arbitrarily close to a cusp, and indeed tend to a cusp as $t \to \infty$ in certain cases.

This picture has been further refined, although not completely rigorously, in [26]. There, rigorous matched expansion estimates are derived to prove the existence of cuspidal behaviour in supercooled (S), but there is as yet no regularity theory for the earlier behaviour of $\partial\Omega$ from which to start the rigorous discussion. By now not surprisingly, the asymptotic estimates in [26] are all developed for (29), for the very same reasons that pertained at the end of section 3. Two noteworthy features emerge:

(i) As hinted in [18], cusp formation (rather than "Sherman" blow-up) is possible in two-dimensional supercooled (S) for arbitrarily small initial undercooling (i.e., u_0 does not have to be anywhere near the critical value -1, as in (10b), just as long as it is suitably negative sufficiently near $\partial\Omega_0$).

(ii) The method is capable of generalisation to three dimensions.

5. Conclusion

A monograph would be required even to list all the known implications of (CG) for other free boundary problems. Here we will simply cite three other aspects not yet mentioned:

1. We have not entered into any discussion of the numerical aspects of (CG). Clearly the variational formulation suggests a variety of finite element schemes, at least for well-posed problems, and other possibilities are discussed in [3]: one of these is the alternating phase truncation method [2], where, as mentioned in the introduction, negative values of c are automatically discarded.

2. We have said almost nothing about two-phase problems, but a two-phase (CG) model has in fact been proposed in [8]. It is, from (20),

$$\frac{\partial c}{\partial t} = \frac{\partial^2 c}{\partial x^2} - 1 + \frac{\partial u}{\partial x}(s+0,t)(s-x),$$

   where u is the temperature of the two-phase material in $x > s$. Although the presence of the last term is inconvenient, to say the least, the negativity of u, i.e., the absence of any superheating, enhances the consumption rate for c.

3. The idea of smoothing "rough" free boundary problems such as (S) by relating them to smooth ones such as (CG) has recently been proposed for generalised Hele–Shaw problems [13] and for problems in slow viscous flow where the field equation is biharmonic [14].

In summary, as the theory of free boundary problems becomes more and more elaborate and widely applicable, the crucial role played by really well-behaved paradigms, of which (CG) is the progenitor, becomes more and more apparent.

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References

[1] D. R. Atthey, A finite difference scheme for melting problems, J. Inst. Math. Appl. 13 (1974) 353–366.

[2] A. E. Berger, M. Ciment and J. C. W. Rogers, SIAM J. Numer Anal. 12 (1975) 646–672.

[3] J. Crank, *Free and Moving Boundary Problems* (Oxford University Press, Oxford, 1984).

[4] J. Crank and R. S. Gupta, A moving boundary problem arising in the diffusion of oxygen in absorbing tissue, J. Inst. Math. Appl. 10 (1972) 19–44.

[5] J. Crank and R. S. Gupta, A method for solving moving boundary problems in heat-flow using cubic splines or polynomials, J. Inst. Math. Appl. 10 (1972) 296–304.
[6] C. M. Elliott and V. Janovsky, A variational inequality approach to Hele–Shaw flow with a moving boundary, Proc. Roy. Soc. Edinburgh Sect. A 88 (1981) 93–107.
[7] C. M. Elliott and J. R. Ockendon, Weak and variational methods for moving boundary problems, Pitman Res. Notes Math. 59 (1982).
[8] A. Fasano, S. D. Howison, M. Primicerio and J. R. Ockendon, Some remarks on the regularisation of supercooled one-phase Stefan problems in one-dimension, Quart. Appl. Math. 48 (1990) 153–168.
[9] A. Fasano and M. Primicerio, A critical case for the solvability of Stefan-like problems, Math. Methods Appl. Sci. 5 (1983) 84–96.
[10] A. Fasano, M. Primicerio and A. A. Lacey, New results on some classical parabolic free-boundary problems, Quart. Appl. Math. 38 (1981) 439–460.
[11] M. Herrero and J. J.-L. Velazquez, Singularity formation in the one-dimensional supercooled Stefan problem, European J. Appl. Math. 7, to appear.
[12] S. D. Howison and J. R. King, Explicit solutions to six free-boundary problems in fluid flow and diffusion, IMA J. Appl. Math. 42 (1989) 155–175.
[13] J. R. King, Development of singularities in some moving boundary problems, European J. Appl. Math. 6 (1995) 491–508.
[14] J. R. King, private communication.
[15] J. R. King, A. A. Lacey and J. J.-L. Velazquez, Persistence of corners in free-boundaries in Hele–Shaw flow, European J. Appl. Math. 6 (1995) 455–490.
[16] A. A. Lacey, Bounds on solutions of one-phase Stefan problems, European J. Appl. Math. 6 (1995) 509–516.
[17] A. A. Lacey, S. D. Howison and J. R. Ockendon, Irregular morphologies in unstable Hele–Shaw free boundary problems, Quart J. Mech. Appl. Math. 43 (1990) 387–405.
[18] A. A. Lacey and J. R. Ockendon, Ill-posed free boundary problems, Control and Cybernet. 14 (1985) 275–296.
[19] P. Ya. Polubarinova-Kochina, *Theory of Groundwater Movement* (Princeton University Press, 1962).
[20] S. Richardson, Hele–Shaw flows with a free boundary produced by the injection of fluid into a narrow channel, J. Fluid Mech. 56 (1972) 609–618.
[21] L. I. Rubinstein, *The Stefan Problem*, Transl. Math. Monographs 27 (Amer. Math. Soc., Providence, RI, 1971).
[22] P. G. Saffman and G. I. Taylor, The penetration of fluid into a porous medium or Hele–Shaw cell, Proc. Roy. Soc. London Ser. A 254 (1958) 312–329.
[23] D. G. Schaeffer, Some examples of singularities in a free boundary, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 4 (1977) 133–144.
[24] A. J. Schatz, J. Math. Anal. Appl. 28 (1969) 569–580.
[25] B. Sherman, A general one-phase Stefan problem, Quart. Appl. Math. 28 (1970) 377–382.
[26] J. J.-L. Velazquez, Cusp formation for the undercooled Stefan problem in two and three dimensions, European J. Appl. Math. 7, to appear.
[27] P. Wilmott, S. D. Howison and J. N. Dewynne, *The Mathematics of Financial Derivatives* (Cambridge University Press, Cambridge, 1995).