Supplemental Material for Section 5.6.
Simplifying Integrals by Substitution

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Introduction.

Substitution is used throughout mathematics to simplify expressions so that they can be worked with more easily. Usually, we start by writing out all of the details of the substitution. Then, with proficiency, we write fewer and fewer details, perhaps for simple cases doing the whole substitution process in our heads. However, when work is to be graded, be sure to write down enough so that the grader can follow your work.

We begin with one of the fundamental formulas of integration.

(1) $$\int u^n du = \frac{u^{n+1}}{n+1} + C, \qquad n \neq -1.$$

This, of course, should be memorized.

We give some examples.

Example 1. Find $\int x^2(2x^3+5)^2 \; dx$.

Solution. Here we have a quantity to a power. We try to put this into the form $\int u^n du$ that we can apply Equation 1. The first principle of substitution is:

(2) When you have a quantity to a power, let u equal that quantity and try to get du.

Warning: $u =$ the quantity, **not** the quantity to the power.

$$\text{Let } u = 2x^3 + 5. \text{ Then } du = 6x^2 dx$$

Since we are trying to get du, we first rewrite the problem with the x^2 next to the dx.

$$\int x^2(2x^3+5)^2 dx = \int (2x^3+5)^2 x^2 dx$$

Next multiply by $\frac{6}{6}$, bring the unwanted $\frac{1}{6}$ outside the integral, and substitute in.

$$\int (2x^3+5)^2 \ \frac{6}{6} \ x^2 dx = \frac{1}{6} \ \int \underbrace{(2x^3+5)^2}_{u^2} \ \underbrace{6x^2 dx}_{du} = \frac{1}{6} \ \int u^2 du$$

We can now use Formula 1 getting

$$\frac{1}{6}\int u^2 du = \frac{1}{6} \ \frac{u^3}{3} + C = \frac{1}{18}(2x^3+5)^3 + C$$

This is the answer, which we can check by differentiating:

$$\left[\frac{1}{18}(2x^3+5)^3 + C\right]' = \frac{1}{18} \ 3(2x^3+5)^2(6x^2) + 0 = x^2(2x^3+5)^2$$

which is the original integrand, verifying that our answer is correct.

Example 2. Find $\displaystyle\int \frac{2\sin 3x}{[5+\cos 3x]^4} \ dx$.

Solution. Again we have a quantity to a power, so we let u equal the quantity and try to get du.

$$u = 5+\cos 3x, \quad du = -3\sin 3x \ dx$$

We work towards putting this in the form $\displaystyle\int u^n du$. To avoid unnecessary confusion, we move the "2" outside the integral and rewrite this as

$$\int \frac{2\sin 3x}{[5+\cos 3x]^4} \ dx = 2\int [5+\cos 3x]^{-4} \ \sin 3x \ dx$$

We proceed as before:

$$\begin{aligned} &= \quad 2\int [5+\cos 3x]^{-4} \ \frac{-3}{-3}\sin 3x \ dx \\ &= \quad -\frac{2}{3}\int \underbrace{[5+\cos 3x]^{-4}}_{u^{-4}} \ \underbrace{(-3\sin 3x \ dx)}_{du} = -\frac{2}{3}\int u^{-4} du \end{aligned}$$

We use Formula 1 and get

$$= -\frac{2}{3} \ \frac{u^{-3}}{-3} + C = \frac{2}{9} \ [5+\cos 3x]^{-3} + C = \frac{2}{9(5+\cos 3x)^3} + C$$

This is the answer which you check by differentiating

$$\left\{\frac{2}{9}[5+\cos 3x]^{-3} + C\right\}' = \frac{2}{9}(-3)[5+\cos 3x]^{-4}(-\sin 3x \cdot 3) + 0 = \frac{2\sin 3x}{[5+\cos 3x]^{-4}}$$

Substitution with Trig Functions.

There are four fundamental trig differentiation formulas:

$$\begin{array}{ll} d(\sin u) = \cos u \; du & d(\tan u) = \sec^2 u \; du \\ d(\cos u) = -\sin u \; du & d(\sec u) = \sec u \tan u \; du \end{array}$$

By going backwards, these lead to four fundamental integration formulas.

$$(3) \qquad \begin{array}{ll} \displaystyle\int \cos u \; du = \sin u + C & \displaystyle\int \sec^2 u \; du = \tan u + C \\ \displaystyle\int \sin u \; du = -\cos u + C & \displaystyle\int \sec u \tan u \; du = \sec u + C \end{array}$$

There are similar formulas involving $\cot u$ and $\csc u$.

Example 3. Find $\displaystyle\int x^3 \sin(5x^4 + 6) dx$.

Solution. Here, we do not have a quantity to a power, so the first principle of substitution (2) does not apply. Hence, we turn to the second principle of substitution.

(4) When you do not have a quantity to a power, but you do have a trig function of a quantity let u equal that quantity and try to get du.

Later, this same principle will equally apply to other kinds of functions, such as exponential, logarithmic, inverse trig, etc.

$$\text{Let } u = 5x^4 + 6. \text{ Then } du = 20x^3 dx$$

We work towards putting the problem in the form $\displaystyle\int \sin u \; du$. We first move the x^3 next to the dx.

$$\int x^3 \sin(5x^4 + 6) dx = \int \sin(5x^4 + 6) x^3 \; dx$$

Next, we multiply by $\dfrac{20}{20}$, bring the unwanted $\dfrac{1}{20}$ outside of the integral, and substitute in

$$\int \sin(5x^4 + 6) \; \frac{20}{20} \; x^3 \; dx = \frac{1}{20} \int \sin \underbrace{(5x^4 + 6)}_{u} \; \underbrace{20x^3 dx}_{du} = \frac{1}{20} \int \sin u \; du$$

We now use the appropriate formula from (2) getting

$$\frac{1}{20}[-\cos u] + C = -\frac{1}{20}\cos(5x^4+6) + C$$

This is the answer, which we can check by differentiating

$$\left[-\frac{1}{20}\cos(5x^4+6) + C\right]' = -\frac{1}{20}\left[-\sin(5x^4+6)(20x^3)\right] + 0 = x^3\sin(5x^4+6)$$

Example 4. Find $\displaystyle\int \frac{9\sec^2(3\sqrt{x}+5)}{\sqrt{x}}\,dx$.

Solution. We recognize that we have a formula for $\displaystyle\int \sec^2 u\ du$ in (3), so we move toward putting the problem into this form.

$$\text{Let } u = 3\sqrt{x}+5 = 3x^{1/2}+5. \text{ Then } du = \frac{3}{2}\ x^{-1/2}dx = \frac{3}{2}\ \frac{1}{\sqrt{x}}\ dx$$

We first move the "9" outside of the integral and move the $\dfrac{1}{\sqrt{x}}$ next to the dx.

$$\int \frac{9\sec^2(3\sqrt{x}+5)}{\sqrt{x}}\ dx = 9\int \sec^2(3\sqrt{x}+5)\ \frac{1}{\sqrt{x}}\ dx$$

Next multiply by $\dfrac{2}{3}\cdot\dfrac{3}{2}$, move the unwanted $\dfrac{2}{3}$ outside of the integral, simplify and substitute.

$$9\int \sec^2(3\sqrt{x}+5)\ \frac{2}{3}\cdot\frac{3}{2}\ \frac{1}{\sqrt{x}}\ dx = 9\cdot\frac{2}{3}\int \sec^2\underbrace{(3\sqrt{x}+5)}_{u}\ \underbrace{\frac{3}{2}\cdot\frac{1}{\sqrt{x}}\ dx}_{du} = 6\int \sec^2 u\ du$$

We apply the appropriate formula from (3) getting

$$6\tan u + C = 6\tan(3\sqrt{x}+5) + C$$

This is the answer, which you should check by differentiating, but we shall not here.

Skipping Steps.

The first five or so times you use substitution, you should write out all of the steps. Thereafter, you should skip steps that you can easily do in your head and that leaving out does not lead to errors.

For instance, in Example 1, $\displaystyle\int x^2(2x^3+5)^2dx$, we let $u = 2x^3+5$ and find $du = 6x^2\ dx$. Many people would not physically move the x^2 next to the dx, nor would they multiply by $\dfrac{6}{6}$, but they would just think these steps. They would proceed by:

$$\int x^2(2x^3+5)^2dx = \frac{1}{6}\int 6x^2(2x^3+5)^2dx = \frac{1}{6}\int u^2du = \frac{1}{6}\frac{u^3}{3} + C = \frac{1}{18}(2x^3+5)^3 + C$$

When you are really experienced and the problem is simple enough, you might want to try doing more in your head. But be careful, and, again, if the work is being graded, make sure you write down enough so that the grader can follow what you do. The balance is, as it is throughout mathematics, you want to write down enough details so that you do not make errors, but not too many details so that the flow of the problem is disrupted in your mind.

Special Case.

Very simple problems like $\int \cos 7x \; dx$ should be handled in a simple way. Remember, you know $\int x^7 dx = \frac{1}{8}x^8 + C$ and not $x^8 + C$. You need the $\frac{1}{8}$ to cancel the "8" when checking $(x^8)' = 8x^7$. This is a similar situation. You know $\int \cos x \; dx = \sin x + C$. As with $\int x^7 dx$, $\int \cos 7x \; dx = \frac{1}{7}\sin 7x + C$ and not $\sin 7x + C$, because when you check, $(\sin 7x)' = 7\cos 7x$ and you need the $\frac{1}{7}$ to cancel the 7. Similarly,

$$\int \sec \frac{x}{2} \tan \frac{x}{2} \; dx = 2 \sec \frac{x}{2} + C$$

since the "2" is needed to cancel the "$\frac{1}{2}$" when you check the answer.

When Substitution Does Not Work.

Sometimes the "obvious" choice of substitution does not work. This method of substitution allows you to see as soon as possible that it does not work, and you must look for an alternative method.

Example 5. Find $\int x(2x^3+5)^2 dx$ (instead of $\int x^2(2x^3+5)^2 dx$ of Example 1).

Solution. We see we have a quantity to a power, so (as in Example 1) we try,

$$u = 2x^3 + 5, \quad du = 6x^2 dx.$$

But right here we see we cannot get x^2 since we only have an x. (Warning: We cannot multiply by $\frac{6x}{6x}$, since we can bring the $\frac{1}{6}$ outside the integral sign, but we cannot bring the $\frac{1}{x}$ outside.) Thus we must try something else. The thing to do here is to square out the quantity and then multiply through with the x.

$$\int x(2x^3+5)^2 dx = \int x(4x^6 + 20x^3 + 25) \; dx = \int (4x^7 + 20x^4 + 25x) \; dx = \cdots$$

Example 6. Find $\int \frac{9\sec^2(3\sqrt{x}+5)}{\sqrt{x}} \; dx$, from Example 4.

Solution. Suppose we see that we have a quantity to a power, and we try

$$u = \sec(3\sqrt{x}+5), \;\; du = \sec(3\sqrt{x}+5)) \tan(3\sqrt{x}+5) \; \frac{3}{2\sqrt{x}}$$

We can see right away that there is no $\sec(\cdot)\tan(\cdot)$. Thus, we see we cannot get du, so we look for an alternative approach, and hope we see the method of Example 4.

We conclude with an example from Calc. II. This will illustrate that we shall learn many methods of integration there, and it is important to see as quickly as possible whether a particular method does or does not work on a given problem.

Here are two formulas that will be derived in Calc. II.

$$\int \frac{du}{u} = \ell n|u| + C, \qquad \int \frac{du}{1+u^2} = \tan^{-1} u + C \tag{5}$$

The first formula is interesting because it tells us what happens in Formula 1 when $n = -1$.

Example 7. Find (a) $\displaystyle\int \frac{x^3}{1+x^4} dx$, (b) $\displaystyle\int \frac{x}{1+x^4} dx$.

Solution. (a) We have a quantity in the denominator, so we try

$$u = 1 + x^4, \quad du = 4x^3 dx$$

We see with the x^3 in the numerator we can get du, and we proceed to do it.

$$\int \frac{x^3}{1+x^4} dx = \frac{1}{4} \int \frac{1}{\underbrace{1+x^4}_{u}} \underbrace{4x^3 dx}_{du} = \frac{1}{4} \int \frac{du}{u}$$

$$= \frac{1}{4} \ \ell n|u| + C = \frac{1}{4} \ \ell n(1+x^4) + C$$

(b) If we were to try $u = 1+x^4$ again, we see with only an x in the numerator, we cannot get $du = 4x^3 dx$. In searching for an alternate approach, we might ask, how do we make use of the single x in the numerator? If we observe x is the crucial part of the derivative of x^2, and that in the denominator $x^4 = (x^2)^2$, then we solve the problem with the surprising substitution

$$u = x^2, \quad du = 2x \ dx$$

Then,

$$\int \frac{x}{1+x^4} \ dx = \frac{1}{2} \int \frac{1}{1+\underbrace{(x^2)^2}_{u^2}} \underbrace{2x \ dx}_{du} = \frac{1}{2} \int \frac{du}{1+u^2}$$

$$= \frac{1}{2} \tan^{-1}(u) + C = \frac{1}{2} \tan^{-1}(x^2) + C$$

This completes the discussion.