

- 1) Determine the power series expansion for the function  $\frac{x^2}{1-2x^2}$  and determine its radius of convergence. Explain your reasoning.

We know  $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1+x+x^2+\dots$ , for  $|x| < 1$

(geometric Series with ratio  $R=1$ ). Thus,

$$\frac{1}{1-(2x^2)} = \sum_{n=0}^{\infty} (2x^2)^n = \sum_{n=0}^{\infty} 2^n \cdot x^{2n}, \text{ for } |2x^2| < 1$$

$$\text{Now, } |2x^2| < 1 \iff |x^2| < \frac{1}{2} \iff -\sqrt{\frac{1}{2}} < x < \sqrt{\frac{1}{2}}$$

So, the open interval of convergence is  $(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$

This means the radius of convergence is

$$R = \frac{1}{\sqrt{2}}$$

Finally,

$$\begin{aligned} \frac{x^2}{1-2x^2} &= x^2 \cdot \frac{1}{1-2x^2} = x^2 \cdot \sum_{n=0}^{\infty} 2^n \cdot x^{2n} \\ &= \boxed{\sum_{n=0}^{\infty} 2^n \cdot x^{2n+2}}. \end{aligned}$$

(Multiplying by  $x^2$  does not change the radius of convergence).

2) Find the radius of convergence and the open interval of convergence for the series

$$\sum_{n=1}^{\infty} \frac{(2x+1)^{2n+1}}{n^2 + 4} = \sum_{n=0}^{\infty} \frac{2^{2n+1} (x - (-\frac{1}{2}))^{2n+1}}{n^2 + 4}$$

The Center is  $a = -\frac{1}{2}$ .

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(2x+1)^{2n+3}}{(n+1)^2 + 4} \cdot \frac{n^2 + 4}{(2x+1)^{2n+1}} \right| \\ &= (2x+1)^2 \cdot \lim_{n \rightarrow \infty} \left| \frac{n^2 + 4}{n^2 + 2n + 5} \right| \\ &= (2x+1)^2 \cdot \lim_{n \rightarrow \infty} \left| \frac{n^2}{n^2} \right| = (2x+1)^2. \end{aligned}$$

$$\text{Now, } (2x+1)^2 < 1 \Leftrightarrow -1 < 2x+1 < 1$$

$$\Leftrightarrow 0 \leq 2x < 2$$

$$\Leftrightarrow 0 < x < 1$$

Therefore, the radius of convergence is  $R = \frac{1}{2}$ , and  
the open interval of convergence is  $(0, 1)$ .