Face Recognition Using $M-$Band Wavelet Analysis

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***Abstract*—In this paper, we propose a new facial recognition algorithm based on wavelet analysis. $M-$Band wavelets are used to decompose face images into M^2 frequency levels. The efficiency of this approach is tested using the eigenface procedure for face recognition. Along with signal decomposition, Wavelet methods allow for data compression, thus reducing computational efforts.**

***Index Terms*—Eigenface; Face Recognition; $M-$Band Wavelets; Wavelet transforms**

I. Introduction

As security needs grow, the capability of positively recognizing individuals becomes a necessity. In an attempt to solve identity issues, researchers have devised numerous recognition algorithms. Among these, we have seen: signature, fingerprint, iris, voiceprint, and many others. However, face recognition is arguably the most convenient method of identification. Biomedical research has shown that human beings often recognize one another through facial characteristics. As early as the late nineteenth century, researchers have tried to identify dominant facial features through automatic methods of classification. Popular face recognition algorithms include: Principal Component Analysis (Eigenface), Linear Discriminate Analysis, Elastic Bunch Graph Matching, Fisherface, and Hidden Markov Chain Models. Of these, the Eigenface Method has shown to have a high recognition rate. In facial recognition, disc space is often an issue, especially when dealing with a large population. Wavelets will not only represent faces as primary and secondary features, but also compress data dramatically and thus save disc space. This makes Wavelets an ideal tool in recognizing facial features.

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II. The Proposed Wavelet Method

For the purpose of this research, we will not introduce 2-Band wavelets. A thorough discussion about these wavelets can be found in [1] and [2].

It is known that $2-$Band wavelets suffer from severe constraint conditions, such as linear-phase $2-$Band wavelets do not exist [3]. $M-$Band Wavelets were designed as an alternative with more freedom and flexibility. This family of wavelets allows for better decomposition of a signal into its components, far superior to the $2-$Band wavelets.

To construct an $M-$Band wavelet ($M \geq 3$), first one must generate a set of M vectors, $h_0, h_1, \cdots, h_{M-1}$, each with components $h_{m,n}$, $0 \leq m \leq M-1,\ 0 \leq n \leq ML-1$, where $m, n, L \in \mathbb{Z}^+$. Researchers in the field of Signal Processing refer to these vectors as filter banks. In this paper, we refer to a set of M filters as an $M-$Band wavelet generator. The choice of L, the regularity, dictates the smoothness of the wavelet transformation: the larger L, the smoother the transformation. These wavelet generators are subject to the following:

Definition 1. A set of M vectors, $\{h_m\}_{m=0}^{M-1}$, where $h_m = \{h_{m,n} \mid m, n, L \in \mathbb{Z}^+,\ 0 \leq n \leq ML-1\}$, is said to be an $M-$Band Wavelet generator if it meets the following:

- Low-Pass and High Pass Condition:
$$\sum_{n=0}^{LM-1} h_{m,n} = \sqrt{M}\delta,\ \delta = \begin{cases} 1 & m=0 \\ 0 & \text{otherwise} \end{cases}.$$
- Orthonormal Condition:
$$h_i \cdot h_j = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}.$$
- Perfect Reconstruction Condition:
$$s = \tilde{s},$$
where s is the original signal and $\tilde{s}$ is the wavelet-based reconstructed signal[3].

Below is a $4-$Band wavelet generator for the case of $L = 2$ [3]:

$$h_0 = \begin{bmatrix} -0.067371764 \\ 0.094195111 \\ 0.40580489 \\ 0.56737176 \\ 0.56737176 \\ 0.40580489 \\ 0.094195111 \\ -0.067371764 \end{bmatrix}, \; h_1 = \begin{bmatrix} -0.094195111 \\ 0.067371764 \\ 0.56737176 \\ 0.40580489 \\ -0.40580489 \\ -0.56737176 \\ -0.067371764 \\ 0.094195111 \end{bmatrix}$$

$$h_2 = \begin{bmatrix} -0.094195111 \\ -0.067371764 \\ 0.56737176 \\ -0.40580489 \\ -0.40580489 \\ 0.56737176 \\ -0.067371764 \\ -0.094195111 \end{bmatrix}, \; h_3 = \begin{bmatrix} -0.067371764 \\ -0.094195111 \\ 0.40580489 \\ -0.56737176 \\ 0.56737176 \\ -0.40580489 \\ 0.094195111 \\ 0.067371764 \end{bmatrix}$$

An excellent outline of this construction process can be found in [3].

Once a wavelet generator has been created, we assemble the transformation matrix W. First, one chooses the size, $N \times N$, of the matrix to use, where $N = M^j$ for some $j \in \mathbb{N}$. We divide the $N \times N$ matrix into M horizontal sub-matrices, each of size $M^{j-1} \times M^j$. Next, we create the i^{th} sub-matrix by cyclically shifting the h_i to the right, M places at a time.

$$\left[\begin{array}{cccccccccccccccc}
h_{0,0} & h_{0,1} & h_{0,2} & h_{0,3} & h_{0,4} & h_{0,5} & h_{0,6} & h_{0,7} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & h_{0,0} & h_{0,1} & h_{0,2} & h_{0,3} & h_{0,4} & h_{0,5} & h_{0,6} & h_{0,7} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & h_{0,0} & h_{0,1} & h_{0,2} & h_{0,3} & h_{0,4} & h_{0,5} & h_{0,6} & h_{0,7} \\
h_{0,4} & h_{0,5} & h_{0,6} & h_{0,7} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & h_{0,0} & h_{0,1} & h_{0,2} & h_{0,3} \\
h_{1,0} & h_{1,1} & h_{1,2} & h_{1,3} & h_{1,4} & h_{1,5} & h_{1,6} & h_{1,7} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & h_{1,0} & h_{1,1} & h_{1,2} & h_{1,3} & h_{1,4} & h_{1,5} & h_{1,6} & h_{1,7} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & h_{1,0} & h_{1,1} & h_{1,2} & h_{1,3} & h_{1,4} & h_{1,5} & h_{1,6} & h_{1,7} \\
h_{1,4} & h_{1,5} & h_{1,6} & h_{1,7} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & h_{1,0} & h_{1,1} & h_{1,2} & h_{1,3} \\
h_{2,0} & h_{2,1} & h_{2,2} & h_{2,3} & h_{2,4} & h_{2,5} & h_{2,6} & h_{2,7} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & h_{2,0} & h_{2,1} & h_{2,2} & h_{2,3} & h_{2,4} & h_{2,5} & h_{2,6} & h_{2,7} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & h_{2,0} & h_{2,1} & h_{2,2} & h_{2,3} & h_{2,4} & h_{2,5} & h_{2,6} & h_{0,7} \\
h_{2,4} & h_{2,5} & h_{2,6} & h_{2,7} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & h_{2,0} & h_{2,1} & h_{2,2} & h_{3,3} \\
h_{3,0} & h_{3,1} & h_{3,2} & h_{3,3} & h_{3,4} & h_{3,5} & h_{3,6} & h_{3,7} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & h_{3,0} & h_{3,1} & h_{3,2} & h_{3,3} & h_{3,4} & h_{3,5} & h_{3,6} & h_{3,7} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & h_{3,0} & h_{3,1} & h_{3,2} & h_{3,3} & h_{3,4} & h_{3,5} & h_{3,6} & h_{0,7} \\
h_{3,4} & h_{3,5} & h_{3,6} & h_{3,7} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & h_{3,0} & h_{3,1} & h_{3,2} & h_{3,3}
\end{array}\right]$$

Figure 1. 16×16, 4−Band, Regularity 2 Wavelet Matrix

Figure 1 displays the structure of a 16×16, 4−Band, Regularity 2 ($L = 2$) Wavelet matrix, while Figure 2 shows a sample face image with its 4−Band Transform.

The following terms and theorems will be used throughout the rest of this paper.

Definition 2. The $M-$Band wavelet transform $f : \mathbb{R}^{N\times N} \to \mathbb{R}^{N\times N}$ is defined by $f(X) = W \cdot X \cdot W^T$, where W is the $M-$Band wavelet matrix constructed above, and $X \in \mathbb{R}^{N\times N}$.

Figure 2 shows a sample face image with its 4−Band transformation.

Figure 2. Example 4-Band Transformation

Definition 3. Let $X \in \mathbb{R}^{N\times N}$ with entries $\{x_{i,j}\}_{i,j=1}^{N}$. We define the function $\mathcal{E} : \mathbb{R}^{N\times N} \to \mathbb{R}$ by:

$$\mathcal{E}(X) = \sum_{i=1}^{N}\sum_{j=1}^{N} x_{i,j}^2.$$

$\mathcal{E}(X)$ is referred to as the energy of X.

Definition 4. Let $A = [\overrightarrow{a_1}\,\overrightarrow{a_2} \cdots \overrightarrow{a_N}]$ be any $N \times N$ matrix, where $\overrightarrow{a_i}$ is the i^{th} column of A. Define a map $g : \mathbb{R}^{N\times N} \to \mathbb{R}^{N^2}$ by $g\left(A\right) = \begin{bmatrix} \overrightarrow{a_1} \\ \vdots \\ \overrightarrow{a_N} \end{bmatrix}$.

Theorem 5. *The map $g : \mathbb{R}^{N\times N} \to \mathbb{R}^{N^2}$ is bijective.*

Proof: Let $A, B \in \mathbb{R}^{N\times N}$, where

$$A = [\overrightarrow{a_1} \cdots \overrightarrow{a_N}],\; B = \left[\overrightarrow{b_1} \cdots \overrightarrow{b_N}\right].$$

Suppose that $g\left(A\right) = g\left(B\right)$. Then,

$$\begin{bmatrix} \overrightarrow{a_1} \\ \vdots \\ \overrightarrow{a_N} \end{bmatrix} = \begin{bmatrix} \overrightarrow{b_1} \\ \vdots \\ \overrightarrow{b_N} \end{bmatrix}.$$

It follows that $\overrightarrow{a_i} = \overrightarrow{b_i}$, $i = 1, \ldots, N$. Hence, $B = [\overrightarrow{a_1} \cdots \overrightarrow{a_N}] = A$. Therefore, g is one-to-one.

Let $V \in \mathbb{R}^{N^2}$, where $V = [v_1 \ldots v_{N^2}]^T$. Let $\overrightarrow{c_i} = \left[v_{(i-1)N+1}, \ldots, v_{iN}\right]^T$, $i = 1, \ldots, N$. Then,

$$V = \begin{bmatrix} \overrightarrow{c_1} \\ \vdots \\ \overrightarrow{c_N} \end{bmatrix}.$$

If we let $C = [\overrightarrow{c_1} \cdots \overrightarrow{c_N}]$, then $g\left(C\right) = V$. Therefore, g is onto.

From what precedes, g is a bijection. ∎

Corollary 6. *The map g is invertible.*

Proof: By Theorem 5, the map g is a bijection. Therefore,

$g^{-1} : \mathbb{R}^{N^2} \to \mathbb{R}^{N\times N}$ exists. ■

Theorem 7. *There exists $a, b \in \mathbb{R}^+$ such that, for any $X, Y \in \mathbb{R}^{N\times N}$, we have*

$$a\|f(X-Y)\| \le \|X-Y\| \le b\|f(X-Y)\|$$

where $\|\cdot\|$ is any norm of $\mathbb{R}^{N\times N}$.

Proof: First recall that $f(X-Y) = W(X-Y)W^T$. Therefore,

$$\begin{aligned} \|f(X-Y)\| &= \|W(X-Y)W^T\| \\ &\le \|W\|\|(X-Y)\|\|W^T\| \\ &= \|W\|\|W^T\|\|(X-Y)\| \end{aligned}$$

Now, dividing both sides of the inequality by $\|W\|\|W^T\|$, we obtain

$$\frac{\|f(X-Y)\|}{\|W\|\|W^T\|} \le \|(X-Y)\|.$$

Notice also that

$$\begin{aligned} \|X-Y\| &= \|W^T\left(W(X-Y)W^T\right)W\| \\ &= \|W^T f(X-Y)W\| \\ &\le \|W^T\|\|f(X-Y)\|\|W\| \\ &= \|W\|\|W^T\|\|f(X-Y)\|. \end{aligned}$$

Hence, $a\,f(\|X-Y\|) \le \|X-Y\| \le b\,f(\|X-Y\|)$ for any $X, Y \in \mathbb{R}^{N\times N}$, where $a = \frac{1}{\|W\|\|W^T\|}$ and $b = \|W\|\,\|W^T\|$. ■

III. The Eigenface Procedure

Developed in 1987 by Sirovich and Kirby [4] and used by Matthew Turk and Alex Pentland [5], Eigenface is considered to be the first successful implementation of facial recognition technology.

Based on Principal Component Analysis (PCA), the eigenface procedure is implemented as follows:

1) Prepare a training set of K face images. This training set consists of p different individuals, each with q different face pictures. Thus, $K = p \cdot q$.
2) Let Γ_i be the matrix corresponding to the i^{th} picture in the database.
3) For each Γ_i, let $\Phi_i = g(\Gamma_i)$. This results in K column vectors of N^2 entries each.
4) Construct a matrix $S = [\Phi_1\, \Phi_2 \cdots \Phi_K]$ with dimension $N^2 \times K$.
5) Calculate the mean face image Ψ by averaging all K face images in the database:

$$\Psi = \frac{1}{K}\sum_{i=1}^{K}\Phi_i$$

Figure 3. Color Database

Figure 4. The Mean Image

6) Calculate the mean μ and standard deviation σ of the database. Note that $0 \le \mu, \sigma \le 255$.
7) Normalize each face picture as follows:

$$\Phi_i' = \frac{\Phi_i - \mu_i}{\sigma_i}$$

where μ_i and σ_i are the mean and standard deviation of Φ_i, respectively, and Φ_i' is the vector representation of the normalized face image Φ_i.

8) Construct a new matrix $A = [\Phi_1'\, \Phi_2' \cdots \Phi_K']$.
9) Construct a covariance matrix. Normally, one would calculate the covariance matrix C as $C = AA^T$. The eigenvectors v_i of C are such that $Cv_i = AA^Tv_i = \lambda_i v_i$, where λ_i is the eigenvalue of C corresponding to v_i. However, since A has dimension $N^2 \times K$, AA^T would have dimension $N^2 \times N^2$. A tremendous computational effort will be required to obtain all N^2 eigenvectors

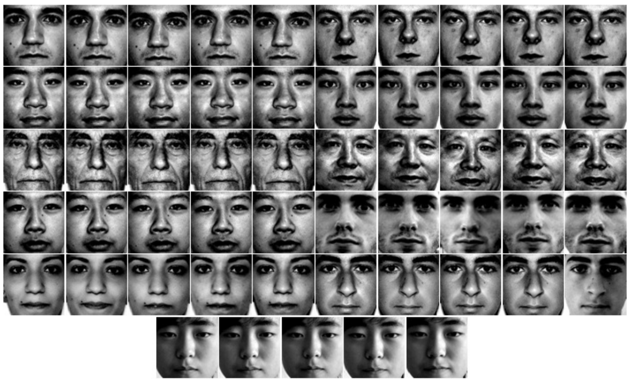

Figure 5. The Normalized Database

v_i and eigenvalues λ_i. To conquer this obstacle, [5] proposes replacing the traditional AA^T matrix with A^TA. If u_i is an eigenvector of A^TA corresponding to the eigenvalue λ_i, then

$$A^TAu_i = \lambda_i u_i \text{ and } AA^TAu_i = \lambda_i Au_i$$

If we let $v_i = Au_i$, then v_i is an eigenvector of C corresponding to the same eigenvalue, λ_i. Note that A^TA now has dimension $K \times K$, which means that it will take less time and effort to obtain the principle components of C.

10) For each eigenvector v_i we can construct the matrix $g^{-1}(v_i) \in \mathbb{R}^{N\times N}$, called eigenface.

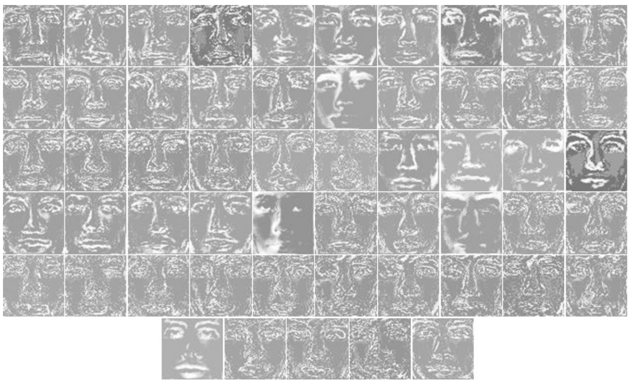

Figure 6. The Eigenface Database

Remark 8. Notice that A^TA is a symmetric matrix. For each eigenvalue λ_i of A^TA with multiplicity m_i, the eigenspace corresponding to λ_i has dimension m_i. By the Gram-Schmidt Procedure, we can find m_i orthonormal eigenvectors of A^TA corresponding to λ_i. Then, these orthonormal bases determine an orthonormal set of K linearly independent eigenvectors of A^TA, which forms an orthonormal basis $\{w_i\}_{i=1}^K$ of $\mathbb{R}^K$. A detailed discussion of properties of symmetric matrices can be found in [6].

Definition 9. We denote by G the set $\left\{g^{-1}(v_i) \mid v_i\text{'s correspond to different eigenvalues } \lambda_i\right\}$. Let $K' \le K$ denote the cardinality of G.

Definition 10. We define the face space $F \subset \mathbb{R}^{N\times N}$ as $F = \text{Span}\{G\}$.

Theorem 11. *The set G forms a basis for F.*

Proof: It is enough to show that G is linearly independent. To do so, let

$$c_1 g^{-1}(v_1) + \cdots + c_{K'} g^{-1}(v_{K'}) = 0_{N\times N}.$$

We need to show that all c_i's are zero. Note that

$$\sum_{i=1}^{K'} c_i g^{-1}(v_i) = g^{-1}\left(\sum_{i=1}^{K'} c_i v_i\right).$$

Since v_i's are eigenvectors of the symmetric matrix AA^T corresponding to different eigenvalues, then they are orthogonal to each other. Hence, v_i's are linearly independent. So,

$$g^{-1}\left(\sum_{i=1}^{K'} c_i v_i\right) = 0_{N\times N} \iff \sum_{i=1}^{K'} c_i v_i = 0_{N^2\times 1}$$

and since v_i's are linearly independent, all c_i's are zero, proving the theorem. ■

Using this theorem, we can now express any face picture as a linear combination of the elements of $\left\{g^{-1}(v_i)\right\}_{i=1}^{K'}$.

Remark 12. In the case of a large population, say a database for a country or a state is being assembled, K could be too large. However, further reduction in computation can be achieved by selecting the K' eigenfaces corresponding to the K' highest eigenvalues, as discussed in [5]. Determining K' is a matter of experimentation. One way of doing this, is through an RMS measure, that calculates the "percentage error" of a reconstructed picture to its original form. For instance, we may chose the maximal percentage error to be 2%.

IV. $M-$Band Wavelets-Based Recognition

Although computational time has been reduced by manipulating the covariance matrix, further reduction is necessary in the case of large databases (high values of K). $M-$band wavelets preserve distances in such a way that if two images are close in distance, then so are their corresponding $M-$band wavelet transforms. This is outlined in Theorem 7.

As mentioned previously, the images are chosen so that their dimension N is an integral power of M, say $N = M^j$ for some $j \in \mathbb{N}$. Then, the low frequency sub-matrix of the i^{th} level $M-$band wavelet transform is of dimension $M^{j-i} \times M^{j-i}$, for $0 < i < j$. This sub-matrix is an approximation of the original image. So, instead of working in the original image domain, namely $\mathbb{R}^{N \times N}$, we work in the 1-level wavelet approximation domain $\mathbb{R}^{\frac{N}{M} \times \frac{N}{M}}$.

Choosing M is experimental: if $M = 3$, the approximated image is still relatively large; if $M = 5$, the approximated image is so small that we lose significant features. $M = 4$, which is used in this research, offers the ideal approximation for a face image.

The implementation of $4-$Band wavelet-based recognition consists of replacing all original images Γ_i in the database with the approximation components $\mathcal{A}_i$, of their $4-$band wavelet transforms $f(\Gamma_i)$. K' remains unchanged, while the dimension of each picture has been reduced to $\frac{N}{4} \times \frac{N}{4}$.

The eigenface procedure is applied as described previously, only this time using the approximation component of $f(\Gamma_i)$ instead of Γ_i itself.

V. Testing

A. Procedure

To recognize an input image X as an individual in the database, proceed as follows:

1) Normalize X as described in the eigenface procedure.
2) Compute $f(X)$ and extract its approximation sub-matrix. Denote this latter by $\mathcal{A}$.
3) We modify the method mentioned in [5], by creating a vector Ω representing the contribution of each of the eigenfaces to the reconstruction of $g(\mathcal{A})$. The m^{th} component of Ω is calculated by:

$$\omega_m = v_m^T \cdot (g(\mathcal{A}) - \Psi)$$

where Ψ is the average of all $g(\mathcal{A}_i)$.

4) We create a face class $\Omega^{(i)}$ for each $g(\mathcal{A}_i)$, by repeating the previous step:

$$\omega_m^{(i)} = v_m^T \cdot (g(\mathcal{A}_i) - \Psi)$$

where $\omega_m^{(i)}$ is the m^{th} component of the face class $\Omega^{(i)}$.

5) For each face class, calculate

$$\epsilon^{(i)} = ||\Omega - \Omega^{(i)}||$$

where $\|\cdot\|$ is the euclidean norm of $\mathbb{R}^{K'}$.

6) Finally, the match is the face Γ_i corresponding to the lowest $\epsilon^{(i)} < \epsilon$, where $\epsilon = \max\left\{||\Omega - \Omega^{(i)}||\right\}_{i=1}^{K'}$ and where Ω corresponds to an existing image in the database.

Remark 13. Although we will always find the lowest $\epsilon^{(i)}$, we want this minimal value to be less than the threshold value ϵ defined above. The reason for this is that the test picture might belong to someone who is not in the database at all; in this case, we should not get a match at all.

B. Results

We have built a database consisting of $p = 11$ individuals with $q = 5$ pictures each, giving a total of $K = 55$ images. A total of 44 images of individuals known to be in the database were tested, along with 10 images of people not in the database. Of the 44 images, 41 matched correctly, yielding a recognition rate of 93.2%. As for the 10 other images, 8 were correctly identified as not belonging to the database, while the other two did incorrectly match to some individuals.

	PCA	PCA + Wavelets
Recognition Rate	88.6%	93.2%

Table I
RECOGNITION RATES VS. METHOD USED

VI. Conclusion

Based on the results of this research, we can conclude that the implementation of $M-$band wavelets in the eigenface procedure can lead to higher accuracy and efficiency. By bringing out the significant features of the face, wavelet transforms make better candidates for recognition algorithms. Computational complexity is also reduced dramatically, allowing for the analysis of larger databases in less time. Furthermore, $M-$band wavelets have the potential of greatly reducing the size of images by means of compression algorithms. Many algorithms are outlined nicely in [1].

As in any research, the algorithm described in this paper is not final. We conjecture that this process can be improved by combining PCA, $M-$band wavelets, and Graph Matching techniques. Apart from $M-$band wavelets, our method can be implemented using curvelets and second generation wavelet packages.

Finally, we express the possibility of performing this algorithm on a quantum computer in the near future. The first commercial version of these machines was sold in June 2011. As time goes on, the true computational capabilities of these machines will be unveiled, invoking faster and more powerful recognition algorithms. Multiple feature detection methods will be performed in less time, thus allowing for unprecedented accuracy. An interesting read on this subject can be found in [7].

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