A Characterization of the Natural Module for some Classical Groups

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May 2, 2003

preliminary version

1 Introduction

Let K be a skew field, R a ring, G a "classical" group defined over K, Z a "long root" group contained in G and N a "natural" KG-module, where N is allowed to be finite or infinite dimensional over K. Further, put $L = C_G([N, Z])$ and let V be an RG-module such that $[V, Z, L] = 0$ and $V = [V, G]$. The main goal of this paper is to prove that any such V has to be of the form $M \otimes_K N$, for some (R, K)-module M with G acting trivally on M. This is achieved in Theorems A and B. We became interested in this problem through the work of J.I.Hall and R.E.Phillips [Ha1], [Ha2], [Ph] on groups of finitary transformations. They classified certain classes of such groups. In Theorem C we are able to classify the corresponding modules. Theorems A, B and C partially generalize similar results found in [Cu],[LP], [Ha3], and [Tf].

In order to state exactly what we mean with "classical" groups, "natural "module" and "long root" groups we now introduce some notations and definitions which will be used throughout the paper:

K is a skew field.
R is a ring.
G is a group.
An (R, K)-module M is an abelian group , which is a left R-module and a right vector space over K such that $(rm)k = r(mk)$ for all $r \in R, k \in K$ and $m \in M$. M is called unitary if R has a unit 1 and $1m = m$ for all $minM$.
An RG-module M is an abelian group , which is a left R-module and a right $\mathbb{Z}G$-module such that $(rm)k = r(mk)$ for all $r \in R, k \in K$ and $m \in M$.
N is a left vector space over K. $\tilde{N}$ is a subspace of the dual space N^* of N with $C_N(\tilde{N}) = 0$, i.e for every $0 \neq n \in N$ there exists $\tilde{n} \in \tilde{N}$ such that $n\tilde{n} \neq 0$. (Note that $\tilde{N}$ is a right vector space over K and so a left vector space over K_{op}, the opposite skew field of K.)

For $n \in N$ and $n^* \in N^*$ define $t(\tilde{n}, n) \in \mathrm{End}_K(N)$ by

$$v.t(n^*, n) = v + vn^* \cdot n$$

For a subspace X of N and a subspace $\tilde{X}$ of $\tilde{N}$ the following subgroup of $GL_K(N)$ was introduced in [CH]

$$T(\tilde{X}, X) = \langle t(\tilde{x}, x) | x \in X, \tilde{x} \in \tilde{X}, x\tilde{x} = 0 \rangle.$$

For the convenience of the reader we will recall the definition of a pseudo quadratic space (see [Ti] and [Gr] for basic properties of pseudo quadratic spaces).

(N, q, f) is a (σ, s)-pseudo quadratic space provided that

(PQ1) σ is an anti-automorphism of K and $0 \neq s \in K$ such that for all $x \in K$

$$x^{\sigma^2} = x^s = s^{-1}xs \text{ and } s^\sigma = s^{-1}.$$

(PQ2) f is a trace-valued (σ, s)-hermitian form on N, i.e

$$f : N \times N \to \text{K is biadditive},$$

$$f(\mu v, \lambda w) = \mu f(v, w)\lambda^\sigma,$$
$$f(v, w) = sf(w, v)^\sigma,$$
$$f(v, v) \in K_+,$$

for all $\mu, \lambda \in K$ and $v, w \in N$. Here K_+ is the additive subgroup of K defined by $K_+ = \{k + sk^\sigma | k \in K\}$.

(PQ3) q is a map from N to $\overline{K} = K/K_-$ so that

$$q(v, w) = q(v) + q(w) + \overline{f(v, w)},$$

$$q(\lambda v) = \lambda * q(v).$$

Here $K_- = \{k - sk^\sigma | k \in K\}$ and $k * \overline{\lambda} = \overline{k\lambda k^\sigma}$.

(N, q, f) is called a quadratic space if $\sigma = id_K$ and $s = 1$, that is if $K_- = 0$. Note that in this case K is necessarily commutative.

We mention some other special cases of pseudo-quadratic spaces. If $\sigma = id_K$, $s = -1$ and char $K \neq 2$, then $K = K_-$ and so $q = 0$. Hence q is redundant and (N, f) is a symplectic space. The symplectic spaces over fields of even characteristic are also included, since they can be written as $N/N^\perp$, where N is an appropriate quadratic space. More generally, any vector space with a trace-valued (σ, s)-hermitian form can be written as $N/N^\perp$, where N is an appropriate pseudo-quadratic space (see [Ti]). Finally, the case $|\sigma| = 2$ and $s = 1$ covers the unitary spaces over fields.

$\mathcal{O} = O(N, q, f)$ is the group of invertible K-linear transformations g of N such that $f(v, w) = f(vg, wg)$ and $q(v) = q(vg)$ for all $v, w \in N$.

If U is a subset of N, then $U^\perp = \{v \in N | f(v, u) = 0$ for all $u \in U\}$. A subspace U of N is called singular provided that the restrictions of q and f to U vanish.

rad $N = \{n \in N^\perp | q(n) = 0$.

N is called degenerate if rad $N \neq 0$ and defective if $N^\perp \neq 0$.

$\mathcal{S}(i)$ is the set of i-dimensional singular subspaces of N.

The Witt index of (N, q, f) is the maximal dimension of a singular subspace in N. Note that the Witt index can be zero, any positive integer or infinite.

For $U \in \mathcal{S}(i)$ let $P_U = N_{\mathcal{O}}(U)$, $Q_U = C_{\mathcal{O}}(U^\perp/U) \cap C_{\mathcal{O}}(U)$, $T_U = C_{\mathcal{O}}(U^\perp)$ and $Z_U = C_{\mathcal{O}}(N/U)$. Let$\Sigma$ be a set of subgroups of G. We will consider the following hypotheses:

Hypothesis (A) 1. *Σ is the set of all $\{T(\tilde{X}, X)$ where X and $\tilde{X}$ are 1-dimensional subspaces of N and $\tilde{N}$, respectively, with $X\tilde{X} = 0$; and $G = \langle\Sigma\rangle = T(\tilde{N}, N)$*

2. *N is at least 3-dimensional over K.*

Hypothesis (B) 1. *$\Sigma = \{T_U | U \in \mathcal{S}(1)\}$ and $G = \langle\Sigma\rangle$ where (N, q, f) is a non-degenerate pseudo-quadratic space with Witt index at least two.*

2. *If (N, q, f) is quadratic, then (N, q, f) defective.*

Hypothesis (C) *$\Sigma = \{Z_U | U \in \mathcal{S}(2)\}$ and $G = \langle\Sigma\rangle$, where (N, q, f) is a nondegenerate quadratic space with Witt index at least two and* $\dim N \geq 5$.

Hypothesis (C*) *Hypothesis (C) holds and if char $K = 2$, then* $\dim N/N^\perp \geq 6$, *and if $|K| = 2$ and* $\dim N = 6$, *then (N, q, f) has Witt index 2.*

We define the following graph on Σ. Let $Z_1, Z_2 \in \Sigma$. Then Z_1 and Z_2 are adjacent if, in case (A), $\langle Z_1, Z_2\rangle \cong SL_2(K)$, or if, in case (B) or (C), $[N, Z_1] \cap [N, Z_2]^\perp = 0$.

For convenience we view N under Hypothesis (A) as a singular pseudo quadratic space. In particular, $\mathcal{O} = GL_K(N)$ and $Q_X = C_{GL_K(N)}(X) \cap C_{GL_K(N)}(N/X)$.

We are now able to state our main results:

Theorem A *Suppose that (G, Σ) fulfills Hypothesis (A),(B) or (C*) from above. Let $Z \in \Sigma$ and put $L = C_G([N, Z])$. Let V be an RG-module with $[V, Z, L] = 0$. Then there exist an (R, K)-module M and an R-submodule C of $M \otimes_K N$ with $[C, G] = 0$ such that one of the following holds:*

1. *$[V, G]$ and $(M \otimes_K N)/C$ are isomorphic as RG-modules.*

2. *G fulfills (B), $|K| = 4$, $\sigma \neq id$,* $\dim N = 4$ *and $[V, G]/C_V(G)$ and $M \otimes_K N$ are isomorphic as RG-modules.*

Theorem B *Assume that* (G, Σ) *fulfills Hypothesis (A). Put* $L_0 = C_G([N, Z]) \cap C_G([\tilde{N}, Z])$ *and let* V_0 *be an RG-module with* $[V_0, Z, L_0] = 0$. *Then there exists an* (R, K)*-module* M *and an* (R, K_{op})*-module* $\tilde{M}$ *such that*

$[V_0, G]$ *is isomorphic to* $M \otimes_K N \oplus \tilde{M} \otimes_{K_{op}} \tilde{N}$ *as RG-modules.*

Theorem C Remark: Need Witt index assumption ? *Suppose that* (G, Σ) *fulfills Hypthesis (A), (B) or (C) and that* $N/N^{\perp}$ *is infinite dimensional over* K. *Let* R *be a divison ring and* V *be a non-trivial irreducible finitary RG-module. Then one of the following holds:*

1. *There exists an irreducible* (R, K)*-module* M, *which is finitely generated over* R *so that* V *is isomorphic to* $M \otimes_K N$ *as RG-module.*

2. *Hypothesis (B) holds and there exists an irreducible* (R, K_{op})*-module* $\tilde{M}$, *which is finitely generated over* R, *so that* V *is isomorphic to* $\tilde{M} \otimes_{K_{op}} \tilde{N}$ *as RG-module.*

Some remarks on Hypothesis' (B) and (C). In case (B) the quadratic space are assumed to be defective to ensure that $T_U \neq 1$ for $U \in \mathcal{S}(1)$ (see 3.2). Note that Hypothesis (C) can be used to characterize the natural module of G for non-defective quadratic spaces. The assumption that N contains 2-dimensional singular subspaces is needed in the proof of 4.8. We do not know whether Theorem A holds also in the case where the maximal singular subspaces are 1-dimensional.

Under hypothesis (C) the assumption that (N, q, f) is quadratic rather then pseudo-quadratic is redundant. Indeed, if N is pseudo-quadratic but not quadratic, it is easy to see that (G, V) fulfills the hypothesis (B). (C*) is needed since, if char $K = 2$ and $\dim N/N^{\perp} < 6$, or if $|K| = 2$, $\dim N = 6$ and N has Witt index three, the half-spin module for G fullfills Hypothesis (C) but not the conclusion of Theorem A.

We conclude this introduction with some remarks on the structure of $M \otimes_K \overline{N}$, where $\overline{N} = N/N^{\perp}$. Since M is a right vector space over K, we can decompose M into a direct sum of 1-dimensional K-subspaces. This leads to a $\mathbb{Z}G$-decomposition of $M \otimes_K \overline{N}$ into a direct sum of copies of N. But as an RG-module $M \otimes_K \overline{N}$ might still be irreducible or indecomposable. For example, if $R = M = K$, ϕ is an embedding of K into itself and M is regarded as an R-module by multiplication from the left and as a K-module by multiplication from the right by $\phi(k)$, then $M \otimes_K \overline{N}$ is an irreducible RG-module. Now if ϕ is not onto, then $M \otimes_K \overline{N}$ is not irreducible as a $\mathbb{Z}G$-module. In particular, $M \otimes_K \overline{N}$ is not isomorphic to N. For another example, let K be a field with a non-trivial derivation δ, (that is a map $\delta : K \to K$ with $\delta(kl) = \delta(k)l + k\delta(l)$), $R = K$ and $M = R^2$, viewed as left vector space over R. Embed K into $\mathrm{Hom}_R(M)$ by mapping k to $\begin{pmatrix} k & 0 \\ \delta(k) & k \end{pmatrix}$. Since δ is a derivation this map is indeed a homomorphism. Via this homomorphism M becomes an (R, K)-module and $M \otimes_K \overline{N}$ is as an RG-module a non-split extension of $\overline{N}$ by $\overline{N}$.

On the otherhand, if $R = K$ and K is an algebraic Galois extension of its ground field, $M \otimes_K \overline{N}$ is as an RG-module the direct sum of modules algebraically conjugate to $\overline{N}$. Indeed, this follows easily from 2.1 below.

2 Preliminaries

Preliminaries

KML

Lemma 2.1 *Let K, L be such that L is isomorphic to a subfield of K and M an indecomposable unitary (K, L)-module.If L is algebraic and Galois over its ground field, then there exist a $\mathbb{Z}$-isomorphims $\alpha : M \to K$ and a field monomorphism $\sigma : L \to K$ such that $\alpha(kml) = k\alpha(m)l^\sigma$ for all $k \in K, m \in M, l \in L$.*

Proof: Let Q be the ground field of L and σ the isomorphism from Q to the ground field of K. View M as a left vector space over K. Since M is unitary, each $q \in Q$ acts as scalar multiplication by q^σ on M. Let $l \in L$ and f the minimal polynomial of l over Q. Then $f^\sigma(l) = 0$ as an element of $End_K(M)$. Since $L : Q$ is Galois, f splits over L and has no double roots. Since K contains a subfield isomorphic to L, f^σ splits over K and has no double roots. Thus as a left vector space over K, M decomposes into the direct sum of the eigenspaces for l on M. Since L is commutative, L normalizes each of the eigenspaces. But M is indecomposable as a (K, L)-module and so l acts as a scalar $l^\sigma \in K$ on M. Since this is true for each $l \in L$, L normalizes all K-subspaces in M and M is 1-dimensional over K. The lemma is now readily verified. $\square$

BwB

Lemma 2.2 *Let S be a group, T a subgroup of S, $B = N_S(T)$, $\omega \in S \setminus B$, W an RS-module and Y an RB-submodule of $C_W(T)$. Suppose that each of the following holds:*

(i) $S = B \cup B\omega T$ *and* $S = \langle T^S \rangle$,

(ii) $[W, T, T] = 0$,

(iii) $W = \langle YS \rangle$.

Then each of the following is true:

1. $W = [W, S] + C_W(S) = Y + Y\omega + C_W(S)$,
2. $C_W(T) = Y + C_W(S) = Y + [W, T] = [W, T] + C_W(S)$,
3. *Put* $\overline{W} = W/C_W(S)$. *Then* $C_{\overline{W}}(S) = 0$.
4. *If* $T = [T, H]$ *and* $[Y, H] = 0$, *for some* $H \leq B$ *with* $H = H^\omega$, *then* $[W, S] = 0$ *and* $W = Y$.

Proof: Put $Y_0 = Y + [W, T]$. By the assumptions T centralizes Y_0. By (a)

$$T^S = T^B \cup T^{B\omega T} = T \cup T^{\omega T}$$

and hence

$$S = \langle T^S \rangle = \langle T, T^\omega \rangle.$$

This implies

$$W = \langle YS\rangle = Y + [W,S] = Y + [W,T] + [W,T^\omega] = Y_0 + [W,T]\omega.$$

Similarly $W = \langle (Y\omega)S\rangle = Y_0\omega + [W,T]$. Form the last two statements and the modular laws

$$Y_0 + C_{[W,T]\omega}(T) = C_W(T) = C_{Y_0\omega}(T) + [W,T].$$

Moreover, $C_{Y_0\omega}(T) \leq C_W(\langle T, T^\omega\rangle) = C_W(S)$ and $C_{Y_0\omega}(T) \leq C_W(S) \cap Y_0\omega \leq Y_0$. So

$$C_W(T) = Y_0 = [W,T] + C_W(S).$$

Since $W = \langle YS\rangle$ we conclude from (a) that

$$W = Y + Y\omega T = Y + Y\omega + [Y\omega, T] = Y\omega + C_W(T).$$

Therefore

$$C_W(T^\omega) = Y\omega + (C_W(T) \cap C_W(T^\omega)) = Y\omega + C_W(S)$$

and

$$W = Y + C_W(T^\omega) = C_W(T^\omega) + C_W(T) = Y\omega + C_W(S) + Y =$$

$$= [W,T] + [W,T]\omega + C_W(S) = [W,S] + C_W(S).$$

This completes the proof of (1) and(2).

Let U be the inverse image of $C_{\overline{W}}(S)$ in W. By (2) applied to $\overline{W}$ we have $\overline{U} \leq \overline{Y}[\overline{W},T]$. Thus $U \leq Y[W,T]C_W(S) \leq C_W(T)$ and $[U,T] = 0$. Since $S = \langle T^S\rangle$ we get $U \leq C_W(S)$ and $\overline{U} = 0$, proving (3).

To prove (4) note that H centralizes $W = Y + Yw + C_W(S)$ and hence $H \leq C_S(W)$. It follows that

$$T = [T,H] \leq C_S(W) \text{ and } S = \langle T^S\rangle \leq C_S(W).$$

So $[W,S] = 0$ and the lemma is proved. $\square$

SL2K

Lemma 2.3 *Suppose that (G,Σ) fulfills (A)(1) with* $\dim N = 2$. *Let $Z \in \Sigma, L = C_G[N,Z]$ and V an RG modules with $[V,Z,L] = 0$, $V = [V,G]$ and $C_V(G) = 0$, then there exists an (R,K)-module M such that $V \cong M \otimes_K N$.*

Proof: We start with defining some elements of $G = SL_2(K)$. Let

$$a(k) = \begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix}, k \in K;\ h(\lambda) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, 0 \neq \lambda \in K;\ \omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

$Z = \{a(k) | k \in K\}$ and $H = \{\begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix} | \lambda \in (K \setminus \{0\}'\}$.

Note that Z as defined above is indeed an element of Σ. We reader might verify that the following relation holds for all $t \in K^{\#}$. (But the relation will follows from some computation below)

$$(*) \qquad \omega^{-1}a(-t)\omega = a(t^{-1})h(t)\omega a(t^{-1}).$$

Since both $a(t^{-1})$ and $a(t)^{\omega}$ are in G, this implies $h(t)\omega \in G$. Hence $h(t)h(r^{-1}) \in G$ for all $t, r \in K^*$ and we conclude that all $h(t)$ and $\omega \in G$. Hence also $h(t)h(r)h(r^{-1}t^{-1} \in H$ and so $H \leq G$. In particular, $ZH \leq L$ (actually L=ZH, but we will not need that). Put $M = [V, Z]$. Since $G = \langle Z, Z^{\omega} \rangle$ we get

$$V = [V, G] = M + M\omega \text{ and } M \cap Mw \leq C_V(G) = 0.$$

Hence,

$$V = M \oplus M\omega.$$

Let $D = \mathrm{Hom}_R(M, M)$ and $M_2(D)$ the ring of 2×2-matrices with coefficients in D. We define a ring isomorphism from $M_2(D)$ to $\mathrm{Hom}_R(V, V)$ by sending $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ to ϕ where ϕ is defined by $(u + v)\phi = ua + v\omega^{-1}c + ub\omega + v\omega^{-1}d\omega$ for $u \in M$ and $v \in M\omega$. Direct computations show that this indeed defines a ring isomorphism. Furthermore we see that:

$$\omega \leftrightarrow \begin{pmatrix} 0 & 1 \\ \epsilon & 0 \end{pmatrix}, \epsilon^2 = 1,$$

$$h(\lambda) \leftrightarrow \begin{pmatrix} y(\lambda) & 0 \\ 0 & z(\lambda) \end{pmatrix},$$

$$a(k) \leftrightarrow \begin{pmatrix} 1 & 0 \\ x(a) & 1 \end{pmatrix}.$$

$a(k)a(l) = a(k + l)$ implies

$$(1) \qquad x(k) + x(l) = x(k + l).$$

Since $\omega^2 \in Z(G)$ we have $\epsilon y(\lambda) = y(\lambda)\epsilon$. From $\omega^{-1}h(\lambda)\omega = h(\lambda^{-1})$ we get

$$(2) \qquad z(\lambda) = y(\lambda^{-1}).$$

Since $h(\lambda)h(\kappa) \equiv h(\lambda\kappa) \pmod{H}$ and $[M, H] = 0$, we have $h(\lambda)h(\kappa) \equiv h(\lambda\kappa)$ (mod $C_G(M)$). Thus

$$(3) \qquad y(\lambda)y(\kappa) = y(\lambda\kappa).$$

Next we make another use of (*). $a(t^{-1})h(t)\omega a(t^{-1})$ corresponds to

$$\begin{aligned}
&\begin{pmatrix} 1 & 0 \\ x(t^{-1}) & 1 \end{pmatrix}\begin{pmatrix} y(t) & 0 \\ 0 & y(t^{-1}) \end{pmatrix}\begin{pmatrix} 0 & 1 \\ \epsilon & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ x(t^{-1}) & 1 \end{pmatrix}\\
&= \begin{pmatrix} y(t) & 0 \\ x(t^{-1})y(t) & y(t^{-1}) \end{pmatrix}\begin{pmatrix} x(t^{-1}) & 1 \\ \epsilon & 0 \end{pmatrix}\\
&= \begin{pmatrix} y(t)x(t^{-1}) & y(t) \\ x(t^{-1})y(t)x(t^{-1}) + y(t^{-1})\epsilon & x(t^{-1})y(t) \end{pmatrix}
\end{aligned}$$

and $\omega^{-1}a(-t)\omega$ to

$$\begin{aligned}
&\begin{pmatrix} 0 & \epsilon \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ x(t^{-1}) & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ \epsilon & 0 \end{pmatrix} = \begin{pmatrix} \epsilon x(-t) & \epsilon \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ \epsilon & 0 \end{pmatrix}\\
&= \begin{pmatrix} 1 & \epsilon x(-t) \\ 0 & 1 \end{pmatrix}
\end{aligned}$$

So $y(t)x(t^{-1}) = x(t^{-1})y(t) = 1$, $x(t^{-1}) = y(t)^{-1} = y(t^{-1})$ and hence

(4) $\qquad x(t) = y(t)$, for all $0 \neq t \in K$.

Moreover, $y(t) = \epsilon x(-t) = -\epsilon x(t) = -\epsilon y(t)$ and since $y(1) = 1$,

(5) $\qquad \epsilon = -1$.

By (1),(3) and (4) $x : K \to D$ is a ring homomorphism. In particular, M is an (R, K)-module. Moreover,

$$\omega \leftrightarrow \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} x(0) & x(1) \\ x(-1) & x(0) \end{pmatrix},$$

$$a(k) \leftrightarrow \begin{pmatrix} 1 & 0 \\ x(a) & 1 \end{pmatrix} = \begin{pmatrix} x(1) & x(0) \\ x(a) & x(1) \end{pmatrix}.$$

Since $G = \langle Z, Z^\omega \rangle = \langle Z, \omega \rangle$ we conclude

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \leftrightarrow \begin{pmatrix} x(a) & x(b) \\ x(c) & x(d) \end{pmatrix}$$

for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$.

On the other hand cleary $M \otimes_K K \cong M$ as R-modules. Since $N \cong K \oplus K$ as K-modules, we conclude that $M \otimes_K N \cong M \oplus M$ as R-modules. Let $e_1 = (1, 0) \in K \oplus K$, $e_2 = (0, 1) \in K \oplus K$a nd $m_1, m_2 \in M$. Direct computations now show that

$$m_1 + m_2\omega \to m_1 \otimes e_1 + m_2 \otimes e_2$$

defines an RG isomorphism from V to $M \otimes_K N$ $\qquad\square$.

$CVG=0$

Lemma 2.4 *Suppose that (G,Σ) fulfills part 1 of Hypothesis (A) with $\dim N = 2$. Let $Z \in \Sigma, L = C_G[N,Z]$ and V an RG modules with $[V,Z,L]=0$ and $V=[V,G]$. If $\operatorname{char} K \neq 2$ or K is not commutative, then $C_V(G)=0$.*

Proof: Put $\overline{V} = V/C_V(G)$. Since $V=[V,G]$, $V=[V,Z]^G$ and we can apply 2.2(c). Thus $C_{\overline{V}}(G)=0$ and by 2.3 $\overline{V} \cong M \otimes_K N$, for some (R,K)-module M.

Assume first that $\operatorname{char} K \neq 2$ amd let $t=-\mathrm{id} \in Z(G)$. Then $\overline{V}=[\overline{V},t]$ and $C_{\overline{V}}(t)=0$. It follows that $V=[V,t]\oplus C_V(G)$ and $V=[V,G]=[V,t]$, and so $C_V(G)=0$.

Next assume that $\operatorname{char} K = 2$ and K is not commutative. We claim that $2v=v+v=0$ for all $v \in V$. Indeed let $X=\{v \in V | 2v=0\}$. Since $2\overline{V}=0$, $[2V,G]=0$. Moreover, $V/X \cong 2V$ and so $V=[V,G]\leq X$, proving the claim.

Let $a(t), \omega$ and Z be as in 2.3. By 2.2b, $C_V(Z)=[V,Z]+C_V(G)$. Pick $\lambda \in (K\setminus 0)'$ with $\lambda \neq 1$ and put $h=\begin{pmatrix} \lambda & 0 \\ 0 & 1\end{pmatrix}$. Note that $h \in G$. Then $a(t)^h = a(t\lambda)$ and since $[N,h]\leq[N,Z]$, $[V,h]\leq C_V(Z)$. So for $v \in V$ and $t \in K$ we get

$$[v,a(t)]h=[vh,a(t)^h]=[v+[v,h],a(t\lambda)]=[v,a(t\lambda)].$$

Hence

$$[v,a(t(1+\lambda))]=[v,a(t)a(t\lambda)]=[v,a(t)]+[v,a(t\lambda)]=$$
$$=[v,a(t)]+[v,a(t)]h=[v,a(t),h],$$

where we used $2V=0$ in the last equality. So $[v,a(t(l+1))]\in[V,Z,h]$. Since every $s \in K$ is of the form $t(\lambda+1)$ for some t we conclude that $[V,Z]\leq[V,Z,h]$. Now $C_{[\bar{V},Z]}(h)=0$ and so $[V,Z,h]\cap C_V(G)=0$. Thus $[V,Z]\cap C_V(G)=0$. On the other hand by 2.2b applied to $Y=[V,Z]$, $C_V(G)\leq[V,Z]$ and so $C_V(G)=0$. □

Remark: we should make some remark on the existence of universal central extensions

$defUE$

Definition 2.5 *Let X be an $\mathbb{Z}G$-module. Then a $\mathbb{Z}G$-module $\hat{X}$ is called a universal central $\mathbb{Z}G$-extension of M provided that*

(a) There exists a $\mathbb{X}G$-homomorphism $\phi:\hat{X}\to X$ with $[\ker\phi,G]=0$.

(b) Whenever W is a $\mathbb{Z}G$-module and $\psi: W \to X$ is a $\mathbb{Z}G$-homomorphism with $[\ker\psi,G]=0$, then there exist a unique $\mathbb{Z}G$-homomorphism $\alpha:\hat{X}\to W$ with $\phi=\psi\alpha$.

UE

Lemma 2.6 *Let X be a KG-module and $(\hat{X},\phi)$ a universal central $\mathbb{Z}G$-extension of X.*

(a) $\hat{X}$ is a KG-module and ϕ is KG-linear.

(b) Let M be an right vector space over K. Then $M\otimes_K \hat{X}$ is a universal central $\mathbb{Z}G$-extension of $M\otimes_K X$.

Proof: (a) Let $0 \neq k \in K$. Then by part (b) of 2.5 applied $W = \hat{X}$ and $\psi = k^{-1}\phi$, there exists a $\mathbb{Z}G$-homomorphism $\alpha_k : \hat{M} \to \hat{M}$ with $\phi = k^{-1}\phi\alpha_k$, i.e such that $k\phi = \phi\alpha_k$. By the uniqueness of α_k, $\alpha_1 = id$,$\alpha_{k+l} = \alpha_k + \alpha_l$ and $\alpha_{kl} = \alpha_k\alpha_l$. Defining $k\hat{x} = \alpha_k(\hat{x})$, $\hat{X}$ becomes a vector space over K and (a) is proved.

(b) Let W be a $\mathbb{Z}G$-module and $\psi : W \to M \otimes X$ be any $\mathbb{Z}G$-homomorphism with $[\ker\psi, G] = 0$. For $0 \neq m \in M$, let W_m be the inverse image of $m \otimes X$ under ψ. The map $x \to m \otimes x$ is a $\mathbb{Z}G$ from X to $m \otimes X$ and we can define a $\mathbb{Z}G$-homomorphism $\psi_m : W_m \to X$ by $\psi(w) = m \otimes \psi_m(w)$. By the universial property of $\hat{X}$ (2.5b) there exists a $\mathbb{Z}G$ homomorphism $\alpha_m : \hat{X} \to W_m$ with $\phi = \psi_m\alpha_m$. Also put $\alpha_0 = 0$. We obtain a map $\alpha : M \times \hat{X} \to W, (m, x) \to \alpha_m(x)$. Clearly, α is additive in the second coordinate and by the uniqueness of α_m it is easy to check that α is additive in the first coordinate and is balanced (that is $\alpha(m, kx) = \alpha(mk, x)$ for all $m \in M, x \in X, k \in K$.) Thus by the universial property of a tensor product α can be extended to a $\mathbb{Z}$-homomorphism $\alpha : M \otimes_K \hat{X} \to W$. Moreover, one readily verifies that α commmutes with G and $\phi = \psi\alpha$. □

rUE

Remark 2.7 *Let $\mathcal{C}$ be a class of $\mathbb{Z}G$-modules. For X in $\mathcal{C}$ define a universial central $\mathcal{C}$ extension of X as in 2.5 except that $\hat{X}$ and W are assumed to be in $\mathcal{C}$. Then (with the same proof) 2.6 is still true for universial central $\mathcal{C}$-extensions provided that $M \otimes_K \hat{X}$ is in $\mathcal{C}$.*

□

3 Some properties of pseudo quadratic spaces

$f(n, n)$

Lemma 3.1 *Let (N, q, f) be a pseudo quadratic space and put $K^+ = \{k \in K | k + sk^\sigma = 0\}$. Then*

(a) $K_- \leq K^+$.

(b) *For all $a \in N$ and $q \in q(a)$, $f(a, a) = q + sq^\sigma$.*

(c) *Suppose that $K_- = K^+$. Then $q(a)$ is uniquely determine by $f(a, a)$. In particular, $f(a, a) = 0$ implies $q(a) = 0$ and rad $N = N^\perp$.*

(d) *Suppose that* char$K \neq 2$ *or that σ acts non trivially on $Z(K)$. Then $K_- = K^+$.*

Proof: (a) $(k - sk^\sigma)^\sigma = k^\sigma - k^{\sigma^2}s^\sigma = k^\sigma - s^{-1}kss^{-1} = k^\sigma - s^{-1}k$ and so $k - sk^\sigma = -s(k - sk^\sigma)^\sigma$, $k - sk^\sigma \in K^+$ and $K_- \leq K^+$.

(b) Let $k, l \in K$. Then on the one hand

$$q(ka + la) = q(ka) + q(la) + \overline{f(ka, la)} = k * q(a) + l * q(a) + \overline{kf(a, a)l^\sigma}$$
$$= \overline{kqk^\sigma + lqlb^\sigma + kf(a, a)l^\sigma}$$

and on the other hand

$$q(ka + la) = q((k + l)a) = (k + l) * q(a) = \overline{kqk^\sigma + kql^\sigma + lqk^\sigma + lql^\sigma}.$$

Thus
$$\overline{kf(a,a)l^\sigma} = \overline{kql^\sigma + lqk^\sigma}.$$
But $lqk^\sigma - ksq^\sigma l^\sigma = lqk^\sigma - s(lqk^\sigma)^\sigma \in K_-$ and so
$$\overline{kf(a,a)l^\sigma} = \overline{k(q+sq^\sigma)l^\sigma}$$
for all $k,l \in K$. If $\overline{K} \neq 0$ this cleary implies $f(a,a) = q + sq^\sigma$. If $\overline{K} = 0$, then $K = K_-$ and so by (a) $K = K^+$. Thus $K_+ = 0$ and $q + sq^\sigma = 0$. Moreover, by (PQ1), f is trace valued and so also $f(a,a) = 0$.

(c) By (b) $q(a)$ is unique up to an element $\overline{K^+}$. Thus (c) holds

(d) If char $K \neq 2$, put $\lambda = 1$ and if char $K = 2$ and σ acts non trivially on $Z(K)$ pick $\lambda \in Z(K)$ with $\lambda \neq \lambda^\sigma$. Then in any case, $\lambda \in Z(K)$ and $\lambda + \lambda^\sigma \neq 0$. Put $\mu = \lambda(\lambda+\lambda^\sigma)^{-1}$. Since $\lambda \in Z(K)$, $\lambda^{\sigma^2} = \lambda$. Thus $\mu + \mu^\sigma = 1$ and $\mu^\sigma \in Z(K)$. Let $k \in K$ with $k = -sk^\sigma$. Then
$$k\mu - s(k\mu)^\sigma = k\mu - s\mu^\sigma k^\sigma = k\mu - sk^\sigma\mu^\sigma = k(\mu+\mu^\sigma) = k.$$
Hence $k \in K_-$, $K^+ \leq K_-$ and (d) follows from (a). $\square$

transvection

Lemma 3.2 *Let N be a left vector space over K, $0 \neq n \in N$ and $0 \neq \phi \in N^*$.*

(a) *$t(\phi,n)$ is invertible if and only if $n\phi \neq -1$, in which case the inverse is given by $t(\phi, -(1+n\phi)^{-1}n)$.*

(b) *Let (q,f) be a (σ,s)-pseudo quadratic form on N. Then $t(\phi,n) \in O(V,q,f)$ if and only if one of the following holds:*

  (b1) *$n \in$ rad N and $n\phi \neq -1$.*

  (b2) *$n \notin N^\perp$ and there exists $0 \neq k \in K$ with $-k^{-1} \in q(n)$ and $v\phi = f(v,n)k$ for all $v \in N$.*

Proof: Put $t = t(n^*, n)$.

(a) If $n\phi = -1$ then $n.t = 0$ and t is not invertible. If $n\phi \neq -1$ it is trivial to verify that $t(\phi, -(1+n\phi)^{-1}.n)$ is an inverse for t.

(b) Recall that $t \in O(N,q,f)$ if and only if $f(at,bt) = f(a,b)$ and $q(at) = q(a)$ for all $a,b \in N$. Since $f(at,bt) = f(a + a\phi.n, b + b\phi.n)$ weget

(1) $$f(at,bt) = f(a,b) + a\phi \cdot f(n,b) + f(a,n).(b\phi)^\sigma + a\phi \cdot f(n,n) \cdot (b\phi)^\sigma.$$

and since $q(at) = q(a\phi.n + a)$

(2) $$q(at) = a\phi * q(n) + \overline{a\phi.f(n,a)} + q(a).$$

In particular, if $n \in \mathrm{rad}N$ and t is invertible, $t \in O(N,q,f)$. Thus we may assume that $n \notin \mathrm{rad}N$.

Assume now that $t \in O(V,q,f)$. Suppose that $n \in N^\perp$. Since $n \notin \mathrm{rad}\ V$, $q(n) \neq 0$. Hence by (2) $a\phi = 0$ for all $a \in N$, a contradiction.

Thus $n \notin N^\perp$. Let $b \in \ker\phi$. Then by (1) $a\phi.f(n,b) = 0$ for all a in N and so $b \in N^\perp$ and $\ker\phi \leq n^\perp$. Since both $\ker\phi$ and $n^\perp$ are hyperplanes in N, $\ker\phi = n^\perp$. Therefore there exists $k \in K$ with

$$(3) \qquad a\phi = f(a,n)k \text{ for all } a \in N.$$

Since $f(n,a) = sf(a,n)^\sigma$, (2) now implies $f(a,n)k * q(n) + \overline{f(a,n)ksf(a,n)^\sigma} = 0$. Thus

$$f(a,n) * (k * q(n)) + f(a,n) * \overline{ks} = 0$$

Hence $k * q(n) = -\overline{ks}$ and $q(n) = k^{-1} * \overline{ks} = -\overline{sk^{-\sigma}}$. Now $k^{-1} - sk^{-\sigma} \in K_-$ and so $\overline{k^{-1}} = \overline{sk^{-\sigma}}$. Thus $q(n) = -\overline{k^{-1}}$ and (b2) holds in this case.

Suppose next that t fulfils (b2). Reading the above calculations backwards we see that $q(a) = q(at)$ for all $a \in N$. Put $q = -k^{-1}$. By assumption $q \in q(n)$ and so by 3.1 $f(n,n) = q + sq^\sigma$. Using (1) it is now readily verified that $f(at,bt) = f(a,b)$ for all $a,b \in N$. It remains to show that t is invertible. Otherwise by (a), $-1 = n\phi = f(n,n)k$ and so $q = k^{-1} = f(n,n)$. Thus $q = f(n,n) = q + sq^\sigma$, $sq^\sigma = 0$ and $q = 0$, a contradiction to the definition of q. $\square$

We denote the element in $O(V,q,f)$ of form $t(\phi,n)$ as in 3.2b2 by $t(k,n)$, that is

$$v.t(k,n) = v + f(v,n)k \cdot n \text{ where } n \in N \setminus N^\perp, k \in K^\#, -k^{-1} \in q(n), v \in N$$

$t(k,n)$ is called a pseudo-transvection with axis Kn. We remark that for any given $n \in N \setminus \mathrm{rad}N$, there exists a pseudo-transvection with axis Kn, unless n is a singular vector in a quadratic space.

transitive

Lemma 3.3 *Let (N,q,f) be a non-degenerate pseudo-quadratic space such that $\mathcal{S}(1) \neq \emptyset$.*

(a) $N = \langle \mathcal{S}(1) \rangle$.

(b) *Let i be a positive integer with $\mathcal{S}(i) \neq \emptyset$ and $U \in \mathcal{S}(i)$. Then Q_U acts transitively on the set of all $U_0 \in \mathcal{S}(i)$ with $U_0 \cap U^\perp = 0$.*

Proof: (a) Let $M = \langle \mathcal{S}(1) \rangle$ and suppose $N \neq M$. Let $n \in N \setminus M$. Then n is not singular and so there exists a pseudo-transvection t in $\mathcal{O}(N,q,f)$ with axis Kn. Since t normalizes M and $n \notin M$, $[M,t] \leq M \cap Kn = 0$ and so n is perpendicular to M. It follows that $M \cup M^\perp = N$, $M^\perp = N$, and $M \leq N^\perp$. Since M is generated by singular vectors, $M \leq \mathrm{rad}N = 0$, a contradiction.

(b) Let $U_1, U_2 \in \mathcal{S}(i)$ with $U^\perp \cap U_k = 0, k = 1,2$. As N is non degenerate, $N \cap U = 0$ and so $\dim N/U^\perp = i$. Thus $N = U^\perp + U_k$. Let $x_1, x_2, \ldots, x_i$ be a basis for U over K and for $k = 1,2$ let $y_1(k), y_2(k), \ldots, y_i(k)$ be the basis for U_k with $f(x_j, y_l(k) = \delta_{jl}$. Note that $U^\perp = U \oplus (U + U_k)^\perp$. Hence for every $x \in (U+U_k)^\perp$ there exists a unique $y \in (U+U_2)^\perp$ with $x + U = y + U$, and the map $x \to y$ is an isometry. Extend this map to $h \in GL_K(N)$ such that $x_j h = x_j$ and $y_j(1)h = y_j(2)$. Then it is easy to see that h is an isometry, $h \in Q_U$ and $U_1 h = U_2$. $\square$

Lemma 3.4 *Let (N, q, f) be a nondegenerate pseudo quadratic space, i a positive integer and $U \in calS(i)$. Then*

(a) $P_U/Q_U \cong GL_K(U) \times O(U^\perp/U, q, f)$.

(b) $Q_U/T_U \cong \mathrm{Hom}_K(U^\perp/U^{\perp\perp}, U) \cong \mathrm{Hom}_K(N/U^\perp, U^perp/U^{\perp\perp}$.

(c) $T_U/Z_U \cong \mathrm{Hom}_K(N/U^\perp, N^\perp)$

(d) $Q_U/Z_U \cong \mathrm{Hom}_K(N/U^\perp, U^\perp/U)$

(e) Z_U *is isomorphic to the additive group of all $i \times i$-matrices M with $M^T = -sM^\sigma$ and $M_{jj} \in K_-$ for all $1 \le j \le i$.*

(f) T_U *is isomorphic to the additive group o fall $i \times i$-matrices M with $M^T = -sM^\sigma$ and $M_{jj} + K_- \in q(N^\perp)$ for all $1 \le j \le i$.*

(g) $Z(Q_U) = T_U$ *(unless $i = 1$ and $K_- = 0$, in which case Q_U is abelian)*

(h) $Q'_U = Z_U$ *(unless $Q_U = T_U$, that is $U^\perp = U^{\perp\perp}$)*

Proof: It follows easily from 3.3a and induction that there exists a singular subspace E in N with $N = U^\perp \oplus E$.

Put $X = (U \oplus E)^\perp$. Since U is finite dimensional and $U \cap N^\perp = 0$, $N = U \oplus X \oplus E$ and $U^{\perp\perp} = U \oplus N^\perp$.

(a) By 3.3b and a Frattini argument, $P_U = N_{P_U}(E)Q_U$. Since $N_{P_U}(E)$ normalizes X, $N_{P_U}(E) \cap Q_U = 1$ and it suffices to show that $N_{P_U}(E) \cong GL_K(U) \times O(X, q, f)$. Let $g \in GL_k(U)$ and $h \in O(X, q, f)$. It is an easy exercise to show that there exists unique $\hat{g} \in GL_K(E)$ with $f(ug, e\hat{g}) = f(u, e)$ for all $u \in U, e \in E$. Moreover, there exists a unique $t \in GL_K(N)$ which acts as g on U, as $\hat{g}$ on E and as h on X. Clearly, $t \in \mathcal{O}$ and so (a) holds.

(b) Let $\alpha \in \mathrm{Hom}_K(X, U)$ with $N^\perp\alpha = 0$. Put $X_\alpha = \{x + x.\alpha | x \in X\}$ and note that $U^\perp = U \oplus X_\alpha$, Moreover, $X_\alpha^\perp = U \oplus N^\perp \oplus E_\alpha$ for some i-dimensional singular subspace E_α. As in 3.3b, there exists $t \in \mathcal{O}$ such that t centralizes U and $xt = x + x\alpha$ for all $x \in X$. Clearly $t \in Q_U$ and so the map

$$Q_U \to Hom_K(X/N^\perp, U)$$

$$q \to (x + N^\perp \to [x, q])$$

is onto. Its kernel is obviously T_U and so the first equality in (b) holds. The second is just the dual version of the first.

Let $\beta \in \mathrm{Hom}_K(N, N^\perp)$ with $U^\perp\beta = 0$ and $\dim N\beta = 1$. Then clearly $(\ker\beta)^\perp = Ku + N^\perp$ for some $0 \ne u \in U$. Let $e \in N$ with $f(e, u) = 1$ and put $r = e\beta$. Then $0 \ne r \in N^\perp$, $q(r) \ne 0$ and so there exists $0 \ne \lambda \in K$ with $-\lambda^{-1} \in q(r)$. Pick $\mu \in K$ with

$\mu^\sigma = \lambda^{-1}$ and put $n = r + \mu \cdot e$. Since e is singular and perpendicular to r, $q(n) = q(e)$ and so we obtain the pseudo-transvection $t(\lambda, n) \in \mathcal{O}$. Moreover,

$$(1) \qquad [e, t(\lambda, n)] = f(e,n)\lambda \cdot n = \mu^\sigma \lambda \cdot (r + \mu u) = r + \mu \cdot u \equiv r \mathrm{mod} U.$$

Note that $U^{\perp\perp} = U \oplus N^\perp$ and $[N, t] \in U^{\perp\perp}$ for all $t \in T_U = C_{\mathcal{O}}(U^\perp)$. Consider the homomorphism ρ;

$$T_U \to \mathrm{Hom}_K(N/U^\perp, U^{\perp\perp}/U)$$

$$t \to (x + U^\perp \to [x,t] + U)$$

Clearly $\ker \rho = Z_U$. Let $\hat{\beta}$ the the compostion of β with the natural isomorphism $N^\perp \to U^{\perp\perp}/U$. Then by (1), ρ maps $t(\lambda, n)$ onto $\hat{\beta}$. As the holds for all β's,ρ is onto and (c) holds.

(d) Cosider the map

$$Q_U \to \mathrm{Hom}_K(N/U^\perp, U^\perp/U)$$
$$t \to (x + U^\perp \to [x,t] + U)$$

By (b) and (c) this map is onto. Its kernel is Z_U and so (d) holds.

(e,f) Let $u_1, u_2, \ldots, u_i$ a basis for U and pick $e_1, e_2, \ldots e_i \in E$ with $f(u_l, e_j) = \delta_{lj}$. Let $z \in GL_k(N)$ with $[N, z] \leq U^{\perp\perp}$ and $[U^\perp, z] = 0$. Define a $i \times i$ matrix $M = (m_{kl})$ and $r_k \in N^\perp$ by $[e_k, z] = \sum_{l=1}^i m_{kl} u_l + r_k$. Then z is uniquely determined by M and (r_k). We need to find necessary and sufficient conditions on M and (r_k) for z to be in $\mathcal{O}$. So we compute

$$f(e_j z, e_k z) = f(e_j + \sum_{l=1}^i m_{jl} u_l, e_k + \sum_{l=1}^i m_{kl} u_l) =$$
$$= f(m_{jk} u_k, e_k) + f(e_j, m_{kj} u_k) = m_{jk} + s m_{kj}^\sigma$$

and

$$q(e_k z) = q(e_k + \sum_{l=1}^i m_{kl} u_l + r_k) = \overline{f(m_{kk} u_k, e_k)} + q(r_k) = \overline{m_{kk}} + q(r_k).$$

Thus $z \in \mathcal{O}$ if and only if $M^T = -sM^\sigma$ and $m_{kk} + K_- \in -q(r_k)$ for all $1 \leq k \leq i$. Note that $q_{N^\perp}$ is injective. Hence for any M with $M^T = -sM^\sigma$ and $m_{kk} + K_- \in q(N^\perp)$ there exists unique $r_k \in N^\perp$ with $m_{kk} + K_- \in q(r_k)$. Thus (f) holds. Moreover, $z \in T_U$ if and only if in addition $r_k = 0$. Thus also (e) holds.

(g,h) By definition of Q_U, $[N, Q_U] \leq U^\perp$ and $[U^\perp, Q_U] \leq U$. Thus $[N, Q_U, Q_U] \leq U$ and the three subgroup lemma implies $[N, Q_U'] \leq U$. Thus $Q_U' \leq Z_U$. Note that $[N, T_U] \leq U^{\perp\perp} = U +$ rad N. Thus $[N, T_U, Q_U] = 0$ and $[N, Q_U, T_U] = 0$. Hence $[Q_U, T_U, N] = 0$ and $T_U \leq Z(Q_U)$. So to prove (g) and (h) we need to show $Z(Q_U) \leq T_U$ and $Z_U \leq Q_U'$.

Let $a, b \in Q_U$. We wish to compute $[a, b]$. For this put $a_i = [e_i, a]$ and $b_i = [e_i, b]$. Then $a_i, b_i \in U^\perp$. Let $x \in U^\perp$. Then $[x, a] \leq U$ and so $0 = f(x, e_i) = f(x.a, e_i.a) = f(x + [x, a], e_i + a_i) = f(x, a_i) + f([x, a], e_i]$. Hence

$$-[x, a] = \sum_{k=1}^{i} f(x, a_k) u_k.$$

A similar formula holds for b. Furthermore, $[e_k, a^{-1}] = -a_k + r_k$ and $[e_k, b^{-1}] = -b_k + s_k$ for some $r_k, s_k \in U^{\perp\perp}$. Also $[x, a^{-1}] = -[x, a]$ and so

$$(e_k.[a, b] = e_k.a^{-1}b^{-1}ab = (e_k - a_k + r_k).b^{-1}ab$$

$$= e_k - b_k + s_k - a_k - \sum_{j=1}^{i} f(a_k, b_j) u_j + r_k).ab$$

Using $(e_k - a_k + r_k).a = e_k$ we get

$$e_k.[a, b] = (e_k - b_k + \sum_{j=1}^{i} f(b_k, a_j) u_j - \sum_{j=1}^{i} f(a_k, b_j) u_j + s_k).b$$

$$= e_k + \sum_{j=1}^{i} f(b_k, a_j) u_j - \sum_{j=1}^{i} f(a_k, b_j) u_j.$$

Put $B = (f(a_k, b_j))$. Then $[a, b] \in Z_U$ corresponds to the matrix $-B + sB^{T\sigma}$.

Suppose there exists $a \in Z(Q_U) \setminus T_U$. Then $a_k \not\leq U^{\perp\perp}$ for some k. Let $1 \leq k, j \leq i$ and $0 \neq \lambda \in K$. By (b) we can choose b such that $b_l \in U^{\perp\perp}$ if $l \neq j$ and $f(a_k, b_j) = \lambda$. Since $a \in Z(Q_U)$ we get $B = sB^{T\sigma}$. Since all but the j'th colummns of B are zero and the $k - j$ spot of B is λ and so not zero, we conclude that $j = k$ and $\lambda = s\lambda^\sigma$. Since j and λ are arbitrary we conclude that $i = 1$ and $K_- = 0$. Also if $i = 1$ and $K_- = 0$, $[a, b] = 1$ for any $a, b \in Q_U$ and so (g) holds.

To complete the proof for (h) we may assume that $U^\perp \neq U^{\perp\perp}$. Fix $1 \leq k, j \leq i$ and $\lambda \in K$ and choose $a, b \in Q_U$ such that $a_l \in U^{\perp\perp}$ if $l \neq k$, $b_l \in U^{\perp\perp}$ if $l \neq j$ and $f(a_k, b_j) = \lambda$. Then B is zero everywhere except in the $k - j$ spot, where it is λ. Clearly every matrix M with $M^T = -sM^\sigma$ and $m_{ll} \in K_-$ for all l is the sum of matices of the form $-B + B^{T\sigma}$, $1 \leq kj \leq i, \lambda \in K$ and so (h) is proved. □

transitive2

Lemma 3.5 *Let (N, q, f) be a non-degenerate pseudo-quadratic space with $\mathcal{S}(1) \neq \emptyset$.*

(a) *Suppose that (N, q, f) is not a 2-dimensional quadratic space.*

- (a1) *$\langle Q_U | U \in \mathcal{S}(1) \rangle$ acts transitively on $\mathcal{S}(1)$.*
- (a2) *If $x, y \in \mathcal{S}(1)$ with $x \neq y$,then $\mathcal{S}(1) \neq (\mathcal{S}(1) \cap x^\perp) \cup (\mathcal{S}(1) \cap y^\perp)$.*

(a3) *Define $x, y \in \mathcal{S}(1)$ to be adjacent if $x \not\perp y$. Then the corresponding graph on $\mathcal{S}(1)$ is connected.*

(b) *$\langle T_U | U \in \mathcal{S}(1)\rangle$ acts transitively on $\mathcal{S}(1)$ unless $N$is a non defective quadratic space.*

Proof: (a1) Let $U_1, U_2 \in \mathcal{S}(1)$ with $U_1 \neq U_2$.

Suppose first that $U_1 \perp U_2$. Pick $U_3 \in \mathcal{S}(1)$ with $U_3 \leq U_1 + U_2$ and $U_1 \neq U_3 \neq U_2$. By 3.4a, Q_{U_3} induces a full unipotent subgroup on $U_1 + U_2$ and so U_2 and U_1 are conjugate under Q_{U_3}.

Suppose next that U_1 and U_2 are not perpendicular. Assume that $\mathcal{S}(2) \neq \emptyset$. Then U_1 is contained in a 2-dimensional singular subspace space W. Put $U_3 = W \cap U_2^{\perp}$. Then U_1 and U_2 are both conjugate to U_3 under $\langle Q_U | U \in \mathcal{S}(1)$. Thus we may assume that $\mathcal{S}(2) = \emptyset$.

Suppose that there exists $U_3 \in \mathcal{S}(1)$ with $U_1 \neq U_3 \neq U_2$. Then by 3.3b U_1 and U_2 are conjugate under Q_{U_3} and we may assume that no such U_3 exists. Thus $\mathcal{S}(1) = \{U_1, U_2\}$, Q_{U_1} centralizes U_2 and so $Q_{U_1} = 1$. By 3.4 this implies $U = U^{\perp}$ and $K_- = 0$, i.e. N is a 2-dimensional quadratic space.

(a2) Let $x, y \in \mathcal{S}(1)$ and suppose that $\mathcal{S}(1) = (\mathcal{S}(1) \cap x^{\perp}) \cup (\mathcal{S}(1) \cap y^{\perp})$.

If $\mathcal{S}(2) = \emptyset$, then $\mathcal{S}(1) = (\mathcal{S}(1) \cap x^{\perp}) \cup (\mathcal{S}(1) \cap y^{\perp}) = \{x, y\}$. As seen in the proof of(a) this implies that (N, q, f) is a 2-dimensional quadratic space,.

So we may assume that $\mathcal{S}(2) \neq \emptyset$. By 3.3a (a) applied to $y^{\perp}/y$, $y^{\perp}$ is generated by its singular subspaces. Thus there exists $z \in \mathcal{S}(1) \cap y^{\perp}$ with $z \neq y$ and $z \not\perp x$. Let $w \in \mathcal{S}(1) \cap z^{\perp}$ with $w \notin y^{\perp}$. Then $w \in x^{\perp}$. So $w + z \in \mathcal{S}(2), (w + z) \cap x^{\perp} = w$ and $(w + z) \cap y^{\perp} = z$. But $w + z$ contains more than two 1-dimensional subspaces contradicting the assumption that $\mathcal{S}(1) = (\mathcal{S}(1) \cap x^{\perp}) \cup (\mathcal{S}(1) \cap y^{\perp})$.

(a3) By (a2) we can choose $z \in \mathcal{S}(1)$ with $z \not\perp x$ and $z \not\perp y$. Thus x is adjacent to z and z is adjacent to y, proving (a3).

(b) Without loss N is not a non defective quadratic space. Let $U_1, U_2 \in \mathcal{S}(1)$ with $U_1 \neq U_2$. We need to show that U_1 and U_2 are conjugate under $\langle T_U | U \in \mathcal{S}(1)\rangle$. By (a3) we may assume that U_1 and U_2 are not perpendicular. Put $R = U_1 + U_2 + N^{\perp}$. Then R itself is a non degenerate, pseudo quadratic space which is not non-defective quadratic. So we may assume $N = R$. By (a1) U_1 and U_2 are conjugated under $\langle Q_U | U \in \mathcal{S}(1)\rangle$. Let $U \in \mathcal{S}(1)$. As $N/N^{\perp}$ is 2-dimensional, $U^{\perp} = U + N^{\perp} = U^{\perp\perp}$ and so $T_U = Q_U$. Thus (b) holds. □

TU

Lemma 3.6 *Let (N, q, f) be a nondegenerate pseudo quadratic space and $0 \neq U$ a finite dimensional singular subspace of N. Then*

(a) $T_U = \langle T_U \cap Q_E | E \in \mathcal{S}(1) \cap U\rangle$

(b) $Q_U = \langle Q_U \cap Q_E | E \in \mathcal{S}(1) \cap E\rangle$

(c) $T_U = \langle T_E | E \in \mathcal{S}(1) \cap E\rangle$*, unless N is a non-defective quadratic space.*

(d) *Let $0 \neq E \leq U$. Then $C_{Q_U}(E^{\perp}/E) = Q_U \cap Q_E$ and $Q_U/Q_U \cap Q_E \cong C_{E^{\perp}/E}(U^{\perp}/U) \cap C_{E^{\perp}/E}(U/E)$*

Proof: Let $u_1, \ldots, u_i$ be a basis for U. We use the correspondence between T_U and certain $i \times i$ matrices established in the proof of 3.4f without further reference.

(a) Let $1 \leq j \leq i$. Then $T_U \cap Q_{Ku_j}$ corresponds to the set of matrices with $m_{kl} = 0$ for all $1 \leq k, l \leq i$ with $k \neq j \neq l$. This implies (a).

(b) Let $E \in \mathcal{S}(1) \cap U$ and $\alpha \in \mathrm{Hom}_K(U^\perp, U)$ with $U^{\perp\perp}\alpha = 0$ and $U^\perp\alpha \leq E$. Extend α to an element β of $\mathrm{Hom}_K(E^\perp, E)$. By 3.4b, there exists $q \in Q_E$ with $\beta = q - 1|_{E^\perp}$. Then clearly $q \in Q_U \cap Q_E$ and $\alpha = q - 1|_{U^\perp}$. Thus

$$Q_U = \langle Q_U \cap Q_E | E \in \mathcal{S}(1) \cap U \rangle T_U$$

and so (b) follows from (a).

(c) Without loss N is not a non-defective quadratic space. Thus $K_- \neq 0$ or $N^\perp \neq 0$. Thus there exists $0 \neq \rho \in K$ with $\bar{\rho} \in q(N^\perp)$. Since f vanishes on $N^\perp$, 3.1 implies $\rho + s\rho^\sigma = 0$. Let λ be an arbitrary element in K and put $\mu = -\lambda\rho^{-1}s\lambda^\sigma$. Then

$$\mu = -\lambda\rho^{-1}(s\rho^\sigma)\rho^{-\sigma}\lambda^\sigma = (\lambda\rho^{-1})\rho((\lambda\rho^{-1})^\sigma$$

and so $\overline{\mu} = (\lambda\rho^{-1}) * \overline{\rho} \in q(N^\perp)$.

Note that

$$\begin{pmatrix} 0 & -s\lambda^\sigma \\ \lambda & 0 \end{pmatrix} = \begin{pmatrix} \rho & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \mu \end{pmatrix} + \begin{pmatrix} -\rho & -s\lambda^\sigma \\ \lambda & \mu \end{pmatrix}$$

and $-\lambda\rho^{-1} \cdot (-\rho, -s\lambda^\sigma) = (\lambda, -\mu)$. In particular all three matrices on the right side of the above equation correspond to transvections and it is easy to see that (c) holds.

(d) Let F be a complement to $E^\perp$ and $X = (E+F)^\perp$. Then clearly $C_{O(X)}(X \cap U^\perp / X \cap U) \cap C_{O(X)}(U \cap X) \cong C_{Q_U}(E+F)$ and (d) is easily verified. □

*connected*1

Lemma 3.7 *Let (N, q, f) be a nondegenerate pseudo quadratic space such that $\mathcal{S}(2) \neq \emptyset$, and define $x, y \in \mathcal{S}(2)$ to be adjacent if $x \cap y^\perp = 0$. Then the corresponding graph on $\mathcal{S}(2)$ is connected, unless (N, q, f) is a four dimensional quadratic space.*

Proof: Without loss (N, q, f) is not a four dimensional quadratic space. Let $x \neq y \in \mathcal{S}(2)$.

Consider first the case that $U = x \cap y \neq 0$. By 3.5a2 applied to $U^\perp/U$ in place of N there exists $z \in \mathcal{S}(2)$ with $U \leq z$, $z \not\perp x$ and $z \not\perp y$. Let $F \in \mathcal{S}(1) \cap z$ with $F \neq U$ and choose $S \in \mathcal{S}(1) \cap F^\perp$ with $S \not\perp U$. Then $F + S \in \mathcal{S}(2)$.

We claim that $F + S$ is adjacent to x. Suppose not. Then $T = x \cap (F+S)^\perp \neq 0$. Since $U \not\perp S$, $U \not\leq T$. Hence $Z = F + U \perp T + U = x$ and $x \perp z$, contradicting our choice of z. Similarly $F + S$ is adjacent to y and so x and y lie in the same connected component.

Next consider the case with $x \cap y = 0$. Let $P_1 \in x \cap \mathcal{S}(1)$ and $P_2 \leq y \cap P^\perp \cap \mathcal{S}(1)$. Then $P_1 + P_2 \in \mathcal{S}(2)$. By the previous paragraph x, $P_1 + P_2$ and y lie in the same connected component and the lemma is proved. □

*connected*2

Lemma 3.8 *Suppose (G, Σ) fulfills Hypothesis' (A),(B) or (C). Then graph on Σ is connected.*

Proof: If (B) holds this is 3.5a3 and if (C) holds this is 3.7. So suppose (A) holds. Let Z_1 and Z_2 be in Σ. Then it is readily verified that there exists Z_3 in Σ such that Z_3 is adjacent to Z_1 and Z_2. $\square$

$Q_U in G$

Lemma 3.9 *Suppose (G, Σ) fulfills Hypothesis (B) or (C).*

(a) *Let $0 \neq U$ be a finite dimensional singular subspace of N. Then $Q_U \leq G$.*

(2) *Let $U \in \mathcal{S}(1)$. Then $C_G(U)$ acts transitively on the 1-dimensional singular subspaces of $U^\perp/U$.*

Proof: (a) Let $U \in \mathcal{S}(1)$. Pick $x \in N \setminus U^\perp$. By 3.4 the map

$$\alpha: \ Q_U/Z_U \to U^\perp/U, \ \ qZ_U \to [x,q] + U$$

is an isomorphism. For $V \in \mathcal{S}(2)$ with $U \leq V$ define $Z(U,V) = Z_V \cap Q_U$. Then under the isomorphism in 3.4e (where we choose the first basis vector for V iin U), $Z(U,V)$ corresponds to the 2×2 matrices M with $M^T = -T^\sigma$, $m_{11} \in K_-$ and $m_{22} = 0$. It follows that $\alpha(Z(U,V)/Z_U) = V/U$. By 3.3a, $U^\perp/U$ is spanned by its 1-dimensional singular subspaces and so

$$Q_U = \langle Z(U,V) | U \leq V \in \mathcal{S}(2) \rangle.$$

We claim that $Z_V \leq G$. Indeed, in case (C) $Z_V \in \Sigma$ and in case (B) 3.6(c) yields $Z_V \leq T_V \leq \langle T_E | E \in \mathcal{S}(1) \rangle \leq \langle \Sigma \rangle$. In particular, $Z(U,V) \leq G$ and so $Q_U \leq G$. Thus (a) follows from 3.6b.

(b) Suppose first that Hypothesis (B) holds. Then by 3.5b applied to $U^\perp/U$, $\langle T_E | E \leq \mathcal{S} \cap U^\perp \rangle$ acts transitively on the singular 1-spaces of $U^\perp/U$.

Under Hypothesis (C) $U^\perp/U$ is at least three dimensional and so by 3.5a1 (applied to $U^\perp/U$) and 3.6d we get that $\langle Q_V | U \leq V \in \mathcal{S}(2) \rangle$ acts transitively on the singular 1-spaces of $U^\perp/U$. Thus (b) holds also in this case. $\square$

$Qtransitive$

Lemma 3.10 *Suppose that (G, Σ) fulfills Hypothesis (A).*

(a) *Let $0 \neq x \in N$ and $X = Kx$. Then $Q_X \cap G = \{t(\phi, x) | \phi \in \tilde{N}, x\phi = 0\}$, and $C_G(x)$ acts transitively on $Q_X^\# \cap G$. Moreover, $Q_X \cap G$ acts transitively on the 1-dimensional subspaces of $\tilde{N}$ outside of $C_{\tilde{N}}(x)$.*

(b) *Let $0 \neq \phi \in \tilde{N}$ and $\Phi = K\phi$. Then $Q_X \cap G = \{t(\phi, x) | x \in \ker \phi\}$ and $C_G(x)$ acts transitively on $Q_\Phi^\# \cap G$. Moreover, $Q_\Phi \cap G$ acts transitively on the 1-dimensional subspaces of N outside of $\ker \phi$.*

(c) *Let $Z \in \Sigma$ and Q be the stabilizer in G of the series $0 \leq [N, Z] \leq C_N(Z) \leq N$. Then Q acts transitively on the set of all $Z_0 \in \Sigma$ adjacent to Z.*

Proof: (a) Let $1 \neq t \in Q_X \cap G$. The clearly x is a transvection with axis Kx and so $t = t(\phi, x)$ for some $\phi \in N^*$ with $x\phi = 0$. Furthermore, $\phi \in [N^*, t] \leq [N^*, G] \leq \tilde{N}$ and so first part of (a) holds. $T(C_{\tilde{N}}(x), N)$ acts transitively on the nonzero vectors of $\tilde{N}$ and so also on $Q_X^\# \cap G$. Finally, let $\phi_1, \phi_2 \in \tilde{N}$ with $x\phi_1 = 1 = x\phi_2$. Then $\phi_1.t(\phi_1 - \phi_2, x) = \phi_2$ and (a) is proved.

(b) Follows by a dual argument or by observing that $T(\tilde{N}, N) = T(N, \tilde{N})$ if we identify N with its copy in $\tilde{N}^*$.

(c) Let $Z = T(\Phi, X)$ and $Z_i = T(\Phi_i, X_i) \in \Sigma$, $i = 1, 2$ with Z_i adjacent to Z. We have to show that Z_1 and Z_2 are conjugate under Q. Since Z_i is adjacent to Z, $X\Phi_i \neq 0$ and $X_i\Phi \neq 0$. Note that $Q_\Phi \leq Q$ and so by (b) we may assume that $X_1 = X_2$. Since $X_i\Phi_i = 0$, the element of $t(\phi_1 - \phi_2, x)$ (found in (a)), conjugates Φ_1 to Φ_2 and fixes $X_1 = X_2$. This proves (c). □

$Q = xCGU$

Lemma 3.11 *Suppose that (G, Σ) fullfils Hypothesis' (A),(B) or (C). Let $Z \in \Sigma$, U a 1-dimensional subspace of $[N, Z]$ and $Q = Q_U \cap G$. Let $x \in Q \setminus Q'$ such that $[N, x]$ is singular. Then $Q = \langle x^{C_G(U)} \rangle$, unless* $\dim N/N^\perp = 4$ *and $q(N^\perp) \neq K/K_-$.*

Proof: Under Hypothesis (A) this follows directly from 3.10a.

So suppose Hypotesis' (B) or (C). Let $0 \neq y \in [N, x] + U/UU$ and A the subgroup of $U^\perp/U$ generated by $y.C_G(U)$. In view of 3.4d,h $Q = \langle x^{C_G(U)} \rangle$ if and only if $A = U^\perp/U$. By 3.9 it suffices to show that A contains a singular 1-space. Also A spans $U^\perp/U$ as a K-space and so f does not vanish on A. Let $z \in A \setminus y^\perp$ and $E = [N, x]U$. By 3.6d, Q_E acts as $Q_{E/U}$ on $U^\perp/U$.

Suppose that $A \cap y^\perp \not\leq E + \mathrm{rad}N/U$. Then by 3.4b, $E/U \leq [A \cap y^\perp, Q_E] \leq A$ and the lemma holds.

So we may assume that $A \cap y^\perp \leq E + \mathrm{rad}N/U$. In particular, $[z, Q_E] \leq E + \mathrm{rad}N/U$ and so by 3.4b, $E^\perp = E + \mathrm{rad}N$. Hence $\dim N/N^\perp = 4$. If $q(N^\perp) = K/K_-$ then by 3.4f, $[z, Q_E] + \mathrm{rad}N + U/U = E + \mathrm{rad}N/U$. Thus $A + \mathrm{rad}(U^\perp/U) = U^\perp/U$. Hence $[(U \perp /U, Q_E] \leq A$. If $|K| = 2, \{0, y\}$ is a singular 1-space and we may assume that $|K| \neq 2$. But then $[(U^\perp/U, Q_E] = E + \mathrm{rad}N/U$, and the lemma is proved. □

$perfect$

Lemma 3.12 *Suppose that Hypothesis (A), (B) or (C) holds, Then G is perfect.*

Proof: Let

□

4 The structure of $[V, G]/[V, G] \cap C_V(G)$.

Throughout this section (G, Σ) fulfills Hypothesis (A),(B) or (C), $Z \in \Sigma$, $L = C_G([N, Z]$ and V is an RG-modules with $[V, Z, L] = 0$. Let $Z_0 \in \Sigma$ be adjacent to Z and let U be a 1-dimensional singular subspace of $[N, Z]$. Put $X = \langle Z, Z_0 \rangle$, and $P = C_G(U)$.

$o + 4$

Lemma 4.1 *Suppose that Hypothesis (C) holds. The there exists $g_1 \in P$ so that the following holds for $X_1 = X^{g_1}$ and $L^* = LX_1$.*

(a) $[X, X_1] = 1$ *and* $[N, X] = [N, X_1]$.

(b) X_1 *nomalizes* $[N, Z]$ *and* L. *In particular* L^* *is a subgroup of* $N_G([N, Z])$.

(c) $[N, Z].g_1 = U + [N, Z_O] \cap U^\perp$, $[N, Z_0].g_1 = [N, Z] \cap [N, Z_0].g_1 + [N, Z_0] \cap [N, Z_0].g_1$.

Proof: Let U_1 be a singular 1-space in $[N, Z]$ different from U. Put $U_2 = [N, Z_0] \cap U^\perp$ and $U_3 = [N, Z_0] \cap U_1^\perp$. Then $U + U_1 + U_2 + U_3 = [N, Z] + [N, Z_0] = [N, X]$ is a 4 dimensional quadratic space "+"-type. Let $E = [N, Z] = U + U_1$, $E_0 = [N, Z_0] = U_2 + U_3$, $E_1 = U + U_2$, $E_2 = U_1 + U_3$, $Z_i = T_{E_i}, i = 1, 2$ and $X_1 = \langle Z_1, Z_2 \rangle$. Note that for $z \in Z$ and $n \in N$, $n.z$ is perpendicular to n. Hence $[U_1, Z_1] \leq U_1^\perp \cap [N, Z_1] = U_1^\perp \cap (U + U_2) = U$. It follows that Z_1 normalizes E and so centralizes Z. By symmetry, also Z_2 normalises E and centralizes Z. So (b) holds. Morover, X_1 centralizes Z and (by symmetry) Z_O. Thus (a) holds. To complete the proof of this lemma it now suffices to find $g_1 \in P$ with $Z^{g_1} = Z_1$ and $Z_O^{g_1} = Z_2$, i.e with $E.g_1 = E_1$ and $E_O.g_1 = E_2$.

By 3.9b there exists $g \in P$ with $E.g = E_1$. Since $E^\perp \cap E_0^= 0$, $E_1^\perp \cap E_0.g = 0$. Since also $E_1 \perp \cap E_2 = 0$ we conclude from 3.3b that $E_0.gq = E_2$ for some $q \in Q_{E_1}$. Put $g_1 = gq$. As g_1 centralizes E_1, we get $g_1 \in P$, $E.g_1 = E_1$ and $E_O.g_1 = E_2$. □

$G = LZ0$

Lemma 4.2 $G = \langle L, Z_0 \rangle = \langle L, X \rangle$.

Proof: Let $\Sigma_0 = Z^{\langle L, Z_0 \rangle}$. By 3.3b, 3.9a and 3.10, Z_0^L contains all elements in Σ adjacent to Z. Further, if Z_1 and Z_2 are adjacent in Σ, then they are conjugated in $\langle Z_1, Z_2 \rangle$. It follows that Σ_0 contains the connected component of Σ which contains Z. So 3.8 implies $\Sigma_0 = \Sigma$. Since $G = \langle \Sigma \rangle$, $G = \langle L, Z_0 \rangle$. □

$[W, Z] = W0$

Lemma 4.3 *Let* $B = N_X(Z)$ *and* W_0 *an* $R(LB)$*-submodule of* V *with* $[W_0, L] = 0$. *Put* $W = \langle W_0.X \rangle$. *Then*

$C_W(X) \leq C_V(G)$, $[W, Z] + C_W(G) = W_0 + C_W(G)$ *and* $W = [W, X] + C_W(G)$.

Proof: By 2.2 we have $C_W(X) \leq W_0 + [W, Z]$. Since $[V, Z, L] = 0$ this implies $[C_W(X), L] = 0$. By 4.2, $\langle L, X \rangle = G$ and so $C_W(X) \leq C_V(G)$. The other assertions now follow from 2.2. □

$[M, Z] = M1$

Lemma 4.4 *Let* M_0 *be an* $RN_G(U)$*-submodule of* V *with* $[M_0, P] = 0$. *Put* $M_1 = M_0$, *in cases (A) and (B), and* $M_1 = \langle M_0.L^* \rangle$,*in case (C). Put* $M = \langle M_0.G \rangle$. *Then* $[M, Z] + C_M(G) = M_1 + C_M(G)$.

Proof: Let $g \in G$. If $[U.g, Z] = 0$, then $Z \leq P^g$ and $[M_0 g, Z] = 0$.

If $[Ug, Z] \neq 0$, we claim that there exists $\omega \in N_X(Z)$ and $h \in L \cap C_G(Z)$ (or, in case (C), $h \in L^*$) with $Ug = U\omega h$. Indeed, in case (A) this follows from 3.10b. In case (B) $U.g$ and $U.\omega$ both are not perpendicular to U and the claim follows from 3.3b. In case (C) we first choose $h_1 \in L_*$ with $Ugh_1 \perp U$. Then both $U.gh_1$ and $U.\omega$ are perpendicular to U and neither Ugh_1 nor $U\omega$ are perpendicular to $[N, Z]$. Thus by 3.6d and 3.3b (the latter applied to $U^\perp/U$) we get $U.gh_1 q = U.\omega$ for some $q \in Q_{[N,Z]}$. This proves the claim. In particular,

$$[M_0.g, Z] = [M_0.\omega h, Z] = [M_0.w, Z]h.$$

By 4.3 we have

$$[M_0.\omega, Z] + C_M(G) = M_0 + C_M(G).$$

Thus $[M_0g, Z] + C_M(G) = M_0h + C_M(G)$ and $M_0h + C_M(G) \leq [M, Z] + C_M(G) \leq M_1 + C_M(G)$. Since L^* normalises $[M, Z]$ the lemma is established. $\square$

CVLG

Lemma 4.5 (a) *$\langle C_V(L).X\rangle = [C_V(L), X] + C_V(G)$.*

(b) *$\langle C_V(L).G\rangle = [V, G] + C_V(G)$.*

(c) *If $V = \langle C_V(L).G\rangle$, then $[V, G] = [V, G, G]$.*

(d) *$[V, G, G, G] = [V, G, G]$.*

(e) *Let D be maximal in V with $[V, G, G] = 0$, then $C_{V/D}(G) = 0$.*

Proof: (a) follows immediately from 4.3 applied to $W_0 = C_V(L)$.

(b) By (a) $\langle C_V(L).G\rangle \leq [V, G] + C_V(L)$. Now $[V, G] = \langle [V, Z].G\rangle \leq \langle C_V(L).G\rangle$ and (b) is proved.

(c) follows from (b). (d) from (c) applied to $[V, G]$ in place of V and (e) from (d) applied the inverse image of $C_{V/D}(G) = 0$ in V. $\square$

VQQ

Lemma 4.6 *Suppose that (C) holds. Then*

(a) $[C_V(L), Q_U, P] = 0$

(b) $[V, Q_U, Q_U, P] = 0$ and $[V, Q, Q] = [V, Q, Z]$.

(c) $[V, Z, L^*] \leq \langle C_V(P)G\rangle$

Proof: Let $g \in P$ with $[N, Z]g \not\leq [N, Z]$. Then by **??** $P = \langle L, L^g\rangle$ and $Q_U = Z^g(Q_U \cap L)$. Thus

$$[V, Z, Q_U] \leq [C_V(L), Q_U] = [C_V(L), Z^g] \leq C_V(\langle L, L^g\rangle \leq C_V(P)$$

. Thus (a) is proved. Now (b) follows from $Q_U = \langle Z^P\rangle$ and (c) from $L^* = \langle Q_U^{L^*}\rangle L$. $\square$

*VZL**

Lemma 4.7 *Suppose that (C*) holds and $[V, Z, L^*] = 0$. Then $[V, G] = 0$*

Proof: By 4.5 we may assume that $C_V(G) = 0$. Let $D = [V, Z]$. Since $G = \langle Z^G\rangle$ we need to show that $D = 0$. Suppose not.

Assume first that char $K \neq 2$. Then $Z(X) = Z(X_1) \leq L_*$. Thus $[D, Z(X) = 0$ and so we conclude from 2.3 that $[D, X] = 0$. By 2.3, $D \leq C_V(G) = 0$ and we are done in this case.

Assume next that char $K = 2$. We may assume without loss that $V = \langle DG\rangle$. We will first prove

(1) $V, Q_U, Q_U] = 0$.

Indeed by 4.6, $[V, Q, Q] = [V, Q, Z]$ is centralized by P and by L^* and so by G.

(2) If N has Witt index at least three, then N is nondefective of dimension 6 and $|K| = 2$.

Pick $E \in calS(2) \cap [N, Z]^{\perp}$ with $E \cap U = 0$. Then $Z_E \leq P$ and so $[C_V(Q_U), Z_E]$ is centralized by $Y = \langle Q_U, L^*E$. Hence if $\dim N > 6$ we conclude from **??** that $Y = G$ and $C_V(G) = 0$i mplies $[C_V(Q_U), Z_E$=0. Since $[V, Z] \leq C_V(Q_U)$, $[V, Z]$ is centralized by $\langle Z_E^P \rangle$ and L^* and so by G. To show that K=2,pick $Z_2 \in P \cap \Sigma$ adjacent to Z_E. Put $S = \langle Z_E, Z_2$. Note that $N_S(Z_E) \leq L^*$ and so centalizes D. Suppose that $|K| \neq 2$. Then $Z_E = [Z_E, N_S(Z_E)]$ and so by 2.2d, S centralise D. But $\langle S, L^* \rangle = G$ and (2) holds.

(3) N has Witt index at least three.

Suppose that N has Witt index 2. Since $Q_{[N,Z} = (Q_{[N,Z} \cap Q_U)(Q_{[N,Z]} \cap Q_U)^g$ for $ginL^* \setminus N_G(U)$ we conclude from (1) that $[C_V(Q_U), Q_{[N,Z]}, Q_{[N,Z]}] = 0$ and we can apply 2.2 to $\langle D^P \rangle$. Since L cenralizes D and since by **??** $Q_E Q_U = [Q_E, L]Q_U$ we conclude from 2.2d that $\langle Q_U^P \rangle$ centralizes $[D]$. So again $[D, G] = 0$.

$CVPG$

Lemma 4.8 *Suppose that (A),(B) or (C*) holds. Then* $\langle C_V(P)G \rangle = [V, G] + C_V(G)$.

Proof: If (A) or (B) holds this is 4.5. So suppose that (C*) holds. Then by the same reference, $C_V(P) \leq C_V(L) \leq [V, G] + C_V(G)$. Moreover, by 4.6c and 4.7 (applied to $V/\langle C_V(P)G \rangle$), $[V, G] \leq \langle C_V(P)G \rangle$.

For the case (B) we need to define a few more subgroups of G. Let $U_1 \in (2)$ with $U \leq U_1$. Let $U2 \in \mathcal{S}(2)$ with $U_1 \cap U_2^{\perp} = 0$ and put $F = U + (U_2 \cap U^{\perp}$. Define $\tilde{X} = C_{\mathcal{O}}((U1{+}U2)^{)} \cap N_{\mathcal{O}}(U_1) \cap N_{\mathcal{O}}(U_2)$, $Z^* = C_{\tilde{X}}(F)$, and $X^* = \langle Z{*}^{\tilde{X}} \rangle$. Note that $X^* \cong SL_2(K)$ and Z^* is a maximal unipotent subgroup of X^*. Moreover, $Z^* \leq Z_F \leq Q_F$ and so by 3.9 both Z^* and X^* are contained in G.

$VZ*Z*$

Lemma 4.9 *Suppose that (B) holds.*

(a) $[V, T_F, C_G(F)) = 0$. *Inparticular,* $[V, Z^*, Z^*] = 0$.

(b) $G = \langle X^*, C_G(F) \rangle$.

(c) 4.3 still holdsif X, Z and L are replaced by X^*, Z^* and $C_G(F)$, respectively.

Proof: By 3.6, $T_F = \langle T_E | E \in \mathcal{S}(1) \cap F \rangle$. Futhermore, $C_G(F) \leq C_G(E) \leq C_G([V, T_E]$ for all E in $\mathcal{S}(1) \cap F$. Thus (a) holds.

(b) $G_0 = \langle X^*, C_G(F) \rangle$ and $\Lambda = FG_0 \subseteq \mathcal{S}(2)$. Since FX^* contains elements adjacent to F (with respect to the graph defined in 3.7)3.3b implies that Λ containes all elements in $calS(2)$ adjacent to F. Since G_0 acts transitive on Λ, Λ is a connected component of $calS(2)$ and so by 3.7 $\Lambda = calS(2)$. Since $T_F \leq G_0$ and $G = \langle \Sigma \rangle$, $G = G_0$.

(c) Using (a) and (b) the proof for 4.3 goes through.

Theorem 4.10 *Suppose that $V = \langle C_V(P)^G \rangle$. Then there an (R,K)-module M such that $V/V_V(G) \cong M \otimes_K N/N^\perp$ as an RG -module.*

Proof: By4.5 we may assume that $C_V(G) = 0$. Let $P_0 = NG(U)$. In case (B) define X^* and Z^* as above. In cases (A) and (C) let $X^* = X$ and $Z^* = Z$. Then $X^* \cap P_0$ acts transitively on $U^\#$ and therefore $P_0 = (X \cap P_0)P$.

Let $W = C_V(P)$ and $W_1 = \langle WX^\rangle$. It is easily checked that $X \leq L, X^*\rangle$ and so by 4.2, $G = \langle L, X^*$. Note that by 4.9b Z^* acts quadratically on W_1. By 4.3 and 4.9c $C_{W_1}(X^*) \leq C_V(G) = 0$, $W_1 = [W_1, X*]$ and $W = [W_1, Z^*]$. Let $U_1 = \langle UX*\rangle$. Then U_1 is a natural module for $X* \cong SL_2(K)$,$U = [U_1, Z^*]$ and $C_{X^*}(U) = P \cap X*$. Hence $[W, C_{X^*}(U)] = 0$ and $[W_1, Z*, C_{X^*}(U)] = 0$. Therefore we can apply 2.3 to X^* and W_1 and find an (R,K)-module M such that

(1) $W_1 \cong M \otimes_K U_1$ as an RX^*- module.

Let $N_0 = M \otimes K(N/N^\perp)$. We will show that N_0 and V are isomorphic as RG- modules. $P_0 = (X^* \cap P_0)P$ and (1) imply that $M \otimes_K U$ and W are isomorphic RP_0-modules. Let M be the RG-module induced from the RP_0-module W. Let

$$\tilde{M} = M/\langle [M, Z, L]^G \rangle \text{ and } \bar{M} = \tilde{M}/C_{\tilde{M}}(G).$$

By assumption $V = \langle WG \rangle$. Furthermore, $N_0 = \langle M \otimes_K G \rangle$ and so by the universal property of induced modules there exist RG-epimorphisms:

$$\phi_1 M \to V \text{ and } \phi_2 M \to N_0.$$

Since $[V, , Z, L] = 0$ and $[N_0, Z, L] = 0$ we get that $\langle [M, Z, L] \rangle \leq \ker \phi_i$ for i=1,2. Further, $C_V(G) = 0$ and $C_N 0(G) = 0$, the latter being true since N_0 as a $\mathbb{Z}G$ module is the direct sum of copies of $N/N^\perp$. Thus ϕ_1 and ϕ_2 induce RG-epimorphism

$$\overline{\phi_1} M \to V \text{ and } \bar{\phi_2} M \to N_0.$$

We now prove that

(2) $\ker \overline{phi_i} \cap [\overline{M}, Z] = 0$.

For this note first that by 4.4 applied to $[\overline{M}, Z] = \langle \overline{W} L^* \rangle$, where we identified W with its canonical image in M. Obviously $\overline{\phi_i}$ restricted to W is one to one. So if $[I, Z] = W$, (2) is proved. Otherwise (C) holds. Recall the definition of $X - 1$ and g_1 at the beginning of this section.Then

$$[\overline{I}, Z] = \langle WL^* \rangle = \langle WX_1 \rangle \langle WX \rangle g_1 = \langle W_1 \rangle g_1$$

Now $[W_1 g_1, Z_1^g] = W g_1 = W$ and so $[W_1 g_1, Z] \cap \ker \overline{\phi_i} = 0$, $[\ker \overline{\phi_i} \cap W_1 g_1, Z] = 0$ and $\ker \overline{\phi_i} \cap W_1 g_1 \leq C_{W_1 g_1}(X_1) = 0$. This proves (2).

By (2) we get that $[\ker \overline{\phi_i}, Z] = 0$ and so $\ker \overline{\phi_i} \leq C_{barM}(G)$. By 4.5c, $C_{\bar{M}}(G) = 0$ and thus $\overline{\phi_i}$ is one to one. Hence

$$V \cong \overline{M} \cong N_0 \text{ as RG-modules,}$$

Theorem 4.10 is established.

5 Determination of $[V, G] \cap C_V(G)$

Retain the assumptions and notation from the previous section. This section is entirely devoted to the proof of

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Proposition 5.1 *Suppose that $V/C_V(G) \cong M \otimes_K N/N^\perp$ for some (R, K)-module M and that $V = [V, G]$. Then one of the follwing holds*

(a) There exists an R-submodule $C \leq M \otimes N^\perp$ such that $V \cong M \otimes_K N/C$ as RG-modules.

(b) $|K| = 4$, $\sigma \neq id$ and $\dim N = 4$.

Proof: Recall the notations introduced in 2.5 and 2.7. Then we are trying to proof that $M \otimes_K N$ is a universial central $\mathcal{C}$ extension, where $\mathcal{C}$ is the class of RG-modules W with $[W, Z, L] = 0$. In view of 2.7 we may assume without loss that $V/C_V(G) \cong N/N^\perp$. We first prove

(1) There exists an RG-module W and RG-submodules C_1 and C_2 of $C_W(G)$ such that $W/C_1 \cong V$,$W/C_2 \cong N$ and $[W, Z, L] = 0$.

Let μ be an RG-isomorphism from $V/C_V(G)$ onto $N/N^\perp$. Put $W = \{(v, n) | v \in V, n \in N, \mu(v + C_V(G)) = n + N^\perp\}$, Let $C_1 = \{0\} \times N^\perp$ and $C_2 = C_V(G) \times \{0\}$. Then C_i is the kernel of the projection of W onto the i'th coordinate and so $W/C_1 \cong V$ and $W/C_2 \cong N$.

In view of (1) we may assume that $C_V(G)$ has a submodule C such that $V/C \cong N$. Pick $x \in Q_U \setminus T_U$ such that $[N, x]$ is singular. Put $\bar{V} = V/C_V(G)$ and $A = [V, P, Q_U]$. Note that $\bar{V} \cong N/N^\perp$.

(2) $C_{\bar{V}}(x) = \overline{C_V(x)}$.

Let R be maximal in G with respect to acting trivially on $[\bar{V}, x], C_{\bar{V}}(x)/[\bar{V}, x]$ and $\bar{V}/C_{\bar{V}}(x)$. Then by 3.4b and3.10a, $[\bar{V}, R] = C_{\bar{V}}(x)$. Note that $[V, x, R, Z]$=0. Indeed, if x is contained in an element of Σ, then this follows from $[V, Z, L] = 0$ and if not, (B) holds and it follows from 4.9a. Moreover, $[R, Z] = 0$ anf the three subgroup lemma implies $[V, R, x] = 0$.

(3) Asssume that (B) holds and $|K| \neq 2, 4$. Then $[V, Z^*] \cap C_V(G) = 0$.

Recall the definitions of $X^*, \tilde{X}$ and Z^* (see before **??**). It is enough to prove that $[V, X^*] \cap C_V(G) = 0$. By (2), $V = [V, X^*]C_V(X^*)$ since the same statement holds for $\bar{V}$ in plcae of V. Thus if K is not commutative or char $K \neq 2$ we are done by 2.4 So assume that K is commutative. Since $|K| \neq 2, 4$ there exist $\lambda\mu \in K\setminus\{0\}$ so that $\lambda\mu \neq 1$ and $\lambda^2 = \mu^{-1}\mu^\sigma$. This in turn yields an element $1 \neq h \in X^*$ with $[h, X^*] = 1$ and acting as $\lambda\mu$ on one of the 2-dimensional singular subspaces of $[N, X^*]$ normalised by X^*. Now $[[V, X^*], h] \cap C_V(G) = 0$, $[V, X^*, X^*] \leq [[V, X^*], h]$ and since X^* is perfect $[V, X^*] \cap C_V(G) = 0$. So (3) is proved.

(4) $A \cap C_V(G) = 0$, except possibly in the case (B), $|K| = 4$, $\dim N = 4$ and $\sigma \neq id$.

Suppose first that $Q = \langle x^P \rangle$. Since $[V, P, x, P] = 0$ we get that $[V, P, Q] = \langle x^P \rangle = \langle [V, P, x] \rangle = [V, P, x]$. Now by (1), $[V, x] \cap C_V(G) = 0$ and (4) holds in this case. Suppose next that the hypothesis of (3) holds. Note that by **??**, $Q = \langle Z^{*P} \rangle$. So replacing x by Z^* in the preceeding argument shows that (4) holds also in this case. Now by **??** we have covered all cases but the one excluded in (4).

(5) If $A \cap C_V(G) = 0$, then $C = 0$.

Let $v \in V \setminus ([V, P] + C_V(G))$. We claim that $[v, Q] \cap (C_V(G) + A) \leq A$. Suppose not and put $\tilde{V} = V/A$. Since $[\tilde{V}, Q, Q] = 0$, we have $[V, Q'] = 0$ and $[\tilde{v}, Q] = \{[\tilde{v}, q] | q \in Q]\}$. Hence there exists $q \in Q$ with $[\tilde{v}, q] \neq 0$ and $[v, q] \in C_V(G) + A$. Reading this equation modulo C and applying the "$Q' = T$"-statement of **??** we get that $q \in Q'$, a contradiction to $[\tilde{v}, q] \neq 0$.

Therefore $([v, Q] + A) \cap C_V(G) = 0$. Since P normalizes $[v, Q] + A$, we conclude that $[v, Q] + A = [V, P, P] = [V, Q]$ and $[V, Q] \cap C_V(G) = 0$. Let $g \in G$ with $U^g \not\leq U^\perp$. Then $V = [V, Q] \oplus C_V(G) \oplus A^g$. $P = (P \cap P^g)Q$ and $[A^g, P \cap P^g] = 0$ imply that $[V, P] = [V, Q]$. Moreover, $[V, Q] = A \oplus [V, P \cap P^g]$ and so $V = [V, G] = [V, P] + [V, P]^g = A \oplus [V, P \cap P^g] \oplus A^g$. Finally, this direct sum remains a direct sum modulo C and C intersects each of the summands trivally. This implies that C = 0, proving (5).

Theorem A is now a direct consequence of (4) and (5).

6 Proof of Theorem B

ProofOfTheorem
Reduction

Theorem 6.1 *Suppose (G, Σ) fulfills the hypothesis of Theorem B. Let $L_1 = C_G([N, Z])$, $L_2 = C_G([\tilde{N}, Z])$, $F = [V_0, Z]$, $F_i = C_F(L_i)$ and $V_i = \langle F_i^G \rangle$, $(i = 1, 2)$. Then for $i = 1, 2$ $[V_i, Z, L_i] = 0$ and $[V_0, G] = W_1 + W_2$.*

Proof: Since $\dim N > 2$ we can choose 1-dimensional subspaces U and $\tilde{U}$ of N and $\tilde{N}$, respectively, so that $\tilde{U}([N, Z]) = 0$, $\tilde{U}(U) = 0$, $[\tilde{N}, Z](U) \neq 0$ and $\tilde{U} \neq [\tilde{N}, Z]$. Let $Z_0 = T(\tilde{U}, [N, Z])$. Then $[V_0, Z_0]$ is centralized by $C_G(\tilde{U}) \cap C_G([N, Z])$ and, in particular, by $Q := C_G(\tilde{U}^\perp) \cap CG(\tilde{})$. Note that Z centralizes$\tilde{U}$ and therefore normalizes Q. Now $[V_0, Z_0] \leq C_{V_0}(Q)$ and so

$$[V_0, Z_0, Z] \leq C_{V_0}(Q).$$

By **??** applied to $L_1/C_{L_1}(C_{\tilde{N}}([N,Z])$ in place of G we have $\langle L_0, Z_0\rangle = L_1$. Hence $L_1 = \langle L_0, Q\rangle$ and

$$[V_0, Z_0, Z] \le C_{V_0}(L_0) \cap C_{V_0}(Q) \le C_{V_0}(L1).$$

Since $\langle Z_0^{L_0}\rangle$ is normalized by $\langle L_0, Z_0\rangle = L_1$, we have $L_1 = \langle Z_0^{L_0}\rangle$ and thus

$$[V_0, \langle Z_0^{L_0}\rangle, Z] = [V_0, L_1, Z] \le C_{V_0}(L_1)$$

By a symmetric argument $[V_0, L_2, Z] \le C_{V_0}(L_2)$. Furthermore,

$$[V_0, G] = [V_0, \langle L_1, L_2\rangle] = [V_0, L_1] + [V_0, L_2]$$

and thus It follows that $[V_0, G] = [V_0, G, G] = V_1 + V_2$.

To complete the proof of 6.1 it is enough to show that $[V_i, Z] \le F_i + C_{V_i}(G)$. A glance at the proofs of **??**, **??** and **??** shows that these lemmas hold with V replaced by V_i and L replaced by L_0. It follows that $[V_i, Z] \le F_i + C_V(G)$. So $[V_i, Z, L_i] = 0$ and 6.1 is established. □

To prove Theorem B we now merely have to apply Theorem A to the modules V_1 and V_2 of 6.1. Note here that we can view N as a subspace of the dual space of $\tilde{N}$ and that then $T(\tilde{,}N) = T(N, \tilde{N})$.

7 Finitary modules for Classical Groups

finitary

Remark: we need to be more precise

Suppose G is one of the groups in the introduction and that N is infinite dimensional over K. Furthermore, let W be a G-module over the integers such that $[W, g]$ has finite rank for all $g \in G$. In case (A) let L_0 be defined as in Theorem B, otherwise let $L_0 = L$. Then in any case $[Z, L_0] = 0$. Now it is well-known(?) that L_0 has no non-central Z-module of finite rang. Hence $[W, z, L_0] = 0$, for all $z \in Z$, and so $[W, Z, L_0] = 0$. Therefore we can apply our main theorems with R the ring of integers to see that $[W, G]/[W, G] \cap C_W(G)$ is a direct sum of natural modules. Thus Theorem C holds.

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