LOWER LIPSCHITZ BOUNDS FOR PHASE RETRIEVAL FROM LOCALLY SUPPORTED MEASUREMENTS

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Abstract. In this short note, we consider the worst case noise robustness of any phase retrieval algorithm which aims to reconstruct all nonvanishing vectors $\mathbf{x} \in \mathbb{C}^d$ (up to a single global phase multiple) from the magnitudes of shifted local correlation measurements. Examples of such measurements include both spectrogram measurements of $\mathbf{x}$ using locally supported windows and masked Fourier transform intensity measurements of $\mathbf{x}$ using bandlimited masks. As a result, the robustness results considered herein apply to a wide range of both ptychographic and Fourier ptychographic imaging scenarios. In particular, the main results imply that the accurate recovery of high-resolution images of extremely large samples using highly localized probes is likely to require an extremely large number of measurements in order to be robust to worst case measurement noise, independent of the recovery algorithm employed. Furthermore, recent pushes to achieve high-speed and high-resolution ptychographic imaging of integrated circuits for process verification and failure analysis will likely need to carefully balance probe design (e.g., their effective time-frequency support) against the total number of measurements acquired in order for their imaging techniques to be stable to measurement noise, no matter what reconstruction algorithms are applied.

1. Introduction and Statement of Results

We consider the robustness of the *finite-dimensional phase retrieval problem* in which one attempts to recover a signal $\mathbf{x} := (\mathbf{x}(1), \dots, \mathbf{x}(d))^T \in \mathbb{C}^d$ from one of two nonlinear measurement maps $\alpha, \beta : \mathbb{C}^d \to \mathbb{R}^N$ given by

$$\alpha(\mathbf{x}) = \{|\langle \mathbf{x}, \mathbf{f_k} \rangle|\}_{k=1}^N \text{ and } \beta(\mathbf{x}) = \{|\langle \mathbf{x}, \mathbf{f_k} \rangle|^2\}_{k=1}^N,$$

where the vectors $\{\mathbf{f_1}, \dots, \mathbf{f_N}\} \subset \mathbb{C}^d$ form a frame (i.e., a spanning set) of $\mathbb{C}^d$. This problem is motivated by inverse problems that arise in several scientific areas including optics [23], astronomy [10], quantum mechanics [9], and audio signal processing [18, 22]. In particular, we will focus on a special class of frame vectors $\mathbf{f_k}$ which have *localized support* (i.e., all of whose nonzero entries are contained in an interval of length at most $\delta \ll d$). Such frames are commonly encountered in applications like ptychographic imaging in which small overlapping regions of a much larger specimen are illuminated one at a time, and a detector captures the intensities of the resulting local diffraction patterns [20].

It is clear that for any $\theta \in \mathbb{R}$ one has $\alpha(\mathbb{e}^{i\theta}\mathbf{x}) = \alpha(\mathbf{x})$ and $\beta(\mathbb{e}^{i\theta}\mathbf{x}) = \beta(\mathbf{x})$. Therefore, we can at best hope to recover $\mathbf{x}$ up to the equivalence relation $\mathbf{x} \sim \mathbf{x}'$, if $\mathbf{x} = \mathbb{e}^{i\theta}\mathbf{x}'$ for some $\theta \in \mathbb{R}$. Following the work of Balan et al. [2, 3], we will consider two commonly used metrics on $\mathbb{C}^d/\sim$: the natural metric

$$D_2(\mathbf{x}, \mathbf{x}') := \min_{\theta \in \mathbb{R}} \|\mathbf{x} - \mathbb{e}^{\mathbb{i}\theta}\mathbf{x}'\|_2,$$

and the matrix-norm induced metric

$$d_1(\mathbf{x}, \mathbf{x}') := \|\mathbf{x}\mathbf{x}^* - \mathbf{x}'\mathbf{x}'^*\|_1 := \sum_k \sigma_k(\mathbf{x}\mathbf{x}^* - \mathbf{x}'\mathbf{x}'^*),$$

where $\sigma_k(\mathbf{x}\mathbf{x}^* - \mathbf{x}'\mathbf{x}'^*)$ is the k-th singular value of the (at most rank-two) matrix $\mathbf{x}\mathbf{x}^* - \mathbf{x}'\mathbf{x}'^*$. In [3], Balan et al. showed that if α and β are injective on $\mathbb{C}^d/\sim$, then β is bi-Lipschitz with respect to d_1, and α is bi-Lipschitz with respect to D_2, where in both cases $\mathbb{R}^N$ is equipped with the Euclidean norm.

Motivated by applications such as (Fourier) ptychography [20, 24] and related numerical methods [15, 16], we will study frames which are constructed as the shifts of a family of locally supported measurement vectors.

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Specifically, we assume that $\{\mathbf{m_1}, \mathbf{m_2}, \ldots, \mathbf{m_K}\}$ is a family of measurement masks in $\mathbb{C}^d$ such that for all $1 \leq k \leq K$ the nonzero entries of $\mathbf{m_k}$ are contained in the set $[\delta] := \{1, \ldots, \delta\}$ for some $\delta \leq \frac{d}{4}$ (although all of our results remain valid if the support of our masks are contained in any interval of length δ).

Definition 1. *Let L be an integer which divides d, such that $a := \frac{d}{L} < \delta$. We define the quadratic nonlinear phaseless measurement map $Y : \mathbb{C}^d \to \mathbb{R}^{K \times L}$ by its coordinate functions*

$$Y_{k,\ell}(\mathbf{x}) := |\langle S_{\ell a}\mathbf{m_k}, \mathbf{x}\rangle|^2. \tag{1.1}$$

Definition 2. *Let L be an integer which divides d, such that $a = \frac{d}{L} < \delta$. We define the first-order nonlinear phaseless measurement map $Z : \mathbb{C}^d \to \mathbb{R}^{K \times L}$ by its coordinate functions*

$$Z_{k,\ell}(\mathbf{x}) := |\langle S_{\ell a}\mathbf{m_k}, \mathbf{x}\rangle|. \tag{1.2}$$

In both Definiton 1 and Definition 2, S_ℓ is the circular shift operator on $\mathbb{C}^d$ defined for all $\ell \in \mathbb{Z}$ by

$$(S_\ell \mathbf{x})(n) := \mathbf{x}\left((n + \ell - 1) \bmod d + 1\right).$$

(The $+\,1$ is needed because we are indexing our vectors from one.) For notational convenience, we will assume that d is even, although our results remain valid, with similar proofs, when d is odd.

The purpose of this paper is to provide lower bounds on the Lipschitz constants of any maps, A and B, which reconstruct $\mathbf{x}$ from Y and Z, respectively. With such lower bounds in hand, one would be better equipped to, e.g., judge the optimality of theoretical noisy reconstruction guarantees for phase retrieval algorithms which utilize locally supported measurements (see, e.g., [15, 16]). Unfortunately, Y and Z are not injective on all of $\mathbb{C}^d/\sim$. For example, if two vectors $\mathbf{x}^\pm \in \mathbb{C}^d$ are defined by

$$\mathbf{x}^\pm(n) := \begin{cases} 1, & 1 \leq n \leq \frac{d}{2} - \delta \\ 0, & \frac{d}{2} - \delta < n \leq \frac{d}{2} \\ \pm 1, & \frac{d}{2} < n \leq d - \delta \\ 0, & d - \delta < n \leq d \end{cases}, \tag{1.3}$$

then $\mathbf{x}^+ \not\sim \mathbf{x}^-$, but $Y(\mathbf{x}^+) = Y(\mathbf{x}^-)$ and $Z(\mathbf{x}^+) = Z(\mathbf{x}^-)$. (If d were odd, we could add an extra entry of 1 to $\mathbf{x}^\pm$.) However, it can be shown [15] that Y and Z are injective when restricted to the subset of $\mathbb{C}^d$ such that $\mathbf{x}(n) \neq 0$ for all $1 \leq n \leq d$, for certain choices of masks in the case where $L = d$. Given this, we will consider the maps Y and Z restricted to the subset

$$\mathcal{C}_{p,q} = \{\mathbf{x} \in \mathbb{C}^d/\sim \text{ such that } p \leq |\mathbf{x}(n)| \leq q \text{ for all } 1 \leq n \leq d\},$$

for some fixed $0 < p \leq q$, and provide lower bounds on the Lipschitz constants of A and B which grow linearly with respect to the ratio $\frac{q}{p}$.

1.1. Related Work and Implications.

Our local measurement maps, defined in (1.1) and (1.2), are closely related to several practical measurement models that have been explored in the phase retrieval literature including, for example, Short-Time Fourier Transform (STFT) magnitude measurements (see, e.g., [5, 17, 19, 21]). In particular, suppose that our STFT magnitude measurements are generated by a compactly supported window $\mathbf{w} \in \mathbb{C}^d$ whose n^{th}-entry $\mathbf{w}(n)$ is nonzero only if $n \in [\delta]$. In this setting, we can use one locally supported mask $\mathbf{m_k}$ to represent each measured frequency $\omega_k \in \Omega \subset [d] := \{1, \ldots, d\}$ by letting $\mathbf{m_k} := W_{\omega_k}\mathbf{w}$ for each frequency index k, where W_{ω_k} is the modulation operator defined on $\mathbb{C}^d$ by

$$(W_{\omega_k}\mathbf{w})(n) := \mathbb{e}^{\frac{2\pi \mathbb{i}(n-1)(\omega_k - 1)}{d}}\mathbf{w}(n).$$

In this case, we have

$$|\langle S_{\ell a}\mathbf{m_k}, \mathbf{x}\rangle| \;=\; |\langle \mathbf{x}, S_{\ell a}W_{\omega_k}\mathbf{w}\rangle| \;=\; \left|\left\langle \mathbf{x}, \mathbb{e}^{\frac{2\pi \mathbb{i}\ell a(\omega_k - 1)}{d}} W_{\omega_k}S_{\ell a}\mathbf{w}\right\rangle\right| \;=\; |\langle \mathbf{x}, W_{\omega_k}S_{\ell a}\mathbf{w}\rangle|$$

for all k and ℓ. Therefore, one can see that the main results below yield lower Lipschitz bounds for any such STFT magnitude measurements in terms of the total number of shifts L, the number of measured frequencies K, and the window $\mathbf{w}$'s support size δ.

Another common model considered in the phase retrieval literature concerns masked Fourier measurements of the form

$$|F\ \mathrm{Diag}(\mathbf{w_k})\ \mathbf{x}|^2, \tag{1.4}$$

where F is the $d \times d$ discrete Fourier transform matrix whose entries are defined by

$$F_{j,k} := \mathbb{e}^{-2\pi \mathbb{i} \frac{(j-1)(k-1)}{d}},$$

and $\{\mathbf{w_1}, \ldots, \mathbf{w_k}\} \subset \mathbb{C}^d$ is a family of measurement vectors (see, e.g., [4, 7, 8, 13]). In this setting one can ask what effect, if any, requiring each $\mathbf{w_k}$ to be bandlimited (i.e., to have support size $\delta \ll d$ in the Fourier basis) might have on the stability of these measurements. Furthermore, one might also consider subsampling each of the masked Fourier measurements in frequency instead of acquiring measurements for all d frequencies. (This may even be a necessity due to, for example, detector limitations.) We will show that our results may also be applied to these types of measurements as a special case.

Suppose for example that each measurement vector $\mathbf{w_k}$ has $\widehat{\mathbf{w_k}}(n) := (F\mathbf{w_k})(n) = 0$ for all $n \notin \{1\} \cup \{d-\delta+2, \ldots, d\}$.[1] For a vector $\mathbf{u} \in \mathbb{C}^d$, let $\tilde{\mathbf{u}} \in \mathbb{C}^d$ be the vector obtained by reflecting the entries of $\mathbf{u}$ about its first entry so that

$$\tilde{\mathbf{u}}(n) := \mathbf{u}\left((1-n) \bmod d+1\right).$$

In this case, we see that the measurements (1.4) are given by the quadratic measurement map defined in (1.1) applied to $\widehat{\mathbf{x}}$ with the locally supported measurement masks $\mathbf{m_k} := \frac{1}{d}\overline{\widetilde{\widehat{\mathbf{w_k}}}}$. Indeed,

$$\begin{aligned}
|\langle S_{\ell a}\mathbf{m_k}, \widehat{\mathbf{x}}\rangle| &= \frac{1}{d}\left|\left\langle \widehat{\mathbf{x}}, S_{\ell a}\overline{\widetilde{\widehat{\mathbf{w_k}}}}\right\rangle\right| = \frac{1}{d}\left|\sum_{n=1}^{d} \widehat{\mathbf{x}}\,(n)\, S_{\ell a}\widetilde{\widehat{\mathbf{w_k}}}\,(n)\right| \\
&= \frac{1}{d}\left|\sum_{n=1}^{d} \widehat{\mathbf{x}}\,(n)\, \widehat{\mathbf{w_k}}\left((1-\ell a-n) \mod d+1\right)\right| \\
&= \frac{1}{d}\left|\left(\widehat{\mathbf{w_k}} * \widehat{\mathbf{x}}\right)\left(-\ell a \bmod d+1\right)\right|, \qquad (1.5)
\end{aligned}$$

where $*$ is circular convolution given by

$$(\mathbf{x} * \mathbf{y})(m) = \sum_{n=1}^{d} \mathbf{x}(n)\mathbf{y}((m-n) \bmod d+1).$$

Continuing from (1.5), we see by the convolution theorem

$$|\langle S_{\ell a}\mathbf{m_k}, \widehat{\mathbf{x}}\rangle| \;=\; \left|F\left(\mathbf{w_k} \circ \mathbf{x}\right)\left((-\ell a \bmod d)+1\right)\right| \;=\; \left|F\left(\mathrm{Diag}\left(\mathbf{w_k}\right)\mathbf{x}\right)\left((-\ell a \bmod d)+1\right)\right|,$$

where $\circ$ represents the Hadamard (componentwise) product.

As a result, we see that recovering a vector $\mathbf{x}$ from masked Fourier measurements of the form (1.4) with bandlimited measurement vectors $\mathbf{w_k}$ is equivalent to recovering $\widehat{\mathbf{x}}$ from the quadratic measurements (1.1) with locally supported measurement masks $\mathbf{m_k}$. Therefore, the main results below also yield lower Lipschitz bounds for any such masked Fourier magnitude measurements in terms of the total number of frequencies L collected per measurement vector, the total number K of measurement vectors used, and the maximum Fourier support size δ of each bandlimited measurement vector.

1.2. **Main Results.** The main results of this paper are the following two theorems which provide lower bounds for the Lipschitz constants of any maps A and B for which $A(Y(\mathbf{x})) = \mathbf{x}$ and $B(Z(\mathbf{x})) = \mathbf{x}$ for all $\mathbf{x} \in \mathcal{C}_{p,q}$.

Theorem 1. *Let $0 < p \leq q$, and consider the map Z, defined as in (1.2), restricted to the subset $\mathcal{C}_{p,q} \subset \mathbb{C}^d/\sim$. Assume that $\delta \leq \frac{d}{4}$ and that $d = aL$ for some integer $1 \leq a < \delta$. Then if B is any Lipschitz map (with respect to D_2) such that $B(Z(\mathbf{x})) = \mathbf{x}$ for all $\mathbf{x} \in \mathcal{C}_{p,q}$, we have that*

$$C_B \geq \frac{1}{8}\frac{q\sqrt{da}}{p\sqrt{K}\|\mathbf{m}\|_\infty \delta^{3/2}} = \frac{1}{8}\frac{qd}{p\sqrt{KL}\|\mathbf{m}\|_\infty \delta^{3/2}}, \qquad (1.6)$$

where C_B is the Lipschitz constant of B, and $\|\mathbf{m}\|_\infty := \max_{1\leq k\leq K}\|\mathbf{m_k}\|_\infty$.

[1] Note that this particular support interval (modulo d) is not particularly special. The same arguments below can be extended to apply to any interval of support of size $\leq \delta$ in a straightforward fashion.

Theorem 2. *Let $0 < p \leq q$, and consider the map Y, defined as in (1.1), restricted to the subset $\mathcal{C}_{p,q} \subset \mathbb{C}^d/\sim$. Assume that $\delta \leq \frac{d}{4}$ and that $d = aL$ for some integer $1 \leq a < \delta$. Then if A is any Lipschitz map (with respect to d_1) such that $A(Y(\mathbf{x})) = \mathbf{x}$ for all $\mathbf{x} \in \mathcal{C}_{p,q}$, we have that*

$$C_A \geq \frac{1}{16}\frac{qd\sqrt{a}}{p\sqrt{K}\|\mathbf{m}\|_\infty^2\delta^{5/2}} = \frac{1}{16}\frac{qd^{3/2}}{p\sqrt{KL}\|\mathbf{m}\|_\infty^2\delta^{5/2}}, \tag{1.7}$$

where C_A is the Lipschitz constant of A, and $\|\mathbf{m}\|_\infty := \max_{1\leq k\leq K}\|\mathbf{m_k}\|_\infty$.

Ideally, we would like a stable phase retrieval algorithm to have have $C_A = \mathcal{O}(1)$ (or $C_B = \mathcal{O}(1)$) while using only $KL = \mathcal{O}(d)$ total measurements, i.e., while having the frame redundancy $\frac{KL}{d} = \frac{K}{a} = \mathcal{O}(1)$. Unfortunately, Theorems 1 and 2 demonstrate that this is impossible when δ, the support size of the masks, is very small. At best, a phase retrieval algorithm that uses only $KL = \mathcal{O}(d)$ local correlation measurements can have global Lipschitz constants that are of size $\mathcal{O}\left(\frac{d}{\delta^{5/2}}\right)$ in the case of the quadratic Y-measurements, and $\mathcal{O}\left(\frac{\sqrt{d}}{\delta^{3/2}}\right)$ in the case of the first-order Z-measurements. This implies that extremely large samples $\mathbf{x}$ (i.e., with d large) cannot be stably recovered from measurements which are noisy and extremely localized (i.e., with δ small) in the worst case using only $\mathcal{O}(d)$ total measurements. To contextualize this in an application setting, one may consider recent research initiatives aimed at achieving the ability to rapidly obtain detailed images of relatively large circuit boards [14]. One approach to solving this problem involves using ptychographic imaging and taking STFT magnitude measurements of the circuit board using a probe (i.e., an STFT window function) with a comparably small effective support size δ. In this context, Theorem 2 implies that the probe's effective support size should not be taken to be too small unless additional measurements are taken in order to help ensure stability to noise.

Algorithms for inverting the quadratic measurements (1.1) were presented in [15] and [16] along with upper bounds for the stability of these algorithms to noise. In particular, in [16], it was shown, in the case that $L = d$, and $K = 2\delta - 1$, that

$$D_2(\mathbf{x},\mathbf{x}') \leq C\kappa\frac{q}{p^2}\left(\frac{d}{\delta}\right)^2\|Y(\mathbf{x}) - Y(\mathbf{x}')\|_2 + Cd^{1/4}\sqrt{\kappa\|Y(\mathbf{x}) - Y(\mathbf{x}')\|_2}, \tag{1.8}$$

where κ is the condition number of a certain linear system which arises in the proposed algorithm. [16] considers two examples of well-conditioned families of masks, and shows that in both cases κ can be bounded as a function of δ. In particular, for the masks considered in section 3.2, it is shown that $\kappa \leq 4\delta$. These upper bounds are not directly comparable to the main results of this paper because, in general, the quadratic measurements (1.1) are not Lipschitz with respect to D_2. However, like Theorems 1 and 2, (1.8) shows that the stability of the measurements detioriates when d is much larger than δ or when q is much larger than p.

As we shall see, the proofs of both Theorems 1 and 2 will depend on signals modeled along the lines of (1.3) whose support sets are composed of two disjoint components separated from one another by at least δ zeroes. In [15] it was noted that phase retrieval of such signals using locally supported masks $\mathbf{m_k}$ of the type proposed herein was impossible, and that recovery of signals with more than δ consecutive small entries appeared to be unstable. Interestingly enough, subsequent work in the infinite-dimensional setting has independently identified such disjointly supported signals as being the principal cause of instability in phase retrieval problems using continuous Gabor measurements as well because they lead to measurements which are supported on disjoint subsets of the time-frequency plane [1, 12]. Similarly, we will use (essentially) disjointly supported signals similar to those in (1.3) to provide lower bounds on the Lipschitz constants of our maps A and B using the fact that they (i) are far apart with respect to the D_2 and d_1 metrics defined above and (ii) produce measurements with respect to our maps Y and Z which are (nearly) identical. While we do not prove that our bounds are sharp, we do note that the signals signals considered below are as close as possible, among elements of $\mathcal{C}_{p,q}$, to those in (1.3). Therefore, we believe our bounds are likely quite close to being sharp, although this remains a conjecture at this point.

2. The Proofs of Theorem 1 and Theorem 2

We are now prepared to prove our main results.

Proof of Theorem 1. First observe that for any $\mathbf{x}, \mathbf{x}' \in \mathcal{C}_{p,q}$,

$$D_2(\mathbf{x},\mathbf{x}') = D_2(B(Z(\mathbf{x})), B(Z(\mathbf{x}'))) \leq C_B\|Z(\mathbf{x}) - Z(\mathbf{x}')\|_2.$$

Therefore,

$$C_B \geq \sup \frac{D_2(\mathbf{x}, \mathbf{x}')}{\|Z(\mathbf{x}) - Z(\mathbf{x}')\|_2}, \tag{2.1}$$

where the supremum is taken over all $\mathbf{x} \not\sim \mathbf{x}' \in \mathcal{C}_{p,q}$. Define $\mathbf{x}^+$ and $\mathbf{x}^- \in \mathbb{C}^d$ by

$$\mathbf{x}^{\pm}(n) := \begin{cases} q, & 1 \leq n \leq \frac{d}{2} - \delta \\ p, & \frac{d}{2} - \delta < n \leq \frac{d}{2} \\ \pm q, & \frac{d}{2} < n \leq d - \delta \\ p, & d - \delta < n \leq d \end{cases}.$$

Note that $D_2(\mathbf{x}^+, \mathbf{x}^-) \geq q\sqrt{d}$ since $\delta < \frac{d}{4}$ and for all $\theta \in \mathbb{R}$,

$$\begin{aligned} \|\mathbf{x}^+ - \mathrm{e}^{\mathrm{i}\theta}\mathbf{x}^-\|_2^2 &\geq \sum_{n=1}^{d/2-\delta} |(1 - \mathrm{e}^{\mathrm{i}\theta})q|^2 + \sum_{n=d/2+1}^{d-\delta} |(1 + \mathrm{e}^{\mathrm{i}\theta})q|^2 \\ &= \left(\frac{d}{2} - \delta\right) q^2 |1 - \mathrm{e}^{\mathrm{i}\theta}|^2 + \left(\frac{d}{2} - \delta\right) q^2 |1 + \mathrm{e}^{\mathrm{i}\theta}|^2 \\ &\geq \frac{d}{4} q^2 \left(|1 - \mathrm{e}^{\mathrm{i}\theta}|^2 + |1 + \mathrm{e}^{\mathrm{i}\theta}|^2\right) = dq^2, \end{aligned}$$

since $|1 - \mathrm{e}^{\mathrm{i}\theta}|^2 + |1 + \mathrm{e}^{\mathrm{i}\theta}|^2 = 4$ for all θ. Let $Z^{\pm} := Z(\mathbf{x}^{\pm})$. We will show that

$$\|Z^+ - Z^-\|_2 \leq 8\sqrt{K} p \|\mathbf{m}\|_\infty \frac{\delta^{3/2}}{\sqrt{a}}. \tag{2.2}$$

Since $B(Z^{\pm}) = \mathbf{x}^{\pm}$, combining this with (2.1) will complete the proof.

Observe that for all k, the support of $S_{\ell a}\mathbf{m_k}$ is contained in $[1 + \ell a, \delta + \ell a]$. Therefore, $Z^+_{k,\ell} = Z^-_{k,\ell}$ except when $1 + \ell a \leq \frac{d}{2} < \delta + \ell a$ or $1 + \ell a \leq d - \delta < \delta + \ell a$ since if the support of $S_{\ell a}\mathbf{m_k}$ does not intersect $(\frac{d}{2}, d - \delta]$, we have that $\langle S_{\ell a}\mathbf{m_k}, \mathbf{x}^+\rangle = \langle S_{\ell a}\mathbf{m_k}, \mathbf{x}^-\rangle$, and if the support of $S_{\ell a}\mathbf{m_k}$ is contained in $(\frac{d}{2}, d - \delta]$, then $\langle S_{\ell a}\mathbf{m_k}, \mathbf{x}^+\rangle = -\langle S_{\ell a}\mathbf{m_k}, \mathbf{x}^-\rangle$.

We will obtain a bound for $|Z^+_{k,\ell} - Z^-_{k,\ell}|$ in the the case where $1 + \ell a \leq d - \delta < \delta + \ell a$. The case where $1 + \ell a \leq \frac{d}{2} < \delta + \ell a$ can be bounded in a similar fashion. For fixed ℓ such that $1 + \ell a \leq d - \delta < \delta + \ell a$, let

$$j := \ell a + 2\delta - d$$

so that the last j nonzero entries of $S_{\ell a}\mathbf{m_k}$ are located in positions greater than $d - \delta$ and the first $\delta - j$ nonzero entries are located in positions less than or equal to $d - \delta$. (Note that $1 \leq j \leq \delta - 1$.) Then,

$$\langle S_{\ell a}\mathbf{m_k}, \mathbf{x}^-\rangle = -q \sum_{n=1}^{\delta-j} \mathbf{m_k}(n) + p \sum_{n=\delta-j+1}^{\delta} \mathbf{m_k}(n) = -\langle S_{\ell a}\mathbf{m_k}, \mathbf{x}^+\rangle + 2p \sum_{n=\delta-j+1}^{\delta} \mathbf{m_k}(n).$$

Therefore,

$$|Z^-_{k,\ell} - Z^+_{k,\ell}| \leq 2jp\|\mathbf{m}\|_\infty. \tag{2.3}$$

Since $1 \leq j \leq \delta - 1$, summing over the set of ℓ such that $1 + \ell a \leq d - \delta < \delta + \ell a$, corresponds to summing over $j = a, 2a, \ldots, \lfloor \frac{\delta-1}{a} \rfloor a$ if a divides $d - 2\delta$, or summing over $j = j_0, j_0 + a, j_0 + 2a, \ldots, j_0 + \lfloor \frac{\delta - j_0 - 1}{a} \rfloor a$ for some $0 < j_0 < a$ otherwise. Therefore, summing over both the terms corresponding to $1 + \ell a \leq d - \delta < \delta + \ell a$ and to $1 + \ell a \leq \frac{d}{2} < \delta + \ell a$,

$$\|Z^+ - Z^-\|_2^2 \leq 2 \cdot 2^2 \|\mathbf{m}\|_\infty^2 p^2 \sum_{k=1}^{K} \sum_{t=1}^{\lfloor \delta/a \rfloor + 1} |at|^2 \leq 8Ka^2 \|\mathbf{m}\|_\infty^2 p^2 \left(\frac{\delta}{a} + 1\right)^3 \leq 64Kp^2 \|\mathbf{m}\|_\infty^2 \frac{\delta^3}{a}, \tag{2.4}$$

since $\frac{\delta}{a} \geq 1$. This proves (2.2) and therefore completes the proof. $\square$

Proof of Theorem 2. Similarly to the proof of Theorem 1,

$$C_A \geq \sup \frac{d_1(\mathbf{x}, \mathbf{x}')}{\|Y(\mathbf{x}) - Y(\mathbf{x}')\|_2}, \tag{2.5}$$

where the supremum is again taken over all $\mathbf{x} \not\sim \mathbf{x}' \in \mathcal{C}_{p,q}$. Let $\mathbf{x}^{\pm}$ be in as in the proof of Theorem 1, and let $Y^{\pm} := Y(\mathbf{x}^{\pm})$. By the same reasoning as in the previous proof, $Y^{+}_{k,\ell} = Y^{-}_{k,\ell}$, unless $1 + \ell a \leq \frac{d}{2} < \delta + \ell a$ or $1 + \ell a \leq d - \delta < \delta + \ell a$.

We will obtain a bound for $|Y^{+}_{k,\ell} - Y^{-}_{k,\ell}|$ in the case where $1 + \ell a \leq d - \delta < \delta + \ell a$. As in the proof of Theorem 1, a similar bound can be obtained in the case where $1 + \ell a \leq \frac{d}{2} < \delta + \ell a$. Let ℓ be such that $1 + \ell a \leq d - \delta < d + \ell a$, and again let $j := \ell a + 2\delta - d$. Since for all k and ℓ, we have

$$|Z^{\pm}_{k,\ell}| \leq q\|\mathbf{m}\|_\infty \delta,$$

we see

$$|Y^{+}_{k,\ell} - Y^{-}_{k,\ell}| = |(Z^{+}_{k,\ell})^2 - (Z^{-}_{k,\ell})^2| = |Z^{+}_{k,\ell} + Z^{-}_{k,\ell}||Z^{+}_{k,\ell} - Z^{-}_{k,\ell}| \leq 4\|\mathbf{m}\|_\infty^2 q\delta pj,$$

by (2.3). Therefore, by the same reasoning as in (2.4),

$$\|Y^+ - Y^-\|_2^2 \leq 32\|\mathbf{m}\|_\infty^4 q^2\delta^2 p^2 \sum_{k=1}^{K} \sum_{t=1}^{\lfloor \delta/a \rfloor + 1} |at|^2 \leq 32K\|\mathbf{m}\|_\infty^4 q^2 \delta^2 p^2 a^2 \left(\frac{\delta}{a} + 1\right)^3 = 256K\|\mathbf{m}\|_\infty^4 q^2 p^2 \frac{\delta^5}{a}.$$

Thus, the proof will follow from (2.5) once we show $d_1(\mathbf{x}^+, \mathbf{x}^-) \geq dq^2$.

For $n, m \in \mathbb{N}$, let $0_{n\times m}$ and $\mathbb{1}_{n\times m}$ denote the $n \times m$ matrices of all zeros and of all ones respectively. With this notation we see that

$$\mathbf{x}^{\pm} = (q\mathbb{1}_{1\times\eta}, p\mathbb{1}_{1\times\delta}, \pm q\mathbb{1}_{1\times\eta}, p\mathbb{1}_{1\times\delta})^T,$$

and

$$\mathbf{x}^{\pm}\mathbf{x}^{\pm *} = \begin{pmatrix} q^2\mathbb{1}_{\eta\times\eta} & qp\mathbb{1}_{\eta\times\delta} & \pm q^2\mathbb{1}_{\eta\times\eta} & qp\mathbb{1}_{\eta\times\delta} \\ qp\mathbb{1}_{\delta\times\eta} & p^2\mathbb{1}_{\delta\times\delta} & \pm qp\mathbb{1}_{\delta\times\eta} & p^2\mathbb{1}_{\delta\times\delta} \\ \pm q^2\mathbb{1}_{\eta\times\eta} & \pm qp\mathbb{1}_{\eta\times\delta} & q^2\mathbb{1}_{\eta\times\eta} & \pm qp\mathbb{1}_{\eta\times\delta} \\ qp\mathbb{1}_{\delta\times\eta} & p^2\mathbb{1}_{\delta\times\delta} & \pm qp\mathbb{1}_{\delta\times\eta} & p^2\mathbb{1}_{\delta\times\delta} \end{pmatrix},$$

where $\eta := \frac{d}{2} - \delta$. Therefore,

$$\mathbf{x}^{+}\mathbf{x}^{+*} - \mathbf{x}^{-}\mathbf{x}^{-*} = 2q \begin{pmatrix} 0_{\eta\times\eta} & 0_{\eta\times\delta} & q\mathbb{1}_{\eta\times\eta} & 0_{\eta\times\delta} \\ 0_{\delta\times\eta} & 0_{\delta\times\delta} & p\mathbb{1}_{\delta\times\eta} & 0_{\delta\times\delta} \\ q\mathbb{1}_{\eta\times\eta} & p\mathbb{1}_{\eta\times\delta} & 0_{\eta\times\eta} & p\mathbb{1}_{\eta\times\delta} \\ 0_{\delta\times\eta} & 0_{\delta\times\delta} & p\mathbb{1}_{\delta\times\eta} & 0_{\delta\times\delta} \end{pmatrix}$$

is a rank-two Hermitian matrix. One may use the identity $\mathbb{1}_{m\times n}\mathbb{1}_{n\times k} = n\mathbb{1}_{m\times k}$, to verify that

$$(\eta q\mathbb{1}_{1\times\eta}, \eta p\mathbb{1}_{1\times\delta}, \pm\sqrt{2\eta\delta p^2 + \eta^2 q^2}\,\mathbb{1}_{1\times\eta}, \eta p\mathbb{1}_{1\times\delta})^T$$

are linearly independent eigenvectors with corresponding eigenvalues $\pm 2q\sqrt{\eta^2 q^2 + 2\eta\delta p^2}$. Therefore, the singular values of $\mathbf{x}^{+}\mathbf{x}^{+*} - \mathbf{x}^{-}\mathbf{x}^{-*}$ are given by

$$\sigma_1 = \sigma_2 = 2q\sqrt{\eta^2 q^2 + 2\eta\delta p^2}.$$

Since $\eta \geq \frac{d}{4}$, this implies $d_1(\mathbf{x}^+, \mathbf{x}^-) = 4q\sqrt{\eta^2 q^2 + 2\eta\delta p^2} \geq dq^2$ as desired and therefore completes the proof. □

3. Examples: Lower Bounds for Specific Measurement Masks

In this section, we will see that the estimates of Theorems 1 and 2 can be improved for specific choices of well-conditioned measurement masks.

3.1. Windowed Fourier Measurement Masks.

In this subsection, we consider a family of masks $\{\mathbf{m_k}\}_{k=1}^{2\delta-1}$, defined by

$$\mathbf{m_k}(n) := \begin{cases} \frac{\mathrm{e}^{-n/b}}{(2\delta-1)^{1/4}} \mathrm{e}^{2\pi \mathrm{i}(k-1)(n-1)/(2\delta-1)} & 1 \leq n \leq \delta \\ 0 & \delta < n \leq d \end{cases}, \tag{3.1}$$

for some fixed parameter $b > 4$. Masks of this form are closely related to those used in ptychographic imaging (see, for example, [15], Section 1.3 and the references provided therein). In [15] it was shown that, with this choice of masks, that the condition number κ of the linear system considered there satisfies

$$\kappa \leq C\delta^2.$$

Combining this with (1.8) shows

$$D_2(\mathbf{x},\mathbf{x}') \le C\frac{q}{p^2}d^2\|Y(\mathbf{x})-Y(\mathbf{x}')\|_2 + Cd^{1/4}\delta\sqrt{\|Y(\mathbf{x})-Y(\mathbf{x}')\|_2}. \tag{3.2}$$

Therefore, the map Y, restricted to the subset of $\mathbb{C}^d$ where $\mathbf{x}(n) \neq 0$ for all $1 \le n \le d$, can be inverted by an algorithm which is both efficient and numerically stable in the case where $L = d$.

We will prove two corollaries to Theorems 1 and 2, which show that the lower bounds for C_B and C_A can be improved for this choice of masks.

Corollary 1. *Let $0 < p \le q$, and consider the map Z, defined as in (1.2), restricted to the subset $\mathcal{C}_{p,q} \subset \mathbb{C}^d/\sim$. Assume that $\delta \le \frac{d}{4}$ and that $d = aL$ for some integer $a < \delta$. Then if $\{\mathbf{m_k}\}_{k=1}^{2\delta-1}$ is the family of masks given by (3.1) and B is any Lipschitz map (with respect to D_2) such $B(Z(\mathbf{x})) = \mathbf{x}$ for all $\mathbf{x} \in \mathcal{C}_{p,q}$, then*

$$C_B \ge \frac{1}{2\sqrt{2}}K_b\frac{q\sqrt{da}}{p(2\delta-1)^{1/4}\delta^{1/2}} = \frac{1}{2\sqrt{2}}K_b\frac{qd}{p\sqrt{L}(2\delta-1)^{1/4}\delta^{1/2}}, \tag{3.3}$$

where $K_b := \mathrm{e}^{1/b} - 1$, and C_B is the Lipschitz constant of B.

Corollary 2. *Let $0 < p \le q$, and consider the map Y, defined as in (1.1), restricted to the subset $\mathcal{C}_{p,q} \subset \mathbb{C}^d/\sim$. Assume that $\delta \le \frac{d}{4}$ and that $d = aL$ for some integer $a < \delta$. Then if $\{\mathbf{m_k}\}_{k=1}^{2\delta-1}$ is the family of masks given by (3.1) and A is any Lipschitz map (with respect to d_1) such $A(Y(\mathbf{x})) = \mathbf{x}$, for all $\mathbf{x} \in \mathcal{C}_{p,q}$, then*

$$C_A \ge \frac{1}{4\sqrt{2}}K_b^2\frac{qd\sqrt{a}}{p\sqrt{\delta}} = \frac{1}{4\sqrt{2}}K_b^2\frac{qd^{3/2}}{p\sqrt{L}\sqrt{\delta}}, \tag{3.4}$$

where $K_b := \mathrm{e}^{1/b} - 1$, and C_A is the Lipschitz constant of A.

Remark 1. *For this choice of masks, $K = 2\delta - 1$ and $\|\mathbf{m}\|_\infty = \mathrm{e}^{-1/b}(2\delta-1)^{-1/4}$. Therefore, the constants obtained in Corollaries 1 and 2 have the same asymptotic behavior with respect to a and d, but are larger with respect to δ than those obtained by directly applying Theorems 1 and 2 to this choice of masks.*

Remark 2. *Similar lower bounds can be derived for any choice of masks along the lines of* (3.1) *whose nonzero entries have magnitudes that form a truncated geometric progression.*

Proof of Corollary 1. Let $\mathbf{x}^\pm$ and $Z^\pm$ be as in the proofs of Theorems 1 and 2. As before, note that $Z^+_{k,\ell} = Z^-_{k,\ell}$ except when either $1 + \ell a \le \frac{d}{2} < \delta + \ell a$ or $1 + \ell a \le d - \delta < \delta + \ell a$. We will again restrict attention to the case where $1 + \ell a \le d - \delta < \delta + \ell a$.

Fix ℓ such that $1 + \ell a \le d - \delta < \delta + \ell a$, and as in the proof of the preceding theorems, let $j := \ell a + 2\delta - d$ so that the last j nonzero entries of $S_{\ell a}\mathbf{m_k}$ are located in positions greater than $d - \delta$ and the first $\delta - j$ nonzero entries are located in positions less than or equal to $d - \delta$. We have seen that

$$Z^\pm_{k,\ell} = \left| \pm q\sum_{n=1}^{\delta-j}\mathbf{m_k}(n) + p\sum_{n=\delta-j+1}^{\delta}\mathbf{m_k}(n)\right|.$$

Therefore,

$$|Z^-_{k,\ell} - Z^+_{k,\ell}| \le 2p\left|\sum_{n=\delta-j+1}^{\delta}\mathbf{m_k}(n)\right| \le 2p\sum_{n=\delta-j+1}^{\delta}|\mathbf{m_k}(n)|. \tag{3.5}$$

To estimate the above sum, we note that $|\mathbf{m_k}(n)| = (2\delta-1)^{-1/4}s^n$, where $s := \mathrm{e}^{-1/b}$. Since $0 < s < 1$,

$$\sum_{n=\delta-j+1}^{\delta}|\mathbf{m_k}(n)| \le (2\delta-1)^{-1/4}\sum_{n=1}^{\delta}s^n \le (2\delta-1)^{-1/4}\frac{s}{1-s}.$$

For each $1 \le k \le 2\delta - 1$, there are at most $\frac{\delta}{a}$ choices of ℓ such that $1 + \ell a \le d - \delta < \delta + \ell a$ and $\frac{\delta}{a}$ choices of ℓ such that $1 + \ell a \le \frac{d}{2} < \delta + \ell a$. Therefore,

$$\|Z^+ - Z^-\|_2^2 \le 8(2\delta-1)\frac{\delta}{a}p^2(2\delta-1)^{-1/2}\left(\frac{s}{1-s}\right)^2$$

$$= 8(2\delta-1)^{1/2}\frac{\delta}{a}p^2\left(\frac{\mathbb{e}^{-1/b}}{1-\mathbb{e}^{-1/b}}\right)^2$$
$$= 8(2\delta-1)^{1/2}\frac{\delta}{a}p^2\left(\frac{1}{\mathbb{e}^{1/b}-1}\right)^2.$$

Recalling that $D_2(\mathbf{x}^+,\mathbf{x}^-) \geq q\sqrt{d}$ as shown in the proof of Theorem 1 and applying (2.1) completes the proof. □

Proof of Corollary 2. Let $\mathbf{x}^\pm$ and $Y^\pm$ be as in the proofs of Theorems 1 and 2. Note that for all k,ℓ,

$$|Z^\pm_{k,\ell}| \leq q\sum_{n=1}^{\delta}|\mathbf{m_k}(n)| \leq q(2\delta-1)^{-1/4}\sum_{n=1}^{\delta}s^n \leq q(2\delta-1)^{-1/4}\frac{s}{1-s}, \tag{3.6}$$

where $s = \mathbb{e}^{-1/b}$ as in the proof of Corollary 1. We again note that $Y^+_{k,\ell} = Y^-_{k,\ell}$ except when either $1+\ell a \leq \frac{d}{2} < \delta+\ell a$ or $1+\ell a \leq d-\delta < \delta+\ell a$ and again restrict attention to the case where $1+\ell a \leq d-\delta < \delta+\ell a$. Combining (3.5) and (3.6) gives

$$|Y^+_{k,\ell} - Y^-_{k,\ell}| = |Z^+_{k,\ell} + Z^-_{k,\ell}||Z^+_{k,\ell} - Z^-_{k,\ell}|$$
$$\leq 4qp(2\delta-1)^{-1/2}\left(\frac{s}{1-s}\right)^2.$$

As in the proof of Corollary 1, for each $1 \leq k \leq 2\delta-1$, there are at most $\frac{\delta}{a}$ choices of ℓ such that $1+\ell a \leq d-\delta < \delta+\ell a$ and $\frac{\delta}{a}$ choices of ℓ such that $1+\ell a \leq \frac{d}{2} < \delta+\ell a$. Therefore,

$$\|Y^+ - Y^-\|_2^2 \leq 32(2\delta-1)\frac{\delta}{a}q^2p^2(2\delta-1)^{-1}\left(\frac{s}{1-s}\right)^4$$
$$\leq 32\frac{\delta}{a}q^2p^2\left(\frac{\mathbb{e}^{-1/b}}{1-\mathbb{e}^{-1/b}}\right)^4$$
$$= 32\frac{\delta}{a}q^2p^2\left(\frac{1}{\mathbb{e}^{1/b}-1}\right)^4.$$

Recalling $d_1(\mathbf{x}^+,\mathbf{x}^-) \geq dq^2$, as shown in the proof of Theorem 2, completes the proof. □

3.2. **Two-Shot Measurement Masks.** Consider the family of masks $\{\mathbf{m_k}\}_{k=1}^{2\delta-1}$ defined by

$$\begin{aligned} \mathbf{m_1} &:= \mathbf{e_1} \\ \mathbf{m_{2j}} &:= \mathbf{e_1} + \mathbf{e_{j+1}} \\ \mathbf{m_{2j+1}} &:= \mathbf{e_1} + \mathbb{i}\mathbf{e_{j+1}} \end{aligned} \tag{3.7}$$

for $1 \leq j \leq \delta-1$, where $\{\mathbf{e_1},\ldots,\mathbf{e_d}\}$ is the standard orthonormal basis for $\mathbb{R}^d$. This family of masks is closely related to the pure-state informationally complete measurements considered by Finklestein in [11]. Similarly to the previous example, in [15] it was shown that the condition number κ of the relevant linear system satisfies

$$\kappa \leq 4\delta.$$

Inserting this into (1.8) yields

$$D_2(\mathbf{x},\mathbf{x}') \leq C\frac{q}{p^2}\frac{d^2}{\delta}\|Y(\mathbf{x}) - Y(\mathbf{x}')\|_2 + Cd^{1/4}\sqrt{\delta\|Y(\mathbf{x}) - Y(\mathbf{x}')\|_2}. \tag{3.8}$$

Thus, the map Y, restricted to the subset of $\mathbb{C}^d$ where $\mathbf{x}(n) \neq 0$ for all $1 \leq n \leq d$, can be inverted by an algorithm which is both efficient and numerically stable in the case where $L = d$.

As in the previous subsection, we will prove two corollaries which improve upon our lower bounds for C_B and C_A for this specific choice of masks.

Corollary 3. *Fix $0 < p \leq q$, and consider the map Z, defined as in (1.2), restricted to the subset $\mathcal{C}_{p,q} \subset \mathbb{C}^d/\sim$. Assume that $\delta \leq \frac{d}{4}$ and that $d = aL$ for some integer $a < \delta$. Then if $\{\mathbf{m_k}\}_{k=1}^{2\delta-1}$ is the family of masks defined by (3.7) and B is any Lipschitz map (with respect to D_2) such that $B(Z(\mathbf{x})) = \mathbf{x}$ for all $\mathbf{x} \in \mathcal{C}_{p,q}$, then*

$$C_B \geq \frac{1}{2\sqrt{2}} \frac{q\sqrt{da}}{p\delta} = \frac{1}{2\sqrt{2}} \frac{qd}{\sqrt{L}p\delta},$$

where C_B is the Lipschitz constant of B.

Corollary 4. *Let $0 < p \leq q$, and consider the map Y, defined as in (1.1), restricted to the subset $\mathcal{C}_{p,q} \subset \mathbb{C}^d/\sim$. Assume that $\delta \leq \frac{d}{4}$ and that $d = aL$ for some integer $a < \delta$. Then if $\{\mathbf{m_k}\}_{k=1}^{2\delta-1}$ is the family of masks defined by (3.7) and A is any Lipschitz map (with respect to d_1) such that $A(Y(\mathbf{x})) = \mathbf{x}$ for all $\mathbf{x} \in \mathcal{C}_{p,q}$, then*

$$C_A \geq \frac{1}{8\sqrt{2}} \frac{qd\sqrt{a}}{p\delta} = \frac{1}{8\sqrt{2}} \frac{qd^{3/2}}{\sqrt{L}p\delta},$$

where C_A is the Lipschitz constant of A.

Remark 3. *Note that for this choice of masks $K = 2\delta - 1$. Therefore, the constants obtained in Corollaries 3 and 4 exhibit the same asympotic behavior with respect to d and are asymptotically larger with respect to δ than those obtained by applying Theorems 1 and 2 to this choice of masks.*

Proof of Corollary 3. Let $\mathbf{x}^{\pm}$ be as in the proof of Theorems 1 and 2. Note that for all $1 \leq n \leq d$, $|\mathbf{x}^+(n)| = |\mathbf{x}^-(n)|$. Therefore, it is clear that for all ℓ,

$$|\langle S_{\ell a}\mathbf{m_1}, \mathbf{x}^+\rangle| = |\mathbf{x}^+(\ell a + 1)| = |\mathbf{x}^-(\ell a + 1)| = |\langle S_{\ell a}\mathbf{m_1}, \mathbf{x}^-\rangle|,$$

and

$$|\langle S_{\ell a}\mathbf{m_{2j+1}}, \mathbf{x}^+\rangle| = |\mathbf{x}^+(\ell a + 1) + \mathbb{i}\mathbf{x}^+(\ell a + j + 1)| = |\mathbf{x}^-(\ell a + 1) + \mathbb{i}\mathbf{x}^-(\ell a + j + 1)| = \langle S_{\ell a}\mathbf{m_{2j+1}}, \mathbf{x}^-\rangle|$$

since the real and imaginary parts of $\langle S_{\ell a}\mathbf{m_{2j+1}}, \mathbf{x}^+\rangle$ and $\langle S_{\ell a}\mathbf{m_{2j+1}}, \mathbf{x}^-\rangle$ have the same absolute values. Therefore, to estimate $\|Z^+ - Z^-\|_2$ we only need to consider the terms $Z^+_{2j,\ell} - Z^-_{2j,\ell}$. Furthermore, it is clear that $Z^+_{2j,\ell}$ will equal $Z^-_{2j,\ell}$, unless ℓ is chosen in such a way that either $\ell a + 1 \leq \frac{d}{2} < \ell a + j + 1$ or $\ell a + 1 \leq d - \delta < \ell a + j + 1$. In either of these cases,

$$|Z^+_{2j,\ell} - Z^-_{2j,\ell}| = 2p. \tag{3.9}$$

Therefore, we will be able to compute $\|Z^+ - Z^-\|_2$ once we estimate the number of ℓ such that $\ell a + 1 \leq \frac{d}{2} < \ell a + j + 1$ or $\ell a + 1 \leq d - \delta < \ell a + j + 1$, which we will do in the following lemma.

Lemma 3. *For fixed j, the number of ℓ such that $\ell a + 1 \leq \frac{d}{2} < \ell a + j + 1$ is less than or equal to $\frac{j}{a}$. Likewise, the number of ℓ such that $\ell a + 1 \leq d - \delta < \ell a + j + 1$ is less than or equal to $\frac{j}{a}$.*

Proof. If $\ell a + 1 \leq \frac{d}{2} < \ell a + j + 1$, then $\frac{d}{2} - j \leq \ell a \leq \frac{d}{2} - 1$, and any set of j consecutive integers can contain at most $\frac{j}{a}$ multiples of a. Likewise, if $\ell a + 1 \leq d - \delta < \ell a + j + 1$, then $d - \delta - j \leq \ell a \leq d - \delta - 1$. □

Combining (3.9) and Lemma 3 gives

$$\|Z^+ - Z^-\|_2^2 \leq \sum_{j=1}^{\delta} \frac{2j}{a}(2p)^2 \leq 8\frac{p^2\delta^2}{a} = 8\frac{Lp^2\delta^2}{d}.$$

Therefore, recalling the fact that $D_2(\mathbf{x}^+, \mathbf{x}^-) \geq \sqrt{d}q$, as shown in the proof of Theorem 1, the proof follows from (2.1). □

Proof of Corollary 4. Since each $\mathbf{m_k}$ has at most two nonzero entries, $|Z^+_{k,\ell} + Z^-_{k,\ell}| \leq 4q$ for all k and ℓ. Therefore, by (3.9) each nonzero entry of $Y^+ - Y^-$ satisfies

$$|Y^+_{k,\ell} - Y^-_{k,\ell}| \leq |Z^+_{k,\ell} + Z^-_{k,\ell}||Z^+_{k,\ell} - Z^-_{k,\ell}| \leq 8qp.$$

Furthermore, similarly to the proof of Corollary 3, $Y^+_{k,\ell} - Y^-_{k,\ell}$ is nonzero if and only if $k = 2j$ for some $1 \leq j \leq \delta - 1$ and $\ell a + 1 \leq \frac{d}{2} < \ell a + j + 1$ or $\ell a + 1 \leq d - \delta < \ell a + j + 1$. Therefore, by Lemma 3,

$$\|Y^+ - Y^-\|_2^2 \leq \sum_{j=1}^{\delta} \frac{2j}{a}(8pq)^2 \leq 128\frac{q^2p^2\delta^2}{a} = 128\frac{Lq^2p^2\delta^2}{d}.$$

Finally, recalling from the proof of Theorem 2 that $d_1(\mathbf{x}^+, \mathbf{x}^-) \geq dq^2$, the result follows from (2.5). □

4. Discussion and Future Work

We believe that this initial work opens up several interesting corridors for further research. First and perhaps most obvious among these is the development of algorithms together with optimal STFT windows, etc., that have Lipschitz upper bounds which match these lower bounds to the extent possible (keeping in mind, of course, that the lower bounds developed here may be gross underestimates). Existing algorithms for local correlation measurements such as [15, 16] yield upper bounds for the measurements Y considered above (1.1) with respect to the D_2-metric, a metric with respect to which an inverse of Y will not generally be Lipschitz [3]. As a result, the upper bounds they provide are not quite appropriate to compare to the lower bounds considered here. Nonetheless, the Lipschitz lower bounds developed here do seem to at least heuristically justify the necessity of, e.g., the d-dependence present in those existing worst case upper bounds.

Another interesting avenue of research would be to explore the extension of the related infinite-dimensional results developed by Alaifari et al. [1, 12] to the finite-dimensional discrete setting. The resulting theory would potentially provide more fine-grained insights into the recovery of samples $\mathbf{x}$ from discrete STFT magnitude measurements, and could also possibly be extended to results concerning general local correlation measurement maps of the type we consider here in a way that would allow for the relaxation of the support assumptions currently made on the masks $\{\mathbf{m_1}, \mathbf{m_2}, \dots, \mathbf{m_K}\}$. Finally, one could also consider local Lipschitz and Hölder lower bounds as opposed to global lower bounds. Though perhaps more difficult to analyze, such lower bounds may be more likely to correspond to achievable upper bounds.

We also remark that in [15], it was shown that the requirement that $p \leq \mathbf{x}(n) \leq q$ can be relaxed slightly. The authors instead consider so-called "m-flat vectors" which essentially require that there be at least one large entry in each each block of consecutive $\lfloor \frac{d}{m} \rfloor$ entries. It is likely that our analysis can be extended to this case. Alternatively, one might be also able to use an interpolation argument similar to [6] to remove the assumption that the $\mathbf{x}(n)$ are non-vanishing.

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