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STABILITY FOR SEMILINEAR PARABOLIC EQUATIONS WITH NONINVERTIBLE LINEAR OPERATOR

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Suppose that

$$x'(t) + Ax(t) = f(t, x(t)), \qquad t \geq 0,$$

is a semilinear parabolic equation, e^{-At} is bounded and f satisfies the usual continuity condition. If for some $0 < \omega \leq 1$, $0 < \alpha < 1$, $\alpha\omega p > 1$, $\gamma > 1$,

$$\|t^{\omega} A e^{-At}\| \leq C, \qquad t \geq 1,$$
$$\|f(t, x)\| \leq C\left(\|A^{\alpha}x\|^{p} + (1 + t)^{-\gamma}\right), \qquad t \geq 0,$$

whenever $\|A^{\alpha}x\| + \|x\|$ is small enough, then for small initial data there exist stable global solutions. Moreover, if the space is reflexive then their limit states exist. Some theorems that are useful for obtaining the above bounds and some examples are also presented.

1. Introduction and the Main Theorem. Assume that A is a sectorial operator [2] on a (real or complex) Banach space X and that there exist $M_1 \geq 1$, $0 < \omega \leq 1$ such that

$$(1) \qquad \begin{aligned} &\text{(i)} \quad \|e^{-At}\| \leq M_1 && \text{for } t \geq 0 \\ &\text{(ii)} \quad \|Ae^{-At}\| \leq M_1 t^{-\omega} && \text{for } t \geq 1. \end{aligned}$$

Some theorems useful in determining ω are presented in §4, and an example is given in §5. For $\beta \geq 0$ let $X^{\beta} = D(A^{\beta})$ and $\|x\|_{\beta} = \|(A + 1)^{\beta}x\|$ for $x \in X^{\beta}$.

Assume that $0 < \alpha < 1$ and that V is an open set in X^{α}. Suppose that f: $[0, \infty) \times V \mapsto X$ is such that for every $t \geq 0$, $x \in V$ there exist $\varepsilon, c \in (0, \infty)$, $0 < \nu \leq 1$, for which

$$\|f(s_1, x_1) - f(s_2, x_2)\| \leq c\left(|s_1 - s_2|^{\nu} + \|x_1 - x_2\|_{\alpha}\right)$$

whenever $s_i \geq 0$, $x_i \in V$ and $|s_i - t| + \|x_i - x\|_{\alpha} < \varepsilon$ for $i = 1, 2$.

For $0 < \tau \leq \infty$ let $S(\tau)$ be the set of continuous functions x: $[0, \tau) \mapsto X$ which satisfy

(i) $x([0, \tau)) \subset V$ and $f(\cdot, x(\cdot)) \in C([0, \tau), X)$

(ii) $x'(t)$ exists (in X), $x(t) \in D(A)$ and $x'(t) + Ax(t) = f(t, x(t))$ for $0 < t < \tau$.

Solutions defined in this way have many known nice properties (see Appendix 2).

Suppose that $\mu > 0$, $p > 1/\alpha\omega$, $\gamma > 1$, $M_2 \geq 0$, $M_3 \geq 0$ are such that if $x \in X^\alpha$ and $\|A^\alpha x\| + \|x\| < \mu$, then $x \in V$ and

$$\|f(t, x)\| \leq M_2 \|A^\alpha x\|^p + M_3 c^\gamma(t), \qquad t \geq 0, \tag{2}$$

where $c(t) = 1$ if $0 \leq t \leq 1$ and $c(t) = t^{-1}$ if $t > 1$. A theorem useful in establishing bounds of this type is given in Appendix 1; an example is analyzed in §5.

In §3 it is shown that if $0 \leq p < 1/\alpha\omega$ then there do not need to exist global solutions for all small initial data.

Observe that there exists $M_4 \geq M_1$ such that

$$\|A^\alpha e^{-At}\| \leq M_4 b(t), \qquad t > 0, \tag{3}$$

where $b(t) = t^{-\alpha}$ for $0 < t < 1$ and $b(t) = t^{-\alpha\omega}$ for $t \geq 1$. For $\beta > 1$ define

$$B(\beta) = \sup\left\{ c^{-\alpha\omega}(t) \int_0^t b(t-s) c^\beta(s)\, ds \,|\, t \geq 0 \right\} \tag{4}$$

and note that $\beta/(\beta - 1) \leq B(\beta) < \infty$.

MAIN THEOREM. *Suppose that* $x_0 \in X^\alpha$, $2N < \mu$ *and* $N^{p-1} p M_2 M_4 B(\alpha\omega p) < 1$, *where*

$$N = \left(\|A^\alpha x_0\| + \|x_0\| + M_3 B(\gamma)\right) p M_4/(p-1).$$

Then

(a) There exists $x \in S(\infty)$ such that $x(0) = x_0$ and, for $t \geq 0$, $\|A^\alpha x(t)\| \leq N c^{\alpha\omega}(t)$, $\|x(t)\| \leq N$.

(b) For each $\varepsilon > 0$ there exists $\delta > 0$ such that if $y_0 \in X^\alpha$ and $\|y_0 - x_0\|_\alpha < \delta$ then there exists $y \in S(\infty)$ with $y(0) = y_0$ and

$$\sup_{t \geq 0} \|x(t) - y(t)\|_\alpha < \varepsilon.$$

(c) If $X = N(A) \oplus \overline{R(A)}$ then there exists $y \in N(A)$ such that $\lim_{t \to \infty} \|x(t) - y\|_\alpha = 0$. ($N(A)$ is the null space of A, $R(A)$ is the range of A.)

REMARK 1. If X is reflexive then $X = N(A) \oplus \overline{R(A)}$ [5].

REMARK 2. Consider the Navier-Stokes equation in an exterior domain. According to [9], A can be taken to be a nonnegative self-adjoint operator, so that $\omega = 1$, and the nonlinear part satisfies

$$\|f(t, x)\| \leq c_1\|A^{1/2}x\|\,\|A^{3/4}x\| \leq c_2\|A^{3/4}x\|^{5/3}\|x\|^{1/3} \quad \text{for } x \in D(A^{3/4}).$$

Hence, all conditions can be satisfied. See also [3].

2. Proof of the Main Theorem.

Part (a). We may assume that in (2), $M_2 > 0$, $M_3 > 0$. Observe that $\|A^\alpha x_0\| + \|x_0\| < \mu$. Let $0 < \tau \leq \infty$, $x \in S(\tau)$ be as in Theorem A2.3 of Appendix 2. Let τ_1 be the biggest number such that $0 < \tau_1 \leq \tau$ and $\|A^\alpha x(t)\| + \|x(t)\| < \mu$ for $0 \leq t < \tau_1$. In the following, assume that $0 \leq t < \tau_1$.

Observe that

$$\|A^\alpha e^{-At}x_0\| \leq M_4\big(\|A^\alpha x_0\| + \|x_0\|\big)c^{\alpha\omega}(t) \equiv M_5 c^{\alpha\omega}(t). \tag{5}$$

Define

$$g(t) = \|A^\alpha x(t)\|, \tag{6}$$

$$h(t) = \sup_{0 \leq s \leq t} g(s)c^{-\alpha\omega}(s). \tag{7}$$

Since

$$x(t) = e^{-At}x_0 + \int_0^t e^{-A(t-s)}f(s, x(s))\, ds \tag{8}$$

we have

$$g(t) \leq M_5 c^{\alpha\omega}(t) + \int_0^t M_4 b(t-s)\big(M_2 g(s)^p + M_3 c^\gamma(s)\big)\, ds,$$

equations (4), (6) and (7) imply that

$$h(t) \leq \tilde{c}h(t)^p + N(p-1)/p, \tag{9}$$

where $\tilde{c} = M_2M_4B(\alpha\omega p)$. Set $L = (p\tilde{c})^{-1/(p-1)}$. Since $0 < N < L$ there exists $0 < L_0 < N$ such that

$$\begin{aligned} s &< \tilde{c}s^p + N(p-1)/p \quad \text{for } 0 \leq s < L_0, \\ s &> \tilde{c}s^p + N(p-1)/p \quad \text{for } L_0 < s \leq L. \end{aligned} \tag{10}$$

Since $h(0) = \|A^\alpha x_0\| < N$ we have by (10) and (9) that $h(0) \leq L_0$ and since h is continuous we have that $h(t) \leq L_0 < N$. Therefore, by (6) and (7)

$$\|A^\alpha x(t)\| \leq L_0 c^{\alpha\omega}(t) < Nc^{\alpha\omega}(t). \tag{11}$$

From (4) and (8) it follows that

$$\|x(t)\| \leq M_4\|x_0\| + M_2 M_4 L_0^p B(\alpha\omega p) + M_3 M_4 B(\gamma),$$

and from (10) it follows that

$$\|x(t)\| < N.$$

This and (11) imply that

$$\|A^\alpha x(t)\| + \|x(t)\| < 2N < \mu. \tag{12}$$

Therefore $\tau_1 = \tau$. Since

$$\|f(t, x(t))\| \leq M_2 N^p + M_3,$$

(12) and Theorem A2.4 imply that $\tau = \infty$.

Part (b). Let $N_1 > N$ be such that $2N_1 < \mu$ and $N_1^{p-1} p M_2 M_4 B(\alpha\omega p) < 1$. Let $\delta_0 > 0$ be such that if $z_0 \in X^\alpha$ and $\|x_0 - z_0\|_\alpha < \delta_0$ then

$$\left(\|A^\alpha z_0\| + \|z_0\| + M_3 B(\gamma)\right) p M_4/(p-1) \leq N_1.$$

Suppose that $z_0 \in X^\alpha$ and $\|x_0 - z_0\|_\alpha < \delta_0$. By Part (a), there exists $z \in S(\infty)$ such that $z(0) = z_0$ and, for $t \geq 0$,

$$\|A^\alpha z(t)\| \leq N_1 c^{\alpha\omega}(t), \qquad \|z(t)\| \leq N_1.$$

Fix any $t \geq \tau + 1 \geq 2$. Then

$$z(t) - e^{-A(t-\tau)} z(\tau) = \int_\tau^t e^{-A(t-s)} f(s, z(s))\, ds$$

and, hence,

$$\|z(t) - e^{-A(t-\tau)} z(\tau)\| \leq M_4 \int_\tau^\infty \left(M_2 N_1^p s^{-\alpha\omega p} + M_3 s^{-\gamma}\right) ds \equiv g(\tau).$$

Similarly we obtain

$$\begin{aligned} \|A^\alpha\left(z(t) - e^{-A(t-\tau)} z(\tau)\right)\| &\leq \frac{1}{1-\alpha} M_4\left(M_2 N_1^p \tau^{-\alpha\omega p} + M_3 \tau^{-\gamma}\right) + g(\tau) \\ &\equiv h(\tau) - g(\tau). \end{aligned}$$

Theorem A1.1 of Appendix 1 gives us a constant $\tilde{c}$ such that

$$\|z(t) - e^{-A(t-\tau)} z(\tau)\|_\alpha \leq \tilde{c} h(\tau).$$

Since z could also be x, it follows that

$$\|z(t) - x(t)\|_\alpha \leq 2\tilde{c} h(\tau) + M_1 \|z(\tau) - x(\tau)\|_\alpha.$$

Therefore,

$$\sup_{s\geq 0}\|x(s)-z(s)\|_\alpha \leq 2\tilde{c}h(\tau) + M_1 \sup_{0\leq s\leq \tau+1}\|z(s)-x(s)\|_\alpha .$$

This and Theorem A2.5 imply Part (b).

Part (c). If $z \in D(A)$ then by (1), $\|e^{-At}Az\| \to 0$ as $t \to \infty$. Therefore, if $z \in \overline{R(A)}$ then $\|e^{-At}z\| \to 0$ as $t \to \infty$, and if $z \in N(A)$ then $e^{-At}z = z$ for $t \geq 0$. Define $Px = \lim_{t\to\infty} e^{-At}x \in N(A)$ for $x \in X$.

Fix any $v \geq t \geq \tau \geq 1$. Then

$$e^{-A(v-t)}x(t) - e^{-A(v-\tau)}x(\tau) = \int_\tau^t e^{-A(v-s)}f(s, x(s))\,ds,$$

$$\text{(13)}\qquad \begin{aligned}&\left\|e^{-A(v-t)}x(t) - e^{-A(v-\tau)}x(\tau)\right\| \\ &\qquad\leq M_1\int_\tau^\omega \left(M_2N^ps^{-\alpha\omega p} + M_3s^{-\gamma}\right)ds \equiv \tilde{g}(\tau).\end{aligned}$$

Therefore

$$\|Px(t) - Px(\tau)\| \leq \tilde{g}(\tau)$$

and, hence, there exists $y \in N(A)$ such that $\|y - Px(\tau)\| \leq \tilde{g}(\tau)$. This and equation (13) give us

$$\|x(t) - y\| \leq 2\tilde{g}(\tau) + \left\|Px(\tau) - e^{-A(t-\tau)}x(\tau)\right\|.$$

3. Counterexamples. In this section assume that A is a sectorial operator on a Banach space X. Suppose also that e^{-At} is bounded for $t \geq 0$, $0 \leq \alpha < 1$, $p \geq 1$ and $f(x) = \|A^\alpha x\|^p x$ for $x \in D(A^\alpha)$.

Clearly, $f\colon X^\alpha \mapsto X$ is locally Lipschitz. Define $S(\tau)$ as in the Introduction.

Suppose that $x_0 \in D(A^\alpha)$. Define $g(t) = \|A^\alpha e^{-At}x_0\|^p$ for $t \geq 0$, and let $0 < \tau \leq \infty$ be such that $\int_0^t g(s)\,ds < 1/p$ for all $0 \leq t < \tau$. For $0 \leq t < \tau$, define

$$x(t) = \left(1 - p\int_0^t g(s)\,ds\right)^{-1/p} e^{-At}x_0.$$

A simple computation shows that $x \in S(\tau)$. Suppose also that x_0 is such that $\int_0^\infty g(t) = \infty$, therefore, for no $\varepsilon > 0$ exists $x_\varepsilon \in S(\infty)$ for which $x_\varepsilon(0) = \varepsilon x_0$. Now, to see that in the Introduction we cannot allow $\alpha\omega p < 1$, we need to find an A that satisfies (1) and x_0 as above. Take $X = L^1(0,\infty)$, $\omega \in (0,1]$, $h(s) = s + is^\omega$ for $s \geq 0$ and let $A = h$—the multiplication operator. Assuming that $\alpha\omega p < 1$ we can find $\beta > 0$ such that $(\alpha+\beta)\omega p < 1$. Now, let the above x_0 be $(x_0)(s) = s^{\beta\omega-1}e^{-s}$ for $s \geq 0$. This is the counterexample in case $p \geq 1$; in case $p \in [0,1)$ replace the above f by $f(x) = \|A^{\alpha p}x\|x$ for $x \in D(A^\alpha)$.

Suppose that $x_0 \in X$, $g(t) = \|A^{\alpha}e^{-At}x_0\|^p$ for $t > 0$ and $\int_t^1 g(s)\,ds \to \infty$ as $t \to 0^+$. Define $x(0) = 0$ and

$$x(t) = \left(1 + p\int_t^1 g(s)\,ds\right)^{-1/p} e^{-At}x_0, \qquad 0 < t < 1.$$

Then $x \not\equiv 0$ and

(a) $x \in C([0,1), X)$.

(b) $x(t) \in D(A)$, $x'(t)$ exists, $x'(t) + Ax(t) = f(x(t))$ for $0 < t < 1$.

(c) For every $0 < \delta < 1$ there exists c such that for all $t, s \in (\delta, 1)$

$$\|f(x(t)) - f(x(s))\| \le c|t - s|.$$

(d) $\int_0^1 \|f(x(s))\|\,ds < \infty$.

To see that such x_0 and A exist, take $X = L^1(0,\infty)$, $h(s) = 1$ if $0 \le s \le 1$, $h(s) = s$ if $s > 1$ and $A = h$. Assume that $\alpha p > 1$ and $0 < \beta < \alpha - 1/p$. Define $(x_0)(s) = 0$ if $0 \le s \le 1$ and $(x_0)(s) = s^{-1-\beta}$ if $s > 1$. Therefore, in the class of solutions that satisfy conditions (a)–(d), one does not need to have uniqueness, stability, etc. [**2**].

4. The linear operator. In this section assume that X is a complex Banach space. Proofs of the following lemmas are presented at the end of the section.

LEMMA 4.1. *Suppose that* $\delta > 0$ *and that* f: $S \equiv \{z \in \mathbf{C} \mid \delta\,\mathrm{Re}(z) > |\mathrm{Im}(z)|\} \mapsto X$ *is holomorphic. Suppose also that* $\beta \ge 1$, $M_1 \ge 0$, $M_2 \ge 0$ *are such that*

$$\|f(z)\| \le M_1 \exp\left(M_2\left|\frac{\mathrm{Im}(z)}{\mathrm{Re}(z)}\right|^{\beta} \mathrm{Re}(z)\right) \quad \text{for } z \in S.$$

If $M_2 = 0$ *set* $\omega = 1$, *and otherwise* $\omega = 1 - 1/\beta$. *Then for some* c

$$\|f'(t)\| \le \begin{cases} ct^{-1} & \text{if } 0 < t < 1, \\ ct^{-\omega} & \text{if } t \ge 1. \end{cases}$$

Using this lemma and the Hille-Yosida Theorem for $e^{i\phi}A$, ϕ small and nonzero, one can easily obtain necessary and sufficient conditions for (1) to hold. Instead of this theorem we shall, following [**8**, **10**], present more illuminating and more useful sufficient conditions.

For f: $\mathbf{R} \mapsto [0,\infty]$, $f(0) = 0$, define Lf: $\mathbf{R} \mapsto [0,\infty]$ by

$$(Lf)(x) = \sup_s \{sx - f(s)\}.$$

Lf is called the Legendre transformation of f.

LEMMA 4.1. *Suppose that*

(i) $f\colon \mathbf{R} \to [0, \infty], f(0) = 0$;
(ii) $A\colon D(A) \subset X \mapsto X$ *is a linear operator*;
(iii) $\lambda_0 > 0$ *and* $R(A + \lambda_0) = X$;
(iv) *for every* $x \in D(A)$ *with* $\|x\| = 1$ *there exists* $l \in X^*$ *such that* $\|l\| = l(x) = 1$ *and* $f(\mathrm{Im}[l(Ax)]) \leq \mathrm{Re}[l(Ax)]$.

Then

$$\left\|((1 + ia)A + (Lf)(a) + z)^{-1}\right\| \leq 1/\mathrm{Re}(z)$$

whenever $z \in \mathbf{C}$, $\mathrm{Re}(z) > 0$, $a \in \mathbf{R}$ *and* $(Lf)(a) < \infty$.

Using this lemma one can immediately obtain the following two theorems.

THEOREM 4.1. *Suppose, in addition to the assumptions of Lemma* 4.2, *that*

(i′) $0 < \phi < \pi/2$, $0 \leq b < \infty$, *are such that* $(Lf)(x) \leq b$ *for* $|x| \leq tg\phi$
(ii′) $\pi/2 - \phi < \alpha < \pi/2$.

Then

$$\left\|(A - \zeta)^{-1}\right\| \leq \frac{1}{|\zeta + b|\cos(\pi - \alpha - \phi)}$$

whenever $\zeta \in \mathbf{C}$, $\zeta \neq -b$ *and* $\alpha \leq |\arg(\zeta + b)| \leq \pi$.

THEOREM 4.2. *Suppose, in addition to the assumptions of Lemma* 4.2, *that*

(i″) $f(x) = f(-x)$ *for all* $x > 0$ *and* $(Lf)(\delta) < \infty$ *for some* $\delta > 0$;
(ii″) A *is densely defined.*

Then A *is a sectorial operator and*

$$\|e^{-Az}\| \leq \exp\left((Lf)\left[\frac{\mathrm{Im}(z)}{\mathrm{Re}(z)}\right]\mathrm{Re}(z)\right)$$

whenever $z \in \mathbf{C}$ *and* $|\mathrm{Im}(z)| < \delta\,\mathrm{Re}(z)$.

Proof of Lemma 4.1. Define $\mu = \delta/2$ and

$$z(x) = \begin{cases} |x| - i\mu x & \text{if } -1 \leq x \leq 1, \\ |x| - i\mu x|x|^{-1/\beta} & \text{if } |x| > 1, \end{cases}$$

and note that for $t > 0$,

$$f'(t) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} (z(x) - t)^{-2} z'(x) f(z(x))\,dx.$$

Hence, for some c_i and all $t > 0$, $\|f'(t)\| \leq c_0(I_1(t) + I_2(t))$, where

$$I_1(t) = \int_0^1 \left((x-t)^2 + (\mu x)^2\right)^{-1} dx \leq c_1 t^{-1},$$

$$I_2(t) = \int_1^\infty \left((x-t)^2 + (\mu x^{1-1/\beta})^2\right)^{-1} dx \leq c_2 t^{-1+1/\beta}.$$

If $M_2 = 0$, take $z(x) = |x| - i\mu x$ for $x \in \mathbf{R}$.

Proof of Lemma 4.2. We may assume that $X \neq \{0\}$. Suppose $x \in D(A)$, $\|x\| = 1$ and $0 \leq s \leq 1$. Let l be as in (iv). Observe that $(Lf)(as) \leq s(Lf)(a) < \infty$ and.

$$(Lf)(as) \geq as\,\mathrm{Im}(l(Ax)) - f[\mathrm{Im}(l(Ax))],$$

$$0 \leq (Lf)(as) + \mathrm{Re}(l[(1+ias)Ax]).$$

Hence, for every $z \in \mathbf{C}$,

$$\mathrm{Re}(z) \leq \mathrm{Re}(l([(1+ias)A + z + (Lf)(as)]x)).$$

Therefore, for every $z \in \mathbf{C}$, $x \in D(A)$, $0 \leq s \leq 1$,

$$\mathrm{Re}(z)\|x\| \leq \|((1+ias)A + (Lf)(as) + z)x\|.$$

In particular, if $\mathrm{Re}(z) > 0$, $x \in D(A)$, $0 \leq s \leq 1$, then

$$\|x\| \leq h(s)\|(A + g(s))x\|,$$

where

$$g(s) = ((Lf)(as) + \lambda_0(1-s) + zs)/(1+ias)$$

and

$$h(s) = \left(1 + (as)^2\right)^{1/2}(\lambda_0(1-s) + s\,\mathrm{Re}(z))^{-1}.$$

Now, increase s from 0 to 1.

5. Examples. By AC we will denote the set of complex-valued functions which are absolutely continuous on $[-a, a]$ for all $a > 0$. Fix $1 \leq p < \infty$ and define

$$T_0 f = f', \qquad f \in D(T_0) = \{g \in L^p \cap \mathrm{AC} \mid g' \in L^p\}.$$

Define $T = -T_0^2$. L^q will stand for $L^q(\mathbf{R})$.

THEOREM 5.1. *Suppose that* $p > 1$ *and that*

(i) $g_2\colon \mathbf{R} \mapsto \mathbf{R}$, $g_2 = h_1 + h_2$ *for some* $h_1 \in L^p$ *and* $h_2 \in L^\infty$.

(ii) $g_1\colon \mathbf{R} \mapsto \mathbf{R}$, $g_1 \in \mathrm{AC} \cap L^\infty$ *and* $pg_2 \geq g_1'$ *a.e.*

Set $A = T + g_1 T_0 + g_2$ *and*

$$h(x) = \frac{1}{4}\left(x\|g_1\|_\infty\right)^2\left(1 - \frac{|x(p-2)|}{2\sqrt{p-1}}\right)^{-1}.$$

Then

(a) *A is sectorial and if* $|\mathrm{Im}(z)(p-2)| < 2(p-1)^{1/2}\mathrm{Re}(z)$ *then*

$$\|e^{-Az}\| \le \exp\left(h\left[\frac{\mathrm{Im}(z)}{\mathrm{Re}(z)}\right]\mathrm{Re}(z)\right).$$

(b) $\sup_{t\ge 1}\|t^{\omega}Ae^{-At}\| < \infty$ *where* $\omega = 1$ *if* $g_1 = 0$ *and* $\omega = 1/2$ *otherwise.*

Proof. It is clear that A is sectorial and that (b) follows from (a) and Lemma 4.1. Suppose that $f \in D(T)$ and that $\|f\|_p = 1$. Let $l = |f|^p/f$. Hence $\int fl = 1 = \|l\|_q$ where $1/q = 1 - 1/p$. Let $c = \int lAf$. Integrations by parts give that $\mathrm{Re}(c) \ge 0$ and

$$|\mathrm{Im}(c)| \le \frac{1}{2}|2\text{-}p|(p-1)^{-1/2}\mathrm{Re}(c) + \|g_1\|_\infty(\mathrm{Re}(c))^{1/2}.$$

An application of Theorem 4.2 completes the proof.

For the operator $T + g_1T_0 + g_2$, we now present some bounds similar to those in equation (2).

LEMMA 5.1. *Suppose that* $p \le r \le \infty$ *and* $\theta = (1/p - 1/r)/2$. *Then for* $f \in D(T)$,

$$\|f\|_r \le 2\|Tf\|_p^{\theta}\|f\|_p^{1-\theta},$$

$$\|f'\|_r \le 2\|Tf\|_p^{\theta+1/2}\|f\|_p^{-\theta+1/2}.$$

Proof. Choose any $z > 0$. Hence

$$f = (1/2z)\left((z-T_0)^{-1} + (T_0+z)^{-1}\right)(Tf + z^2f)$$

and

$$f' = \tfrac{1}{2}\left((z-T_0)^{-1} - (T_0+z)^{-1}\right)(Tf + z^2f).$$

Since $\|(z \pm T_0)^{-1}g\|_r \le z^{2\theta-1}\|g\|_p$ for $g \in L^p$ we have

$$\|f\|_r \le z^{2\theta-2}\|Tf\|_p + z^{2\theta}\|f\|_p,$$

$$\|f'\|_r \le z^{2\theta-1}\|Tf\|_p + z^{2\theta+1}\|f\|_p.$$

Theorem A1.2 implies the following lemma.

LEMMA 5.2. *Suppose* $p \le r \le \infty$. *Then*

(a) *If* $2\gamma > 1/p - 1/r$ *and* $\alpha = (1/p - 1/r)/(2\gamma)$, *then for some* c,

$$\|f\|_r \le c\|T^{\gamma}f\|_p^{\alpha}\|f\|_p^{1-\alpha}, \qquad f \in D(T^{\gamma}).$$

(b) *If* $2\gamma > 1 + 1/p - 1/r$ *and* $\alpha = (1 + 1/p - 1/r)/(2\gamma)$, *then for some* c,

$$\|f'\|_r \le c\|T^\gamma f\|_p^\alpha \|f\|_p^{1-\alpha}, \qquad f \in D(T^\gamma).$$

Hölder inequalities imply

LEMMA 5.3 *Suppose that*

(i) $q, t \in [p, \infty]$ *and* $1/p = 1/q + 1/t$.

(ii) $r, s \in [0, \infty)$, $r + s > 0$ *and* $(r + s)t \ge p$.

(iii) *If* $s > 0$ *then* $2\gamma > 1 + 1/p - 1/v$, *where* $v = \max\{p, ts\}$.

(iv) *If* $s = 0$ *then* $2\gamma > 1/p - 1/(rt)$.

Then for some c *and all* $f \in D(T^\gamma)$, $g \in L^q$

$$\left\|g|f|^r|f'|^s\right\|_p \le c\|g\|_q\|T^\gamma f\|_p^\alpha \|f\|_p^{r+s-\alpha},$$

where $\alpha\gamma = (s + (r + s - 1)/p + 1/q)/2$.

LEMMA 5.4. *Set* $\sigma = 1 - 1/p$ *and assume that*

(i) $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbf{C}$ *are such that if*

$$(\lambda^2 + \alpha_1\lambda + \alpha_2)(\lambda^2 + \beta_1\lambda + \beta_2) = 0$$

and $\mathrm{Re}(\lambda) = 0$, *then* $\lambda = 0$. *Define* $g_{11}(x) = \alpha_1$, $g_{21}(x) = \alpha_2$ *for* $x \ge 0$ *and* $g_{11}(x) = \beta_1$, $g_{21}(x) = \beta_2$ *for* $x < 0$.

(ii) g_{12}, g_{22}: $\mathbf{R} \mapsto \mathbf{C}$ *are such that both* $x \mapsto (1 + |x|)^\sigma g_{12}(x)$ *and* $x \mapsto (1 + |x|)^{\sigma+1} g_{22}(x)$ *are in* L^p. *Define* $g_1 = g_{11} + g_{12}$, $g_2 = g_{21} + g_{22}$.

(iii) *There is no* $c \in \mathbf{C}$ *such that* $g_1(x) = (c - x)g_2(x)$ *a.e.*

(iv) *If* $p = 1$ *then* $g_1 \in \mathrm{AC}$.

(v) *If* $f \in \mathrm{AC}$, $f' \in \mathrm{AC}$, $\sup_x(1 + |x|)^{-\sigma}|f'(x)| < \infty$ *and* $f'' + g_1 f' + g_2 f = 0$, *the* f *is a constant.*

Then for some c *and all* $f \in D(T)$,

$$\|Tf\|_p \le c\|Tf - g_1 T_0 f - g_2 f\|_p.$$

The obvious consequence of this lemma is

THEOREM 5.2. *Suppose that the assumptions of Lemma* 5.4 *are satisfied and that* $T - g_1 T_0 - g_2$ *is a generator of a bounded strongly continuous semigroup. Then Lemma* 5.2 *and Lemma* 5.3 *hold with* T *replaced by* $T - g_1 T_0 - g_2$.

Proof of Lemma 5.4. If one expresses f'' in terms of $f(0)$, $f'(0)$ and $f'' + g_{11}f' + g_{21}f$ then a direct computation shows that

C1. (i) implies that for some c_1 and all $f \in D(T)$,

$$\|f''\|_p \leq c_1\big(\|f'' + g_{11}f' + g_{21}f\|_p + |f'(0)| + \big(|\alpha_2| + |\beta_2|\big)|f(0)|\big).$$

On the other hand, one can show that

C2. (i), (ii), (iii) imply that if $f_n \in D(T)$, $n = 1, 2, \ldots$ and

$$\sup_n\big(\|f_n''\|_p + \|f_n'' + g_1 f_n' + g_2 f_n\|_p\big) < \infty,$$

then for some c_2 and all n, $|f_n'(x)| \leq c_2(1 + |x|)^{\sigma}$ a.e. Moreover, if $g_2 \neq 0$ a.e., then for some c_3 and all n, $|f_n(x)| \leq c_3(1 + |x|)^{\sigma+1}$ a.e.

Suppose that the conclusion is false. Then there exist $f_n \in D(T)$, $n = 1, 2, \ldots$ such that $\|f_n'' + g_1 f_n' + g_2 f_n\|_p < 1/n$ and $\|f_n''\|_p = 1$. We shall distinguish four cases: Case 1 ($p > 1$, $g_2 \neq 0$ a.e.), Case 2 ($p > 1$, $g_2 = 0$ a.e.), Case 3 ($p = 1$, $g_2 \neq 0$ a.e.), Case 4 ($p = 1$, $g_2 = 0$ a.e.). Since in all cases one arrives at the contradiction in a similar way, only Cases 1 and 4 will be analyzed here.

Case 1. Since $|f_n'(x) - f_n'(y)| \leq |x - y|^{\sigma}$ we have by Ascoli's Theorem that there exists $f \in \mathrm{AC}$ such that f' is continuous, and for all $x \in \mathbf{R}$, $|f(x)| \leq c_3(1 + |x|)^{\sigma+1}$, $|f'(x)| \leq c_2(1 + |x|)^{\sigma}$. Moreover, for some subsequence $\{n_k\}$ and all $x \in \mathbf{R}$, $f_{n_k}(x) \to f(x)$, $f_{n_k}'(x) \to f'(x)$ as $k \to \infty$. Therefore as $k \to \infty$,

$$\big\|g_{12}\big(f_{n_k}' - f'\big)\big\|_p \to 0, \qquad \big\|g_{22}\big(f_{n_k} - f\big)\big\|_p \to 0$$

and

$$\big\|f_{n_k}'' + g_{11}f_{n_k}' + g_{21}f_{n_k} + g_{12}f' + g_{22}f\big\|_p \to 0.$$

Therefore all $x, y \in \mathbf{R}$,

$$f_{n_k}'(x) - f_{n_k}'(y) + \int_y^x \big(g_{11}f_{n_k}' + g_{21}f_{n_k} + g_{12}f' + g_{22}f\big) \to 0 \quad (k \to \infty),$$

which implies that $f' \in \mathrm{AC}$ and $f'' + g_1 f' + g_2 f = 0$. Hence f is a constant and since $g_2 \neq 0$, we have $f = 0$. C1 implies that $\|f_{n_k}''\|_p \to 0$, contradiction.

Case 4. Define $h_n(x) = \int_0^x f_n'$. By Ascoli's Theorem there exist $z \in \mathbf{C}$, a continuous function f and a subsequence $\{n_k\}$ such that $f_{n_k}'(0) \to z$ and $h_{n_k}(x) \to f(x)$ for all x. Since for all x, n,

$$\left|f_n'(x) - f_n'(0) + g_1(x)h_n(x) - \int_0^x g_1' h_n\right| < \frac{1}{n},$$

we have that $f \in AC$, $f' \in AC$ and $f'' + g_1 f' = 0$. Since $f(0) = 0$ we have that $f = 0$ and hence $f'_{n_k}(x) \to 0$. Therefore $\|f''_{n_k} + g_{11} f'_{n_k}\|_1 \to 0$ and since $\alpha_2 = \beta_2 = 0$, C1 leads to a contradiction.

Appendix 1. In this appendix we present a precise definition of the fractional powers, some of their properties and a (possibly) new result (Theorem A1.2). A very thorough analysis of fractional powers was done by H. Komatsu in a series of papers [**5**, **6**...]. Details omitted here can be found in [**5**, **6**].

Throughout this appendix it will be assumed that A is a generator of a strongly continuous semi-group on a (real or complex) Banach space X and that $\|e^{-At}\| \le M < \infty$ for all $t \ge 0$.

For $\lambda > 0$, $\alpha > 0$ define $(A + \lambda)^{-\alpha}$ by

$$(A + \lambda)^{-\alpha} x = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha - 1} e^{-\lambda t} e^{-At} x \, dt, \qquad x \in X.$$

Hence $\|(A + \lambda)^{-\alpha}\| \le M\lambda^{-\alpha}$, $(A + \lambda)^{-\alpha}$ is one-to-one and its range is dense in X. $(A + \lambda)^{\alpha}$ is defined to be the inverse of $(A + \lambda)^{-\alpha}$ and $(A + \lambda)^0$ is the identity map.

Let $\alpha \ge 0$. It was shown in [**5**, **6**] that $D((A + \lambda)^\alpha)$ is independent of $\lambda > 0$ and that $\lim_{\lambda \to 0^+} (A + \lambda)^\alpha x$ exists (in norm) for all $x \in D((A + 1)^\alpha)$. Define $A^\alpha x = \lim_{\lambda \to 0^+} (A + \lambda)^\alpha x$ for $x \in D(A^\alpha) \equiv D((A + 1)^\alpha)$.

THEOREM A1.1. *Suppose that either* $\alpha, \beta, \gamma \in \mathbf{R}$ *and* $\lambda > 0$ *or* $\alpha, \beta, \gamma \in [0, \infty)$ *and* $\lambda = 0$. *Then*:

(1) *If* α *is an integer then* $(A + \lambda)^\alpha$ *agrees with the usual definition.*

(2) $(A + \lambda)^\alpha$ *is closed and densely defined.*

(3) *If* $x \in D(A + \lambda)^\alpha$ *and* $t \ge 0$ *then*

$$(A + \lambda)^\alpha e^{-At} x = e^{-At} (A + \lambda)^\alpha x.$$

(4) *If* $0 \le \alpha \le \beta$ *then* $D(A^\beta) \subset D(A^\alpha)$.

(5) *If* $x \in D((A + \lambda)^\beta) \cap D((A + \lambda)^{\alpha + \beta})$ *then*

$$(A + \lambda)^\alpha (A + \lambda)^\beta x = (A + \lambda)^{\alpha + \beta} x.$$

(6) *If* $\alpha < \beta < \gamma$ *and* $\theta = (\beta - \alpha)/(\gamma - \alpha)$ *then there exists* c *such that*

$$\|(A + \lambda)^\beta x\| \le c \|(A + \lambda)^\gamma x\|^\theta \|(A + \lambda)^\alpha x\|^{1 - \theta}$$

for all $x \in D(A + \lambda)^\gamma$.

(7) *If* $\alpha \in [0, 1]$, $t \ge 0$, $x \in D(A^\alpha)$, *then*

$$\|x - e^{-\lambda t} e^{-At} x\| \le 2(M + 1)^2 t^\alpha \|(A + \lambda)^\alpha x\|.$$

(8) *If* $\lambda > 0$ *and* $\alpha \in [0,1]$ *then there exist* c_1, c_2 *such that for all* $x \in D(A^\alpha)$,

$$\|A^\alpha x\| + \|x\| \le c_1\|(A+\lambda)^\alpha x\| \le c_2(\|A^\alpha x\| + \|x\|).$$

(9) *If* $\alpha \in (0,1)$ *then the limit (in norm) of*

$$\frac{1}{\Gamma(-\alpha)}\int_\varepsilon^\infty t^{-\alpha-1}(e^{-\lambda t}e^{-At} - 1)x\,dt$$

as $\varepsilon \to 0^+$ *exists if and only if* $x \in D(A^\alpha)$. *The limit is* $(A+\lambda)^\alpha x$.

The following theorem is very useful in getting control over nonlinear terms in semilinear parabolic equations and it is well known when $\|e^{-At}\|$ decays exponentially in t [**7, 4, 5, 1, 2**].

THEOREM A1.2. *Suppose that*

(i) *Y is a Banach space with the same scalar field as X.*

(ii) *B: $X \mapsto Y$, $D(A) \subset D(B)$ and B is a closable linear operator. Let $\bar{B}$ be a closed extension of B.*

(iii) *$\beta \in (0,1]$ and $c \ge 0$ are such that $\|Bx\|_Y \le c\|Ax\|^\beta\|x\|^{1-\beta}$ for all $x \in D(A)$.*

(iv) $0 \le \alpha < \beta < \gamma$ *and* $\theta = (\beta - \alpha)/(\gamma - \alpha)$.

Then $D(A^\gamma) \subset D(\bar{B})$ and there exists c_1 such that for all $x \in D(A^\gamma)$,

$$\|\bar{B}x\|_Y \le c_1\|A^\gamma x\|^\theta\|A^\alpha x\|^{1-\theta}.$$

Proof. If $\beta = 1$ then the conclusion is obvious. Assume that $\beta < 1$. Choose δ so that $\beta < \delta < (\beta - \alpha + \alpha\beta)/\beta$ and $\delta < \gamma$. Choose $\lambda > 0$ and $x \in D(A^\delta)$. Then

$$x = (A+\lambda)^{1-\delta}(A+\lambda)^{\delta-1}x = \frac{1}{\Gamma(\delta-1)}\int_0^\infty f(t)\,dt,$$

where $f(t) = t^{\delta-2}(e^{-\lambda t}e^{-At} - 1)(A-\lambda)^{\delta-1}x$. Note that f and Bf are continuous on $(0,\infty)$ and

$$\begin{aligned}\|Bf(t)\|_Y \le{}& c(M+1)^{2\beta}t^{\delta-2}\|(A+\lambda)^\delta x\|^\beta \\ &\times\|(e^{-\lambda t}e^{-At} - 1)(A+\lambda)^{\delta-1}x\|^{1-\beta}.\end{aligned}$$

Two bounds on the last term lead to

$$\|Bf(t)\|_Y \le c_2 t^{\delta-\beta-1}\|(A+\lambda)^\delta x\|,$$

$$\|Bf(t)\|_Y \le c_2 t^{-\mu-1}\|(A+\lambda)^\delta x\|^\beta\|(A+\lambda)^\alpha x\|^{1-\beta},$$

where $c_2 = 2c(M+1)^2$ and $\mu = \beta - \alpha + \alpha\beta - \delta\beta$. Therefore for all $\varepsilon > 0$,

$$\int_0^\infty \|Bf(t)\|_Y dt$$
$$\leq c_2\left(\frac{1}{\delta-\beta}\varepsilon^{\delta-\beta}\left\|(A+\lambda)^\delta x\right\| + \frac{1}{\mu}\varepsilon^{-\mu}\left\|(A+\lambda)^\delta x\right\|^\beta\left\|(A+\lambda)^\alpha x\right\|^{1-\beta}\right),$$

hence $x \in D(\overline{B})$ and,

$$\|\overline{B}x\|_Y \leq \frac{c_2}{|\Gamma(\delta-1)|}\left(\frac{1}{\delta-\beta} + \frac{1}{\mu}\right)\left\|(A+\lambda)^\delta x\right\|^\eta\left\|(A+\lambda)^\alpha x\right\|^{1-\eta},$$

where $\eta = (\beta-\alpha)/(\delta-\alpha)$. Now, let $\lambda \to 0^+$ and bound $\|A^\delta x\|$ by $\|A^\alpha x\|$ and $\|A^\gamma x\|$.

Appendix 2. Our approach to semilinear parabolic equations is similar to the one used by D. Henry [**2**]. However, Henry's definition of a solution [**2**; 3.3.1] needs a minor modification (see a counterexample in §3), otherwise one does not need to have uniqueness of solutions, which in turn messes up many other theorems (e.g., stability). Almost all of his proofs apply unchanged under the new definition of a solution. Here we shall present theorems needed in the main part of the paper.

A linear operator A in a complex Banach space is said to be a sectorial operator if it is a closed densely defined operator and if there exist $a \in \mathbf{R}$, $M \geq 0$ and $0 < \phi < \pi/2$ such that $z \notin \sigma(A)$ and

$$\left\|(A-z)^{-1}\right\| \leq M/|a-z|$$

whenever $z \in \mathbf{C}$, $z \neq a$ and $\phi \leq |\arg(z-a)| \leq \pi$.

A linear operator A on a real Banach space X is said to be sectorial if the natural extension of A on the complexification of X is sectorial.

Assume that A is a sectorial operator on a Banach space X. Fix an $a \in \mathbf{R}$ so that $\|e^{-At}\| \leq Me^{-(a+\delta)t}$ for some $M \geq 0$, $\delta > 0$ and all $t \geq 0$. For $\beta \geq 0$ define $X^\beta = D((A-a)^\beta)$ and $\|x\|_\beta = \|(A-a)^\beta x\|$ for $x \in X^\beta$.

Fix $0 \leq \alpha < 1$, $-\infty < t_0 < t_1 \leq \infty$ and assume that V is open in X^α. Assume that f: $[t_0, t_1) \times V \mapsto X$ is such that for every $t_0 \leq t < t_1$, $x \in V$ there exist $\delta, M \in (0,\infty)$ and $\nu \in (0,1]$ such that

$$\|f(s_1,x_1) - f(s_2,x_2)\| \leq M\left(|s_1-s_2|^\nu + \|x_1-x_2\|_\alpha\right)$$

whenever $t_0 \leq s_i < t_1$, $x_i \in V$ and $|s_i - t| + \|x_i - x\|_\alpha < \delta$ for $i = 1, 2$.

For every $t_0 < \tau \leq t_1$, let $S(\tau)$ denote the set of continuous functions x: $[t_0,\tau) \mapsto X$ such that

(i) $x([t_0,\tau)) \subset V$ and $f(\cdot, x(\cdot)) \in C([t_0,\tau), X)$

(ii) $x'(t)$ exists (in X), $x(t) \in D(A)$ and $x'(t) + Ax(t) = f(t, x(t))$ for $t_0 < t < \tau$.

THEOREM A2.1. *Suppose that $t_0 < \tau \le t_1$. Then $x \in S(\tau)$ if and only if*
(i) $x([t_0, \tau)) \subset V$ *and* $f(\cdot, x(\cdot)) \in C([t_0, \tau), X)$
(ii)

$$x(t) = e^{-A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{-A(t-s)}f(s, x(s))\, ds \quad \textit{for } t_0 \le t < \tau.$$

THEOREM A2.2. *Suppose that $t_0 < \tau \le t_1$ and $x \in S(\tau)$. Then*
(a) *$f(\cdot, x(\cdot))$, Ax, x, x': $(t_0, \tau) \mapsto X$ are locally Hölder continuous functions.*
(b) *If $\alpha \le \beta \le 1$ and $x(t_0) \in X^\beta$ then $x \in C([t_0, \tau), X^\beta)$.*

THEOREM A2.3. *Suppose that $x_0 \in V$. Then there exists $t_0 < \tau \le t_1$ such that*
(a) *There is an $x \in S(\tau)$ such that $x(t_0) = x_0$.*
(b) *If $t_0 < t^* \le t_1$, $y \in S(t^*)$ and $y(t_0) = x_0$ then $t^* \le \tau$ and $y(t) = x(t)$ for $t_0 \le t < t^*$.*

THEOREM A2.4. *Suppose that $t_0 < \tau < t_1$, $x \in S(\tau)$ and $\sup\{\|f(s, x(s))\| \mid t_0 \le s < \tau\} < \infty$. Then there exists $y \in X^\alpha$ such that $\lim_{t \to \tau^-} \|x(t) - y\|_\beta = 0$ for all $0 \le \beta < 1$. Moreover, if $y \in V$ then there exist $\tau < \tau_1 \le t_1$ and $z \in S(\tau_1)$ such that $z(t_0) = x(t_0)$.*

THEOREM A2.5. *Suppose that $t_0 < \tau^* < \tau \le t_1$ and $x \in S(\tau)$. Then there exist $\mu > 0$, $c \ge 0$ such that if $y_0 \in X^\alpha$ and $\|x(t_0) - y_0\|_\alpha < \mu$ then there exists $t > \tau^*$, $y \in S(t)$ for which $y(t_0) = y_0$ and*

$$\|x(s) - y(s)\|_\alpha \le c\|y_0 - x(t_0)\|_\alpha$$

for $t_0 \le s \le \tau^*$.

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