Standard Response Questions. Show all work to receive credit. Please $\boxed{\textbf{BOX}}$ your final answer.

1. Determine if the following series converge or diverge. **If the series converges, also compute the sum**. You must show all of your work and support your conclusions.

(a) (7 points) $\displaystyle\sum_{n=0}^{\infty} \frac{3^{n-1}}{2^{2n}} = \sum_{n=0}^{\infty} \frac{1}{3}\frac{3^n}{4^n} = \sum_{n=0}^{\infty} \frac{1}{3}\left(\frac{3}{4}\right)^n.$

This is a geometric series with a ratio $r = \frac{3}{4}$, $|r|<1$.

Thus it converges, and its sum is

$$\frac{1^{st} \text{ term}}{1-r} = \frac{\frac{1}{3}\left(\frac{3}{4}\right)^0}{1-\frac{3}{4}} = \frac{\frac{1}{3}}{\frac{1}{4}} = \boxed{\frac{4}{3}}$$

(b) (7 points) $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n^2+5}}{3n+4}$

$$\lim_{n\to\infty} \frac{\sqrt{n^2+5}}{3n+4} = \lim_{n\to\infty} \frac{\sqrt{n^2(1+5/n^2)}}{n\left(3+\frac{4}{n}\right)} = \lim_{n\to\infty} \frac{\cancel{n}\sqrt{1+5/n^2}}{\cancel{n}(3+4/n)}$$

$$= \frac{\sqrt{1+0}}{3+0} = \boxed{\frac{1}{3} \neq 0}$$

By the n^{th} term test for divergence, the seriesabove diverges.

2. Determine if the following series converge or diverge. You must show all of your work and justify your use of any series convergence tests.

(a) (7 points) $\sum_{n=1}^{\infty} \frac{3n^2+5}{2^n}$ Let $a_n = \frac{3n^2+5}{2^n}$; Then $a_{n+1} = \frac{3(n+1)^2+5}{2^{n+1}}$, and

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{3(n+1)^2+5}{2^{n+1}} \cdot \frac{2^n}{3n^2+5}\right| = \frac{1}{2}\cdot\lim_{n\to\infty}\left|\frac{3(n^2+2n+1)+5}{3n^2+5}\right|$$

$$= \frac{1}{2}\lim_{n\to\infty}\left|\frac{\cancel{n^2}\left(3+\frac{6}{n}+\frac{8}{n^2}\right)}{\cancel{n^2}\left(3+\frac{5}{n^2}\right)}\right| = \frac{1}{2}\cdot\left|\frac{3}{3}\right| = \boxed{\frac{1}{2} < 1}$$

By the ratio test, $\sum a_n$ converges absolutely, and therefore **converges.**

(b) (7 points) $\sum_{n=1}^{\infty} \frac{1}{3^n - 1}$ Let $a_n = \frac{1}{3^n-1}$, $b_n = \frac{1}{2^n} = \left(\frac{1}{2}\right)^n$. Then

$a_n > 0$, $b_n > 0$ for all $n \geq 1$, and $a_n < b_n$ for all $n \geq 1$.

Also, $\sum b_n = \sum\left(\frac{1}{2}\right)^n$ converges (geometric Series with ratio $r = \frac{1}{2}$, $|r| < 1$). Therefore, by the direct comparison test, $\sum a_n$ **converges** as well.

3. For each of the following functions, find its 3rd degree Taylor polynomial centered at the given a.

(a) (7 points) $f(x) = \sin(5x)$, centered at $a = 0$.

We know $\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$

Therefore, $\sin(5x) = \sum_{n=0}^{\infty} (-1)^n \frac{(5x)^{2n+1}}{(2n+1)!} = (5x) - \frac{(5x)^3}{3!} + \frac{(5x)^5}{5!} - \cdots$

So, $\boxed{T_3(x) = 5x - \frac{5^3x^3}{3!}}$

(b) (7 points) $g(x) = \ln(x)$, centered at $a = 2$.

$g(x) = \ln(x)$, $\quad g(2) = \ln(2)$

$g'(x) = \frac{1}{x}$, $\quad g'(2) = \frac{1}{2}$

$g''(x) = \frac{-1}{x^2}$, $\quad g''(2) = -\frac{1}{4}$

$g'''(x) = \frac{2}{x^3}$, $\quad g'''(2) = \frac{2}{8} = \frac{1}{4}$.

$$T_3(x) = g(2) + \frac{g'(2)}{1!}(x-2) + \frac{g''(2)}{2!}(x-2)^2 + \frac{g'''(2)}{3!}(x-2)^3$$

$$= \boxed{\ln(2) + \frac{1}{2}(x-2) - \frac{1}{4\cdot 2!}(x-2)^2 + \frac{1}{4\cdot 3!}(x-2)^3.}$$

4. (7 points) Find the Maclaurin series (Taylor series centered at $a = 0$) representation of $f(x) = \dfrac{3x^2}{1+x^2}$. Express your answer in sigma notation.

We know $\dfrac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots$ for all $|x|<1$.

Thus $\dfrac{3x^2}{1+x^2} = 3x^2 \cdot \dfrac{1}{1-(-x^2)} = 3x^2 \cdot \sum_{n=0}^{\infty} (-x^2)^n$

$$= \sum_{n=0}^{\infty} 3x^2 \cdot (-1)^n x^{2n}$$

$$= \boxed{\sum_{n=0}^{\infty} 3(-1)^n x^{2n+2}}.$$

5. (7 points) Find the interval of convergence for the power series $\sum_{n=3}^{\infty} \dfrac{(7x-2)^n}{(n+3)^2}$. (Leave your answer as an open interval; you do not have to test the end points for convergence.)

Let $a_n = \dfrac{(7x-2)^n}{(n+3)^2}$. We want $\lim_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| < 1$ for convergence, by the ratio test.

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(7x-2)^{n+1}}{(n+4)^2} \cdot \frac{(n+3)^2}{(7x-2)^n}\right| = \lim_{n\to\infty} \left|(7x-2) \cdot \left(\frac{n+3}{n+4}\right)^2\right|$$

$$= |7x-2| \cdot \lim_{n\to\infty} \left|\left(\frac{1+\frac{3}{n}}{1+\frac{4}{n}}\right)^2\right| = |7x-2| \cdot 1^2 = |7x-2|.$$

Now, $|7x-2| < 1 \iff -1 < 7x-2 < 1 \iff 1 < 7x < 3$

$$\iff \frac{1}{7} < x < \frac{3}{7}$$

The open interval of convergence is therefore $\boxed{\left(\frac{1}{7}, \frac{3}{7}\right)}$.

Multiple Choice. Circle the best answer. No work needed. No partial credit available.

6. (4 points) Find the limit of the **sequence** a_n where the n^{th} term is given by $a_n = \dfrac{3n+\cos(3n)}{4n}$.

A. 0
B. $\frac{3}{4}$
C. 3
D. 4
E. The sequence does not have a limit.

$$\underbrace{\frac{3n-1}{4n}}_{\to 3/4 \text{ as } n\to\infty} < \underbrace{\frac{3n+\cos(3n)}{4n}}_{\downarrow 3/4} < \underbrace{\frac{3n+1}{4n}}_{\to \frac{3}{4} \text{ as } n\to\infty}$$

by the squeeze theorem

7. (4 points) Which statement about the series $\sum_{n=2}^{\infty} \frac{\ln(3n)}{\sqrt{n^2-1}}$ is true?

A. It **diverges** by using a comparison test with $\sum_{n=1}^{\infty} \frac{1}{n}$.
B. It **converges** by using a comparison test with $\sum_{n=1}^{\infty} \frac{1}{n}$.
C. It **diverges** by using a comparison test with $\sum_{n=1}^{\infty} \frac{1}{n^2}$.
D. It **converges** by using a comparison test with $\sum_{n=1}^{\infty} \frac{1}{n^2}$.
E. It **diverges** by the ratio test.

$\ln(3n) > 1$ for $n \geq 2$

$\sqrt{n^2-1} < \sqrt{n^2} = n$ for $n \geq 2$

Thus $\frac{\ln(3n)}{\sqrt{n^2-1}} > \frac{1}{n}$ for $n \geq 2$

$\sum \frac{1}{n}$ div

8. (4 points) Which statement is true about the series $\sum_{n=2}^{\infty} \frac{1}{n(\ln(n))^p}$?

A. The **integral test** shows that the series converges for all p.
B. The **integral test** shows that the series diverges for $p \leq 1$.
C. The **integral test** hypotheses <u>are not</u> met by this series, so it cannot be applied.
D. The **integral test** hypotheses <u>are</u> met by this series, however the test is inconclusive.
E. None of the above are true.

$u = \ln(n)$, $du = \frac{1}{n}dn$

$\int_{\ln(2)}^{\infty} \frac{1}{u^p}du = \infty$ if $p \leq 1$.

9. (4 points) Find the radius of convergence of $\sum_{n=1}^{\infty} \frac{(2x+3)^{4n+5}}{(n-1)!}$.

   A. 1/4

   B. 1/3

   C. 3

   D. 4

   E. $+\infty$

$\left|\frac{(2x+3)^{4n+9}}{n!} \cdot \frac{(n-1)!}{(2x+3)^{4n+5}}\right| = \left|(2x+3)^4\right| \cdot \left|\frac{1}{n}\right| \to 0$ as $n \to \infty$ for all x.

Thus $R = \infty$

10. (4 points) Determine whether the following series are absolutely convergent, conditionally convergent, or divergent:

$$(1) \sum_{n=3}^{\infty} \frac{1}{n^2 \ln(n)} \quad \text{and} \quad (2) \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$$

conv. by Alt. Series test
does not conv. abs. by p-Series, $p=1$
(So cond. conv.)

   A. (1) is absolutely convergent; (2) is divergent.

   B. (1) is conditionally convergent; (2) is divergent.

   C. (1) is absolutely convergent; (2) is conditionally convergent.

   D. (1) is divergent; (2) is conditionally convergent.

   E. Both (1) and (2) are conditionally convergent.

$\ln(n) > 1$ for $n \geq 3$

$n^2 \ln(n) > n^2$

$\frac{1}{n^2 \ln(n)} < \frac{1}{n^2} \leftarrow \sum \frac{1}{n^2}$ conv.

conv abs $\left(\left|\frac{1}{n^2 \ln(n)}\right| = \frac{1}{n^2 \ln(n)}\right)$

11. (4 points) Which statement is true about the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$?

$\lim_{n\to\infty} \left|\frac{n^2}{(n+1)^2}\right| = 1$

$\sum \frac{1}{n^2}$ Conv. by p-Series, $p = 2 > 1$.

   A. By the ratio test, the series converges.

   B. By the ratio test, the series diverges.

   C. The ratio test is inconclusive for this series, but the series converges by another test.

   D. The ratio test is inconclusive for this series, but the series diverges by another test.

   E. None of the above are true.

12. (4 points) By Taylor's Inequality, on the interval $[-2,2]$, the difference between e^x and its degree 2 Taylor polynomial centered at $x=0$ is at most:

A. $\frac{|x|^2}{2}$.

B. $\frac{|x|^3}{6}$.

C. $\frac{e^2|x|^3}{6}$

D. $\frac{|x|^4}{24}$.

E. $\frac{e^3|x|^4}{24}$.

$$R_2(x) \leq \left| \frac{f'''(x)}{3!}(x-0)^3 \right| = \left| \frac{e^x}{6} \cdot x^3 \right| \leq \frac{e^2}{6}|x|^3$$

13. (4 points) The Taylor series of the function $f(x)$, centered at $a=2$, is given by $\sum_{n=0}^{\infty} \frac{n^2+5}{n!}(x-2)^n$.

What is the value of the third derivative $f'''(2)$?

A. 9/2

B. 9

C. 5

D. 14/6

E. 14

$$\sum \cdots = \frac{5}{0!}(x-2)^0 + \frac{1^2+5}{1!}(x-2)^1 + \frac{2^2+5}{2!}(x-2)^2 + \frac{3^2+5}{3!}(x-2)^3$$

$$\frac{f'''(2)}{3!}(x-2)^3 = \frac{3^2+5}{3!}(x-2)^3$$

Thus $f'''(2) = 3^2+5 = 9+5 = 14$

14. (4 points) The Taylor series up to order 4, centered at $a=0$, for $f(x) = \ln(1+x) - x\sin(x)$ is

A. $1 + x - \frac{3x^2}{2} + \frac{x^3}{2} - \frac{x^4}{4}$

B. $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}$

C. $x - \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{4}$

D. $x - \frac{3x^2}{2} + \frac{x^3}{3} - \frac{x^4}{12}$

E. $x - \frac{3x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4!}$

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} \cdots\cdots$$

$$x\sin x = x \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} \cdots\right)$$

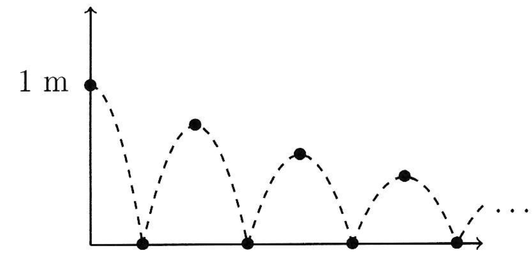
$$= x^2 - \frac{x^4}{3!} + \frac{x^6}{5!} \cdots\cdots$$

So, $\ln(1+x) - x\sin x = x - \frac{x^2}{2} - x^2 + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^4}{6} \cdots$

$$= x - \frac{3}{2}x^2 + \frac{x^3}{3} - \frac{x^4}{12} + \cdots$$

More Challenging Questions. Show all work to receive credit. Please $\boxed{\textbf{BOX}}$ your final answer.

15. (a) (4 points) A ball falls from a height of 1m and continues bouncing forever. Each time it hits the floor, it bounces back to $\frac{3}{4}$ of the previous height. Write a (numerical) series that represents the total *vertical* distance traveled by the ball (ignore any horizontal displacement). (Hint: don't forget to include the distance traveled before the first bounce.)



1st bounce $= 1$

2nd bounce $= 1 \times \frac{3}{4} \times 2$

3rd bounce $= 1 \times \frac{3}{4} \times \frac{3}{4} \times 2 = 1 \times \left(\frac{3}{4}\right)^2 \times 2$.

total distance $= 1 + \sum_{n=1}^{\infty} 2\left(\frac{3}{4}\right)^n = 1 + 2\left(\frac{3}{4}\right) + 2\left(\frac{3}{4}\right)^2 + 2\left(\frac{3}{4}\right)^3 + \cdots$

(b) (4 points) Compute the sum of the series you found in part (a).

geometric Series with ratio $r = \frac{3}{4}$, $|r| < 1$. Converges to

$1 + \frac{1^{\text{st}} \text{ term}}{1 - r} = 1 + \frac{2\left(\frac{3}{4}\right)}{1 - \frac{3}{4}} = \boxed{7}$

16. (6 points) By the ratio test, the radius of convergence of the power series $\sum_{n=1}^{\infty} a_n x^n$ is found to be $R = 2$.

If $b_n = (a_n)^2$, does the power series $\sum_{n=1}^{\infty} b_n x^n$ converge at $x = 3$? Justify your reasoning.

$\sum a_n x^n$ has $R = 2 \Rightarrow \sum a_n \cdot (\sqrt{3})^n$ Converges $\Rightarrow \lim_{n\to\infty} \left| \frac{a_{n+1} \cdot (\sqrt{3})^{n+1}}{a_n \cdot (\sqrt{3})^n} \right| < 1$.

So $\lim_{n\to\infty} \left| \frac{(a_{n+1})^2}{(a_n)^2} \cdot (\sqrt{3})^2 \right| < 1 \Rightarrow \lim_{n\to\infty} \left| \frac{b_{n+1}}{b_n} \cdot 3 \right| < 1 \Rightarrow \lim_{n\to\infty} \left| \frac{b_{n+1} \cdot 3^{n+1}}{b_n \cdot 3^n} \right| < 1$

By the Ratio test, $\sum b_n x^n$ Converges at $x = 3$.