Solution

Math 133, Quiz #6

1. Compute the integral
$$\int_{-1}^{0} \frac{5x}{x^2-2x+1}\,dx$$

2. Determine whether the following improper integral converges or not. Explain your reasoning.
$$\int_{1}^{\infty} \frac{dx}{\sqrt{e^x+x}}$$

3. Compute the integral
$$\int_{-\pi/2}^{\pi/2} \cos^3 x\,dx$$

① $\frac{5x}{x^2-2x+1} = \frac{5x}{(x-1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2} \Rightarrow 5x = A(x-1)+B.$

$x=1 \Rightarrow \underline{B=5}$; $x=0 \Rightarrow 0=-A+B \Rightarrow A=B=5$. Thus

$$\int_{-1}^{0} \frac{5x}{x^2-2x+1}\,dx = \int_{-1}^{0} \frac{5}{x-1} + \frac{5}{(x-1)^2}\,dx = 5\ln|x-1| - \frac{5}{x-1}\Big|_{-1}^{0}$$

$$= 5\ln|1| - \frac{5}{-1} - \left(5\ln|2| - \frac{5}{-2}\right) = \boxed{\frac{5}{2} - 5\ln(2).}$$

② for any $x \geq 1$, $0 \leq \frac{1}{\sqrt{e^x+x}} \leq \frac{1}{\sqrt{e^x}} = e^{-\frac{x}{2}}$, and $\frac{1}{\sqrt{e^x+x}}$, $\frac{1}{\sqrt{e^x}} = e^{-x/2}$ are continuous on $[1,\infty)$. Observe that

$$\int_1^{\infty} e^{-x/2}\,dx = \lim_{t\to\infty}\left(-2e^{-\frac{x}{2}}\right)\Big|_1^t = \frac{2}{\sqrt{e}},$$ So $\int_1^{\infty} e^{-\frac{x}{2}}\,dx$ converges;

Since $\frac{1}{\sqrt{e^x+x}} \leq \frac{1}{\sqrt{e^x}}$, Then $\int_1^{\infty} \frac{dx}{\sqrt{e^x+x}}$ $\boxed{\text{Converges}}$ as well

By the Direct Comparison test.

(3) $$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^3 x \, dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2 x \cos x \, dx$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1-\sin^2 x) \cos x \, dx$$

$u = \sin x$

$du = \cos x \, dx$

$$= \int_{-1}^{1} 1-u^2 \, du$$

$$= u - \frac{u^3}{3} \Big|_{-1}^{1}$$

$$= \left(1 - \frac{1}{3}\right) - \left(-1 - \frac{(-1)^3}{3}\right)$$

$$= \boxed{\frac{4}{3}}$$