

Standard Response Questions. Show all work to receive credit. Please **BOX** your final answer.

1. Evaluate the following integrals. Show all work.

(a) (7 points) $\int \frac{1}{1+9x^2} dx$

$$\int \frac{1}{1+9x^2} dx = \int \frac{dx}{1+(3x)^2} \cdot \begin{cases} \text{Let } u=3x, \\ du=3dx \end{cases}$$

$$= \frac{1}{3} \int \frac{du}{1+u^2}$$

$$= \frac{1}{3} \arctan(u) + C = \boxed{\frac{1}{3} \arctan(3x) + C}$$

(b) (7 points) $\int x \cosh(x) dx$.

Let $u=x$, $du=dx$, $dv=\cosh(x)dx$, $v=\sinh(x)$.

Then $\int x \cosh(x) dx = \int u dv = uv - \int v du$

$$= x \sinh(x) - \int \sinh(x) dx$$

$$= \boxed{x \sinh(x) - \cosh(x) + C}$$

2. Evaluate the following limits. (If you use L'Hopital's Rule, explicitly state your reasoning.)

(a) (7 points) $\lim_{x \rightarrow \infty} \ln(x^{1/\sqrt{x}})$

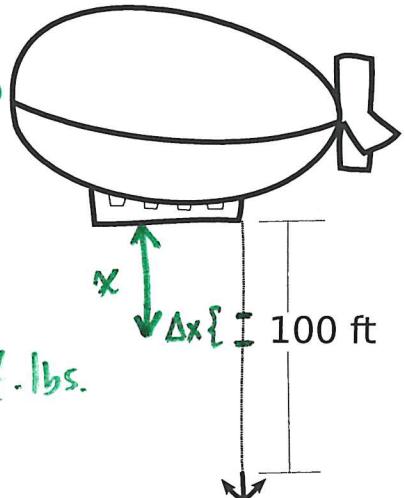
$$\begin{aligned} \lim_{x \rightarrow \infty} \ln(x^{1/\sqrt{x}}) &= \lim_{x \rightarrow \infty} \frac{\ln(x)}{\sqrt{x}} \quad \left(\frac{\infty}{\infty}, \text{L}'\text{Hop.} \right) \\ &= \lim_{x \rightarrow \infty} \frac{\frac{1}{x}}{\frac{1}{2\sqrt{x}}} = \lim_{x \rightarrow \infty} \frac{2\sqrt{x}}{x} \\ &= \lim_{x \rightarrow \infty} \left(\frac{2}{\sqrt{x}} \right) = \frac{2}{\infty} = \boxed{0} \end{aligned}$$

(b) (7 points) $\lim_{y \rightarrow 0} \frac{1 - \cos(4y)}{e^{8y} - 1}$

$$\begin{aligned} \lim_{y \rightarrow 0} \frac{1 - \cos(4y)}{e^{8y} - 1} &\quad \left(\frac{0}{0}, \text{L}'\text{Hop.} \right) \\ &= \lim_{y \rightarrow 0} \frac{4 \sin(4y)}{8e^{8y}} \\ &= \frac{4 \cdot 0}{8 \cdot e^0} = \boxed{0} \end{aligned}$$

3. (7 points) A 500 lb anchor hangs off of an airship from a chain that is 100 ft long. The chain itself weighs another 200 lbs (in addition to the weight of the anchor). How much work will it take to pull the anchor up onto the deck of the airship by the chain?

$$W_{\text{anchor}} = \text{Weight}_{\text{anchor}} \times \text{distance} = 500 \times 100 = 50,000 \text{ ft-lb}$$



A (Δx) ft piece of chain weighs $\frac{200}{100} \Delta x = 2(\Delta x)$ lbs.

its work is $2(\Delta x) \cdot x = 2x \cdot \Delta x$ ft-lbs

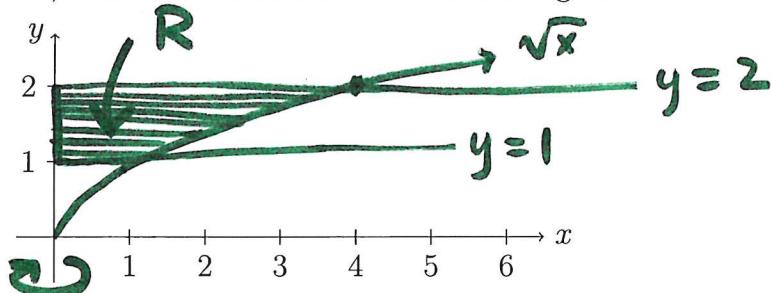
$$W_{\text{chain}} = \int_0^{100} 2x \, dx = x^2 \Big|_0^{100} = 100^2 = 10,000 \text{ ft-lbs.}$$

Total Work is $W = 50,000 + 10,000$

$$= 60,000 \text{ ft-lbs.}$$

4. Let R be the region bounded by the curves $y = \sqrt{x}$, $y = 1$, $y = 2$, and the y -axis.

- (a) (2 points) Sketch the region R ; make sure to shade and label the region R .



- (b) (5 points) Find the volume of the solid formed by revolving R around the y -axis.

A horizontal cross-section at height y is circular, of radius $r(y) = y^2$, and area $A(y) = \pi r^2(y) = \pi y^4$.

The volume is therefore

$$V = \int_1^2 A(y) dy = \int_1^2 \pi y^4 dy = \frac{\pi}{5} y^5 \Big|_1^2 = \boxed{\frac{\pi}{5} (2^5 - 1)}$$

5. (7 points) Give the partial fraction decomposition of $\frac{x}{x^2 + 3x + 2}$.

$$\frac{x}{x^2 + 3x + 2} = \frac{x}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2}; \text{ So}$$

$$x = A(x+2) + B(x+1); \text{ Now,}$$

$$\begin{aligned} x = -2 &\Rightarrow -2 = -B \Rightarrow B = 2, \text{ and} \\ x = -1 &\Rightarrow -1 = A \Rightarrow A = -1. \text{ So,} \end{aligned}$$

$$\boxed{\frac{x}{x^2 + 3x + 2} = \frac{-1}{x+1} + \frac{2}{x+2}}$$

6. (7 points) Evaluate the integral $\int_2^\infty \frac{1}{x-1} - \frac{1}{x+1} dx$.

$$\textcircled{1} \quad \int \frac{1}{x-1} - \frac{1}{x+1} dx = \ln|x-1| - \ln|x+1| + C = \ln\left|\frac{x-1}{x+1}\right| + C.$$

$$\textcircled{2} \quad \int_2^\infty \frac{1}{x-1} + \frac{1}{x+1} dx = \lim_{t \rightarrow \infty} \int_2^t \frac{1}{x-1} + \frac{1}{x+1} dx$$

$$= \lim_{t \rightarrow \infty} \left(\ln\left|\frac{x-1}{x+1}\right| \Big|_{x=2}^{x=t} \right) = \lim_{t \rightarrow \infty} \left(\ln\left|\frac{t-1}{t+1}\right| \right) - \ln\left|\frac{1}{3}\right|$$

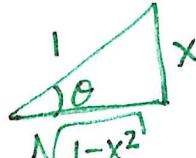
$$= \ln\left(\lim_{t \rightarrow \infty} \left|\frac{t-1}{t+1}\right|\right) + \ln(3)$$

$$= \ln(1) + \ln(3) = \boxed{\ln(3)}$$

Multiple Choice. Circle the best answer. No work needed. No partial credit available.

7. (4 points) Evaluate the integral $\int \frac{1}{x^2\sqrt{1-x^2}} dx$.

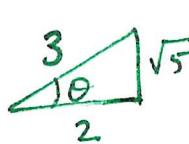
$x = \sin\theta$ A. $\frac{\sqrt{1-x^2}}{x^2} + C$
 $\cos\theta d\theta$ B. $-\frac{\sqrt{1-x^2}}{x^2} + C$
C. $\frac{\sqrt{1-x^2}}{x} + C$
D. $-\frac{\sqrt{1-x^2}}{x} + C$
E. $-x\sqrt{1-x^2} + C$



$$\begin{aligned} &= \int \frac{\cos\theta d\theta}{\sin^2\theta \sqrt{1-\sin^2\theta}} \\ &= \int \frac{\cos\theta d\theta}{\sin^2\theta \cdot \cos\theta} = \int \frac{1}{\sin^2\theta} d\theta \\ &= \int \csc^2\theta d\theta \\ &= -\cot(\theta) + C = -\frac{\cos(\theta)}{\sin\theta} + C \\ &= -\frac{\sqrt{1-x^2}}{x} + C \end{aligned}$$

8. (4 points) $\sin\left(\tan^{-1}\left(\frac{\sqrt{5}}{2}\right)\right) = ?$

- A. $\frac{\sqrt{5}}{3}$
B. $\frac{3}{2}$
C. $\frac{2}{\sqrt{5}}$
D. $\frac{2}{3}$
E. $\frac{3}{\sqrt{5}}$



Let $\theta = \tan^{-1}\left(\frac{\sqrt{5}}{2}\right)$. Then $\tan\theta = \frac{\sqrt{5}}{2}$

So, $\sin\theta = \frac{\sqrt{5}}{3}$

9. (4 points) The derivative of $f(x) = x^{\cos(x)}$ is

- A. $\left(-\sin(x)\ln x + \frac{\cos(x)}{x}\right)x^{\cos(x)} + C$
B. $\left(\sin(x)\ln x + \frac{\cos(x)}{x}\right)x^{\cos(x)}$
C. $\left(-\sin(x)\ln x + \frac{\cos(x)}{x}\right)x^{\cos(x)}$
D. $\sin(x)\ln x + \frac{\cos(x)}{x}$
E. $-\sin(x)\ln x + \frac{\cos(x)}{x}$

Let $y = x^{\cos x}$, $\ln(y) = \cos x \cdot \ln x$
 $\frac{y'}{y} = \underbrace{-\sin x \cdot \ln x}_{y'} + \underbrace{\cos x \cdot \frac{1}{x}}_{\text{Product Rule}}$
 $y' = x^{\cos x} \left(\quad \checkmark \quad \right)$

10. (4 points) Solve the initial value problem $\frac{dy}{dx} = 1 - 2y$ with initial value $y(0) = 0$.

A. $y = \frac{1 - e^{-2x}}{2}$.

B. $y = \ln(1 - 2x)$.

C. $y = -\frac{1}{2} \ln(1 - 2x)$.

D. $y = 1 - e^{-x}$.

E. $y = \frac{e^{-\frac{1}{2}x} - 1}{2}$.

$$\int \frac{dy}{1-2y} = \int dx, \text{ or } \frac{1}{2} \int \frac{dy}{y-\frac{1}{2}} = -\int dx$$

$$\Rightarrow \frac{1}{2} \ln \left| y - \frac{1}{2} \right| = -x + C$$

$$\Rightarrow \ln \left| y - \frac{1}{2} \right| = -2x + C; \quad y(0) = 0 \Rightarrow C = \ln \left(\frac{1}{2} \right)$$

$$\text{So } \ln \left| y - \frac{1}{2} \right| = -2x + \ln \left(\frac{1}{2} \right) \Rightarrow \left| y - \frac{1}{2} \right| = e^{-2x} \cdot e^{\ln \left(\frac{1}{2} \right)} = \frac{1}{2} e^{-2x}$$

$$y(0) = 0 \Rightarrow y < \frac{1}{2} \Rightarrow \left| y - \frac{1}{2} \right| = \frac{1}{2} - y \Rightarrow \frac{1}{2} - y = \frac{1}{2} e^{-2x} \Rightarrow y = \boxed{\frac{1 - e^{-2x}}{2}}$$

11. (4 points) Evaluate $\int_0^{\frac{\pi}{2}} \sin^7 x \cos^3 x \, dx$.

A. $-1/24$

B. $1/24$

C. $-1/40$

D. $1/40$

E. 0

$$\begin{aligned} \int_0^{\frac{\pi}{2}} \sin^7 x \cos^2 x \cos x \, dx &= \int_0^{\frac{\pi}{2}} \sin^7 x (1 - \sin^2 x) \cos x \, dx \\ &= \int_0^1 u^7 (1 - u^2) \, du = \int_0^1 u^7 - u^9 \, du = \left[\frac{u^8}{8} - \frac{u^{10}}{10} \right]_0^1 \\ &= \frac{1}{8} - \frac{1}{10} = \frac{5-4}{40} = \boxed{\frac{1}{40}} \end{aligned}$$

12. (4 points) Compute $f'(x)$ if $f(x) = 2^{\tan^{-1}(x)}$.

A. $2^{\tan^{-1}(x)} \cdot \frac{\ln 2}{1+x^2}$

B. $2^{\tan^{-1}(x)} \cdot \ln(2) \cdot \sec^2(x)$

C. $2^{\tan^{-1}(x)-1} \cdot \frac{\ln 2}{1+x^2}$

D. $2^{\tan^{-1}(x)} \cdot \frac{1}{1+x^2}$

E. $2^{\tan^{-1}(x)-1} \cdot \frac{1}{1+x^2}$

$$f'(x) = 2^{\tan^{-1} x} \cdot \ln(2) \cdot \frac{d}{dx} \tan^{-1} x$$

$$= \boxed{2^{\tan^{-1} x} \cdot \ln 2 \cdot \frac{1}{1+x^2}}$$

13. (4 points) 5 days ago, a sample of radioactive substance is measured at 200 grams. Today, the sample measures at 100 grams. How many days from now will there be only 25 grams of the sample remaining?

- A. 15
 B. 10

- C. 7.5
D. 5
E. 3.75

$$m(t) = m(0) e^{kt} = 100 e^{kt}$$

$$m(-5) = 200 = 100 e^{-5k} \Rightarrow e^{-5k} = 2 \Rightarrow k = \frac{\ln(2)}{-5}$$

$$m(t) = 100 e^{-\frac{\ln(2)}{5}t} \quad / \quad 25 = 100 e^{\frac{-\ln 2}{5}t} \Rightarrow \frac{1}{4} = e^{\frac{-\ln(2)}{5}t}$$

$$\Rightarrow \ln\left(\frac{1}{4}\right) = t \cdot -\frac{\ln(2)}{5} \Rightarrow -\ln(2^2) = -t \cdot \frac{\ln(2)}{5} \Rightarrow t = \frac{2 \cdot \ln(2) \cdot 5}{\ln 2} = \boxed{10}$$

14. (4 points) Let $f(x) = \frac{e^{-x} + 2}{x}$. Which of the following statements is correct concerning the improper integral $\int_1^\infty f(x) dx$?

A. Since $f(x) \geq \frac{1}{x^2}$, by the comparison test the integral converges.

B. Since $f(x) \leq 2 + e^{-x}$, by the comparison test the integral converges.

C. The comparison test cannot be used.

D. Since $f(x) \leq \frac{3}{x}$, by the comparison test the integral converges.

E. Since $f(x) \geq \frac{1}{x}$, by the comparison test the integral diverges.

for any $x > 1$,
 $e^{-x} > 0$, so

$e^{-x} + 2 > 2 > 1$,

so $\frac{e^{-x} + 2}{x} > \frac{1}{x}$.
 $\int_1^\infty \frac{1}{x} dx$ diverges.
 $(p=1)$

15. (4 points) If $f(x) = 4x + \cos(x)$, find $(f^{-1})'(1)$, knowing that $f(0) = 1$.

- A. $\frac{1}{2}$.
B. $-\frac{1}{2}$.
C. $\frac{1}{3}$.
 D. $\frac{1}{4}$.
E. $-\frac{1}{4}$.

$$f(0) = 1 \Rightarrow f^{-1}(1) = 0$$

$$f'(x) = 4 - \sin(x).$$

$$(f^{-1})'(1) = \frac{1}{f'(f^{-1}(1))} = \frac{1}{f'(0)} = \frac{1}{4-0} = \boxed{\frac{1}{4}}$$

More Challenging Questions. Show all work to receive credit. Please **BOX** your final answer.

16. (7 points) You and your friend both evaluate an integral and get different answers. You get that the integral is $\tan^2(x)/2 + C$, and your friend gets that the integral is $\sec^2(x)/2 + C$. Your TA tells the two of you that you are both correct. Use the constant of integration to explain how this can be the case even though $\tan(x) \neq \sec(x)$.

Note That $\tan^2 x + 1 = \sec^2 x$ for all x . So,

$$\begin{aligned}\frac{\sec^2 x}{2} + C &= \frac{\tan^2 x + 1}{2} + C = \frac{\tan^2 x}{2} + \left(C + \frac{1}{2}\right) \\ &= \frac{\tan^2 x}{2} + C',\end{aligned}$$

where $C' = C + \frac{1}{2}$ is just a constant.

17. (7 points) Explain why $\int_{-100}^{100} \sinh(\sin(x)) dx = 0$. Hint: Don't try to evaluate this integral. Instead, start by thinking about how $\sin(-x)$ relates to $\sin(x)$.

Recall That Sin and Sinh are odd functions;
i.e. $\sin(-x) = -\sin(x)$, $\sinh(-x) = -\sinh(x)$.

Now, $\sinh(\sin(-x)) = \sinh(-\sin x) = -\sinh(\sin x)$.

Thus $\sinh(\sin(x))$ is also odd. Since The interval $[-100, 100]$ is symmetric about 0, The integral $\int_{-100}^{100} \sinh(\sin(x)) dx = 0$ (Zero Signed area).