

Solution

MATH 133, QUIZ #8

- (1) Do the following infinite series converge or not? Name the convergence test you are using and show all your work (7.5 points):

(a) $\sum_{n=1}^{\infty} \frac{n+3}{2n^2+1}$ Let $a_n = \frac{n+3}{2n^2+1}$, $b_n = \frac{1}{n}$. Then $a_n > 0, b_n > 0$

for all n , and $\sum b_n = \sum \frac{1}{n}$ diverges by p-Series, $p=1$.

Now, $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \left(\frac{n+3}{2n^2+1} \cdot \frac{n}{1} \right) = \lim_{n \rightarrow \infty} \left(\frac{n^2}{2n^2} \right) = \frac{1}{2}$. Since

$0 < \frac{1}{2} < \infty$, and $\sum b_n$ diverges, then by the LCT, $\sum a_n$ diverges also.

(b) $\sum_{n=2}^{\infty} \frac{n \ln n}{4^n}$ Let $a_n = \frac{n \cdot \ln(n)}{4^n}$. Then $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+1) \ln(n+1)}{4^{n+1}} \cdot \frac{4^n}{n \ln(n)} \right|$

$$= \lim_{n \rightarrow \infty} \left| \frac{n+1}{n} \cdot \frac{\ln(n+1)}{\ln(n)} \cdot \frac{4^n}{4^{n+1}} \right| = \frac{1}{4} \cdot \lim_{n \rightarrow \infty} \left| \frac{\ln(n+1)}{\ln(n)} \right|$$

$$= \frac{1}{4} \cdot 1 \cdot \left| \lim_{n \rightarrow \infty} \frac{\ln(n+1)}{\ln(n)} \left(\frac{\infty}{\infty}, \text{L.H.} \right) \right| = \frac{1}{4} \cdot 1 \cdot \lim_{n \rightarrow \infty} \left| \frac{\frac{1}{n+1}}{\frac{1}{n}} \right|$$

$$= \frac{1}{4} \cdot 1 \cdot \lim_{n \rightarrow \infty} \left| \frac{n}{n+1} \right| = \frac{1}{4} \cdot 1 \cdot 1 = \frac{1}{4} < 1. \quad \sum a_n \text{ conv. absolutely}$$

by ratio test, and therefore converges.

Let $a_n = \frac{\ln(n)}{n}$, $b_n = \frac{1}{n}$. note that $a_n > 0, b_n > 0$ for all $n \geq 2$, and

$\ln(n) > 1$ for all $n \geq 3$. Therefore $a_n > b_n$ for all $n \geq 3$.

Moreover, $\sum b_n$ diverges by p-Series, $p=1$. Thus, by the D.C.T, $\sum a_n$ diverges as well.

(2) (2.5 points). Consider the series $\sum_{n=1}^{\infty} a_n$ where $a_n = \frac{\cos(n\pi)}{n}$.

(a) Apply the Ratio Test to the series. Evaluate $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|$. What is the conclusion?

Note

$$a_n = \frac{\cos(n\pi)}{n} = \frac{(-1)^n}{n}$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+1}}{n+1} \cdot \frac{n}{(-1)^n} \right| = \lim_{n \rightarrow \infty} \left| \frac{n}{n+1} \right| = 1.$$

The ratio test is inconclusive.

(b) Does the series converge absolutely, conditionally or does it diverge? Justify your answer.

$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges by the alternating series test

($b_n = \frac{1}{n}$: $b_n > 0$, $b_{n+1} < b_n$, $\lim_{n \rightarrow \infty} b_n = 0$).

However, the series $\sum |a_n| = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges by p-Series, $p=1$.

Therefore, $\sum \frac{(-1)^n}{n}$ is conditionally convergent.