Non Finitary Locally Finite Simple Groups

Ulrich Meierfrankenfeld
Michigan State University
East Lansing, Mi 48824

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1 Introduction

Define a LFS-group to be an infinite, locally finite, simple group. This paper is a contribution to the general theory of LFS-groups. Recall that a group G is finitary if there exist a field K and a faithful KG-module V such that $[V, g]$ is finite dimensional for all g in G. The infinite alternating groups and all finitary classical groups defined over locally finite fields provide examples of finitary LFS-groups and it is conjectured and almost proved by J.I. Hall that every finitary non-linear LFS-groups is of that kind. On the contrast allthough many examples exist not much is known about general non-finitary LFS-groups. One purpose of this paper is to demonstrate that the division of LFS-group in finitary and non-finitary groups is natural and allows to obtain a considered amount of information about the non-finitary LFS-groups.

Fundamental to the study of locally finite groups is the concept of Kegel covers. Let G be a locally finite group.

A set of pairs $\{(H_i, M_i)|i \in I\}$ is called a Kegel cover for G if, for all i in I, H_i is a finite subgroup of G and M_i is a maximal normal subgroup of H_i, and if for each finite subgroup H of G there exists $i \in I$ with $H \leq H_i$ and $H \cap M_i = 1$. The groups H_i/M_i, $i \in I$, are called the factors of the Kegel cover.

It has been proven in [5, 4.3, p113] that every LFS-group has a Kegel cover. Using Kegel covers many questions about LFS-groups can be transfered to questions about finite simple groups, which in turn may be answered using the classification of finite simple groups. Define a LFS-group to be of alternating type if it is non-finitary and posseses a Kegel cover all of whose factors are alternating groups. If p is a prime, define a LFS-group G to be of p-type if G is non-finitary and every Kegel cover for G has a factor which is isomorphic to a classical group defined over a field in characteristic p. In 3.3 we prove

Theorem A *Let G be a LFS-group. Then one of the following holds:*

(a) G is finitary.

(b) G is of alternating type.

(c) There exists a prime p and a Kegel cover $\{(H_i, M_i) | i \in I\}$ for G such that G is of p-type, and for all $i \in I$, $H_i/O_p(H_i)$ is the central product of perfect central extension of classical groups defined over a field in characteristic p and H_i/M_i is a projective special linear group.

Using Theorem A many question about non-finitary LFS-groups can now be transfered to questions about alternating and projective special linear groups. As an example we prove in section 4:

Theorem B *Let G be a countable, non-finitary LFS-group and H a finite subgroup of G. Then H is contained in a maximal subgroup of G. In particular, G has maximal subgroups.*

In 5.5 we give an affirmative answer to questions 1 and 2 raised in J. Hall's and B. Hartley's paper [1] on a characterization of finitary LFS-groups. In section 6 we provided examples of countable LFS-groups which are not absolutely simple. We remark that by [7] every Kegel cover for a countable LFS-group which is not absolutely simple has fairly complicated structure. In particular, no such group has a Kegel cover where each of the $H_i's$ are a central product of quasisimple groups and, using Theorem A, any such group is of alternating type.

We hope that Theorem A will draw attention to LFS-groups of p-type. In [1, Proposition 1] non-finitary LFS-groups G have been constructed which have an element x of order q, q an odd prime, such that $< x^S >$ is abelian for every q-subgroup S of G containing x. It follows from 5.5 that no such group is of alternating type and thus provided examples of LFS-groups of p-type. It seems plausible that the restricted structure of the Kegel covers for LFS-groups of p-type provided by Theorem A might lead to a classification of LFS-groups of p-type. It also seems worthwhile to examine the $GF(p)$-module arising from the above Kegel cover via the ultrafilter construction in [5, 1L Appendix]. It might be possible to characterize LFS-groups of p-type in terms of that module.

We remark that 3.4b has been proven independently and with different methods by C. Praeger and A.Zalesskii in [8, Theorem 1.7]

I would like to thank Jon Hall and Dick Phillips for many fruitful discussions on the topics in this paper.

We finish the introduction with a list of some of the notations used throughout. Let K be a field, V a vector space over K and q a symplectic, orthogonal or unitary form on V. Then $O(V, q)$ is the largest subgroup of $GL_K(V)$ preserving q, $\Omega(V, q) = O(V, q)'$ and $P\Omega(V, q) = \Omega(V, q)/Z(\Omega(V, q))$. A singular subspace of V is K-subspace of V on which q vanishes. $\mathcal{F}$ is the class of finite simple groups isomorphic to $P\Omega(V, q)$ for some finite field K, some finite dimensional vector space V over K and some non-degenerated symplectic, quadratic or unitary form q on V. $\mathcal{L}$ is the class of finite simple groups isomorphic to $PSL_K(V)$ for some finite field K and some finite dimensional vector space V over K. $\mathcal{C} = \mathcal{L} \cup \mathcal{F}$. $\mathcal{A}$ is the class of finite simple alternating groups. If p is a prime, then $\mathcal{C}(p)$, $\mathcal{F}(p)$ and $\mathcal{L}(p)$ are defined similar to the above only that the defining fields are assumed to be in characteristic p. If x is acting on V, then $\deg_V(x) = \dim_K[V, x]$, $\mathrm{pdeg}_V(x) = \min\{\deg_V(kx)|0 \neq k \in K\}$ and $\mathrm{pdeg}_V(Z(GL_K(V))x) = \mathrm{pdeg}_V(x)$. If Ω is a set and x acts on Ω, $\mathrm{pdeg}_\Omega(x)$ and $\deg_\Omega(x)$ both are the number of elements in Ω not fixed by x. Also if $G = PGL_K(V), PSL_K(V), P\Omega(V, q)$ or $Alt(\Omega)$ and x induces an inner automorphism y on G, then $\mathrm{pdeg}_G(x) = \mathrm{pdeg}_V(y)$ and $\mathrm{pdeg}_G(x) = \mathrm{pdeg}_\Omega(y)$, respectively.

Let G be a locally finite group. A set of pairs $\{(H_i, M_i)|i \in I\}$ is called a sectional cover for G if, for all i in I, H_i is a finite subgroup of G and M_i is a normal subgroup of H_i, and if, for each finite subgroup H of G, there exists i in I with $H \leq H_i$ and $H \cap M_i = 1$.

The groups H_i/M_i, $i \in I$, are called the factors of the sectional cover.

A set of subgroups $\{H_i|i \in I\}$ of G is called a sectional cover for G if each finite subgroup of G is contained in one of the $H_i's$, i.e. if $\{(H_i, 1)|i \in I\}$ is a sectional cover for G. A Kegel sequence for G is a Kegel cover $\{(H_i, M_i)|i \geq 1\}$ for G such that for all i, $H_i \leq H_{i+1}$ and $H_i \cap M_{i+1} = 1$.

For a finite group H let $w^*(H)$ be the number isomorphism types of transitive permutation representations for H or equally $w^*(H)$ the number of conjugacy classes of subgroups of H. Let $w(H) = \max\{w^*(T) \mid T \leq H\}$.

Let G be a group acting on a set Ω. Let I be a subset of Ω. Then $N_G(I) = \{g \in G | i^g \in I \text{ for all } i \in I\}$ and $C_G(I) = \{g \in G | i^g = i \text{ for all } i \in I\}$.

Let Δ be a set of subsets of Ω. Then $N_G(\Delta) = \{g \in G | D^g \in \Delta \text{ for all } D \in \Delta\}$ and $C_G(\Delta) = \{g \in G | D^g = D \text{ for all } D \in \Delta\}$.

A system of imprimitivity Δ for G on Ω is a set of proper subsets of Ω such that $D^g \in \Delta$, for all $D \in \Delta$ and $g \in G$, and Ω is the disjoint union of the members of Δ. For $|\Omega| > 1$, $\{\{\omega\} \mid \omega \in \Omega\}$ is a system of imprimitivity. All others systems of imprimitivity are called proper. We say the G acts primitively on I if G acts transitively on I and G has no proper system of imprimitivity on I.

If G is a group acting on a vector space V, then a system of imprimitivity Δ for G on V is a set of proper subspaces of V such that $D^g \in \Delta$, for all $D \in \Delta$ and $g \in G$, and V is the direct sum of the members of Δ. We say the G acts primitively on V if G acts irreducibly on V and G has no system of imprimitivity on V.

If Δ_1 and Δ_2 are systems of imprimitivity for G on I or V we say that $\Delta_1 \prec \Delta_2$, if for each $D_1 \in \Delta_1$ there exists D_2 in Δ_2 with $D_1 \subseteq D_2$. Note that if Δ is a maximal system of imprimitivity for G then either G acts primitively on the set Δ or $|\Delta| = 2$ and G acts trivially on Δ.

Suppose that $G/M \cong Alt(\Sigma)$ for some set Σ and some normal subgroup M of G. Let t be a positive integer with $t \leq |\Sigma|/2$. Then G acts t-pseudo natural on Ω with respect to M if G acts transitively on Ω and if there exists a G-invariant partion Δ for G on Ω such that $C_G(\Delta) = M$ and the action of G on Δ is isomorphic to the action of G on subsets of size t of Σ. G acts pseudo naturally on Ω with respect to M, if G acts 1-pseudo natural on Ω. G acts essentially on Ω with respect to M if $C_G(\Omega) \leq M$. We remark that if G/M is perfect (that is $|\Sigma| \geq 5$) and R is the minimal normal supplement to M in G (see 2.8), then Ω is essential if and only if R acts non-trivally on Ω. The reader should notice that in the case $M = 1$ a pseudo natural action does not have to be natural. In fact it is easy to see that even the right regular permutation action is pseudo natural.

If p is a prime, then $d(p) = 1$, if $p \neq 2$, and $d(p) = 2$, if $p = 2$.

For a real number x, let $[x]$ be the largest integer less or equal to x.

For a group G let $G^{(0)} = G$ and inductively, $G^{(i+1)} = (G^{(i)})'$. Put $G^\infty = \bigcap_{i=1}^\infty G^{(i)}$ and note that if G is finite, G^∞ is the largest perfect subgroup of G. If G is solvable, let $\mathrm{der}(G)$ be the smallest non-negative integer i with $G^{(i)} = 1$.

2 Preliminaries

Throughout this chapter K is a finite field, V a finite dimensional vector space over K and $Q = GL_K(V)$.

Lemma 2.1 *Let σ be a field automorphism of K of order 1 or 2 and s a σ-sesquilinear form on V.*

(a) There exists a subspace U of V such that $\dim U \geq (\dim V - 4)/4$ *and* $s|_{U\times U} = 0$.

(b) Let $G \leq Q$ such that for all $g \in G$ there exists $\lambda_g \in K$ with $s(u^g, v^g) = \lambda_g s(u,v)$ for all u, v in V. Let $Z = Z(Q) \cap G$, $m = |G/Z|$ and W a subspace of V. Then there exists a subspace X of W with

$$\dim X \geq (\dim W - 4\frac{4^m - 1}{3})/4^m$$

such that $s|_{U\times U} = 0$, where $U = \langle X^G \rangle$.

Proof: (a) Define $t : V \times V \to K$ by $t(u, v) = s(u, v) + s(v, u)^\sigma$. Then t is a symmetric or unitary form on V. Therefore there exists a subspace W of V with $\dim W \geq (\dim V - 2)/2$ such that $t|_{W\times W} = 0$. Indeed, the worst possible case is when V has dimension $2k$ and t has Witt index $k-1$. If $\sigma \neq 1$, pick λ in K with $\lambda^\sigma = -\lambda$, otherwise let $\lambda = 1$. Then restricted to W, λs is a skew symmetric or unitary form and there exists a subspace U of W with $\dim U \geq (\dim W - 1)/2$ such that $\lambda s|_{U\times U} = 0$. Here the worst possible cases occur if s restricted to W

is unitary and W is odd dimensional, or if char $K = 2$ and s restricted to W is symmetric but not alternating. This proves (a).

(b) Let $g \in G$ and Y a subspace of V. Define a σ-sesquilinear form s_g on Y by $s_g(u, v) = s(u, v^g)$. By (a) there exists a subspace R of Y with $s_g|_{R \times R} = 0$ and $\dim R \geq (\dim Y - 4)/4$. Let $T = \{g_1, ..., g_m\}$ be a transversal to Z in G. Then by an easy induction proof there exists a subspace X in V with $\dim X \geq (\dim W - 4\frac{4^m-1}{3})/4^m$ and $s_{g_i}|_{X \times X} = 0$ for all $1 \leq i \leq m$. Put $U = \langle X^G \rangle$ and note that $U = \langle X^T \rangle$. Let $u, v \in X$. Then $s(u, v^{g_i}) = s_{g_i}(u, v) = 0$. Thus $s(u, w) = 0$ for all $u \in X$ and $w \in U$. Since s is G-invariant, $s(u, v) = 0$ for all $u, v \in U$.

Lemma 2.2 *There exists an increasing function f defined on the positive integers and independent from K and V with the following property: Let $G \leq Q$, $Z = G \cap Z(Q)$ and q is a quadratic, symplectic or unitary form on V such that for all $g \in G$, $q^g = \lambda_g q$ for some $\lambda_g \in K$. If X is a subspace of V of dimension at least $f(|G/Z|)$, then there exists $0 \neq x \in X$ such that $K\langle x^G \rangle$ is singular with respect to q.*

Proof: Let $f(m) = 2 \cdot 4^m + 4\frac{4^m-1}{3})$. If q is symplectic or unitary, the assertion follows directly from 2.1b. The same is true if q is quadratic and char K is odd. So suppose q is quadratic and char $K = 2$. Let s be the symplectic form associate to q. Then 2.1b provides a subspace Y in X of dimension at least two such that s vanishes on $\langle Y^G \rangle$. Pick $0 \neq x \in Y$ with $q(x) = 0$. Then q vanishes on $\langle x^G \rangle$.

Lemma 2.3 *Let $f, G, Z, V,$ and q be as in 2.2. Put $m = |G/Z|$ and let $\mathcal{S}$ be the set of G-invariant singular subspaces of V.*

(a) Let M be a maximal element in $\mathcal{S}$. Then $\dim M > (\dim V - f(m))/2$.

(b) If $\dim V \geq 2m + f(m)$ and $x \in G$ with $\operatorname{pdeg}_V(x) \geq f(m)$, then there exists $M \in \mathcal{S}$ such that x does not act as a scalar on M.

(c) Let t be a positive integer and H a subset of G such that for all $x \in H$ $\operatorname{pdeg}_V(x) \geq 2m(|H|t - 1) + f(m)$. If $\dim V \geq 2tm|H| + f(m)$, then there exists $M \in \mathcal{S}$ with $\dim M \leq t|H|m$ and $\operatorname{pdeg}_M(x) \geq t$ for all $x \in H$.

(d) Put $g(m) = 2m^2 - 2m + f(m)$. If $\operatorname{pdeg}_V(x) \geq g(m)$ for all $x \in G \setminus Z$, then there exists $U \in \mathcal{S}$, such that no element of $G \setminus Z$ acts as a scalar on U.

Proof: (a) Let M be a maximal element in $\mathcal{S}$. Then G normalizes no non-trivial singular subspace of $M^\perp/M$. By 2.2, $\dim M^\perp/M < f(m)$ and so

$$\dim V = \dim V/M^\perp + \dim M^\perp/M + \dim M < \dim M + f(m) + \dim M.$$

Thus (a) holds.

(b) Suppose that x acts as a scalar on every element of $\mathcal{S}$ and let M and N be maximal elements of $\mathcal{S}$. Let $0 \neq u \in N$ and put $U = K\langle u^G \rangle$. Then clearly $\dim U \leq m$. By (a) and since $\dim V \geq 2m + f(m)$, $\dim M > m$. Hence $M \cap U^\perp \neq 0$. Since $(M \cap U^\perp) + U$ is singular, we conclude that x acts as the same scalar on $M, (M \cap U^\perp) + U$ and N. Since this is true for all such M and

N, x acts as a scalar on $\langle \mathcal{S} \rangle$. By 2.2 applied with X being a complement to $\langle \mathcal{S} \rangle$ in V, $\dim V/\langle \mathcal{S} \rangle < f(m)$ and thus $\mathrm{pdeg}(x) < f(m)$, a contradiction.

(c) Suppose first that $|H| = 1$ and let $H = \{x\}$. By induction on t, there exists N in $\mathcal{S}$ with $\dim N \leq (t-1)m$ such that $\mathrm{pdeg}_N(x) \geq t-1$ (choose $N = 0$ if $t = 1$). Let $W = N^{\perp}/N$. Then $\dim W \geq \dim V - 2\dim N > 2m + f(m)$ and $\mathrm{pdeg}_W(x) \geq \mathrm{pdeg}_V(x) - 2\dim N > f(m)$. So by (b) there exists a G-invariant singular subspace X in W such that x does not act as scalar on X. Replacing X by $K\langle u^G \rangle$, where $u \in X$ with $u^x \notin Ku$ we may assume that $\dim X < m$. Let M be the inverse image of X in V. Then $\mathrm{pdeg}_M(x) \geq \mathrm{pdeg}_N(x) + \mathrm{pdeg}_X(x) \geq (t-1) + 1 = t$.

Suppose now that $|H| > 1$. Let $x \in H$ and put $H^* = H \setminus \{x\}$. Then by induction on $|H|$ where exists $N \in \mathcal{S}$ with $\dim N \leq t|H^*|m$ and $\mathrm{pdeg}_N(h) \geq t$ for all $h \in H^*$. Let $W = N^{\perp}/N$. Then $\dim W \geq 2m + f(m)$ and $\mathrm{pdeg}_W(x) \geq 2m(t-1) + f(m)$. Thus (c) follows by applying the $|H| = 1$ case to W and x and take inverse images in $N^{\perp}$.

(d) Let H be a set of representatives of the non-trivial cosets of Z in G. Note that $\mathrm{pdeg}_V(x) > g(m)$ implies $\dim V > g(m)$. Apply (c) with t = 1.

Lemma 2.4 *Let $H = PGL_K(V)$ or a finite symmetric group. Then there exist increasing functions h, k defined on the positive integers and independent from H such that each of the following two statements hold.*

(a) If $x \in S \leq H$ with S solvable, then $\mathrm{der}(\langle x^S \rangle) \leq h(\mathrm{pdeg}_H(x))$.

(b) Let $x \in H'$ with $|x| = p$, p a prime, and, if $H = PGL_K(V)$, $|x| = 2$. If $\mathrm{pdeg}_{H'}(x) \geq k(t)$, then there exists a p-subgroup S of H' with $x \in S$ and $\mathrm{der}(\langle x^S \rangle) \geq t$.

Proof: For (a) see [6, Proposition 1] and for (b) see [1, 2.1,2.4].

Lemma 2.5 *Let $H = Q$ or a finite symmetric group. There exists an increasing function l defined on the positive integers and independent from H with the following property:*

Let $G \leq H$ and $N \lhd G$ with $G/N \in \mathcal{A} \cup \mathcal{L}$. Let $x \in G$ with $|x| = p$, p a prime, and if $G/N \in \mathcal{L}$, $|x| = 2$. If $\mathrm{pdeg}_{G/N}(x) \geq l(m)$ then $\mathrm{pdeg}_H(x) > m$.

Proof: Let k, h be the functions given by 2.4 and define $l(m) = k(h(m) + 2)$. If $\mathrm{pdeg}_{G/N}(x) \geq l(m) + 1$, then by 2.4b there exists a p-group S in G with $x \in S$ and $\mathrm{der}(\langle x^S \rangle N/N) \geq h(m) + 2$. So $\mathrm{der}(\langle x^S \rangle) > h(m) + 1$ and by 2.4a, $\mathrm{pdeg}_H(x) > m$.

Lemma 2.6 *Let $G \leq Q$ and Δ a system of imprimitivity for G on V. Let $U \in \Delta$ and E a subgroup of $GL_K(U)$ with $N_G(U)/C_G(U) \leq E$. Suppose that G acts transitively on Δ. Then there exists $H \leq N_Q(\Delta)$ with $G \leq H$, $N_H(U)/C_H(U) = E$ and $H \cong E \wr Sym(\Delta)$.*

Proof: Let I be a transversal to $N_G(U)$ in G with $1 \in I$. Define $F \leq Q$ by F normalizes U, $F/C_F(U) = E$ and $[W, F] = 0$ for all $W \in \Delta \setminus \{U\}$. Define an action of $Sym(I)$ on V as follows. For $\pi \in Sym(I)$ and $u_i \in U$, $i \in I$, let

$$(\sum_{i\in I} u_i^i)^\pi = \sum_{i\in I} u_i^{i^\pi}.$$

Let $H = \langle F, Sym(I)\rangle$, where we view $Sym(I)$ as a subgroup of Q by the above action. Then clearly H normalizes Δ, $N_H(U)/C_H(U) = E$, $Sym(I) \cong Sym(\Delta)$ and $H \cong E \wr Sym(\Delta)$. It remains to prove that $G \leq H$. Let $g \in G$ and define $\pi \in Sym(I)$ and $n_i \in N_G(U)$, $i \in I$, by $ig = n_i i^\pi$. Let $h = g\pi^{-1}$. Then

$$(\sum_{i\in I} u_i^i)^h = (\sum_{i\in I} u_i^{ig})^{\pi^{-1}} = (\sum_{i\in I} u_i^{n_i i^\pi})^{\pi^{-1}} = \sum_{i\in I} u_i^{n_i i}.$$

Pick f_i in F such that $f_i n_i^{-1}$ centralizes U and $\pi_i \in Sym(I)$ with $1^{\pi_i} = i$. Then since $u_i^{i\pi_i^{-1}} = u_i^{i^{\pi_i^{-1}}} = u_i^1 = u_i$,

$$u_i^{i(\pi_i^{-1} f_i \pi_i)} = u_i^{f_i \pi_i} = u_i^{n_i \pi_i} = u_i^{n_i i}.$$

It follows that $h = \prod_{i\in I} \pi_i^{-1} f_i \pi_i$. So $h \in H$ and $G \leq H$.

Lemma 2.7 *Let $M \leq Q$, Δ be the set of components of M and $H = N_Q(M)$. Assume that $M = \langle \Delta \rangle$, that H acts irreducibly and primitively on V and that H acts transitively on Δ. Then there exist a cyclic group S and $m \geq 1$ such that $H/C_H(\Delta) \cong Sym(m) \wr S$, where the wreath product is build via the regular permutation representation of S.*

Proof: Let U be an irreducible KM-submodule in V. Since H is primitive on V, V is a direct sum of KM-submodules isomorphic to U. Put $D = \mathrm{Hom}_{KM}(V, V)$, $h = \dim_K V / \dim_K(U)$ and $E = \mathrm{Hom}_{KM}(U, U)$. Then E is a field and $D \cong \mathrm{Hom}_E(E^h, E^h)$ as rings. Put $F = \mathrm{Hom}_{DM}(V, V)$. Then F is a finite field isomorphic to E, and $H/C_H(F)$ acts as a group of field automorphisms on F. Hence $H/C_H(F)$ is cyclic. Pick L in Δ and define $S = H/C_H(F)N_H(L)$ and $m = |\Delta|/|S|$. Since $H/C_H(F)$ is cyclic and H acts transitively on Δ, S is cyclic and independent from the choice of L. Moreover, $|S|$ is the number of orbits of $C_H(F)$ on Δ, $C_H(F)$ and $C_H(F)N_H(L)$ have the same orbits on Δ and m is the length of each of those orbits. Note that V is a vector space over F, the elements of H act semilinear with respect to F and every irreducible FM-submodule in V is isomorphic to U as FM-module. Let Γ be an orbit for $C_H(F)$ on Δ. We will prove next that :

(*) Let $\pi \in Sym(\Delta)$ such that π fixes all elements in $\Delta \setminus \Gamma$. Then π is induced by some element $g \in C_H(F)$.

Note that as FM-modules $U \cong \bigotimes_{x\in\Delta} Y_x$, where Y_x is an irreducible Fx submodule in U and the tensor product is build over F. Fix $L \in \Gamma$ and for each $x \in \Gamma$ pick $h(x)$ in $C_H(F)$ with $x = L^{h(x)}$. Put $Y = Y_L$ and $Z = \bigotimes_{x\in\Gamma} Y_x$. Then Y_x and $Y^{h(x)}$ are both irreducible Fx-submodules of U and so isomorphic as Fx-modules. Put $W = \bigotimes_{x\in\Gamma} Y^{h(x)}$. Then Z and W are isomorphic as $F\langle\Gamma\rangle$-modules. Define a map α by

$$\alpha: \prod_{x\in\Gamma} Y^{h(x)} \rightarrow W$$

$$\{y_x^{h(x)}\}_{x\in\Gamma} \rightarrow \otimes_{x\in\Gamma} y_{x^{\pi^{-1}}}^{h(x)}$$

where $y_x \in Y$ for each $x \in \Gamma$. Then it is readily verified that α is F-linear in each of the coordinates and so induces a F-linear map β from W to W. Let $T = \bigotimes_{x\in\Delta\setminus\Gamma} Y_x$. Define $\gamma : T \bigotimes W \rightarrow T \bigotimes W$ by $(t\otimes w)^g = t \otimes w^\beta$. It is easily checked that β normalizes $\langle\Gamma\rangle$ in $GL_F(W)$ and so γ normalizes M in $GL_F(T \bigotimes W)$ and acts as π on Δ. Since U and $T \bigotimes W$ are isomorphic as FM-modules and V is the direct sum of FM-modules isomorphic to U, there exists g in $GL_F(V)$ such that g normalizes M and acts as π on Δ. Since $N_{GL_F(V)}(M) = C_H(F)$, (*) is proved.

By (*), $C_H(F)$ induces all possible permutations of Δ which normalize the orbits of $C_H(F)$ on Δ. Hence the same is true for $C_H(F)N_H(L)$ in place of $C_H(F)$, $C_H(F)N_H(L)/C_H(\Delta) \cong Sym(m)^{|S|}$ and $H/C_H(\Delta) \cong Sym(m)\wr S$.

Lemma 2.8 *Let G be a finite group and N a normal subgroup G such that G/N is perfect. Then there exists a unique subnormal subgroup R of G which is minimal with respect to $G = RN$.*

Proof: Let R_1 and R_2 be minimal subnormal supplements to N in G. Let K_i be proper normal subgroups of G with $R_i \leq K_i$ for $i = 1, 2$. Then $G = K_1N = K_2N$. Since G/N is perfect, $G = [K_1, K_2]N$. Let R be a minimal subnormal supplement to $[K_1, K_2] \cap N$ in $[K_1, K_2]$. By induction on $|G|$, R and R_i are both the unique minimal subnormal supplement to $N \cap K_i$ in K_i for $i = 1, 2$. Thus $R_1 = R = R_2$.

Lemma 2.9 *Let G_1, G, N_1 and N be subgroups of Q such that $N_1 \lhd G_1$, $N \lhd G$, G_1/N_1 and G/N are perfect and simple, $G/N \in \mathcal{L}$, $G_1 \leq G$ and $G_1 \cap N \leq N_1$. Let R_1 be a minimal subnormal supplement to N_1 in G_1. Suppose that each of the following two statements hold:*

(i) there exists x in R_1 with $|xZ(Q)| = 2$ and $\mathrm{pdeg}_{G/N}(x) \geq l(2m + f(m))$, where $m = |G_1/G_1 \cap Z(Q)|$ and f and l are as in 2.2 and 2.5, respectively.

(ii) $C_G(N/O_p(G)) \leq N$, where p is the characteristic of K.

Then there exists $G_2 \leq Q$ and $N_2 \lhd G_2$ such that $G_1 \leq G_2$, $G_1 \cap N_2 \leq N_1$ and all non-abelian composition factors of G_2/N_2 are alternating groups.

Proof: The proof is by induction on $|V|$. Let R be a minimal subnormal supplement to N in G. We assume without loss that $G = G_1R$. By 2.8, R is unique and so normal in G. Since G/N is simple, we conclude R is contained in every subnormal subgroup H of G with $H \not\leq N$. Moreover, since G/N is perfect, $R'N = G$ and so $R = R'$. Hence the three subgroup lemma implies:

(*) If M is a normal subgroup of G with $[M, R] \neq 1$, then $[[M, R], R] \neq 1$.

Assume first that G acts reducibly on V. By (ii), $[R,N] \not\leq O_p(G)$ and so $[R,N]$ does not act unipotently on V. Hence there exists a chief factor W for G on V with $[W,[R,N]] \neq 0$. Let $\bar{G} = G/C_G(W)$. Then $O_p(\bar{G}) = 1$. Suppose that $C_G(W) \not\leq N$, then $R \leq C_G(W)$, a contradiction. Hence $C_G(W) \leq N$ and $\bar{G}/\bar{N} \cong G/N$. If $\bar{G_1} = \bar{N_1}$, $G_1 \leq C_{G_1}(W)N_1 \leq (G_1 \cap N)N_1 = N_1$, a contradiction. Hence $\bar{G_1}/\bar{N_1} \cong G_1/N_1$. By choice of W, $\bar{R} \not\leq C_{\bar{G}}(\bar{N})$ and so $C_{\bar{G}}(\bar{N}) \leq \bar{N}$. It is now easy to verify that $(\bar{G_1}, \bar{N_1}, \bar{G}, \bar{N}, \bar{R_1}, \bar{x}, W)$ fulfills the assumptions of the lemma. So by induction there exists a subgroup $\bar{G_2}$ of $GL_K(W)$ and $\bar{N_2} \lhd \bar{G_2}$ such that $\bar{G_1} \leq \bar{G_2}$, $\bar{G_1} \cap \bar{N_2} \leq \bar{N_1}$ and all non-abelian composition factors of $\bar{G_2}/\bar{N_2}$ are alternating groups. Let $W = X/Y$ for some KG-submodules X, Y in V. Let G_2 and N_2 be the largest subgroups of Q which normalize X and Y and such that $G_2/C_{G_2}(W) = \bar{G_2}$ and $N_2/C_{N_2}(W) = \bar{N_2}$. Then (G_2, N_2) fulfills the conclusion of the lemma.

Assume next that G acts irreducibly but imprimitively on V. Let Δ be a system of imprimitivity for G on V.

Suppose that $R_1 \not\leq C_G(\Delta)$. Then $C_{G_1}(\Delta) \leq N_1$. Note that $N_Q(\Delta)/C_Q(\Delta) \cong Sym(\Delta)$. Put $G_2 = N_Q(\Delta)$ and $N_2 = C_Q(\Delta)$. Then (G_2, N_2) fulfills the conclusion of the lemma.

So we may assume that $R_1 \leq C_G(\Delta)$. Since $G_1 \cap N \leq N_1$, $R_1 \not\leq N$ and we get $C_G(\Delta) \not\leq N$ and $R \leq C_G(\Delta)$. Pick U in Δ and for any $X \subseteq G$ put $\bar{X} = N_X(U)C_G(U)/C_G(U)$. Let S be the minimal subnormal supplement to $N_N(U)$ in $N_G(U)$. Since $RN_N(U) = N_G(U)$, $S \leq R$. In particular, S is normal in R and so subnormal in G. Since $S \not\leq N$, we get $S = R$ by minimality of R.

Suppose that $C_{\bar{G}}(\bar{N}) \not\leq \bar{N}$. Then since $R = S$, $\bar{R} \leq C_{\bar{G}}(\bar{N})$. So $[N,R,R]$ centralizes U and since G acts irreducibly on V and $[N,R,R]$ is normal in G, $[N,R,R] = 1$, a contradiction to (*). Thus $C_{\bar{G}}(\bar{N}) \leq \bar{N}$. If $\bar{G} = \bar{N}$, $N_G(U) \leq C_G(U)N_N(U)$ and so $R = S \leq C_G(U)$, a contradiction. Thus $\bar{G}/\bar{N} \cong G/N$. If $\bar{G_1} = \bar{N_1}$, then $R_1 \leq N_{G_1}(U) \leq C_{G_1}(U)N_{N_1}(U) \leq C_G(U)N_N(U)$. Hence $C_G(U) \not\leq N$ and $\bar{G} = \bar{N}$, contradiction. Thus $\bar{G_1}/\bar{N_1} \cong G_1/N_1$. It is now easily verified that $(\bar{G_1}, \bar{N_1}, \bar{G}, \bar{N}, \bar{R_1}, \bar{x}, U)$ fulfills the assumptions of the lemma. So by induction there exists a subgroup $\bar{G_2}$ of $GL_K(U)$ and $\bar{N_2} \lhd \bar{G_2}$ such that $\bar{G_1} \leq \bar{G_2}$, $\bar{G_1} \cap \bar{N_2} \leq \bar{N_1}$ and all non-abelian composition factors of $\bar{G_2}/\bar{N_2}$ are alternating groups. Apply 2.6 to G_1 in place of G and with $E = \bar{G_2}$. Put $G_2 = H$ and $H_2 = C_{G_2}(\Delta)$ and let N_2 be the largest normal subgroup of G_2 contained in H_2 with $N_2C_{G_2}(U)/C_{G_2}(U) = \bar{N_2}$. Then $H_2/N_2 \cong (\bar{G_2}/\bar{N_2})^{|\Delta|}$ and so every non abelian composition factor of G_2/N_2 is an alternating group. Note that $\bar{G_1} \cap \bar{N_2} \leq \bar{N_1}$. Since $C_{G_1}(U) \leq C_G(U) \cap G_1 \leq N \cap G_1 \leq N_1$, we conclude that $G_1 \cap N_2 \leq N_1$. Therefore (G_2, N_2) fulfills the conclusion of the lemma.

Assume last that G acts irreducibly and primitively on V. Let X be any normal subgroup of G. If X has more than one Wedderburn component on V, these Wedderburn components would form a system of imprimitivity for G on V. Thus V is a direct sum of isomorphic irreducible KX-submodules in V. In particular, $Z(X)$ is cyclic. Let M be a normal subgroup of G in N minimal with respect $[M,R] \neq 1$. By (*) $[M,R,R] \neq 1$ and so $M=[M,R] \leq R$. It follows that $C_M(R) \leq Z(M)$. Since $Z(M)$ is cyclic and R is perfect, $[Z(M),R]=1$ and so $C_M(R)=Z(M)$. Put $\bar{M}=M/Z(M)$. Then $\bar{M}$ is a minimal normal subgroup of $G/Z(M)$ and so $\bar{M}$ is the direct product of isomorphic simple groups.

Suppose first $\bar{M}$ is perfect. Then $M=E(M)$. Let Δ be the set of components of M and note that G acts transitively on Δ. Assume that $R \leq C_G(\Delta)$. Since R is perfect and the outer automorphism group of any finite simple group is solvable we conclude that R induces inner automorphisms on $\bar{M}$. Thus $R \leq MC_G(\bar{M})$, $C_G(\bar{M}) \not\leq N$, $R \leq C_G(\bar{M})$ and $[M,R,R]=1$, a contradicton to $M=[M,R]$. Therefore $R \not\leq C_G(\Delta)$ and so $C_G(\Delta) \leq N$. Put $G_2=N_Q(M)$ and $N_2=C_{G_2}(\Delta)$. Then $G_1 \cap N_2 \leq C_G(\Delta) \leq N$ and so $G_1 \cap N_2 \leq G_1 \cap N \leq N_1$. By 2.7, every non-abelian composition factor of G_2/N_2 is an alternating group and so (G_2,N_2) fulfills the conclusion of the lemma.

Suppose next that $\bar{M}$ is an elementary abelian q-group for some prime q. Since $M' \leq Z(M)$, M' is elementary abelian and cyclic. Thus $|M'|=q$. Define a symplectic form on $\bar{M}$ by $s(a,b)=[a,b]$. By 2.5 applied to $H=GL_{GF(q)}(\bar{M})$ we have that $\mathrm{pdeg}_{\bar{M}}(x) > 2m+f(m)$. Now $m \geq |G_1/C_{G_1}(\bar{M})|$ and so by 2.3b there exists a G_1-invariant subspace $\bar{A}$ in $\bar{M}$ such that $[\bar{A},x] \neq 1$ and s vanishes on $\bar{A}$. Let A be the inverse image of $\bar{A}$ in M. Then A is abelian. Let $\Delta=\{C_V(B) | B \leq A, A/B \text{ cyclic and } C_V(B) \neq 0\}$ and note that V is the direct sum of the elements of Δ. Let $G_2=N_Q(\Delta)$ and $N_2=C_Q(\Delta)$. Then G_2/N_2 is the direct product of symmetric groups and $G_1M \leq G_2$.

Suppose that $G_1 \cap N_2 \not\leq N_1$. Then $R_1 \leq N_2$ and $[R_1,M] \leq N_2$. Pick $D \in \Delta$. Then $[R_1,M]$ normalizes $C_A(D)$ and so $[[R_1,M],C_A(D)] \leq C_M(D) \cap M'=1$. Since also $[C_A(D),M,R_1]=1$ the three subgroup lemma yields, $[C_A(D),R_1,M]=1$. Thus $[C_A(D),R_1] \leq C_M(D) \cap Z(M)=1$. Furthermore, R_1 is perfect and $A/C_A(D)$ is cyclic. Thus R_1 centralizes A, a contradiction to $[\bar{A},x] \neq 1$.

Hence $G_1 \cap N_2 \leq N_1$, (G_2,N_2) fulfills the conclusion of the lemma and the proof of 2.9 is completed.

Corollary 2.10 *2.9 remains true if Q is replaced by $PSL_K(V)$.*

Proof: Apply 2.9 to the inverse images in $GL_K(V)$, intersect the resulting G_2 and N_2 with $SL_K(V)$ and then look at the images in $PSL_K(V)$.

Lemma 2.11 *Let I be a finite set, for $i \in I$ let L_i be a perfect simple group and let $M \leq \prod_{i\in I} L_i$. For $J \subseteq I$, let $L_J=\prod_{j\in J} L_j$, $M_J=M \cap L_J$ and let M^J be the projection of M onto L_J.*

(a) If $M^i=L_i$ for all $i \in I$, then there exists a partition Π of I such that $M=\prod_{\pi\in\Pi} M_\pi$ and for all $\pi \in \Pi$ and $i \in \pi$, the projection of M_π onto L_i is an isomorphism.

(b) Put $J=\{i \in I | M^i=L_i\}$ and $K=I \setminus J$. If L_i is finite and $L_i \cong L_j$ for all $i,j \in I$, then $M=M_J \times M_K$.

Proof: (a) Let $\Pi=\{J \subseteq I | M_J \neq 1 \text{ and } M_K=1 \text{ for all } K \subset J\}$. Let $i \in \pi \in \Pi$ and let ϕ be the projection map from M_π to L_i. Then $\ker\phi \leq M_{\pi\setminus\{i\}}=1$

and ϕ is one to one. Moreover, M and so L_i normalizes the image of ϕ. Since L_i is simple, we conclude that ϕ is onto and so ϕ is an isomorphism. If $\pi' \in \Pi$ with $\pi \cap \pi' \neq \emptyset$, then $1 \neq [M_\pi, M_{\pi'}] \leq M_{\pi\cap\pi'}$ and so $\pi = \pi \cap \pi' = \pi'$.

It remains to show that $M = M^*$, where $M^* = \prod_{\pi\in\Pi} M_\pi$. For m in L_I let $S(m) = \{i \in I | m_i \neq 1\}$. We will prove by induction on $|S(m)|$ that $m \in M^*$. Without loss $m \neq 1$. Then $M_{S(m)} \neq 1$ and so there exists $\pi \in \Pi$ with $\pi \subseteq S(m)$. Pick $i \in \pi$ and $n \in M_\pi$ with $n_i = m_i$. Then $S(mn^{-1}) \subseteq S(m) \setminus \{i\}$ and so by induction $mn^{-1} \in M^*$. Clearly $n \in M^*$ and so $m \in M^*$.

(b) By (a) M/M_K is a direct product of simple groups isomorphic to L_i. Note that M/M_J is a subdirect product of proper subgroups of L_i and so has no composition factor isomorphic to L_i. So no non-trivial factor group of M/M_K is isomorphic to a factorgroup of M/M_J. Thus $M/M_K M_J = 1$, $M = M_K M_J$ and $M = M_J \times M_K$.

Lemma 2.12 *Let L be a perfect, finite, simple group, n a positive integer, $T = L^n$, h an automorphism of T of order q, q a prime, I the set of components of T, t the number of non-trivial orbits for h on I, M an h-invariant subgroup of T and $K = \{g \in T | [g, h] \in M\}$. Then one of the following holds:*

(i) M contains a component of T.

(ii) $|K| \leq (\frac{3}{4})^t |T|$.

Proof: We assume without loss that M does not contain a component of T and $t > 0$. We use the notation introduced in 2.11 with $L_D = D$ for all $D \in I$.

We may assume that

(*) If J is an h-invariant subset of I such that M^J does not contain a component of T and either $M^{I\setminus J}$ does not contain a component of T or h acts trivally on $I \setminus J$, then $J = \emptyset$ or $J = I$.

Indeed suppose that (*) is false. Then by induction on n, $|K^J| \leq (\frac{3}{4})^s |L_J|$ and $|K^{I\setminus J}| \leq (\frac{3}{4})^{t-s} |L_{I\setminus J}|$, where s is the number of non-trivial orbits for h on J. Hence (ii) holds in this case.

Using 2.11 we will prove next that one of the following holds:

(1) The projection of M to L is not onto and h acts transitively on I.

(2) The projection of M to L is an isomorphism and h acts transitively on I.

(3) There exists an h-invariant partition Π of I such that $\Pi \neq \{I\}$, h acts transitively on Π, $M = \prod_{\pi\in\Pi} M_\pi$ and if $D \in \pi \in \Pi$, the projection of M_π to D is an isomorphism.

Indeed, put $J = \{D \in I | M^D = D\}$. Then by 2.11b, $M = M_J \times M_{I\setminus J}$. So $M^{I\setminus J} = M_{I\setminus J}$ and neither M^J nor $M^{I\setminus J}$ contains a component of T. Thus by (*) $I = J$ or $J = \emptyset$. If $J = \emptyset$, let J^* be any h-orbit on I. Then by (*) $J^* = I$ and (1) holds. So we may assume $I = J$ and thus $M^D = D$ for all $D \in I$. Let Π be the partion of I given by 2.11. Then Π is clearly h-invariant. Let Δ be an h-orbit on Π and put $J' = \bigcup \Delta$. Similarly as above neither $M^{J'}$ nor $M^{I \setminus J'}$ does contain a component of T and so by (*), $J' = I$. If $\Pi \neq \{I\}$, (3) holds. So assume $\Pi = \{I\}$ and let J_* be any non trivial h-orbit on I. If $|I \setminus J_*| \geq 2$, $M^{I \setminus J_*}$ does not contain a component of T and if $|I \setminus J_*| \leq 1$, h acts trivally on $I \setminus J_*$. Thus in any case $I = J_*$ by (*), and (2) holds.

Suppose first that (1) or (3) holds. In case (1) put $\Pi = I$. Pick $\pi \in \Pi$ and let $g \in T$. Then $g = g_1 g_2^h ... g_q^{h^{q-1}}$ for some $g_i \in L_\pi$. So

$$[g, h] = (g_1^{-1} g_q)(g_2^{-1} g_1)^h \dots (g_q^{-1} g_{q-1})^{h^{q-1}}.$$

Thus if $g \in K$ we conclude from $M = \prod_{\rho \in \Pi} M^\rho$ that

$$g_1 M^\pi = g_2 M^\pi = \ldots = g_q M^\pi.$$

Hence $|K| \leq |M^\pi|^q |L_\pi / M^\pi| = |T| / |L_\pi / M^\pi|^{q-1}$. It remains to show that $|L_\pi / M^\pi|^{q-1} \geq (4/3)^t$. If (1) holds, $t = 1$ and this is obvious. If (3) holds, $t = |\pi|$. Since M contains no components of T, $|\pi| > 1$ and so $t > 1$. Moreover, $|M_\pi| = |L|$ and so $|L_\pi / M_\pi|^{q-1} = |L|^{(t-1)(q-1)} \geq |L|^{t-1} \geq 4^{t-1} = (4/3)^t 3^t / 4 \geq (4/3)^t$.

Suppose next that (2) holds. Then $M \cong L$, $t = 1$ and $C_T(h) \cong L$. Put $Y = \{[k, h] | k \in K\}$. Then $Y \subseteq M$ and $|Y| \leq |L|$. Let $k, l \in T$. Then $[k, h] = [l, h]$ if and only if $lk^{-1} \in C_T(h)$. Thus $|K| = |Y||C_T(h)| = |Y||L| \leq |L|^2$. If $q > 2$ we conclude that $|K| \leq |L|^2 \leq 3/4 |L||L|^2 \leq 3/4 |L|^q$. If $q = 2$, h inverts all elements of Y. So since M is not abelian it follows from a well-known exercise [3, 2.9 #12, p71], that $|Y| \leq 3/4 |M|$ and so $|K| \leq 3/4 |L|^2 = 3/4 |T|$, completing the proof of the lemma.

Lemma 2.13 *Let Ω be a finite set, $H \leq Sym(\Omega)$, $H^* \subseteq H \setminus \{1\}$ and h, k positive integers. If $\deg_\Omega(x) \geq hkw^*(H)|H||H^*|$ for all $x \in H^*$, then there exists a subset Γ of Ω and a partition Δ of Γ in subsets of size h such that H normalizes Δ and $\deg_\Delta(x) \geq k$ for all $x \in H^*$.*

Proof: Let $\mathcal{W}$ be a set of representatives for the isomorphism classes of transitive permutation representations for H. We note that $|O| \leq |H|$ for all $O \in \mathcal{W}$. For $O \in \mathcal{W}$, let $r(O)$ be the number of H-orbits on Ω isomorphic to O.

Let $x \in H^*$. We claim that there exists $O_x \in \mathcal{W}$ with $x \notin C_H(O_x)$ and $r(O_x) \geq hk|H^*|$. Indeed, let $\mathcal{W}_x = \{O \in \mathcal{W} | x \notin C_H(O)\}$. Then $\deg_\Omega(x) = \sum_{O \in \mathcal{W}_x} r(O) \deg_O(x)$. Since $\deg_O(x) \leq |O| \leq |H|$, $|\mathcal{W}_x| \leq |\mathcal{W}| = w^*(H)$ and $\deg_\Omega(x) \geq hkw^*(H)|H||H^*|$, we conclude that $r(O_x) \geq hk|H^*|$ for at least one O_x in $\mathcal{W}_x$.

Let Y be subset of H^* maximal such that there exist pairwise distinct H-orbits $O(y, i, j)$, $y \in Y$, $1 \leq i \leq h, 1 \leq j \leq k$, in Ω such that $O(y, i, j)$ is isomorphic to O_y. Suppose that $Y \neq H^*$ and pick u in $H^* \setminus Y$. Since $r(O_u) \geq hk|H^*| \geq hk|Y| + hk$ there are at least hk H-orbits on Ω which are isomorphic to O_u and distinct from the $O(y, i, j)'s$, a contradiction to the maximal choice of Y.

Thus $Y = H^*$. Let $\phi(x,i,j)$ be an H-isomorphism from O_x to $O(x,i,j)$. For $d \in O_x$, $x \in H^*$ and $1 \leq j \leq k$ put $D(x,j,d) = \{\phi(x,i,j)(d) | 1 \leq i \leq h\}$. Put $\Delta = \{D(x,j,d) | x \in H^*, 1 \leq j \leq k, d \in O_x\}$. Note that $D(x,j,d)^g = D(x,j,d^g)$ for all $g \in H$ and so H normalizes Δ. For $x \in H^*$ pick d in O_x with $d^x \neq d$. Then $D(x,j,d)^x \neq D(x,j,d)$ for all $1 \leq j \leq k$ and so $\deg_\Delta(x) \geq k$.

Lemma 2.14 *Let Ω be a finite set, $G \leq Sym(\Omega)$, $N \lhd G$ with $G/N \cong Alt(n), n \geq 5$, $H \leq G$ with $H \cap N = 1$, u a positive integer with $u \geq (|H| \log_{4/3} |H|) + 2$ and $k = \max(l(u), 5uw(H)|H|^2, 9|H|^2)$, where l is as in 2.5. If $\mathrm{pdeg}_{G/N}(x) \geq k$ for all $1 \neq x \in H$ then one of the following holds:*

(a) H has a regular orbit on Ω.

(b) G has a t-pseudo natural orbit on Ω with respect to N, where t is a positive integer with $t \leq |H| - 2$.

Proof: Consider a counter example with $|\Omega|$ minimal. Let R be the minimal subnormal supplement to N in G provided by 2.8. Let O be a G-orbit on Ω with $R \not\leq C_G(O)$. Then $(G/C_G(O), NC_G(O)/C_G(O), HC_G(O)/C_G(O), O)$ fulfill the assumption of the lemma. We conclude that $\Omega = O$ and so G acts transitively on Ω.

Suppose that G acts imprimitively on Ω and let Δ be a maximal system of imprimitivity for G on Ω. If N is not transitive on Ω, the orbits of N on Ω form a system of imprimitivity for G on Ω and we can and do choose Δ such that $N \leq C_G(\Delta)$.

If $C_G(\Delta) \leq N$, then $(G/C_G(\Delta), NC_G(\Delta)/C_G(\Delta), HC_G(\Delta)/C_G(\Delta), \Delta)$ fulfills the assumption of the lemma. But then (a) or (b) holds for Δ and so also for Ω.

Hence $C_G(\Delta) \not\leq N$, $R \leq C_G(\Delta), N \not\leq C_G(\Delta)$, N is transitive on Ω and R is not transitive on Ω. Let O be an orbit for R on Ω. For $X \leq G$ put $X_0 = N_X(O)$. Let R^* be the minimal subnormal supplement to N_0 in G_0. Then $R^* \trianglelefteq R$, R^* is a subnormal supplement to N in G and thus $R = R^*$. If $C_{G_0}(O) \not\leq N_0$, $R = R^* \leq C_{G_0}(O)$. Since R is normal in G and G is transitive we conclude that $R = 1$, a contradiction. Thus $C_{G_0}(O) \leq N_0$. If H_0 has a regular orbit on O, H has a regular orbit on Ω. Hence, by minimality of $|\Omega|$, there exists G-invariant partition Γ_0 for G_0 on O such that $N_0 = C_{G_0}(\Gamma_0)$. It follows that N_0 is not transitive on O and so N is not transitive on Ω, a contradiction.

We proved that G acts primitively on Ω. Let M be the stabilizer in G of some point in Ω. Since H has no regular orbit on Ω, $H \cap M^g \neq 1$ for all $g \in G$. Let T be a minimal normal subgroup of G with $T \leq R$. Then $H \cap M^t \neq 1$ for all $t \in T$. By the pigeon hole principal there exists $1 \neq h \in H$ with $|S| \geq |T|/(|H| - 1)$, where $S = \{s \in T | h \in M^{s^{-1}}\}$. Without loss $|h| = p$, p a prime. Pick $s_0 \in S$. Replacing M by $M^{s_0^{-1}}$ and S by Ss_0^{-1} we may assume that $1 \in S$ and so $h \in M$. Let $s \in S$. Then $h \in M^{s^{-1}}$ and so $h^s \in M$. It follows that $[s,h] = h^{-s}h \in M \cap T$ for all $s \in S$.

Suppose that T is an elementary abelian q-group for some prime q. Then T acts regularly on Ω and so $C_G(T) \leq T$ and $T \cap M = 1$. In particular, $S \leq C_T(h)$ and since $|S| \geq |T|/(|H|-1)$, $|T/C_T(h)| \leq |T/S| \leq |H|$. Thus

$$\mathrm{pdeg}_T(h) \leq \deg_T(h) = \log_q |T/C_T(h)| \leq \log_q |H| \leq \log_{4/3} |H|.$$

On the otherhand by assumption $\mathrm{pdeg}_{G/N}(h) \geq l(u)$ and so by 2.5 applied to "$H = GL_{GF(q)}(T)$" and "$x = h$", $\mathrm{pdeg}_T(x) \geq u > \log_{4/3}(|H|)$, a contradiction.

Hence T is the direct product of perfect simple groups. Let I be the set of components of T. Note that T acts transitively on Ω and so $G = MT$ and M acts transitively on I. In particular, M contains no component of T.

Suppose that $C_G(I) \leq N$. Let t be the number of non trivial orbits of h on I. Then by 2.12, $|S| \leq (3/4)^t |T|$. Since $|S| \geq |T|/|H|$ we conclude $t \leq \log_{4/3}(|T|/|S|) \leq \log_{4/3} |H|$. Since $t = \frac{1}{p}\mathrm{pdeg}_I(h) \geq \frac{1}{|H|}\mathrm{pdeg}_I(h)$ we conclude that $\mathrm{pdeg}_I(h) \leq |H| \log_{4/3} |H|$, a contradiction to $\mathrm{pdeg}_{G/N}(h) \geq l(|H| \log_{4/3} |H| + 2)$ and 2.5.

Thus $C_G(I) \not\leq N$ and so $R \leq C_G(I)$, i.e R normalizes all components of T. Since the outer automorphism group of every finite simple group is solvable and since R is perfect we conclude that R induces inner automorphism on T. Thus $R \leq TC_G(T)$ and $TC_G(T) \not\leq N$. Recall that $T \leq R$. If $C_G(T) \not\leq N$ we get $T \leq R \leq C_G(T)$, a contradiction. Hence $C_G(T) \leq N$, $T \not\leq N$ and $R \leq T$. It follows that $R = T$, $R \cap N = 1$, $R \cong Alt(n)$ and $G = R \times N$.

Suppose that $N \neq 1$. Then, since G is primitive, N is transitive and R is regular. So $R \cap M = 1$ and $S \subseteq C_R(h)$. Since $|T|/|S| \leq |H| - 1$, $|R/C_R(h)| \leq |H| - 1$. Since $R \cong Alt(n)$, $n \geq 5$, R has no subgroup of index less than n. So $|H| - 1 \geq n \geq \mathrm{pdeg}_{G/N}(h) \geq |H|$, a contradiction.

Therefore $N = 1$ and $G \cong Alt(n)$. Let $\Lambda = \{1, 2, \ldots, n\}$ with G acting naturally on Λ. Let $1 \neq x \in H$. Then $\deg_\Lambda(x) = \mathrm{pdeg}_{G/N}(x) \geq 5uw(H)|H|^2$ and so by 2.13 applied to $(H, H \setminus \{1\}, 5, u, \Lambda)$ in place of (H, H^*, h, k, Ω) there exists a subset Γ of Λ and a partition Δ of Γ in subsets of size five such that H normalizes Δ and $\deg_\Delta(x) \geq u$ for all $1 \neq x \in H$. Let $T^* = (C_{Alt(\Lambda)}(\Delta) \cap C_{Alt(\Lambda)}(\Lambda \setminus \Gamma))'$ and $M^* = M \cap T^*$,(where M now is some conjugate of the M above). Note that T^* is the direct product of alternating groups of degree five and the action of H on the components of T^* is isomorphic to the action on Δ. If M^* does not contain a component of T^*, then using 2.12 we get the same contradiction as in the case $C_G(I) \leq N$ with T replaced by T^*. Hence M contains a component of T^* and in particular an element acting as a three cycle on Λ. If M acts primitively on Λ, we conclude from [4, II 4.5(c)], that $M = G$, a contradiction.

Thus M does not act primitively on Λ. Note that $\deg_\Lambda(x) > 8|H|$ for all $1 \neq x \in H$. It follows that there exist $4|H|$ pairwise distinct elements a_x, b_x, c_x, d_x, $x \in H \setminus \{1\}$, in Λ such that $a_x^x = c_x$ and $b_x^x = d_x$.

Assume that M acts transitively and imprimitively on Λ, and let Θ be a system of imprimitivity for M on Λ. Put $k = |\Lambda|/|\Theta|$. If $|\Theta| \geq 3|H|$, we can can choose disjoint sets $\alpha_x, \gamma_x, \delta_x$, $x \in H \setminus \{1\}$, of size k in Λ with $a_x, b_x \in \alpha_x$, $c_x \in \gamma_x$ and $d_x \in \delta_x$. Choose $g \in G$ such that Θ^g contains α_x, γ_x and δ_x, for all

$x \in H\setminus\{1\}$. Since $H\cap M^g \neq 1$, there exists $1 \neq x \in H$ with $x \in M^g \leq N_G(\Theta^g)$. Since $a_x^x = c_x$, $\alpha_x^x = \gamma_x$ and since $b_x^x = d_x$, $\alpha_x^x = \delta_x$, a contradiction to $\gamma_x \neq \delta_x$. Thus $|\Theta| \leq 3|H|$ and since $|\Lambda| \geq \deg_\Lambda(x) \geq 9|H|^2$, $k \geq 3|H|$. It follows that we can choose disjoint sets α and δ of size k in Λ with $a_x, b_x, c_x \in \alpha$ and $d_x \in \delta$, for all $x \in H \setminus \{1\}$. As above, choose $g \in G$ with $\alpha, \delta \in \Theta^g$, pick $1 \neq x \in H$ with $x \in M^g \leq N_G(\Theta^g)$ and conclude that $\alpha = \alpha^x = \delta$, contradiction.

Thus M does not act transitively on Λ, so $M = N_G(\Theta)$ for some $\Theta \subseteq \Lambda$ with $|\Theta| \leq |\Lambda \setminus \Theta|$. Suppose that $|\Theta| \geq |H| - 1$. Then there exists $g \in G$ with $a_x \in \Theta^g$ and $c_x \in \Lambda \setminus \Theta^g$ for all $x \in H \setminus \{1\}$. It follows that $H \cap M^g = 1$, a contradiction.

The following example shows that there is no absolute bound for t in case (b) of the preceeding Lemma. Let $G = Alt(n)$ with $n = pmk$ and put $I = \{1, 2, \ldots, n\}$. Also let $H \leq G$ with $H = \langle x_i \mid 1 \leq i \leq k\rangle \cong C_p^k$ such that the supports of the x_i, $1 \leq i \leq k$, form a partion of I into subsets of size pm. Let Ω be the set of subsets of size $k-1$ in I. Then for each $J \in \Omega$ there exists $1 \leq i \leq k$ with $J \leq C_I(x_i)$ and so H has no regular orbit on Ω. For fixed k and p we can choose m large enough such that the assumption of 2.14 are fullfilled. Also 2.14(b) holds with $t = k-1$. But note that $k-1 = \log_p |H| - 1$, so probably our bound $t \leq |H| - 2$ can be improved.

Lemma 2.15 *Let G be a LFS-group and $\{(G_i, N_i)|i \in I\}$ a sectional cover for G.*

(a) There exists a Kegel cover $\{(H_j, M_j)|j \in J\}$ such that for all $j \in J$ there exists $i \in I$ with $N_i \leq M_j \trianglelefteq H_j \trianglelefteq G_i$.

(b) For $i \in I$ let M_i be a normal subgroup of G_i. Then at least one of $\{(G_i, M_i)|i \in I\}$ and $\{(M_i, M_i \cap N_i)|i \in I\}$ is a sectional cover for G.

(c) $\{(G_i^\infty, G_i^\infty \cap N_i)|i \in I\}$ is a sectional cover for G.

(d) Let $\mathcal{E}$ be a class of groups such that $K \in \mathcal{E}$ for each $i \in I$ and each non abelian composition-factor K of G_i/N_i, then there exists a Kegel cover for G all of whose factors are in $\mathcal{E}$.

Proof: (a) Let E be a non trivial finite subgroup of G and $1 \neq e \in E$. Since G is simple, $E \leq \langle e^G\rangle$ and since G is locally finite, $E \leq \langle e^F\rangle$ for some finite subgroup F of G. Similarly $F \leq \langle e^{T_e}\rangle$ for some finite subgroup T_e of G. Then $E \leq \langle e^{\langle e^{T_e}\rangle}\rangle$. Let T be the finite subgroup of G generated by E and all the T_e, $1 \neq e \in E$. Pick $i \in I$ with $T \leq G_i$ and $T \cap N_i = 1$. Put $H_E = \langle E^{G_i}\rangle N_i$. Clearly EN_i does not lie in any normal subgroup of G_i properly contained in H_E and, in particular, not in the intersection of the maximal normal subgroup of H_E containing N_i. Thus there exists a maximal normal subgroup M_E of H_E with $E \not\leq M_E$ and $N_i \leq M_E$. Suppose that $1 \neq e \in E$ with $e \in M_E$. Then $\langle e^{G_i}\rangle \leq H_E$ and $E \leq \langle e^{\langle e^{T_e}\rangle}\rangle \leq \langle e^{\langle e^{G_i}\rangle}\rangle \leq \langle e^{H_e}\rangle \leq M_E$, a contradiction. Thus $E \cap M_E = 1$. It follows that $\{(H_E, M_E)|E$ a non trivial finite subgroup of $G\}$ is a Kegel cover that fulfills (a).

(b) Assume that $\{(G_i, M_i)|i \in I\}$ is not a sectional cover for G. Then there exists a finite subgroup H of G such that $H \cap M_i \neq 1$ for all $i \in I$ with $H \leq G_i$.

Without loss $H \leq G_i$ for all $i \in I$. For $1 \neq h \in H$ let $I_h = \{i \in I | h \in M_i\}$. Then I is the finite union of these I_h and so there exists $1 \neq h \in H$ such that $\{(G_i, N_i) | i \in I_h\}$ is a sectional cover for G. We may assume that $I = I_h$. Let E be any finite subgroup of G. Since G is LFS, there exists a finite subgroup T in G with $h \in T$ and $E \leq \langle h^T \rangle$. Pick $i \in I$ with $T \leq G_i$ and $T \cap N_i = 1$. Then $E \leq \langle h^T \rangle \leq M_i$ and $E \cap (N_i \cap M_i) \leq T \cap N_i = 1$. Thus $\{(M_i, M_i \cap N_i) | i \in I\}$ is a sectional cover for G.

(c) Otherwise we conclude from (b) that $\{(G_i, G_i^\infty) | i \in I\}$ is a sectional cover for G. Hence by (a) G has a Kegel cover all of whose factor are of prime order. Thus G is of prime order, a contradiction since G is infinite.

(d) By (a) there exists a Kegel cover all of whose factors are abelian or lie in $\mathcal{E}$. Since G is not abelian, (d) holds.

3 Kegel covers

Throughout this chapter G is a non-finitary LFS-group and $\mathcal{K}$ is a Kegel cover for G. For $K \in \mathcal{K}$ let H_K and M_K be defined by $K = (H_K, M_K)$ and put $\bar{K} = H_K/M_K$. If $\mathcal{E}$ is a class of groups, $\mathcal{K}_\mathcal{E} = \{K \in \mathcal{K} | \bar{K} \in \mathcal{E}\}$. For $K \in \mathcal{K}_\mathcal{F}$ pick a finite field F_K, a finite dimensional vector space V_K over F_K and a non-degenerated symplectic, quadratic or unitary form q_K on V_K such that $\bar{K} \cong P\Omega_{F_K}(V_K)$. For $K \in \mathcal{K}_\mathcal{L}$, pick a finite field F_K and a finite dimensional vector space V_K over F_K such that $\bar{K} \cong PSL_{F_K}(V_K)$. We view V_K as a projective module for H_K. For $K \in \mathcal{K}_\mathcal{A}$ pick a set Ω_K with $\bar{K} \cong Alt(\Omega_K)$. For a finite subset T of G let $\mathcal{K}(T) = \{(H, M) \in \mathcal{K} | T \subseteq H \text{ and } T \cap M \subseteq \{1\}\}$.

Lemma 3.1 *Let k be a positive integer and $X \subseteq G \setminus \{1\}$ with $|X|$ finite. Put*

$$\mathcal{J} = \{K \in \mathcal{K}_{\mathcal{C} \cup \mathcal{A}}(X) | \mathrm{pdeg}_{\bar{K}}(x) \geq k \text{ for all } x \in X\}.$$

Then $\mathcal{J}$ is a Kegel cover for G. In particular, at least one of $\mathcal{K}_\mathcal{A}, \mathcal{K}_\mathcal{L}$ and $\mathcal{K}_\mathcal{F}$ is a Kegel cover for G.

Proof: Without loss $\mathcal{K} = \mathcal{K}(X)$. By induction on $|X|$ we may assume that $|X| = 1$. Let $x \in X$ and suppose that $\mathcal{J}$ is not a Kegel cover for G. Then $\mathcal{K} \setminus \mathcal{J}$ is a Kegel cover for G. By the classification of finite simple groups there exists a natural number t such that every finite simple group not contained in $\mathcal{C} \cup \mathcal{A}$ has a faithful projectice representation of dimension at most t. Let $s = \max\{k, t\}$. Then $\mathrm{pdeg}_{\bar{K}}(x) \leq s$ for all $K \in \mathcal{K} \setminus \mathcal{J}$. Thus by [2, (3.1)], G has a faithful projectice representation U with $\mathrm{pdeg}_U(x) \leq s$. Since $G = \langle x^G \rangle$, G is finitary, a contradiction.

Proposition 3.2 *(a) Suppose $\mathcal{K} = \mathcal{K}_\mathcal{F}$. Let $J = \{(K, U) | K \in \mathcal{K}$ and U is a singular subspace of $V_K\}$, and for $j = (K, U) \in J$ put $H_j = N_{H_K}(U)$ and $M_j = \{x \in H_j | x$ acts as a scalar on $U\}$. Then $PSL_{F_K}(U) \leq H_j/M_j \leq PGL_{F_K}(U)$, $\{(H_j, M_j) | j \in J\}$ is a sectional cover for G and $\{(H_j^\infty, H_j^\infty \cap M_j) | j \in J\}$ is a Kegel cover for G with all factors in $\mathcal{L}$.*

(b) G has a Kegel cover $\mathcal{J}$ with $\mathcal{J} = \mathcal{J}_\mathcal{A}$ or $\mathcal{J} = \mathcal{J}_\mathcal{L}$.

Proof: (a) Let T be a finite subgroup of G and $k = g(|T|)$, where the function g is defined in 2.3d. By 3.1 there exists $K \in \mathcal{K}$ with $T \leq H_K$ and $\mathrm{pdeg}_{V_K}(t) \geq k$ for all $1 \neq t \in T$. By 2.3d there exists a T-invariant singular subspace U in V_K such that no element of $T \setminus \{1\}$ acts as a scalar on U. Thus $T \leq H_{(K,U)}$ and $T \cap M_{(K,U)} = 1$. So $\{(H_j, M_j)|j \in J\}$ is a sectional cover for G.

Using Witt's theorem we get $PSL_{F_K}(U) \leq H_j/M_j \leq PGL_{F_K}(U)$ for all $j \in J$. The last claim in (a) follows from 2.15c and since $PGL_{F_K}(U)^\infty \in \mathcal{L} \cup \{1\}$.

(b) By 3.1 we may assume that $\mathcal{K}_\mathcal{F}$ is a Kegel cover. Then (a) provides a Kegel cover all of whose factors are in $\mathcal{L}$.

Theorem 3.3 *Let G be an LFS-group, which is neither finitary nor of alternating type. Then G is of p-type for some prime p and there exists a Kegel cover $\mathcal{K}$ for G such that $\mathcal{K} = \mathcal{K}_{\mathcal{L}(p)}$ and for all $K \in \mathcal{K}$, $H_K/O_p(H_K)$ is the central product of perfect central extension of groups in $\mathcal{C}(p)$.*

Proof: Put $\mathcal{P} = \{(H, M)|H \leq G, M \lhd H, H$ is finite and H/M is perfect and simple$\}$ and note that $\mathcal{P}$ is a Kegel cover for G. By assumption G is not of alternating type and so $\mathcal{P}_\mathcal{A}$ is not a Kegel cover.

(1) $\{(H, M)|H \leq G, M \lhd H, H$ finite and every non-abelian composition factor of H/M is in $\mathcal{A}\}$ is not a sectional cover for G.

Otherwise 2.15 provides a Kegel cover with alternating factors.

(2) If $\mathcal{T}$ is a Kegel cover for G, then $\mathcal{T}_\mathcal{C}$ is a Kegel cover as well.

By 3.1 $\mathcal{T}_\mathcal{A}$ or $\mathcal{T}_\mathcal{C}$ is a Kegel cover. By assumption $\mathcal{T}_\mathcal{A}$ is not a Kegel cover.

Let $\mathcal{J}$ be the set of all pairs $(H, M) \in \mathcal{P}_\mathcal{L}$ such that there exists a prime p with $C_H(M/O_p(H)) \leq M$ and $\mathcal{P}_{\mathcal{L}(p)}(H) \neq \emptyset$.

(3) $\mathcal{J}$ is not a Kegel cover.

Suppose that $\mathcal{J}$ is a Kegel cover. Using 2.9 and its Corollary 2.10 we will derive a contradiction to (1). Let T be any finite subgroup of G and pick $(G_1, N_1) \in \mathcal{J}(T)$. Let R_1 be the minimal normal supplement to N_1 in G_1. Since R_1 is of even order there exists $x \in R_1$ with $|x| = 2$. Let $k = l(2|G_1| + f(|G_1|))$, where l and f are as in 2.5 and 2.2. By 3.1 there exists (G_0, N_0) in $\mathcal{J}(G_1)$ with $\mathrm{pdeg}_{G_0/N_0}(x) \geq k$. By definition of $\mathcal{J}$ there exists a prime p with $C_{G_0}(N_0/O_p(G_0)) \leq N_0$ and $K \in \mathcal{P}_{\mathcal{L}(p)}(G_0)$ Hence by 2.10 applied to $PSL_{F_K}(V_K) \cong \bar{K}$ and the images of G_0, N_0, G_1, N_1 in $PSL_{F_K}(V_K)$ in place of G, N, G_1, N_1 there exists $G_2 \leq H_K$ and $N_2 \lhd G_2$ such that $G_1 \leq G_2$, $G_1 \cap N_2 \leq N_1$ and every non- abelian composition-factor of G_2/N_2 is in $\mathcal{A}$. Note that $T \leq G_1 \leq G_2$ and $T \cap N_2 \leq T \cap N_1 = 1$. Since T was an arbitrary finite subgroup of G, we get a contradiction to (1).

(4) There exists a prime p such that G is of p-type. In particular, $\mathcal{P}_{\mathcal{L}(p)}$ is a Kegel cover.

Note that G is of p-type if and only if $\mathcal{P} \setminus \mathcal{P}_{\mathcal{C}(p)}$ is a not Kegel cover. Suppose that for all primes p, $\mathcal{P} \setminus \mathcal{P}_{\mathcal{C}(p)}$ is a Kegel cover. Then by (2) and 3.2, $\mathcal{P}_{\mathcal{L} \setminus \mathcal{L}(p)}$ is a Kegel cover for all primes p. Let T_0 be a finite subgroup of G and note that by 2.15c there exists perfect finite subgroup T of G with $T_0 \leq T$. By 3.1a, there exists $K \in \mathcal{P}_{\mathcal{L}}(T)$ with $\dim_{F_K} V_K > |T|^2$. For $1 \neq t \in T$, pick $v_t \in V_K$ with $v_t^t \notin F_K v_t$. Put $U = F_K \langle v_t^T | t \in T \setminus \{1\} \rangle$. Then $\dim_{F_K} U \leq |T|^2$ and U is a proper T-invariant subspace of V_K such that no non-trivial element of T acts as a scalar on U. Let $H = N_{H_K}(U)^\infty$ and let M be all the elements in H which act as a scalar on U. Since T is perfect, $T \leq H$ and so $(H, M) \in \mathcal{P}_{\mathcal{L}}(T)$. Let $q = \mathrm{char}\, F_K$. Since $\mathcal{P}_{\mathcal{L} \setminus \mathcal{L}(q)}$ is a Kegel cover for G, there exists a p a prime distinct from q such that $\mathcal{P}_{\mathcal{L}(p)}(H) \neq \emptyset$. Note that $O_p(H/H \cap M_K) = 1$ and $C_H(M/H \cap M_K) \leq M$. Thus $C_H(M/O_p(H)) \leq M$ and $(H, M) \in \mathcal{J}(T) \subseteq \mathcal{J}(T_0)$. It follows that $\mathcal{J}$ is a Kegel cover, a contradiction to (3).

Hence there exists a prime p such that $\mathcal{P} \setminus \mathcal{P}_{\mathcal{C}(p)}$ is a not a Kegel cover. This implies that $\mathcal{P}_{\mathcal{C}(p)}$ is a Kegel cover and so by 3.2a, $\mathcal{P}_{\mathcal{L}(p)}$ is a Kegel cover.

For a finite group H and a prime p let $F_p^*(H)$ be defined by $F^*(H/O_p(H)) = F_p^*(H)/O_p(H)$.

(5) $\{(H, F_p^*(H)) | H \leq G, H \text{ finite}\}$ is not a sectional cover for G.

Suppose $\{(H, F_p^*(H)) | H \leq G, H \text{ finite}\}$ is a sectional cover for G. Note that $C_H(F_p^*(H)/O_p(H)) \leq F_p^*(H)$. Hence we conclude from 2.15a that $\mathcal{I} = \{(H, N) \in \mathcal{P} | C_H(N/O_p(H)) \leq N\}$ is a Kegel-cover for G. If $\mathcal{I}_{\mathcal{F}}$ is a Kegel cover, then the Kegel cover provided by 3.2a is contained in $\mathcal{I}_{\mathcal{L}}$. Thus in any case $\mathcal{I}_{\mathcal{L}}$ is a Kegel cover and so by (4), $\mathcal{I}_{\mathcal{L}(p)}$ is a Kegel cover. In particular, $\mathcal{P}_{\mathcal{L}(p)}(H) \neq \emptyset$ for all finite subgroups H of G and $\mathcal{I}_{\mathcal{L}(p)} \subseteq \mathcal{J}$, a contradiction to (3)

(6) $\{F_p^*(H) | H \leq G, H \text{ finite}\}$ is a sectional cover for G.

This follows immediately from 2.15b and (5).

Put $\mathcal{M} = \{H \leq G | H \text{ finite and perfect, } H = F_p^*(H)\}$. By (6) and 2.15c, $\mathcal{M}$ is a sectional cover for G. For $H \in \mathcal{M}$, let $Sol(H)$ be the largest solvable normal subgroup of H. Then $H/Sol(H)$ is the direct product of non abelian simple groups. Let $H_p/Sol(H)$ be the product of the components of $H/Sol(H)$ contained in $\mathcal{C}(p)$.

(7) $\{H_p | H \in \mathcal{M}\}$ is a sectional cover for G and $\{(H, H_p) | H \in \mathcal{M}\}$ is not a sectional cover.

Otherwise we conclude from (6) and 2.15b that $\{(H, H_p)|H \in \mathcal{M}\}$ is a sectional cover for G. Hence by 2.15a there exists a Kegel cover for G none of whose factors are in $\mathcal{C}(p)$, a contradiction to (4).

Applying 2.15a to the sectional cover $\{H_p|H \in \mathcal{M}\}$ we get a Kegel cover $\mathcal{K}$ for G such that $H/O_p(H)$ is the central product of perfect central extensions of elements in $\mathcal{C}(p)$ for all $(H,N) \in \mathcal{K}$. Using 3.2a we can choose $\mathcal{K}$ such that in addition $\mathcal{K} = \mathcal{K}_{\mathcal{L}(p)}$. Thus (b) holds.

Theorem 3.4 *Let G be of alternating type and $\mathcal{K}$ a Kegel cover for G with $\mathcal{K} = \mathcal{K}_{\mathcal{A}}$.*

(a) One of the following holds:

(a1) There exists a Kegel cover $\mathcal{J} \subseteq \mathcal{K}$ such that for all $J, K \in \mathcal{J}$ with $H_J \leq H_K$ every essential orbit of H_J on Ω_K is pseudo natural with respect to M_J.

(a2) For all finite subgroups T in G, $\mathcal{K}_R(T) = \{K \in \mathcal{K}(T)|T$ has a regular orbit on $\Omega_K\}$ is a Kegel cover for G.

(b) If G is countable, there exists a Kegel sequence $\{K_n|n \geq 1\} \subseteq \mathcal{K}$ for G such that one of the following holds:

(b1) For all $n < m$ all essential orbits of H_{K_n} on Ω_{K_m} are pseudo natural with respect to M_{K_n}

(b2) For all $n < m$ H_{K_n} has a regular orbit on Ω_{K_m}.

Proof: Suppose (a) is false. Then there exists a finite subgroup H of G such that $\mathcal{K}_R(H)$ is not a Kegel cover. Hence $\mathcal{K} \setminus \mathcal{K}_R(H)$ is a Kegel cover and we may assume that $\mathcal{K}_R(H) = \emptyset$. Let k be defined as in 2.14. By 3.1 we may assume that $\mathcal{K} = \mathcal{K}(H)$ and $\deg_{\Omega_K}(h) \geq k$ for all $h \in H \setminus \{1\}$ and all $K \in \mathcal{K}$. Let $\mathcal{J}$ be the set of all $K \in \mathcal{K}$ such that for all $L \in \mathcal{K}$ with $H_K \leq H_L$ all essential orbits of H_K on Ω_L are pseudo natural with respect to M_K

Since (a1) does not hold, $\mathcal{J}$ is not a Kegel cover and so there exists a finite subgroup T of G with $\mathcal{J}(T) = \emptyset$. Then $\mathcal{K}(T) \cap \mathcal{J} = \emptyset$ and so for all $K \in \mathcal{K}(T)$ there exists $X(K) \in \mathcal{K}$ such that $H_K \leq H_{X(K)}$ and not all essential orbits of H_K on $\Omega_{X(K)}$ are pseudo natural. Since H has no regular orbit on $\Omega_{X(K)}$, we conclude from 2.14 that all essential orbits for H_K on $\Omega_{X(K)}$ are t-pseudo natural for some t. Hence H_K has a t-pseudo natural orbit on $\Omega_{X(K)}$ with $t > 1$. Pick $K_0 \in \mathcal{K}(T)$ and inductively define $K_n = X(K_{n-1})$ for $n \geq 1$. Put $\Omega_n = \Omega_{K_n}$ and $H_n = H_{K_n}$. Pick $n \geq 1$ and $\omega \in \Omega_n$ with $|C_H(\omega)|$ minimal. Since H has no regular orbit on Ω_n, there exists $1 \neq h \in C_H(\omega)$. Let O be a t-pseudo natural orbit for H_n on Ω_{n+1} with $t > 1$ and Δ an H_n-invariant partition on O such that the action of H_n on Δ is isomorphic to the action of H_n on subsets of size t in Ω_n, where $2 \leq t \leq |\Omega_n|/2$. Note that $\deg_{\Omega_n}(h) \geq k \geq 2|H|$ and thus there exists $a \in \Omega_n \setminus \omega^H$ with $a^h \neq a$. Moreover, $|\Omega_n| - t \geq |H|$ and there exists a subset U of size t in Ω_n with $a \in U$, $a^h \notin U$ and $\omega^H \cap U = \{\omega\}$. Then $N_H(U) \leq C_H(\omega)$ and $h \notin N_H(U)$. Let D be the element of Δ corresponding to U and pick $\sigma \in D$. Then $C_H(\sigma) \leq N_H(D) = N_H(U) < C_H(\omega)$, a contradiction to the minimality of $|C_H(\omega)|$. This completes the proof of (a).

(b) Note that every Kegel cover for a countable group contains a Kegel sequence. Hence (a1) implies (b1) and we may assume that (a2) holds. Let $G = \{g_n | n \geq 1\}$. Pick $K_1 \in \mathcal{K}_R(\langle g_1 \rangle)$, and inductively $K_n \in \mathcal{K}_R(\langle H_{K_{n-1}}, g_n \rangle)$, $n \geq 2$. Then $\{K_n | n \geq 1\}$ is a Kegel sequence for G which fulfills (b2).

Remark 3.5 *Let H be a group and $N \lhd H$ with $H/N \cong Alt(n)$. Then every regular orbit for H is pseudo natural with respect to N. It follows that under the assumption of 3.4b, (b2) is always true if regular is replaced by pseudo natural.*

4 Maximal Subgroups

Theorem 4.1 *Let G be a countable, non-finitary LFS-group and H a finite subgroup of G. Then H is contained in a maximal subgroup of G. In particular, G has maximal subgroups.*

Proof: The idea is to find a Kegel sequence $\{(H_n, M_n) | n \geq 1\}$ of G and maximal subgroups T_n of H_n such that $H \leq T_1$, $T_n \leq T_{n+1}$ and $H_n \not\leq T_{n+1}$. Given such a sequence put $T = \bigcup_{n=1}^{\infty} T_n$. Then T is a maximal subgroup of G. Indeed, since T_n is maximal in H_n and $H_n \not\leq T_{n+1}$, $H_n \cap T_{n+1} = T_n$. Thus $H_n \cap T = T_n$ and in particular, $T \neq G$. Let $x \in G \setminus T$. Then for all n with $x \in H_n$, $H_n = \langle x, T_n \rangle \leq \langle x, T \rangle$ and so $G = \langle x, T \rangle$.

By 3.3 and 3.5 we can find a Kegel sequence $\mathcal{K} = \{K_n | n \geq 1\}$ of G such that one of the following holds (with $X_n = X_{K_n}$ for $X = H, M, V, F$ and Ω).

(1) There exists a prime p such that $\mathcal{K} = \mathcal{K}_{\mathcal{L}(p)}$ and for all $n \geq 1$, $H_n / O_p(H_n)$ is the central product of perfect central extension of groups in $\mathcal{C}(p)$.

(2) For all $n < m$, H_n has a pseudo natural orbit on Ω_m with respect to M_n.

Suppose first that (1) holds. By 3.1 we can find n such that $\dim_{F_n} V_n > |H|$. Without loss $n = 1$. Then H does not act irreducible on V_1 and so there exists a proper H-submodule U in V_1. Put $T_1 = N_{H_1}(U)$. Then T_1 is a maximal subgroup of H_1 containing H. Moreover, by the structure of H_1, $O_p(T_1) \neq O_p(H_1)$. Inductively, we will find a maximal subgroup T_{n+1} of H_{n+1} such that $T_n \leq T_{n+1}, H_n \not\leq T_{n+1}$ and $O_p(T_{n+1}) \neq O_p(H_{n+1})$. Since $O_p(T_n) \neq O_p(H_n)$, there exists a chief factor for H_n on V_{n+1} not centralized by $O_p(T_n)$, so T_n is not irreducible on this chief factor and there exists a T_n-submodule U_n in V_{n+1} wich is not normalized by H_n. Put $T_{n+1} = N_{H_{n+1}}(U_n)$. Then T_{n+1} has all the desired properties and $\bigcup_{n=1}^{\infty} T_n$ is a maximal subgroup of G containing H.

Suppose next that (2) holds. By 3.1 we may assume that for all n, $|\Omega_{n+1}| > 2|H_n|$ and $|\Omega_1| > 2|H|$. Thus H normalizes a proper subset Γ_0 in Ω_1 with $|\Gamma_0| < |\Omega_1|/2$. Let $T_1 = N_{H_1}(\Gamma_0)$. Then T_1 is a maximal subgroup of H_1 which contains H and does not act transitively on Ω_1. Inductively we will find a maximal subgroup T_{n+1} of H_{n+1} such that $T_n \leq T_{n+1}$, $H_n \not\leq T_{n+1}$ and T_{n+1} does not act transitively on Ω_{n+1}. Let O be a pseudo natural orbit for H_n on Ω_{n+1} with respect to M_n. Since T_n does not act transitively on Ω_n, T_n does not act transitively on O. Hence there exists a T_n-orbit Γ_n of O such that H_n does not normalize Γ_n. Then $|\Gamma_n| \leq |T_n| \leq |H_n| < |\Omega_{n+1}|/2$. Put $T_{n+1} = N_{H_{n+1}}(\Gamma_n)$. Then T_{n+1} has all the desired properties and $\bigcup_{n=1}^{\infty} T_n$ is a maximal subgroup of G containing H.

5 An abstract characterization of finitary, locally finite, simple

Lemma 5.1 *Let p be a prime, G a group and S a Sylow p-subgroup of G.*

(a) If $G = PSL_n(p^k)$, then $\mathrm{der}(S) = [\log_2 n]$.

(b) If $G = Sym(n)$, then $\mathrm{der}(S) = [\log_p n]$.

(c) If $G = Alt(n)$ and $p = 2$ then $\mathrm{der}(S) = [\log_2 n]$, *if $n \geq 6$, and* $\mathrm{der}(S) = [\log_2 n] - 1$, *if $n \leq 5$.*

Proof: For (a) and (b) see [4, III 16.3, 15.3] while the proof for (c) is similar to the one for (b) in [4].

Lemma 5.2 *Let p be a prime, F a finite field with char $F = p$, V a finite dimensional vector space over F, P a p-subgroup of $GL_F(V)$, $k \geq 1$ and $x \in P$ with* $\deg_V(x) \geq |P|2^k$. *Then there exists a p-subgroup P^* of $GL_F(V)$ containing P with* $\mathrm{der}(\langle x^{P^*}\rangle) \geq k$.

Proof: Pick a chain $0 = V_0 \leq V_1 \leq V_2 \leq \ldots \leq V_l$ of FP-submodules in V of maximal length with respect to $\dim V_i/V_{i-1} \leq |P|$ and $[V_i/V_{i-1}, x] \neq 0$ for all $1 \leq i \leq l$. Supppose that $l < 2^k$. Then $\dim V_l \leq l|P| < 2^k|P| \leq \deg_V(x)$ and so $[V, x] \not\leq V_l$. Pick $v \in V$ with $[v, x] \notin V_l$ and put $V_{l+1} = F\langle v^P\rangle + V_l$. Then $\dim V_{l+1}/V_l \leq |P|$ and $[V_{l+1}/V_l, x] \neq 0$, a contradiction to the maximal length of the chain.

Thus $l \geq 2^k$. Pick $x_i \in V_i \backslash V_{i-1}$ with $[x_i, x] \notin V_{i-1}$ and put $y_i = [x_i, x]$. Then there exists a basis $v_1, v_2, \ldots$ of V such that P normalizes the corresponding flag and such that there exists indices $i_1 < i_2 < \ldots < i_{2l}$ with $v_{i_{2j-1}} = y_j$ and $v_{i_{2j}} = x_j$. Let P^* be the full stabilizer of this flag and let S be the largest subgroup of $GL_F(V)$ with the following properties:

S centralizes all v_i with $i \notin \{i_1, \ldots, i_{2l}\}$,
S centralises $x_j + y_j$ for all j and
S stabilizes the flag $0 \leq Fx_1 \leq F\langle x_1, x_2\rangle \leq \ldots \leq F\langle x_1, \ldots, x_l\rangle$.

Put $X = F\langle x_1, \ldots, x_l\rangle$. Then S normalizes X, S is isomorphic to a Sylow p-subgroup of $GL_F(X)$ and $[y_j, S] = [x_j, S] \leq F\langle x_1, \ldots, x_{j-1}\rangle \leq F\langle v_1, \ldots, v_{i_{2(j-1)}}\rangle$. It follows that $S \leq P^*$. Since $x_j^x = x_j + y_j$, S centralizes X^x . Thus $\langle S, S^x\rangle$ normalizes X^x, $[S, x]C_{P^*}(X^x) = S^x C_{P^*}(X^x)$ and $[S, x]$ acts as a full Sylow p-subgroup of $GL_F(X)$ on X^x. By 5.1, $\mathrm{der}([S, x]) \geq [\log_2 l]$. Since $l \geq 2^k$, we conclude that $\mathrm{der}(\langle x^{P^*}\rangle) \geq \mathrm{der}([S, x]) \geq k$.

Lemma 5.3 *Let p be a prime, Ω a finite set, $G = Alt(\Omega)$ or $Sym(\Omega)$, P a p-subgroup of G, $k \geq 1$ and $x \in P$ with* $\deg_\Omega(x) \geq 2w(P)|P|p^k$. *Then there exists a p-subgroup P^* of G containing P with* $\mathrm{der}(\langle x^{P^*}\rangle) \geq k$.

Proof: By 2.13 there exists a subset Γ of Ω and a partition Δ of Γ in subset of size $2p^k$ such that P normalizes Δ and x acts non-trivially on Δ. Let $D \in \Delta$ with $D^x \neq D$. Pick $S \leq G$ such that S centralizes $\Omega \setminus D$, S acts as a full Sylow p-subgroup of $Alt(D)$ on D and $N_P(D)$ normalizes S. Put $P^* = \langle S, P\rangle$. Then P^* is a p-group and $[S, x]$ acts as a full Sylow p-subgroup of $Alt(D^x)$ on D^x. Since $|D| = 2p^k$ we conclude from 5.1b,c that $\mathrm{der}([S, x]) \geq k$. Thus $\mathrm{der}(\langle x^{P^*}\rangle) \geq k$.

Lemma 5.4 *Let $G = SL_F(V)$, F a finite field and V a finite dimensional vector space over F. Let p be a prime with char $F \neq p$ and put $d = d(p)$. If P is p-subgroup of G, $x \in P$, $k \geq 1$, and $y \in \langle x^P\rangle^{(d)}$ with $\deg_V(y) \geq 2w(P)|P|^2p^{k+1}$, then there exists a p-subgroup P^* of G with $P \leq P^*$ and $\mathrm{der}(\langle x^{P^*}\rangle) \geq k$.*

Proof: Let T be a Sylow p-subgroup of $GL_F(V)$ with $P \leq T$ and $e \geq 1$ minimal with pd dividing $|F|^e - 1$. Then there exists an FT-submodule U in V with $\dim V/U < e$ and a system of imprimivity Δ for T on U such that $\dim D = e$ for all $D \in \Delta$. Note that $e \leq p$. Let $D \in \Delta$ with $\langle x^P\rangle \leq N_T(D)$. Since $N_T(D)/C_T(D)$ has derived length at most d and $y \in \langle x^P\rangle^{(d)}$ we conclude that $[D, y] = 0$. Hence

$$2w(P)|P|^2p^{k+1} \leq \deg_V(y) \leq e|\Delta \setminus C_\Delta(\langle x^P\rangle)| \leq p|\Delta \setminus C_\Delta(\langle x^P\rangle)|.$$

On the other hand, $|\Delta \setminus C_\Delta(\langle x^P\rangle)| \leq |P| \deg_\Delta(x)$ and therefore $\deg_\Delta(x) \geq 2w(P)|P|p^k$. Let $H = N_G(\Delta)$ and $\bar{H} = H/C_G(\Delta)$. Then $\bar{H} \cong Sym(\Delta)$ or $Alt(\Delta)$ and by 5.3 there exists a p-subgroup $\bar{P}^*$ of $\bar{H}$ with $\bar{P} \leq \bar{P}^*$ and $\mathrm{der}(\langle \bar{x}^{\bar{P}^*}\rangle \geq k$. Let P^* be a Sylow p-subgroup of the inverse image of $\bar{P}^*$ in H with $P \leq P^*$. Then $\mathrm{der}(\langle x^{P^*}\rangle) \geq k$ and 5.4 is proved.

Theorem 5.5 *Let G be a LFS-group, p a prime, and $x \in G$ with $|x|$ a power of p. If G has no Kegel cover with all of its factors in $\mathcal{A} \cup \mathcal{L}(p)$, put $d = d(p)$ and assume that $\langle x^S\rangle^{(d)} \neq 1$ for some p-subgroup S of G with $x \in S$. Then G is finitary if and only if $\langle x^T\rangle$ is solvable for all p-subgroups T of G with $x \in T$.*

Proof: If G is finitary, then by [6, Prop 1], $\langle x^T\rangle$ is solvable for all p-subgroups T of G with $x \in T$. So suppose that G is not finitary. If G has a Kegel cover with all of its factors in $\mathcal{A} \cup \mathcal{L}(p)$, put $S = \langle x\rangle$ and $d = 0$. By 3.2 there exists a Kegel cover $\mathcal{K}$ for G with $\mathcal{K} = \mathcal{A} \cup \mathcal{L}$ and, if $d = 0$, $\mathcal{K} = \mathcal{A} \cup \mathcal{L}(p)$. We will first prove :

(*) If P is a finite p-subgroup of G with $S \leq P$, then there exists a finite p-subgroup P^* of G with $P \leq P^*$ and $\mathrm{der}(\langle x^P >) < \mathrm{der}(\langle x^{P^*}\rangle)$.

Let $k = \mathrm{der}(\langle x^P\rangle)+1$ and pick $1 \neq y \in \langle x^P\rangle^{(d)}$. By 3.1 there exists $K \in \mathcal{K}(P)$ with $\mathrm{pdeg}_{\bar{K}}(y) \geq 2w(P)|P|^2p^{k+1}$. Thus 5.2, 5.3 and 5.4 provide a p-subgroup $\bar{P}^*$ of $\bar{K}$ with $\bar{P} \leq \bar{P}^*$ and $\mathrm{der}(\langle \bar{x}^{\bar{P}^*}\rangle \geq k$. Let P^* be a Sylow p-subgroup of the inverse image of $\bar{P}^*$ in H_K with $P \leq P^*$. Then $\mathrm{der}(\langle x^{P^*}\rangle) \geq k$ and (*) is proved.

Let $P_0 = S$ and inductively define $P_{i+1} = P_i^*$. Put $T = \bigcup_{i=1}^{\infty} P_i$. Then T is a p-subgroup of G such that $\langle x^T \rangle$ is not solvable.

6 Countable LFS- groups which are not absolutely simple

In this chaper we will construct countable LFS-groups G which possess a ascending series

$$M_1 \trianglelefteq\ M_2 \trianglelefteq M_3 \trianglelefteq ...$$

of proper subgroups in G such that $G = \bigcup_{i=1}^{\infty} M_i$. The main step in the construction is the following lemma:

Lemma 6.1 *Let H be a perfect finite group. Then there exists a perfect finite group H^* containing H and function X which associates to each subgroup A of H a subgroup $X(A)$ of H^* such that*
(a) $H \leq \langle h^{H^*} \rangle$ for all $1 \neq h \in H$.
(b) $X(A) \cap H = A$ for all $A \leq H$.
(c) If $A \leq B \leq H$, then $A \trianglelefteq B$ if and only if $X(A) \trianglelefteq X(B)$.
(d) $X(H) \trianglelefteq \langle X(H)^{H^*} \rangle$.

Proof: Let S be any finite simple group such that there exists monomorphism $\alpha : H \to S$ and let T be any non trivial finite perfect group. Furthermore, let S and T act transitively and non-trivally on the sets I and J, respectively. We assume that $0 \in I$ and $\{0, 1\} \subseteq J$. Let $K = H \wr_I S$. For $i \in I$ let $\beta_i : H \to K$ be the canonical isomorphism between H and the i'th component of the base group of K and let β be the canonical monomorphism from S to K. Let $H^* = K \wr_J T$ and for $j \in J$ let $\gamma_j : K \to H^*$ be the canonical isomorphism between K and the j'th component of the base group of H^*. Define $\rho : H \to H^*$ by $\rho(h) = \gamma_0(\beta_0(h))\gamma_1(\beta(\alpha(h)))$. Then ρ is clearly a monomorphism. For $A \leq H$ let $X(A)$ be the set of elements in the base group of H^* such that the projection onto the $0'$th-component is contained in $\gamma_0(\prod_{i \in I} \beta_i(A))$. Identifying H with $\rho(H)$ we see immediately that (b) and (c) hold. Now $\langle X(H)^{H^*} \rangle$ is the base group of H^* and so (d) holds. Moreover, (a) is readily verified.

We are now able to construct locally finite simple groups which are not absolutely simple. Let G_1 be any nontrivial perfect finite group, and inductively let $G_{i+1} = G_i^*$ and X_i any function from the subgroups of G_i to the subgroups of G_{i+1} which fulfills 6.1. Let $G = \bigcup_{i=1}^{\infty} G_i$. Then by (a) in 6.1, G is a locally finite simple group.

Put $M_{1,1} = 1$, $M_{1,2} = G_1$ and inductively, $M_{n+1,j} = X_n(M_{n,j})$, for $1 \leq j \leq 2n$, $M_{n+1,2n+1} = \langle X_n(G_n)^{G_{n+1}} \rangle$ and $M_{n+1,2n+2} = G_{n+1}$. Then by induction and 6.1, $M_{n,i} \triangleleft M_{n,i+1}$ for all $1 \leq i \leq 2n+1$ and $M_{n+1,i} \cap G_n = M_{n,i}$ for all $1 \leq i \leq 2n$. Put $M_i = \bigcup_{n \geq \frac{i}{2}} M_{n,i}$. Then $G_n \leq M_{2n}$, $G_n \cap M_i = M_{n,i}$ for all $i \leq 2n$, $M_i \triangleleft M_{i+1}$ and $G = \bigcup_{i=1}^{\infty} M_i$.

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