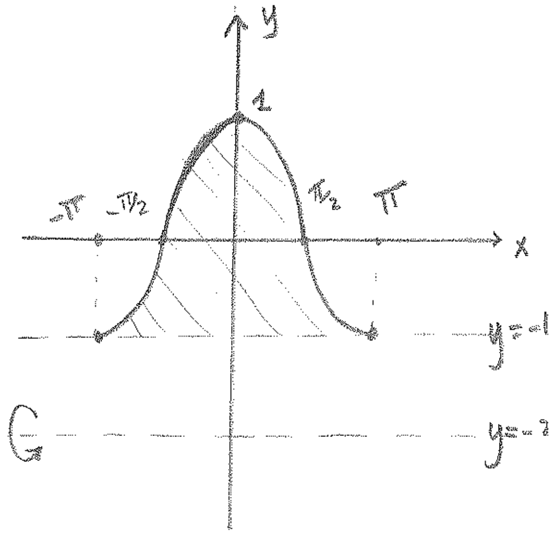
Exam 1 Review

5.2 Volumes.

Find the volume of the solid of revolution obtained by rotating the region bounded by $y = \cos x$, $y = -1$, $x = -\pi$ and $x = \pi$ about the line $y = -2$.

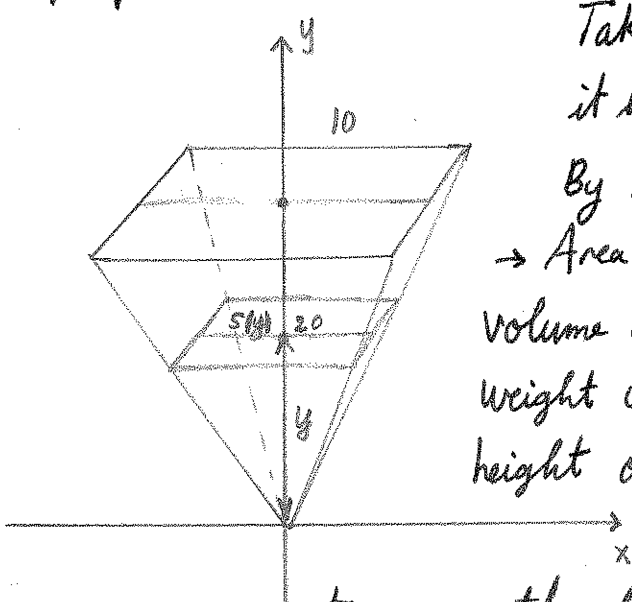


We use the washer method (see figure). we have
$R(x) = \cos(x) - (-2) = \cos x + 2$, and $r(x) = -1-(-2) = 1$

Therefore The vertical cross-section is a circular washer of area $A(x) = \pi\left(R^2(x) - r^2(x)\right) = \pi\left[(\cos x+2)^2 - 1^2\right]$.
Finally $V = \int_{-\pi}^{\pi} \pi\left[\cos^2 x + 4\cos x + 3\right] dx = \text{------}$.

5.4 Work.

A tank of the shape of a downward pointing pyramid, with a square base of side length 10ft and height 20ft, is filled with a liquid weighing 6 lbs/ft^3. Calculate the work required to pump the liquid to a height of 2 feet above the top of the tank.



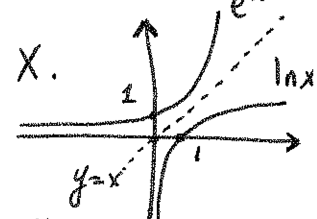
Take a horizontal cross-section at height y; it has a square shape of side length $S(y)$.
By similar triangles $\frac{S(y)}{y} = \frac{10}{20} \Rightarrow S(y) = \frac{1}{2}y$.
$\to$ Area of slice is $A(y) = S^2(y) = \frac{1}{4}y^2$ and so volume of slice is $V(y) = A(y)\cdot\Delta y = \frac{1}{4}y^2\cdot\Delta y$. Thus, weight of slice is $w(y) = V(y)\cdot 6\,\text{lbs}/\text{ft}^3 = \frac{6}{4}y^2\,\Delta y$.
height of slice at y is $y \Rightarrow$ distance to the pump is $d(y) = 22 - y$. Therefore, the work needed to pump the slice at y is $W(y) = w(y)\cdot d(y) = \frac{6}{4}y^2\cdot(22-y)\cdot\Delta y$.

Finally, total work is $W = \int_0^{20} \frac{6}{4}y^2(22-y)\,dy = 28000\ J$.

6.1_ Inverse functions: $f(x)=y \Leftrightarrow f^{-1}(y)=x$. $f(f^{-1}(x)) = f^{-1}(f(x)) = x$.
Graphs of f and f^{-1} are symmetric w.r.t. $y=x$.
In order for f to have an inverse, it must be one-to-one (passes the horizontal line test).
Derivative of f^{-1}: $(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$; Domain (f) = Range (f^{-1}) and Vice-Versa.
. to find a formula for f^{-1}, swap x and y and solve for y (if possible).

6.2_ Natural log (ln): Domain = $(0,\infty)$, Range = $\mathbb{R}$. $\ln(1)=0$, $\ln(e)=1$
$\ln(x \cdot y) = \ln x + \ln y$, $\ln(x/y) = \ln x - \ln y$, $\ln(x^r) = r\ln(x)$.
$\frac{d}{dx}\ln x = \frac{1}{x}$, $\int \frac{1}{x}\,dx = \ln|x| + C$, $\int \ln x\,dx = x\ln x - x + C$ (integration by Parts).

6.3_ Natural exp. (e^x) Domain = $\mathbb{R}$, Range = $(0,\infty)$. $e^{\ln x} = \ln(e^x) = x$.
$\frac{d}{dx}e^x = e^x$, $\int e^x\,dx = e^x + C$.



6.4_ General log and exp. $\log_a x = \frac{\ln x}{\ln a}$, $a^x = e^{x \cdot \ln(a)}$, $\log_a(a^x) = a^{\log_a x} = x$.
$\frac{d}{dx}a^x = a^x \ln a$, $\frac{d}{dx}\log_a x = \frac{1}{x \cdot \ln a}$, $\int a^x\,dx = \frac{a^x}{\ln a} + C$

$\int \tan x\,dx = -\ln|\cos x| + C = \ln|\sec x| + C$ (u-sub, $u = \cos x$).
Log-differentiation: find $\frac{d}{dx}(x^2+x)^{x-2}$. let $y = (x^2+x)^{x-2}$; $\ln(y) = (x-2)\ln(x^2+x)$
Then $\frac{y'}{y} = \frac{1}{x^2+x}\cdot(2x+1)\cdot(x-2) + 1\cdot\ln(x^2+x)$. Then multiply by y to get y'.

9.3_ Separable equations. Solve the initial value problem $(x^2+1)y' = y$, $y(0) = \frac{1}{e}$.
$(x^2+1)\frac{dy}{dx} = y \Rightarrow \frac{dy}{y} = \frac{dx}{x^2+1} \Rightarrow \int\frac{dy}{y} = \int\frac{dx}{x^2+1}$ and so,

$\ln|y| = \tan^{-1}(x) + C$. $y(0) = e^{-1} \Rightarrow \ln e^{-1} = \tan^{-1}0 + C \Rightarrow C = -1-0 = -1$.
$\ln|y| = \tan^{-1}(x) - 1 \Rightarrow y = e^{\tan^{-1}x - 1}$.

6.5_ Exp Growth and Decay. Rate of change of a quantity P is proportional to its current size: $\frac{dP}{dt} = kP$; k = relative growth rate.
Solution: $P(t) = P(0)e^{kt}$. At half life, $P(t) = \frac{1}{2}P(0)$. $P(0)$ = initial size.

Continuously Compounded interest: $A(t) = A_0 e^{rt}$. A_0 = principle, r = interest Rate.

6.6 - Inverse trig. functions. Most important functions are

$\sin^{-1}(x)$, Range $= [-\pi/2, \pi/2]$, $\frac{d}{dx}(\sin^{-1}(x)) = \frac{1}{\sqrt{1-x^2}}$
$\cos^{-1}(x)$, Range $= [0, \pi]$, $\frac{d}{dx}(\cos^{-1}(x)) = \frac{-1}{\sqrt{1-x^2}}$
$\tan^{-1}(x)$, Range $= (-\pi/2, \pi/2)$, $\frac{d}{dx}(\tan^{-1}(x)) = \frac{1}{1+x^2}$, $\lim_{x\to\infty} \tan^{-1}(x) = \frac{\pi}{2}$
$\sec^{-1}(x)$, Range $= [0, \frac{\pi}{2}) \cup [\pi, \frac{3\pi}{2})$, $f'(x) = \frac{1}{x\sqrt{x^2-1}}$. $\lim_{x\to-\infty} \tan^{-1}(x) = -\frac{\pi}{2}$.

6.7 - Hyperbolic functions $\sinh x = \frac{e^x - e^{-x}}{2}$, $\cosh x = \frac{e^x + e^{-x}}{2}$, $\tanh x = \frac{\sinh x}{\cosh x}$.
$\cosh^2 x - \sinh^2 x = 1$, $1 - \tanh^2 x = \operatorname{sech}^2 x$. $(\sinh x)' = \cosh x$, $(\cosh x)' = \sinh x$.

6.8 - L'Hospital's Rule. Works on $\frac{\infty}{\infty}$ or $\frac{0}{0}$ type limits: $\lim_{x\to a} \frac{f}{g} = \lim_{x\to a} \frac{f'}{g'}$.
- Differences: $\infty - \infty$ → write as a quotient (common denominator).
- products: $\infty * 0$ → write as a quotient: $f*g = f \div \frac{1}{g}$.
- powers: 0^0, ∞^0, $1^{\pm\infty}$: Use the natural log.

7.1 Integration by Parts. $\int u\,dv = uv - \int v\,du$. Choose u and dv so that u is easy to differentiate, and/or dv is easy to integrate. Examples: $\int \ln x\,dx$, $\int \sqrt{x}\ln x\,dx$, $\int x^2\cos x\,dx$, $\int x^3 e^{-5x}\,dx$, $\int e^x \sin x\,dx$, $\int e^x\cos x\,dx$, $\int \tan^{-1}x\,dx$, $\int \sec^3 x\,dx$, etc ....

7.2 trig. integrals. $\int \sin^n(x)\cos^m(x)\,dx$, $\int \tan^n(x)\sec^m(x)\,dx$. Use identities $\cos^2 x + \sin^2 x = 1$, $\tan^2 x + 1 = \sec^2 x$, $\sec^2 x - 1 = \tan^2 x$, $\sin 2x = 2\sin x\cos x$, $\cos^2 x = \frac{1+\cos 2x}{2}$, $\sin^2 x = \frac{1-\cos 2x}{2}$. Examples include: $\int \sin^3 x\cos^2 x\,dx$, $\int \cos^3 x\,dx$, $\int \sec^2 x\tan^2 x\,dx$, $\int \sin^4 x\,dx$, $\int \sin^5 x\cos^{-3} x\,dx$, etc. .....

7.3 Trig. substitution: Use $\sin^2 x + \cos^2 x = 1$, $\sec^2 x - 1 = \tan^2 x$, $\tan^2 x + 1 = \sec^2 x$ on:
$\sqrt{a^2 - x^2}$: $x = a\sin\theta$, $dx = a\cos\theta\,d\theta$
$\sqrt{a^2 + x^2}$: $x = a\tan\theta$, $dx = a\sec^2\theta\,d\theta$
$\sqrt{x^2 - a^2}$: $x = a\sec\theta$, $dx = a\sec\theta\tan\theta\,d\theta$

7.4 Partial fractions: $\int \frac{P(x)}{Q(x)} dx$, P, Q are polynomials.

- Step zero: long division if degree (P) ≥ degree (Q)
- Step 1: factor Q completely. If Q doesn't factor → Complete the square.
- Step 2: set-up partial fractions:
  - linear factors $(ax+b)^n$ in $Q \longrightarrow \frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_n}{(ax+b)^n}$.
  - Irreducible quadratic factors in Q: $(x^2+a^2)^n$: $\frac{A_1x+B_1}{(x^2+a^2)} + \cdots + \frac{(A_nx+B_n)}{(x^2+a^2)^n}$.

7.5 - Strategy for integration.

1- Simplify the integrand: $\int \sqrt{x}(1+\sqrt{x})dx = \int x+\sqrt{x}\, dx$
2- Look for obvious substitutions
3- Classify the integrand according to its form:

ⓐ trig function ⓑ rational fcn ⓒ Integ. by parts ⓓ radicals $\sqrt{\pm x^2 \pm a^2}$

7.8 - Improper Integrals

Type I: $\int_a^\infty$, $\int_{-\infty}^b$, $\int_{-\infty}^\infty$. Use limits to replace ∞ and/or $-\infty$

Type II: discontinuous integrands $\int_a^b f(x)\,dx$, f discntns at a, and/or b, and/or a point c in between: split integral at c and use limits (1-sided).

Ex: $\int_{-1}^1 \frac{1}{x^2}dx = \int_{-1}^0 \frac{1}{x^2}dx + \int_0^1 \frac{1}{x^2}dx = \lim_{t\to 0^-}\int_{-1}^t \frac{1}{x^2}dx + \lim_{t\to 0^+}\int_t^1 \frac{1}{x^2}dx$.

Theorem: $\int_1^\infty \frac{1}{x^p}dx$ Converges if $p>1$ and diverges if $p\le 1$.

Direct Comparison test: if $f(x) \ge g(x) \ge 0$, then $\int f(x)dx$ diverges if $\int g(x)dx$ div, and $\int g(x)dx$ converges if $\int f(x)dx$ conv.

Limit Comparison test: if $f(x), g(x) \ge 0$ and $\lim_{x\to\infty} \frac{f(x)}{g(x)} = L$, $0<L<\infty$, then

$\int f(x)dx$ and $\int g(x)dx$ either both converge or both diverge.