

- 1) Determine the Taylor polynomial of degree 4 of $f(x) = \ln(1+x) - x^2 \sin x$ at $a = 0$. (Hint: It may help to use the Maclaurin series of $\sin x$ and the geometric series with sum $\frac{1}{1+x}$. Note that the Maclaurin series of $\sin x$ is $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$.)

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \Rightarrow \frac{1}{1+x} = \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-1)^n x^n \quad |x| < 1$$

$$\ln(1+x) = \int \frac{1}{1+x} dx = \sum_{n=0}^{\infty} \int (-1)^n x^n dx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} + C \quad |x| < 1$$

for $x=0$ $\ln(1+0) = 0 = \sum_{n=0}^{\infty} 0 + C \Rightarrow C = 0$

$$\text{So } \ln(1+x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

$$\text{Now, } x^2 \sin x = x^2 \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x^2 \left(x - \frac{x^3}{3!} + \dots \right) = x^3 - \frac{x^5}{3!} + \dots$$

Therefore $T_4(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} - (x^3)$

$$= \boxed{x - \frac{x^2}{2} - \frac{2}{3}x^3 - \frac{x^4}{4}}$$

2) Determine an upper bound for the error in approximating the function $f(x) = \cos x$ by $1 - \frac{x^2}{2} + \frac{x^4}{24}$ on the interval $[-\pi/2, \pi/2]$.

Let $f(x) = \cos x$. Then $T_4(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24}$ is the 4th degree Taylor polynomial of f . The upper bound on the error is given by

$$R_4(x) \leq \left| \frac{f^{(5)}(x)}{5!} (x-0)^5 \right|$$

$$\leq \left| \frac{\sin(x)}{5!} x^5 \right|$$

$$\leq \left| \frac{1}{5!} \cdot \left(\frac{\pi}{2}\right)^5 \right|$$

$$= \boxed{\frac{\pi^5}{2^5 \cdot 5!}}$$

$\sin\left(\frac{\pi}{2}\right) = 1$ is the largest value of $\sin(x)$ on $[-\frac{\pi}{2}, \frac{\pi}{2}]$