A Computer-free Construction of the Lyons Group

U. Meierfrankenfeld
Department of Mathematics
Michigan State University
East Lansing, Michigan 48824
USA
meier@math.msu.edu

and

C.W. Parker School of Mathematics and Statistics
University of Birmingham
Birmingham B15 2TT
UK
cwp@for.mat.bham.ac.uk

May 18, 2002

Abstract

A finite group L is of Ly-type provided that L is simple and has an involution z with $C_L(z)$ isomorphic to the double cover of Alt(11), $2{}^{\cdot}\mathrm{Alt}(11)$. In this paper we give a computer-free construction of a group of Ly-type.

1 Introduction

The first evidence for the existence of a sporadic simple group with centralizer of an involution isomorphic to the double cover of the alternating group of degree 11 was given by Lyons [Ly] in 1972. Subsequently, Sims [Si] constructed such a group and proved its uniqueness. His proof provides generators and relations for the group and uses a computer to determine the index of a subgroup which is isomorphic to $\mathrm{G}_2(5)$. The linear representations of the Lyons group such as the Meyer–Neutsch–Parker 111-dimensional GF(5) representation [MNP] or the Jansen–Wilson 651-dimensional GF(3) representation [JW] ultimately depend on the relations that Sims produced to show that the group their matrices generate is the Lyons group. In [CFYT] the 111-dimensional GF(5) matrix group was used to obtain permutations which generate the Lyons group and using these permutations a further computer based existence proof has been given by Gollan [Gol]. A more elementary, though still computer dependent, uniqueness proof can be found in Wilson [Wi]. In [AschS] Aschbacher and Segev gave the first computer-free uniqueness proof of a group of Ly-type. The purpose of this paper is to give a hand construction of a group of Ly-type. Thus our main theorem is

Theorem 1.1 *There exists a group of* Ly*-type.*

Formally, for a natural number n, we follow Ivanov [Iv] and define an **amalgam $\mathcal{A}$ of rank n** to be a set M such that for each $1 \le i \le n$ there are subsets M_i and a binary operation $*_i$ defined upon M_i such that the following hold:

(i) $M = \cup_{i=1}^n M_i$;

(ii) $\cap_{i=1}^n M_i \neq \emptyset$;

(iii) $(M_i, *_i)$ is a group; and

(iv) if $x, y \in M_i \cap M_j$, then $x *_i y = x *_j y$.

Generally, $\mathcal{A}$ is denoted by the n-tuple of groups $(M_1, \ldots, M_n)$. And, for $J \subseteq \{1, \ldots, n\}$, we use M_J to denote the subset $\cap_{j \in J} M_j$. A **completion** of the amalgam $(M_1, \ldots, M_n)$ is a group G and a mapping θ from $\cup_{i=1}^n M_i$ to G such that

(i) $G = \langle \theta(M) \rangle$;

(ii) for $1 \leq i \leq n$, $\theta|_{M_i}$ is a group homomorphism.

Note that every amalgam has the trivial group as a completion. Every amalgam also has a **universal completion**, this completion maps surjectively on to every other completion (see [Iv, 1.3.2]). Note that a non-trivial amalgam may have the trivial group as its universal completion. A completion is called **faithful** provided that θ is injective.

Our approach to the proof of Theorem 1.1 is similar to that followed by Ivanov and Meierfrankenfeld in [IM] when they constructed J_4: we show that the universal completion of a certain amalgam of groups is a group of Ly-type or is trivial. Then we prove that the given amalgam exists in $\mathrm{GL}_{111}(5)$.

The amalgams that we shall be concerned with are defined as follows:

Definition 1.2 *An amalgam (M_1, M_2, M_3) is called a* Ly**-amalgam** *provided that*

1. $M_1 \sim 3.\mathrm{McL}.2$, $M_2 \sim 3^6.2^3.\mathrm{Sym}(5)$ *and* $M_3 \sim 3^5.2.\mathrm{Mat}_{11}$.
2. $|M_2 : M_{12}| = 2$, $|M_2 : M_{23}| = 10$ *and* $|M_3 : M_{13}| = 11$.
3. $|M_{23} : M_{123}| = 2$.
4. *No non-trivial subgroup of M_{123} is normal in M_1, M_2 and M_3.*

Notice that, *a priori*, there may be a multitude of Ly-amalgams up to isomorphism.

Our main theorem follows from the more specific result

Theorem 1.3 *Suppose that $\mathcal{T}$ is a* Ly*-amalgam. Then*

(a) *$\mathcal{T}$ is unique up to isomorphism.*

(b) *There exists an amalgam isomorphic to $\mathcal{T}$ in* $\mathrm{GL}_{111}(5)$.

(c) *The universal completion of $\mathcal{T}$ is a group of* Ly*-type.*

In particular, there exists a group of Ly*-type.*

The definition of a Ly-amalgam is motivated by [Ly, Propositions 2.3, 2.5 and 2.7] from which it can be deduced that, if there exists a group of Ly-type, then it is a completion of a Ly-amalgam. Using this fact and Theorem 1.3 we obtain

Corollary 1.4 *There exists a unique group of* Ly*-type.*

The first five sections of this article contain the preparatory results needed for the proof of 1.3. Specifically, Section 2 contains general results about amalgams, information about the 5-dimensional $\mathrm{GF}(3)\mathrm{Mat}_{11}$-modules and facts about representations of extraspecial groups. Section 3 contains a characterization of the alternating groups, 3.2, which will be used in Section 7 to show that, if there is a faithful completion of a Ly-amalgam, then it contains a subgroup isomorphic to $2{}^{\cdot}\mathrm{Alt}(11)$. In Section 4 we gather a myriad of facts about McL and Aut(McL). Section 5 is given over to the construction of a group $H \cong 3{}^{\cdot}\mathrm{McL}$ and a 90-dimensional representation for $3{}^{\cdot}\mathrm{Aut}(\mathrm{McL})$ over GF(5). This section is necessary because the current proof of the existence of the triple cover of McL relies on the existence of the Lyons group.

In Section 6 we get to grips with Ly-amalgams. In 6.3 we prove that a Ly-amalgam exists and in 6.4 we prove that, up to isomorphism, there is only one. Note that this is not the same thing as showing that the universal completion of a Ly-amalgam is not trivial. We need the results of Section 6 for the construction of a Ly-amalgam in $\mathrm{GL}_{111}(5)$ in Section 8. In Section 7 we suppose that M is a faithful completion of a Ly-amalgam and analyse the coset graph Γ of M_1, M_2 and M_3 in M; our aim is to determine the orbits of M_1 on its cosets in M. We quickly find five orbits, 7.3, and the remainder of the section is devoted to proving that these are the only orbits. Of particular consequence is 7.4, where a subgroup H of M isomorphic to $2{}^{\cdot}\mathrm{Alt}(11)$ is uncovered (though, of course, at this stage we do not know that it is the centralizer of an involution in the whole of M). Not only is this positive progress towards showing that M is of Ly-type, but this subgroup is very easy to calculate in! We add the cosets of this subgroup to our coset graph with incidence defined in the usual way. Setting $Z = Z(H)$, in 7.7, we show a certain connected component of the subgraph of Γ which is fixed by Z, in fact has H as its stabilizer in M. This then allows us to prove that if M_1 and M_1h are in the same connected component of the fixed graph of a conjugate of Z, then M_1h lies in one of the orbits that we have already found. Next we focus on the normalizer of a certain subgroup F of order five. We find that M contains a subgroup N of shape $5^{1+4}.4.\mathrm{Sym}(6)$, a connected component of the fixed graph of F has stabilizer N and that, once again, if M_1 and M_1h are in the same connected component of the fixed graph of F, then M_1h is already accounted for in one of our five orbits. Finally in 7.15 we show that there are just the five orbits found in the very beginning of the endeavour. The detailed theorem of Section 7 is

Theorem 1.5 *Suppose that M is a faithful completion of a* Ly*-amalgam of groups (M_1, M_2, M_3). Then*

(a) *M_1 has five orbits on the right cosets of M_1 in M. Moreover, the orbit* *stabilisers have shape* $3{}^{\cdot}\mathrm{Aut}(\mathrm{McL})$, $3^{2+4}.4.\mathrm{Sym}(5)$, $2.\mathrm{Sym}(7)$, $4.\mathrm{Sym}(6)$ *and* $5^{1+2}.\mathrm{Sym}(3)$.

(b) *M has order* $2^8 \cdot 3^7 \cdot 5^6 \cdot 7 \cdot 11 \cdot 31 \cdot 37 \cdot 67$.

(c) *M has a unique class of involutions and* $C_M(t) \cong 2{}^{\cdot}\mathrm{Alt}(11)$ *for any involution t in M.*

(d) *For* $i = 1, 2, 3$, *M_i is a maximal 3-local subgroup of M.*

(e) *M has a subgroup F of order five with $N_M(F)$ of shape* $5^{1+4}.4.\mathrm{Sym}(6)$.

(f) *M is a group of* Ly*-type.*

In our final section we construct a Ly-amalgam in $\mathrm{GL}_{111}(5)$. Once this is done we have completed the proof of 1.3.

We close the introduction with some comments on our notation. Suppose that Γ is a graph with varying types of vertices. If $e \in \Gamma$, then $\Gamma(e)$ is the set of neighbours of e. If i is a type of a vertex of Γ, then $\Gamma_i(e) = \{x \in \Gamma(e) \mid x \text{ has type } i\}$. If A operates as a group of automorphisms on Γ, then Γ^A represents the induced subgraph of Γ consisting of the vertices fixed by A. Also for $G \leq \mathrm{Aut}(\Gamma)$ and $e \in \Gamma$, G_e will be the stabilizer in G of e.

The alternating, symmetric and Mathieu groups of degree n are represented by $\mathrm{Alt}(n)$, $\mathrm{Sym}(n)$ and Mat_n respectively. We shall use Mat_9 to represent the point-stabiliser in Mat_{10}. We use D_n, Q_n and SD_n to represent the dihedral, quaternion and semidihedral groups of order n respectively. We have been terribly inconsistent with our notation for cyclic groups; we either use C_n or when appropriate simply n (rarely seen alone) for a cyclic group of order n. For a prime p, we use p^{a+b+c} to represent a p-group of order p^{a+b+c}. Often the summands in the superscript indicate that a group operating on the p-group has a chief factor of that order on the group. Beware the group p^{a+b+c} may be abelian so, for example, p^{1+4} does not necessarily mean an extraspecial group of order p^5. We write $G \sim A.B.\dots.Z$ or say that G has shape $A.B.\dots.Z$ when G has a normal series with factors isomorphic to $A, B, \dots, Z$. Thus for example $M_2 \sim 3^6.2^3.\mathrm{Sym}(5)$ indicates that M_2 contains normal subgroups of order 3^6 and $3^6.2^3$ and a factor group isomorphic to $\mathrm{Sym}(5)$. It does not specify the structure of the normal 3-subgroup or that of the Sylow 2-subgroup of the normal subgroup of order $3^6.2^3$. However, $M_2 \sim 3^6.\mathrm{C}_8.\mathrm{Sym}(5)$ does specify the structure of Sylow 2-subgroup of the subgroup of order $3^6.2^3$. We follow Atlas notation and write '${}^{\cdot}$' to mean non-split extension; thus $2{}^{\cdot}\mathrm{Alt}(5) \cong \mathrm{SL}_2(5)$. Also, on some occasions, we shall use the symbol ':' to denote a split extension such as $2^2 : \mathrm{Sym}(3) \cong \mathrm{Sym}(4)$. Hopefully the remainder of our notation is standard and can be found in [Go].

Acknowledgement We thank Gerhard Hiss for his contribution to this project. Particularly, he provided the original proof on which the presented

proof of Theorem 5.2 is based. Moreover, he has read the entire manuscript and has suggested a number of clarrifications.

2 Preliminaries

Lemma 2.1 *Suppose that Γ is a graph, G is a group acting on Γ and $A \trianglelefteq H \leq G$. Assume that $\mathcal{F}$ is a non-empty connected subset of Γ^A such that*

1. *$N_{G_e}(A) = H_e$ for all $e \in \mathcal{F}$.*
2. *If e in $\mathcal{F}$ and $d \in \Gamma^A(e)$, then d is H-conjugate to some element in $\mathcal{F}$.*
3. *$H = \langle H_e \mid e \in \mathcal{F} \rangle$.*
4. *No two distinct elements of $\mathcal{F}$ are conjugate under $N_G(A)$.*

Let Ψ be the connected component of Γ^A containing $\mathcal{F}$. Then $\Psi = \{e^h \mid e \in \mathcal{F},\ h \in H\}$ and $H = N_G(A) \cap \mathrm{Stab}_G(\Psi)$.

Proof: Let $\Psi^* = \{e^h \mid e \in \mathcal{F},\ h \in H\}$. Since Ψ is a connected component of Γ^A, each H_e, $e \in \mathcal{F}$, stabilizes Ψ and so (3) implies that H stabilizes Ψ. Thus $\Psi^* \subseteq \Psi$. Suppose now that $a \in \Psi$ and $d(\mathcal{F}, a) = n$. If $n = 1$, then there is an $e \in \mathcal{F}$ such that $a \in \Psi(e)$ and (2) implies that $\Psi(e) \subseteq \Psi^*$. Hence $a \in \Psi^*$. We now proceed by induction on n. Let $b \in \Psi(a)$ with $d(\mathcal{F}, b) = n - 1$. Then by induction there exists $h \in H$ such that $b^h \in \mathcal{F}$. But then either $a^h \in \mathcal{F}$ (so $a \in \Psi^*$) or $\Psi(a^h) \cap \mathcal{F}$ is non-empty. In the latter case (2) shows that there is an h_1 in H such that $a^{hh_1} \in \mathcal{F}$. Therefore in any case $a \in \Psi^*$ and so $\Psi = \Psi^*$. Let $e \in \mathcal{F}$ and $g \in N_G(A) \cap \mathrm{Stab}_G(\Psi)$. Then, as $\Psi = \Psi^*$, there exist $d \in \mathcal{F}$ and $h \in H$ such that $e^g = d^h$. But then (4) implies $e = d$ and (1) gives $gh^{-1} \in H$. Therefore, $g \in H$ and so $N_G(A) \cap \mathrm{Stab}_G(\Psi) \leq H$. Since trivially $H \leq N_G(A) \cap \mathrm{Stab}_G(\Psi)$, we have $N_G(A) \cap \mathrm{Stab}_G(\Psi) = H$ as claimed.

Remark 2.2 *A set $\mathcal{F}$ as in 2.1 can often be found using (variations of) the following procedure:*

0. *Pick some $e \in \Gamma^A$ and put $\mathcal{F} = \{e\} = \mathcal{N}$.*
1. *For all $e \in \mathcal{N}$ verify that $N_{G_e}(A) = H_e$. This for example can be done by finding a subset X of $\mathcal{F}$ such that $N_{G_x}(A) = H_x$ for all $x \in X$ and $N_{G_e}(A) = \langle N_{G_{ex}}(A) \mid x \in X \rangle$.*
2. *For all $e \in \mathcal{N}$ determine a set of representatives for the orbits of $N_{G_e}(A)$ on $\Gamma^A(e)$.*
3. *Replace $\mathcal{N}$ by the orbit representatives found in (2) which are not H-conjugate to a member of $\mathcal{F}$.*

4. *Replace $\mathcal{F}$ by $\mathcal{F} \cup \mathcal{N}$. Finally, if elements of $\mathcal{F}$ are conjugate under H, remove all but one of them from $\mathcal{F}$.*
5. *If $\mathcal{N} \neq \emptyset$ go back to (1).*
6. *Verify that no two distinct elements of $\mathcal{F}$ are conjugate under $N_G(A)$.*

The above procedure will be followed in 7.7 and 7.12 in Section 7.

Lemma 2.3 *Let G and T be groups, $L \trianglelefteq G$ and $H \leq G$ with $G = LH$. Assume $\phi : H \to T$ is a homomorphism and suppose that there exists a unique homomorphism $\psi : L \to T$ with $\psi\mid_{H\cap L} = \phi\mid_{H\cap L}$. Then there exists a unique homomorphism $\Phi : G \to T$ with $\Phi\mid_H = \phi$.*

Proof: Clearly if Φ exists then it is unique and $\Phi(lh) = \psi(l)\phi(h)$ for all $l \in L$ and $h \in H$. It remains to verify that Φ is well-defined and is a homomorphism.

So suppose that $lh = \tilde{l}\tilde{h}$. Define $k = h\tilde{h}^{-1} = l^{-1}\tilde{l}$ and observe that $k \in H \cap L$. Then

$$\begin{aligned}
\Phi(\tilde{l}\tilde{h}) &= \psi(\tilde{l})\phi(\tilde{h}) = \psi(lk)\phi(k^{-1}h) \\
&= \psi(l)\psi(k)\phi(k^{-1})\phi(h) = \psi(l)\psi(k)\psi(k^{-1})\phi(h) \\
&= \psi(l)\phi(h) = \Phi(hl).
\end{aligned}$$

Thus Φ is well-defined.

An elementary calculation shows that Φ is an homomorphism if and only if **(2.3.1)** for all $l \in L, h \in H$ we have

$$\psi(l) = \phi(h)^{-1}\psi(l^{h^{-1}})\phi(h).$$

Fix $h \in H$ and define $\psi^* : L \to T$ by $\psi^*(l) = \phi(h)^{-1}\psi(l^{h^{-1}})\phi(h)$. Then ψ^* is the composition of three homomorphisms, namely conjugation by h^{-1} followed by ψ and finally conjugation by $\phi(h)$. Therefore, ψ^* is a homomorphism from L to T. Clearly ψ^* equals ϕ when restricted to $H \cap L$ and thus, by the uniqueness of ψ, $\psi = \psi^*$ and (2.3.1) is demonstrated. Hence Φ is a homomorphism and the lemma is proved.

The next lemma is extracted from [IM, 2.2]. It will be used in Section 8 when we construct a representation of a Ly-amalgam.

Lemma 2.4 *Let (M_1, M_2, M_3) be an amalgam, H a group and A a subgroup of* $\mathrm{Aut}(H)$. *Suppose that, for each $i \in \{1, 2, 3\}$, there exist homomorphisms*

$$\alpha_i : M_i \to H$$

and elements $a_i \in A$ such that the homomorphisms

$$\begin{aligned} \alpha_1|_{M_{13}} a_2 &= \alpha_3|_{M_{13}} \\ \alpha_2|_{M_{12}} a_3 &= \alpha_1|_{M_{12}} \\ \alpha_3|_{M_{23}} a_1 &= \alpha_2|_{M_{23}}. \end{aligned}$$

Put $M_{23}^* = M_{23}^{\alpha_3 a_2^{-1}}$, $M_{13}^* = M_{13}^{\alpha_1}$, $M_{12}^* = M_{12}^{\alpha_1}$ and $B^* = M_{123}^{\alpha_1}$. Then

(a) *The following two statements are equivalent:*

  (a1) *There exist $b_1, b_2, b_3 \in A$ such that for all $\{i, j\} \subset \{1, 2, 3\}$*

$$\alpha_i b_i|_{M_{ij}} = \alpha_j b_j|_{M_{ij}}.$$

  (a2) $a_2 a_1 a_3 \in C_A(M_{23}^*) C_A(M_{13}^*) C_A(M_{12}^*)$.

(b) $B^* \leq \bigcap_{\{i,j\} \subset \{1,2,3\}} M_{ij}^*$ *and $a_2 a_1 a_3 \in C_A(B^*)$. In particular, (a1) and (a2) hold if*

$$C_A(B^*) = C_A(M_{23}^*) C_A(M_{13}^*) C_A(M_{12}^*).$$

(c) *Assume that (a1) holds and that for each $i \in \{1, 2, 3\}$, α_i is one-to-one. For each $i \in \{1, 2, 3\}$ set $M_i^* = M^{\alpha_i b_i}$. If $M_i^* \cap M_j^* = M_{ij}^{\alpha_i b_i}$ for all $1 \leq i < j \leq 3$, then (M_1^*, M_2^*, M_3^*) is an amalgam isomorphic to (M_1, M_2, M_3).*

Next we present some well-known facts about the GF(3)-representations of Mat_{10} and Mat_{11}.

Lemma 2.5 *Suppose that $H \cong \mathrm{Mat}_{11}$, $K \leq H$ with $K \cong \mathrm{Mat}_{10}$ and E is the field with 3 elements.*

(a) *There is a unique irreducible EH-module, V, of dimension 5 in which K leaves invariant a 1-space.*

(b) *There are two irreducible EK'-modules of dimension 4. One is obtained from the other by tensoring with the -1-representation.*

(c) *When restricted to K, V is an indecomposable EK-module with composition factors of dimension 1 and 4.*

(d) *The orbits of H on the 1-dimensional subspaces of V have lengths 11 and 110 with stabilisers K and $3^2{:}\mathrm{Q}_8$ respectively. Moreover, K operates non-trivially on the 1-space fixed by K.*

(e) *K has orbits of length 10 and 30 on the 1-spaces and hyperplanes of either of the 4-dimensional irreducible modules. The 1-space stabilisers are $\mathrm{Mat}_9 \cong 3^2{:}\mathrm{Q}_8$ and $\mathrm{Sym}(4)$ respectively.*

(f) *Suppose that W is an irreducible 4-dimensional EK-module. Then, up to isomorphism, there is a unique indecomposable EK-module of dimension 5 with a quotient isomorphic to W and a submodule of dimension 1 which is inverted by K. Furthermore,* $\dim_E \mathrm{H}^1(K,W)=1$.

(g) *Let f be an element of order 4 in H. Then* $\dim_E(C_V(f)) = 1$ *and* $\dim_E(C_V(f^2)) = 3$.

Proof: ¿From [Jam, Theorem 7.1] we see that the 11-dimensional EH-module $W=(-1)_K^H$ has composition factors 1, 5 and $\bar{5}$ (using the notation from [Jam]). Moreover, from [Jam, Section 7] we have that 5 and $\bar{5}$ are the only 5-dimensional irreducible modules; they are dual to one another. Clearly 1 is not a quotient of W as K centralizes it and, similarly, W has no submodule centralized by K. Hence W is a uniserial module $5/1/\bar{5}$. This proves (a).

¿From [Be, pg. 208] or [MATLAS] we have that $K' \cong \mathrm{Alt}(6)$ has a unique irreducible E-module of dimension 4. This module extends to K and so (b) holds.

Let V be as in part (a). Then V restricted to K has at least one 1-dimensional composition factor. Since Mat_{10} has no E-modules of dimension less than 4, we have (c).

¿From (a), we see that H has an orbit of length 11 on the 1-spaces of V. Let $\{v_1,\dots,v_{11}\}$ be this orbit. Then the normalizer of the subspace $U=\langle v_1,v_2\rangle$ is $L\cong \mathrm{Mat}_9 \sim 3^2{:}\mathrm{Q}_8$ and L inverts U. Hence L fixes the 1-space $\langle v_1+v_2\rangle$. It follows that $\langle v_1+v_2\rangle$ is a representative for an orbit of length 110. Since there are only 121 subspaces of dimension 1 in V, we conclude that (d) holds.

For part (e) we set $U=V/\langle v_1\rangle$. Since $(\langle v_1+v_2\rangle+\langle v_1\rangle)/\langle v_1\rangle$ is a 1-space fixed by L, we conclude that K has an orbit of length 10 on W. Since $\mathrm{Sym}(4)$ fixes a 1-space, we also have an orbit of length 30.

Let R be the 4-dimensional $E\mathrm{Alt}(6)$-module. Then it is easy to show that $\mathrm{H}^1(\mathrm{Alt}(6),R)$ has dimension at most 2. Since the 5-dimensional submodule of the permutation module for $\mathrm{Sym}(6)$ over E has centralisers of elements of order 3 and cycle shape $(1^3,3)$ of dimension 3 and of elements of cycle shape (3^2) of dimension 2, we see that this 5-dimensional module does not admit K. Since $\mathrm{Aut}(\mathrm{Alt}(6))$ acts on $\mathrm{H}^1(\mathrm{Alt}(6),R)$, we conclude that $\mathrm{H}^1(\mathrm{Alt}(6),R)$ has dimension 2 and that $\mathrm{Out}(\mathrm{Alt}(6))$ acts non-trivially on this space. We deduce that $\mathrm{H}_1(K,W)$ has dimension 1. Tensoring with the -1-representation of K now delivers (f).

We obtain part (g) from the ordinary character table of Mat_{11} [ATLAS] or [Frob] and the decomposition matrix [Jam, Theorem 7.1].

Lemma 2.6 *Suppose that p is an odd prime, P is an extraspecial p-group of exponent p and order p^{1+2n}. Let $K=C_{\mathrm{Out}(P)}(Z(P))$, $H=P{:}K\sim p^{1+2n}{:}\mathrm{Sp}_{2n}(p)$, $E\le Q$ be elementary abelian of maximal order and $\mathcal{E}$ be the set of maximal* *subgroups of E which complement $Z(P)$. Assume that r is a prime with $r \neq p$ and let V be an irreducible $\mathrm{GF}(r)P$-module of dimension dp^n where d is the smallest integer such that $\mathrm{GF}(r^d)$ contains a pth root of unity. Then*

(i) $V = \oplus_{E^* \in \mathcal{E}} C_V(E^*)$ *where, for each $E^* \in \mathcal{E}$, $C_V(E^*)$ has dimension d and admits $Z(P)$ irreducibly.*

(ii) *V extends uniquely to a $\mathrm{GF}(r)H$-module.*

(iii) *Suppose that $E^* \in \mathcal{E}$ and let $L = N_K(E^*)/C_K(E^*)$. Then $L \cong \mathrm{GL}_n(p)$ and acts on $C_V(E^*)$. Furthermore, for $x \in L$, either $\det_{E^*}(x)$ is a square in $\mathrm{GF}(p)$ and centralizes $C_V(E^*)$, or x inverts $C_V(E^*)$.*

Proof: Part (i) is well known. Since the irreducible $\mathrm{GF}(r)P$-representations are uniquely determined by the action of $Z(P)$ and H centralizes $Z(P)$, the irreducible modules for P all extend uniquely to H. Thus (ii) holds.

This brings us to (iii). We have $N_K(E)$ is a subgroup of K isomorphic to $p^{\binom{n}{2}+n}{:}\mathrm{GL}_n(p)$. Set $K^* = N_K(E^*)$ and $L = K^*/C_{K^*}(E^*)$. Then $L \cong \mathrm{GL}_n(p)$. Now L acts on $C_V(E^*)$ and, as K^* commutes with $Z(P)$, the only non-trivial action comes from the cyclic group $L/L' \cong \mathrm{GF}(p)^*$ of order $p-1$. Select an element λ of L which projects to a generator of L/L' and such that $P_1 = [P, \lambda]$ is an extraspecial group of order p^3. Then $P_2 = C_P(\lambda)$ is an extraspecial p-group of order $p^{1+2(n-1)}$. Note that $P_1\langle\lambda\rangle$ commutes with P_2. Thus Clifford's Theorem implies that V restricted to $P_1\langle\lambda\rangle$ is a single homogeneous component. So V restricted to $P_1\langle\lambda\rangle$ is a direct sum of p^{n-1} isomorphic irreducible modules of dimension dp. Let $V_1 = \langle C_V(E^*)^{P_1}\rangle$. Then V_1 is a submodule of V of dimension dp. Applying (i) to V_1, we see that V_1 is a sum of p d-dimensional spaces $\mathcal{T}$ and on these subspaces $\langle\lambda\rangle$ acts with orbits of length 1 and $p-1$. So λ induces a $p-1$-cycle π_λ on $\mathcal{T} \setminus \{C_V(E^*)\}$. As π_λ is an odd permutation, $\det_{V_1}(\lambda) = -\det_{C_V(E^*)}(\lambda)$. Since V is a direct sum of p^{n-1} $P_1\langle\lambda\rangle$-modules isomorphic to V_1 and $\det_V(\lambda) = 1$, we deduce that $\det_{V_1}(\lambda) = -1$. This proves (iii).

Lemma 2.7 *Suppose that Q is an extraspecial group of order 3^5, $K \leq \mathrm{Aut}(Q)$ with $K \cong 2{\cdot}\mathrm{Alt}(5) \cong \mathrm{SL}_2(5)$, $H = Q : K$. Let B be the normalizer of a Sylow 3-subgroup of H and $i \in \{18, 36\}$. Then*

(a) *There exists a unique irreducible faithful i-dimensional $\mathrm{GF}(5)H$-module V_i.*

(b) *$V_{36} \cong V_{18} \otimes V_2$, where V_2 is the unique 2-dimensional $\mathrm{GF}(5)H$-module.*

(c) *V_i is uniquely determined by its restriction to B. More precisely, if ϕ and ψ are two irreducible embeddings of H into $\mathrm{GL}_i(5)$ with $\phi_B = \psi_B$, then $\phi = \psi$.*

(d) *Let E be the unique normal subgroup of order 3^3 in B and let E^* be a complement to $Z(Q)$ in E. Assume that $T \cong \mathrm{C}_4$ is a Sylow 2-subgroup of*

$N_B(E)$ *and let* $t \in T$ *be an involution. Then* T *centralizes* $C_{V_{18}}(E^*)$ *and* t *inverts* $C_{V_{36}}(E^*)$.

Proof: We first prove parts (a) and (b). Note that, by 2.6 (ii), H has a faithful irreducible 18-dimensional $\mathrm{GF}(5)H$-module. Tensoring this module by the natural $\mathrm{GF}(5)K$-module V_2 gives an irreducible $\mathrm{GF}(5)H$-module of dimension 36. Our task is to show that these are the unique irreducible modules of these dimensions. To do this we show that there is a unique (up to conjugacy) embedding of H in $\mathrm{GL}(V_i)$. Suppose now that $i \in \{18, 36\}$ and V_i is an irreducible $\mathrm{GF}(5)H$-module of dimension i. Then $V_i|_Q$ is the direct sum of $j = \frac{i}{18}$ (isomorphic) irreducible $\mathrm{GF}(5)Q$-modules of dimension 18. In particular, V_i is uniquely determined, up to isomorphism, as a $\mathrm{GF}(5)Q$-module. Note that $N_{\mathrm{GL}(V_i)}(Q) \sim (3^{1+4} : \mathrm{Sp}_4(3) \circ \mathrm{GL}_j(25)).2$ (where $\circ$ denotes central product) and so $N_{\mathrm{GL}(V_i)}(Q)^\infty = L_1 \times L_2$, with $L_1 \sim 3^{1+4}.\mathrm{Sp}_4(3)$ and $L_2 \sim \mathrm{SL}_j(25)$. Since H is perfect and $C_H(Q) \leq Z(H)$, the Three Subgroups Lemma implies that H has no non-trivial automorphism centralizing Q. Thus the embedding of Q into L_1 can be extended uniquely to an embedding of H into L_1. Moreover, if $j = 2$ there exists a non-trivial homomorphism of $H/Q \cong \mathrm{SL}_2(5)$ into L_2 and this homomorphism is unique up to conjugation in $\mathrm{GL}_2(25)$. Thus (a) and (b) hold.

We now move on to part (c). By (a) $\psi = \phi\rho$ for some $\rho \in \mathrm{Inn}(\mathrm{GL}(V_i))$. Since, by hypothesis, $\psi_B = \phi_B$, ρ centralizes B^ϕ. As B acts irreducibly on V_i, $\mathrm{End}_B(V_i) = \mathrm{End}_H(V_i) \cong \mathrm{GF}(25)$. Thus $\rho \in \mathrm{End}_H(V_i)$ and so ρ centralizes H^ϕ. Therefore, $\phi = \psi$.

Finally we prove part (d). By 2.6 (iii), T centralizes $C_{V_{18}}(E)$. Moreover, t inverts V_2 and so by (b), t inverts $C_{V_{36}}(E)$.

3 A Characterization of the non-Abelian Simple Alternating Groups

In this section we show that certain amalgams of groups characterize the alternating groups. We will employ this characterization in Section 7 to identify a subgroup isomorphic to $2{\cdot}\mathrm{Alt}(11)$ in a faithful completion of a Ly-amalgam of groups. We begin with a definition.

Definition 3.1 *Let* k *and* l *be integers greater than* 2, $n = k + l$, Ω *be a set of size* n *and for* $i \in \{k-1, k, k+1\}$, Ω_i *be a subset of* Ω *of size* i *such that* $\Omega_{k-1} \subset \Omega_k \subset \Omega_{k+1}$. *Then an amalgam of* **type** (k, l)-$\mathbf{Alt}(n)$ *is an amalgam of groups isomorphic to the amalgam* $(\mathrm{Stab}_{\mathrm{Alt}(\Omega)}(\Omega_i) \mid k-1 \leq i \leq k+1)$ *in* $\mathrm{Alt}(\Omega)$.

Observe that $\mathrm{Stab}_{\mathrm{Alt}(\Omega)}(\Omega_i) = \mathrm{Stab}_{\mathrm{Alt}(\Omega)}(\Omega \setminus \Omega_i)$ and thus the definition of an amalgam of type (k,l)-$\mathrm{Alt}(n)$ is symmetric in k and l.

Theorem 3.2 *Suppose that M is a faithful completion of an amalgam (M_1, M_2, M_3) of type (k,l)-$\mathrm{Alt}(n)$. If $n \geq 5$, then $M \cong \mathrm{Alt}(n)$.*

Proof: Without loss of generality we may assume that M is the universal completion of the amalgam. Then there exists a homomorphism $^{-} : M \to \mathrm{Alt}(\Omega)$ such that

$$(\overline{M_j} \mid 1 \leq j \leq 3) = (\mathrm{Stab}_{\mathrm{Alt}(\Omega)}(\Omega_i) \mid k-1 \leq i \leq k+1),$$

where Ω and Ω_i are as in 3.1.

Suppose first that $k = 2$. Then $M_1 \cong \mathrm{Alt}(n-1)$, $M_2 \cong \mathrm{Sym}(n-2) \curlywedge \mathrm{Sym}(2)$ and $M_3 \cong \mathrm{Sym}(n-3) \curlywedge \mathrm{Sym}(3)$. Observe that $M_{12} \cong \mathrm{Alt}(n-2)$ has index 2 in M_2 and is normal therein. Also $M_{123} \cong \mathrm{Alt}(n-3)$, M_{13}/M_{123} and M_{23}/M_{123} are two different groups of order two in M_3/M_{123}. Thus, as $M_3/M_{123} \cong \mathrm{Sym}(3)$, $M_3 = \langle M_{13}, M_{23} \rangle$. Therefore, $M = \langle M_1, M_2 \rangle$.

Choose an involution $t \in M_{23} \setminus M_1$. Then, as $M_{12} \trianglelefteq M_2$, $M_{12} \leq M_1 \cap M_1^t$ and, as $\overline{M_1} \cap \overline{M_1^t} = \overline{M_{12}}$, $M_{12} = M_1 \cap M_1^t$. Let Γ be the graph with vertices the cosets of M_1 in M and edges the translates of $(M_1, M_1 t)$ by right multiplication with elements from M. Then M operates faithfully on Γ and, as $M = \langle M_1, M_2 \rangle = \langle M_1, t \rangle$, Γ is connected.

Since $|M_3 : M_{13}| = 3$, $M_1 M_3$ contains three cosets of M_1: M_1, $M_1 t$ and $M_1 s$ where s and t are involutions (t being the one above in M_{23}). Moreover, M_3 acts 2-transitively on these three cosets and so $M_1 s$ is adjacent to both M_1 and $M_1 t$; in particular, $\Gamma(M_1 t) \cap \Gamma(M_1) \neq \emptyset$. Now M_1^t is the stabilizer in M of the vertex $M_1 t$ and (as $n \geq 5$) M_1^t operates 2-transitively on $\Gamma(M_1 t)$. Hence, as $\Gamma(M_1 t) \cap \Gamma(M_1) \neq \emptyset$, every neighbour of $M_1 t$ is a neighbour of M_1. Since M_1 has exactly $n-1$ neighbours, we find that $|\Gamma| \leq 1 + (n-1) = n$. Thus $|M| \leq n \cdot |M_1| = \frac{n!}{2}$ and, as $\mathrm{Alt}(n)$ is an image of M, we finally conclude that $M \cong \mathrm{Alt}(n)$ in this case.

We now assume that $k \geq 3$ and by symmetry also that $l \geq 3$. Let Ω_{k-2} be a set of size $k-2$ contained in Ω_{k-1}. For $i = 1, 2, 3$ let P_i be the subgroup of M_i which maps to the subgroup of $\overline{M_i}$ which fixes Ω_{k-2} pointwise. Then $(P_i | 1 \leq i \leq 3)$ is of type $(2,l)$-$\mathrm{Alt}(l+2)$. Let $P = \langle P_i \mid 1 \leq i \leq 3 \rangle$. As $l \geq 3$, $l + 2 \geq 5$ and so, by the $k = 2$ case, $P \cong \mathrm{Alt}(l+2)$. Again using $l \geq 3$, we have $\mathrm{Stab}_{M_{123}}(\Omega_{k-2})$ induces $\mathrm{Sym}(k-2)$ on Ω_{k-2} and normalizes every generating subgroup of P and hence P. Put $M_0 = \mathrm{Stab}_{M_{123}}(\Omega_{k-2})P$. Then $M_0 \sim \mathrm{Alt}(l+2).\mathrm{Sym}(k-2)$, $M_0 \cong \overline{M_0} = \mathrm{Stab}_{\mathrm{Alt}(\Omega)}(\Omega_{k-2})$, and for $i = 1, 2$, $M_0 \cap M_i = \mathrm{Stab}_{M_{123}}(\Omega_{k-2})P_i \cong \overline{M_0} \cap \overline{M_i}$. Thus $(M_i \mid 0 \leq i \leq 2)$ is of type $(k-1, l+1)$-$\mathrm{Alt}(n)$ and so, by induction on k, $M^* = \langle M_i \mid 0 \leq i \leq 2 \rangle \cong \mathrm{Alt}(n)$.

Since $M_3 = \langle M_{13}, M_{23}\rangle \leq M^*$, we finally have $M = M^*$ and the lemma is proved.

Lemma 3.3 *Let M be a group, A a finite subgroup of M, $D \in \mathrm{Syl}_3(A)$ and L_D a subgroup of $N_M(D)$ containing D. Suppose that*

1. $A \cong \mathrm{Alt}(5)$.
2. $L_D \cong \mathrm{Sym}(3) \wr \mathrm{Sym}(n)$, $n \geq 2$.
3. $L_D \cap A = N_A(D) \cong \mathrm{Sym}(3)$.
4. *For $g \in A$ and $E = D^g$ define $L_E = L_D^g$ (well-defined by (3)). Then*
$$L_D \cap L_E \cong \begin{cases} \mathrm{Sym}(n-1) & \text{if } \langle D, E\rangle \cong \mathrm{Alt}(4) \\ \mathrm{C}_2 \times \mathrm{Sym}(n-2) & \text{if } \langle D, E\rangle = A \end{cases}.$$
5. $M = \langle A, L_D\rangle$.

Then $M \cong \mathrm{Alt}(n+3)$.

Proof: We shall show that M is a completion of a $(4, n-1)$-$\mathrm{Alt}(n+3)$ amalgam of groups. The result will then follow from 3.2.

Select $E, F \in \mathrm{Syl}_3(A)$ with $B = \langle D, E\rangle \cong \mathrm{Alt}(4)$ and $A = \langle D, F\rangle$ and define

$$\begin{aligned} M_1 &= L_D, \\ M_2 &= B(L_D \cap L_E) \text{ and} \\ M_3 &= A(L_D \cap L_F). \end{aligned}$$

Notice that, by (3), for $X \in \mathrm{Syl}_3(A)$,

$$L_D \cap L_X \cap A = N_A(X) \cap N_A(D).$$

In particular, by (1), $L_D \cap L_E \cap A = 1$ and $|L_D \cap L_F \cap A| = 2$ and $L_D \cap L_E$ and $L_D \cap L_F$ are uniquely determined up to automorphisms of L_D in L_D. Also we see that $L_D \cap L_E$ inverts D and $L_D \cap L_F$ contains an element which centralizes F and inverts D. It now follows from (4) that $M_2 \cong \mathrm{Sym}(4) \wr \mathrm{Sym}(n-1)$ and $M_3 \cong \mathrm{Sym}(5) \wr \mathrm{Sym}(n-2)$. We now calculate that

$$\begin{aligned} M_{12} &= M_1 \cap M_2 = L_D \cap B(L_D \cap L_E) = (L_D \cap B)(L_D \cap L_E) \\ &= D(L_D \cap L_E) \cong \mathrm{Sym}(3) \wr \mathrm{Sym}(n-1). \end{aligned}$$

and, as $|L_D \cap L_F \cap A| = 2$,

$$M_{13} = (L_D \cap A)(L_D \cap L_F) = D(L_D \cap L_F) \cong \mathrm{Sym}(3) \times \mathrm{Sym}(n-2).$$

Furthermore, we now determine M_{123} in M_1 and get that $M_{123} = M_{12} \cap M_{13} \cong \mathrm{Sym}(3) \wr \mathrm{Sym}(n-2)$. Finally, since $M_2 > M_{23} \geq BM_{123}$ we have $M_{23} = BM_{123} \cong \mathrm{Sym}(4) \wr \mathrm{Sym}(n-2)$. It follows that (M_1, M_2, M_3) of type $(4, n-1)$-$\mathrm{Alt}(n+3)$. Hence $M \cong \mathrm{Alt}(n+3)$.

4 Properties of McL

In this section $M = \mathrm{Aut}(\mathrm{McL})$, $U \le M$ with $U \sim \mathrm{U}_4(3).2$, $K \le M$ with $K \sim \mathrm{U}_3(5).2$ and $F \le K$, with $|F| = 5$ and $C_M(F)$ divisible by 3. Set $L = N_M(F) \sim 5^{1+2}.3.8.2$. We define Γ_2 to be the set of 3-central groups of order three in M, Γ_3 the set of elementary abelian groups of order 3^4 in M and Γ_4 the set of 2-central groups of order 2 in M. Let $\Gamma = \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$. We make Γ into a graph by stipulating that for $i \in \{2,3,4\}$ and $a_i \in \Gamma_i$, a_2 is adjacent to a_3 if $a_2 \le a_3$, a_2 is adjacent to a_4 if $[a_2, a_4] = 1$ and a_3 is adjacent to a_4 if a_4 normalizes a_3. Let Λ the left cosets of U in M. We make Λ into a graph by stipulating that $x, y \in \Lambda$ are adjacent if and only if $M_x \cap M_y \sim 3^4.\mathrm{Mat}_{10}$. Thus Λ is the graph, discovered by McLaughlin, which admits M as a primitive rank 3 permutation group and has parameters $k = 112$, $l = 162$, $\lambda = 30$ and $\mu = 56$ [McL]. Finally we let Δ be the set of cyclic subgroups of order 5 which are not equal to F in $N_M(F)$. We have gleaned much of the the information about the subgroup structure of M from the Atlas [ATLAS] and [Fink].

Lemma 4.1 (a) *A Sylow* 3*-subgroup of* M *contains a unique elementary abelian subgroup,* Q*, of order* 3^4*.*

(b) M *operates transitively on* Γ_i *for each* $i \in \{2,3,4\}$*.*

(c) $N_M(Q)$ *operates naturally as* Mat_{10} *on* $\Gamma_2 \cap Q$ *which has order* 10*.*

(d) *Let* $x \in \Gamma_2$*. Then* $\Gamma_4 \cap N_M(x) \subset O_{2,3}(N_{M'}(x)) \sim 3^{1+4} : 2$*; in particular, if* $z \in \Gamma_4 \cap N_M(x)$*, then* z *normalizes all the Sylow 3-subgroups with centre* x*.*

Proof: Let S be a Sylow 3-subgroup of M. Then S contains an elementary abelian subgroup Q of order 3^4 with normalizer $3^4 : (\mathrm{Mat}_{10} \times \mathrm{C}_2)$. Let Q_1 be a further elementary abelian subgroup of S of order 3^4. Then, as Mat_{10} does not possess transvections when operating on a 4-dimensional GF(3)-space, we have $S = QQ_1$ and $Z(S) \ge Q_1 \cap Q$ which has order 9. However, the centre of S has order 3 and so we have a contradiction; no such Q_1 exists. Hence (a) holds and (c) follows from 2.5 (e) as $N_M(Q) \ge S$. Part (b) follows immediately from (a) and the fact that the centres of a Sylow 2-subgroup of M and a Sylow 3-subgroup of M have orders 2 and 3 respectively. Suppose next that $x \in \Gamma_2$ and let $X = N_{M'}(x)$. Then $X \sim 3^{1+4}.2.\mathrm{Sym}(5)$. Let K be a complement to $O_3(X)$. Then K centralizes $O_2(K) := \langle z \rangle \in \Gamma_4$ and so $K \le C_{M'}(z) \cong 2{\cdot}\mathrm{Alt}(8)$ and normalizes a 3-cycle in the quotient Alt(8). Thus we deduce that all the involutions of $K/\langle z \rangle$ are products of exactly two transpositions. Each of these involutions therefore lifts to an element of order 4 in K and so z is the unique involution of K and the final part of the lemma holds.

Lemma 4.2 (a) *The cliques of maximal size in* Γ *have three elements, any clique in* Γ *lies in a clique of maximal size and* M' *acts transitively on the set of cliques of maximal size.*

(b) *Let* $\{a,b,c\}$ *be a maximal clique with* a,b,c *of type* $2,3,4$ *respectively. Then*

(ba) $M_a \sim 3^{1+4}.4.\mathrm{Sym}(5)$, $M_b \sim 3^4.(\mathrm{C}_2 \times \mathrm{Mat}_{10})$ *and* $M_c \sim 2{\cdot}\mathrm{Sym}(8)$.

(bb) $M_{ab} \sim 3^{1+2+1}.(\mathrm{C}_2 \times 3^2.\mathrm{Q}_8)$, $M_{ac} \sim 2.(\mathrm{Sym}(3) \times \mathrm{Sym}(5))$ *and* $M_{bc} \sim 3^2.(\mathrm{C}_2 \times \mathrm{SD}_{16})$.

(bc) $M_{abc} \sim 2.(\mathrm{Sym}(3) \times \mathrm{C}_2 \times \mathrm{Sym}(3))$.

(c) *Let* $c \in \Gamma_4$. *Then* M_c *acts transitively on* $(\Gamma_4)^c \backslash \{c\}$ *and, for* $d \in (\Gamma_4)^c \backslash \{c\}$, $M_{cd} \sim 2.2^{1+2+2}.\mathrm{Sym}(3)$.

(d) *Let* $c \in \Gamma_4$ *and* $i \in \{2,3\}$. *Then* $\Gamma_i^c = \Gamma_i(c)$. *In particular,* M_c *acts transitively on* Γ_i^c.

Proof: Let S be a Sylow 3-subgroup of M and $x = Z(S) \in \Gamma_2$. By 4.1 (a), x is incident to a unique $y \in \Gamma_3$ which is contained in S and by 4.1 (d) the normalizer of S contains $z \in \Gamma_4$ which normalizes y and centralizes x. Thus $\{x,y,z\}$ is a clique and, as the maximal size of a clique is three, $\{x,y,z\}$ is a maximal clique. Moreover, we have shown that any clique with an element of type 2 and an element of type 3 is contained in a maximal clique with three elements. If we have a clique $\{y,z\}$ of type $3,4$, then $z \le N_{M'}(y)$. By 4.1 (c), $N_{M'}(y)$ acts as Mat_{10} on $\Gamma_2 \cap y$ and z induces a cycle of type 2^4. It follows that z centralizes exactly two elements of $\Gamma_2 \cap y$. Thus $\{y,z\}$ extends to a clique with three elements. Now suppose that $\{x,z\}$ is a clique of type $2,4$ with $x = Z(S)$. Then 4.1 (d) indicates that $z \le N_M(S)$ and so 4.1 (a) implies that there is a $y \in \Gamma_3$ which is normalized by z and centralized by x. Thus $\{x,z\}$ also extends to a clique with three elements. This shows that all maximal cliques have size three. Now suppose that $\{a,b,c\}$ and $\{a_1,b_1,c_1\}$ are cliques with $a,a_1 \in \Gamma_2$, $b,b_1 \in \Gamma_3$ and $c,c_1 \in \Gamma_4$. Without loss of generality we may assume that $b = b_1$. Then, as M_b operates transitively on the 10 elements of type 2 in b we can conjugate a to a_1 by an element which fixes b. Hence we may assume $a = a_1$. Now $M_{ab} \sim 3^4.3^2(\mathrm{C}_2 \times \mathrm{Q}_8) \sim 3^4(2 \times \mathrm{Mat}_9)$ by 2.5 (e). Hence, by 4.1 (d), and Sylow's theorem, M_{ab} operates transitively on $\Gamma_4 \cap M_{ab}$ and so M acts transitively on maximal cliques. We obtain part (ba) directly from [ATLAS]. Also we have $M_{ab} \sim 3^4.(\mathrm{C}_2 \times \mathrm{Mat}_9) \sim 3^{1+2+1}.(\mathrm{C}_2 \times 3^2.\mathrm{Q}_8)$ and $M_{bc} = C_{M_b}(c) \sim 3^2.(\mathrm{C}_2 \times \mathrm{SD}_{16})$. Additionally, as a projects to a 3-cycle in M_c/c, $M_{ac} = N_{M_c}(a) \sim 2.(\mathrm{Sym}(3) \times \mathrm{Sym}(5))$. Finally we calculate in $2{\cdot}\mathrm{Sym}(8)$ that $M_{abc} \sim 2.(\mathrm{Sym}(3) \times \mathrm{Sym}(3) \times \mathrm{C}_2)$. Therefore, parts (bb) and (bc) hold. Let $c \in \Gamma_4$. Then Γ_4^c are the conjugates of c which lie in $M_c' \sim 2{\cdot}\mathrm{Alt}(8)$. As there is a unique class of involutions in $2{\cdot}\mathrm{Alt}(8)$ (those corresponding to a product of 4 commuting transpositions), we conclude that part (c) holds. Finally, part (d) follows directly from the definition of incidence in Γ.

Lemma 4.3 *Suppose that $t \in M \setminus M'$ is an involution. Then $C_M(t) \cong \mathrm{C}_2 \times \mathrm{Mat}_{11}$ and, for $i = 2, 4$, $C_M(t)$ acts transitively on Γ_i^t and has two orbits on Γ_3^t. Moreover, setting $C_t = C_M(t)$ we have*

(a) *If $x \in \Gamma_3^t$, then either $M_x \cap C_t \sim \mathrm{C}_2 \times \mathrm{Mat}_{10}$ and t inverts $O_3(x)$ or $M_x \cap C_t \sim \mathrm{C}_2 \times 3^2 : \mathrm{SD}_{16}$.*

(b) *If $x \in \Gamma_4^t$, then $M_x \cap C_t \sim 2^{\cdot}((\mathrm{Sym}(2) \wr 2^3) : \mathrm{Sym}(3))$.*

Proof: That $C_t = C_M(t) \cong \mathrm{C}_2 \times \mathrm{Mat}_{11}$ can be read from the Atlas [ATLAS]. Since Mat_{11} contains a unique class of involutions, we see that C_t is transitive on Γ_4^t. Let $x \in \Gamma_2^t$. Then x is inverted by t and there is a $z \in \Gamma_4^t \cap N_M(x)$. Thus $\langle t, x \rangle \le C_M(z) \sim 2^{\cdot}\mathrm{Sym}(8)$. Since C_t contains no subgroup isomorphic to $2^{\cdot}\mathrm{Alt}(6)$, we see that t projects to an involution of cycle shape $(2, 2, 2)$ in $\mathrm{C}_M(z)/\langle z \rangle$. It follows that $C_{C_M(z)}(t) \sim 2^{\cdot}((\mathrm{Sym}(2) \wr 2^3) : \mathrm{Sym}(3))$ which is transitive on all the subgroups of order 3 which are normalized by t and generated by elements which project to 3-cycles in $C_M(z)/\langle z \rangle$. As C_t acts transitively on Γ_4^t, we conclude that C_t acts transitively on Γ_2^t. Now suppose that $x \in \Gamma_3^t$. Then t is an involution in $N_M(x) \sim 3^4 : (2 \times \mathrm{Mat}_{10})$ which is not in M'. Thus either t inverts $O_3(x)$ or t is a diagonal involution. In the former case we have that $K = C_t \cap N_M(x) \sim \mathrm{C}_2 \times \mathrm{Mat}_{10}$ with $O_2(K) = \langle t \rangle$. Since $N_M(x)$ has a unique class of complements to $O_3(N_M(x))$ and C_t has a unique class of subgroups isomorphic to $\mathrm{C}_2 \times \mathrm{Mat}_{10}$, we conclude that C_t acts transitively on $x \in \Gamma_3^t$ with $O_3(x)$ inverted. In the case that t is a diagonal involution in M_x, we have $C_t \cap M_x \sim \mathrm{C}_2 \times 3^2 : \mathrm{SD}_{16}$. This subgroup is the normaliser in C_t of a Sylow 3-subgroup and so is also unique up to conjugacy in C_t and once again $\langle t \rangle = O_2(C_t \cap M_x)$. It follows that C_t acts transitively on Γ_3^t in this case as well.

Lemma 4.4 *Let $a \in \Gamma_2$. Then M_a has five orbits on Γ_2, $X_i(a), 0 \le i \le 4$. Moreover, the orbits and some of their properties may be described as follows:*

(a) *$X_0(a) = \{a\}$.*

(b) *If $b \in X_1(a)$, then $[a, b] = 1$, $M_{ab} \sim 3^4.(\mathrm{C}_2 \times \mathrm{Q}_8)$ and there exists a unique $c \in \Gamma_3(a) \cap \Gamma_3(b)$.*

(c) *If $b \in X_2(a)$, then $\langle a, b \rangle \sim 2.\mathrm{Alt}(4)$, $M_{ab} \sim 4.\mathrm{Sym}(4)$ and $|X_1(a) \cap X_1(b)| = 4$.*

(d) *If $b \in X_3(a)$, then $\langle a, b \rangle \sim 2.\mathrm{Alt}(5)$, $M_{ab} \sim (\mathrm{Q}_8 \times \mathrm{C}_3).2$ and $|X_1(a) \cap X_1(b)| = 1$.*

(e) *If $b \in X_4(a)$, then $\langle a, b \rangle \sim 5^{1+2}.3$, $M_{ab} \sim \mathrm{D}_{10}$ and $X_1(a) \cap X_1(b) = \emptyset$.*

Proof: Let t be an involution in M' (so an element of Γ_4). Then $C_M(t) \sim 2^{\cdot}\mathrm{Sym}(8)$ and a group of order three in $C_M(t)$ is in Γ_2 if and only if it projects to a 3-cycle in Sym(8). Furthermore, as alluded to in the description of F above, we can find elements from Γ_2 in $C_M(F)$. Thus we see that two elements from Γ_2 may generate the groups indicated in parts (b)-(e) of the lemma. Hence M_a has at least 5 orbits on Γ_2. We next determine the stabilisers of these configurations. Assume that $a, b \in \Gamma_2$ with $a \neq b$ and suppose that $c \in \Gamma_3$ and $\langle a, b\rangle \leq c$. Then, by 4.1 (c), $M_{ab} \sim 3^4 : (\mathrm{C}_2 \times \mathrm{Q}_8)$ and $c = O_3(M_{ab})$. In particular, $\Gamma_3(a) \cap \Gamma_3(b) = \{c\}$.

In the cases when $\langle a, b\rangle = 2^{\cdot}\mathrm{Alt}(4)$ or $2^{\cdot}\mathrm{Alt}(5)$, M_{ab} centralizes the central involution of $\langle a, b\rangle$ and we may therefore calculate in $2^{\cdot}\mathrm{Sym}(8)$ to find the structure of M_{ab}. Finally, in case (e), we have $M_{ab} \leq L \sim 5^{1+2}.3.8.2$, from which we calculate $M_{ab} \cong \mathrm{D}_{10}$. Thus we have the orders of each $X_i(a)$. Since $|\Gamma_2| = 15400 = \sum_{i=0}^{4} |X_i(a)|$, we deduce that $\{X_i(a) \mid 0 \leq i \leq 4\}$ is the set of orbits of M_a on Γ_2. Observing that $X_1(a) \cap X_1(b) = (M_{ab} \cap \Gamma_2) \setminus \{a, b\}$ now completes all the parts of the lemma.

Our next lemma helps us to recognize the orbits of M_a on Γ_2 in terms of the action of elements of Γ_2 on Λ.

Lemma 4.5 (a) *Every maximal clique in Λ has size 5 and is of the form Λ^a for some $a \in \Gamma_2$. Every clique of size three lies in a unique maximal clique and every clique of size 2 lies in 10 maximal cliques.*

(b) *Let Φ be a maximal clique in Λ and $x \in \Lambda \setminus \Phi$. Then x is adjacent to exactly two elements of Φ.*

(c) *Suppose that $a, b \in \Gamma_2$. Then*

(ca) $\Lambda^a = \Lambda^b$ if and only if $a = b$;

(cb) $|\Lambda^a \cap \Lambda^b| = 2$ if and only if $b \in X_1(a)$;

(cc) $|\Lambda^a \cap \Lambda^b| = 1$ if and only if $b \in X_2(a)$;

(cd) if $\Lambda^a \cap \Lambda^b = \emptyset$ and there exists a clique of size four intersecting Λ^a and Λ^b in sets of size two, then $b \in X_3(a)$; and

(ce) if $\Lambda^a \cap \Lambda^b = \emptyset$ and no clique of size four intersects Λ^a and Λ^b in sets of size two, then $b \in X_4(a)$.

Proof: Let $x \in \Lambda$. Then from [McL] we know that $\Lambda(x)$ consists of 112 vertices and for $y \in \Lambda(x)$, $|\Lambda(x) \cap \Lambda(y)| = 30$. Hence the induced subgraph on $\Lambda(x)$ is isomorphic to the graph on the singular 1-spaces in a 6-dimensional non-degenerate orthogonal space of "$-$"-type over GF(3) in which two 1-spaces are incident if and only if they are perpendicular. Since maximal cliques in the graph $\Lambda(x)$ have just 4 members, a maximal clique in Λ has exactly 5-vertices. Furthermore, since every clique of size two in $\Lambda(x)$ is contained in a unique maximal clique in $\Lambda(x)$ we have that every clique of size three is contained in a unique maximal clique. Similarly, we have that every clique of size two is contained in exactly ten maximal cliques.

Since the elements of $\Gamma_2 \cap M_x$ fix a clique of size 4 in $\Lambda(x)$, we have that for $a \in \Gamma_2$, Λ^a contains a clique $\Phi = \{a\} \cup \Lambda(x)^a$ of size 5. Assume that there is a $y \in \Lambda^a \setminus \Phi$. Then $y \notin \Lambda(x)$. From the structure of Λ [McL], y is incident to exactly 56 elements of $\Lambda(x)^a$. Since 56 is not divisible by 3, y is incident to some $z \in \Lambda(x)^a \subset \Phi$. But then $|\Lambda(z)^a| = 5$, a contradiction. Using the fact that for $a, b \in \Gamma_2 \cap U$, $\langle a, b\rangle \sim 3^2$ or $2{\cdot}\mathrm{Alt}(4)$ we deduce that the first three statements in (c) are also true.

Let $a, b \in \Gamma_2$ with $\Lambda^a \cap \Lambda^b = \emptyset$. Note that $X_1(a) \cap X_1(b) \neq \emptyset$ if and only if there exists a clique of size five, Λ^c intersecting Λ^a and Λ^b in sets of size two. Thus, using 4.4 (e), the last two statements in (c) hold.

To prove (b) we note first that by (a), x is adjacent to at most two elements of Φ. Let P be one of the ten sets of size two in Φ. Then P lies in 9 cliques of size 5 different to Φ. Each of these cliques contains three elements not contained in Φ. Hence we have found $10 \cdot 9 \cdot 3 = 270$ distinct elements in Λ which are adjacent to exactly two elements of Φ. Since $|\Lambda \setminus \Phi| = 275 - 5 = 270$, (b) is proved.

Recall that $L = N_M(F) \sim 5^{1+2}.3.8.2$ and Δ is the set of subgroups of $O_5(L)$ of order 5 which are not equal to F.

Lemma 4.6 *L has two orbits Ξ and Θ on Λ. Furthermore,*

(a) $|\Xi| = 125$, $\Xi = \bigcup_{d\in\Gamma_2^F} \Lambda^d$ and $O_5(L)$ acts regularly on Ξ; and

(b) $|\Theta| = 150$, $\Theta = \bigcup_{e\in\Delta} \Lambda^e$, $O_5(L)$ has 6 orbits of length 25 on Θ and $O_{5,2}(L)$ has three orbits $\Lambda_i(L), 1 \leq i \leq 3$ each of length 50 on Θ.

Proof: Let E be a group of order five in U. Then, as L contains a Sylow 5-subgroup of M, without loss of generality we may assume that $E \leq L$. Notice also that F does not fuse into U, and hence F operates semiregularly on Λ. Since there are 6 choices for $\langle E, F\rangle$ in $O_5(L)$ and L acts transitively on such subgroups, it follows that $O_5(L)$ has a multiple of 6 orbits of length 25 on Λ and the remaining orbits are regular. We conclude that $O_5(L)$ has one regular orbit and 6 orbits of length 25; the latter orbits are all fused into a single orbit by L. Hence L has two orbits on Λ. Let Ξ be the orbit of length 125 and Θ be the orbit of length 150. Notice that $\bigcup_{a\in\Gamma_2^F} \Lambda^a$ and $\bigcup_{e\in\Delta} \Lambda^e$ are both L-invariant. By 4.5 (c), for $a, b \in \Gamma_2^F$, $\Lambda^a \cap \Lambda^b = \emptyset$ and so $\left|\bigcup_{a\in\Gamma_2^F} \Lambda^a\right| = 5|\Gamma_2^F| = 5 \cdot 25 = 125$. Therefore, (a) holds. Since also $\bigcup_{e\in\Delta} \Lambda^e$ is L-invariant and $O_5(L)$ is regular on Ξ, we must have $\Theta = \bigcup_{e\in\Delta} \Lambda^e$. Thus (b) also holds.

Remember that $K \sim \mathrm{U}_3(5).2$ and that the *Hoffman-Singleton graph* $\mathcal{H}$ may be constructed on the cosets K'/A where A is any subgroup of K' isomorphic to Alt(7) and two cosets $x, y \in K'/A$ are incident if and only if $K'_{xy} \cong \mathrm{Alt}(6)$. Note that, as $F \leq K$, $O_5(L) \leq K$.

Lemma 4.7 *Let $A_i, 1 \le i \le 3$, be representatives for the three conjugacy classes of subgroups isomorphic to* Alt(7) *in K' and let $\mathcal{H}$ be the Hoffman-Singleton graph with vertex stabilisers the conjugates of A_1. Then*

(a) *$\mathcal{H}$ has diameter 2, has no cycles of length less than five and K' acts transitively on paths of length three and on cycles of length five.*

(b) *The orbits of A_1 on $\mathcal{H}$ have lengths $1, 7$ and 42 and the orbits of A_2 and A_3 on $\mathcal{H}$ have lengths 15 and 35.*

(c) (ca) *$O_5(L)$ has two orbits $\mathcal{L}^1$ and $\mathcal{L}^2$ on $\mathcal{H}$.*

(cb) *For $i = 1, 2$, F has five orbits $\mathcal{L}^i_k, 1 \le k \le 5$ on $\mathcal{L}^i$. Each $\mathcal{L}^i_k$ is, as a subgraph of $\mathcal{H}$, a cycle of length five.*

(cc) *For a fixed $i \in \{1, 2\}$ and for $j, k \in \{1, \ldots, 5\}$ with $j \ne k$, no two elements of $\mathcal{L}^i_j$ and $\mathcal{L}^i_k$ are adjacent.*

(cd) *Let $1 \le j \le k \le 5$. Then $\mathcal{L}^1_j \cup \mathcal{L}^2_k$ is a Petersen graph.*

Proof: Part (a) is well-known and also easily verified. The first part of (b) follows from (a) so to prove (b) it remains to determine the orbits of A_2 and A_3 on $\mathcal{H}$. Note that A_1 and A_2 both contain a Sylow 3-subgroup and a Sylow 7-subgroup of K'. Hence, as no proper subgroup of Alt(7) contains both a Sylow 3-Subgroup and a Sylow 7-subgroup, there exist orbits of A_2 on $\mathcal{H}$ (respectively A_3 on $\mathcal{H}$) one which has length divisible by 3 and not by 7 and one which has length divisible by 7 and not by 3. On the other hand, the Sylow 5-subgroups of A_1 and A_2 are not conjugate in K' and so all orbits of A_2 on $\mathcal{H}$ (respectively A_3 on $\mathcal{H}$) have length divisible by 5. This gives (b).

Since $|\mathcal{H}| = 50$, we see that $O_5(L)$ has 2 orbits $\mathcal{L}^1$ and $\mathcal{L}^2$ on $\mathcal{H}$. So (ca) holds. Let t be an involution in $C_{K'}(F)$. Then $O_5(L)\langle t\rangle$ operates transitively on $\mathcal{H}$. Moreover, $C_{K'}(t) \sim 2.\mathrm{Sym}(5)$, $C_{A_1}(t) \sim 2.(\mathrm{Sym}(3) \times \mathrm{Sym}(2))$ and $C_{\mathrm{Alt}(6)}(t) \sim 2.(\mathrm{Sym}(2) \times \mathrm{Sym}(2))$. Thus $\mathrm{Fix}_{\mathcal{H}}(t)$ is a Petersen graph, and, as $F \le C_{K'}(t)$, t fixes pointwise one F-orbit in $\mathcal{L}^1$ and one F-orbit in $\mathcal{L}^2$. As there are exactly 25 involutions in $O_5(L)\langle t\rangle$ and no two involutions in $O_5(L)\langle t\rangle$ have the same fixed points we deduce that (cb) and (cd) are true. Finally, since the valency of every vertex is 7, and 7 neighbours to each vertex have been accounted for in (cb) and (cd), we deduce that (cc) also holds.

Lemma 4.8 *Suppose that $\mathcal{H}$ is a Hoffman-Singleton graph for K' and $\mathcal{K}$ is the graph with vertices the edges of $\mathcal{H}$ and two vertices incident if and only if they are the first and last edges of a path of length 3 in $\mathcal{H}$.*

(a) *Every clique of size 2 in $\mathcal{K}$ is contained in 10 maximal cliques.*

(b) *Let Ψ be a maximal clique in $\mathcal{K}$. Then either $|\Psi| = 5$ or $|\Psi| = 3$ and there exists a cycle $(a_1, a_2, a_3, a_4, a_5, a_6)$ of length 6 in $\mathcal{H}$ such that Ψ corresponds to $\{(a_1, a_2), (a_3, a_4), (a_5, a_6)\}$.*

Proof: Suppose that $c = \{c_1, c_2\}$ is a clique of size 2 in $\mathcal{K}$. Then c corresponds to a path of length 3 in $\mathcal{H}$ and, as such, by 4.7 (a), is unique up to conjugation. Thus, by 4.7 (cb), we may suppose that c is a path of length 3 in $\mathcal{L}_1^1$. Now, by 4.7 (cc), no element of c is incident to any other edge in $\mathcal{L}_1 \setminus \mathcal{L}_1^1$. Thus any maximal clique containing c must be contained in $P = \mathcal{L}_1^1 \cup \mathcal{L}_k^2$ for some $k \in \{1, \ldots, 5\}$, which, by 4.7 (cd), is a Petersen graph. Calculation in P shows that c is contained in exactly two maximal cliques: one of size 5 and one which consists of every other edge in a cycle of length 6. Hence parts (a) and (b) hold.

Lemma 4.9 (a) *On Λ, K has two orbits $\Lambda_0(K)$ and $\Lambda_1(K)$ of lengths* 100 *and* 175 *respectively. K' acts transitively on $\Lambda_1(K)$ and has two orbits of length* 50*, $\Lambda_2(K)$ and $\Lambda_3(K)$, on $\Lambda_0(K)$.*

(b) *As a K-set $\Lambda_1(K)$ is isomorphic to the edges in a Hoffman-Singleton graph $\mathcal{H}$. Two such edges are adjacent in $\Lambda_1(K)$ if they are the first and the last edge of some path of length three in $\mathcal{H}$.*

(c) *$\Lambda_2(K)$ and $\Lambda_3(K)$ are Hoffman-Singleton graphs and as K'-sets neither isomorphic to each other nor to $\mathcal{H}$.*

(d) *$\Lambda_1(K) = \Xi \cup \Lambda_i(L)$ for some $i \in \{1, 2, 3\}$.*

(e) *Suppose that Ψ is a maximal clique in Λ. Then $|\Psi \cap \Lambda_1(K)| = 1$, 3 or 5.*

Proof: It follows from 4.6 (a) and (b) that $L \cap K$ has orbits of length $125, 50$ and 100 on Λ. As $M = \langle L, K \rangle$, K does not normalize the orbit of length 125. Furthermore, since 275 does not divide $|K|$, K does not act transitively on Λ. Also K does not have a subgroup of index 225 and so we deduce that the orbits of K must have length 175 and 100. In particular, 4.6 implies that part (d) holds. Since K has a unique conjugacy class of subgroups of index 175, $\Lambda_1(K)$ is as a K-set isomorphic to the set of edges of a Hoffman-Singleton graph $\mathcal{H}$. Therefore, K' is transitive on $\Lambda_1(K)$. Moreover, K' has no subgroup of index 100 and hence K' has two orbits each of length 50 on $\Lambda_0(K)$. So part (a) holds. Since elements of order five have at most five fixed points on Λ, $\Lambda_2(K), \Lambda_3(K)$ and $\mathcal{H}$ are as K'-sets pairwise non-isomorphic. Let $i \in \{2, 3\}$. Then $\Lambda_i(K)$ is, as a graph, either isomorphic to the Hoffman-Singleton graph or its complement. But the latter has cliques of size larger than five which is against 4.5 (a). So (c) holds.

To prove the second statement of (b) let x, y be two distinct elements of $\Lambda_1(K)$. Define $\delta(x, y)$ to be the minimal distance of a vertex of $\mathcal{H}$ on x (remember x corresponds to an edge of $\mathcal{H}$) to a vertex of $\mathcal{H}$ on y. Assume now in addition that x, y are adjacent in Λ. If $\delta(x, y) = 0$, the seven edges on a vertex would form a clique of size 7 in Λ, which is impossible as a maximal clique has size 5. So assume that $\delta(x, y) = 2$. We use the notation of 4.7 (c). Let x_k be an edge in $\mathcal{L}_k^1$ and x_0 some edge in $\mathcal{L}_1^1$ different from x_1. Then $\{x_1, x_2, x_3, x_4, x_5\}$

and $\{x_0, x_2, x_3, x_4, x_5\}$ are cliques of size 5 in Λ which intersect in a clique of size 4, a contradiction to 4.5 (a). Thus x and y are adjacent if and only if $\delta(x, y) = 1$ and so (b) holds.

Finally, suppose that Ψ is a maximal clique in Λ. As, by 4.5 (a), each clique of size 2 in Λ is contained in exactly 10 maximal cliques and as a clique of size 2 in $\Lambda_1(K)$ is contained in 10 maximal cliques of $\Lambda_1(K)$ by 4.8, we see that $\Psi \cap \Lambda_1(K)$ has size 0, 1, 3 or 5. Because, by (c), $\Lambda_0(K)$ is the union of two Hoffman-Singleton graphs, 4.7 (a) implies that $\Psi \not\subseteq \Lambda_0(K)$. Hence $|\Psi \cap \Lambda_1(K)| = 0$ is impossible. Therefore, part (e) holds.

The next proposition is the ultimate aim of our preliminary calculations within Aut(McL).

Proposition 4.10 *Suppose that $a \in \Gamma_2$. Then $\Gamma_2^F \not\subseteq X_4(a)$.*

Proof: We use the notation of 4.6 and 4.9. If there exists $u \in \Xi \cap \Lambda^a$, then, by 4.6 (a), there exists $b \in \Gamma_2^F$ with $u \in \Lambda^b$. Thus $\Lambda^a \cap \Lambda^b \neq \emptyset$ and 4.5 (c) implies that $b \notin X_4(a)$ and the proposition holds.

Therefore we may assume that $\Xi \cap \Lambda^a = \emptyset$ and so 4.6 implies $\Lambda^a \subseteq \Theta$. Choose notation in accordance with 4.9 so that $\Lambda_1(K) = \Xi \cup \Lambda_1(L)$ and $\Lambda_0(K) = \Lambda_2(L) \cup \Lambda_3(L)$. Because $\Lambda^a \cap \Theta = \emptyset$, 4.5 (a) and 4.9 (e) imply that $|\Lambda^a \cap \Lambda_1(L)| = |\Lambda^a \cap \Lambda_1(K)| = 1, 3$ or 5. Conjugation by elements of L permutes $\{\Lambda_i(L) \mid 1 \leq i \leq 3\}$ and yields $|\Lambda^a \cap \Lambda_i(L)| = 1, 3$ or 5 for all $1 \leq i \leq 3$. Hence Λ^a intersects one of the $\Lambda_i(L)$ in a set of size 3 and the other 2 in sets of size one. Without loss of generality we may assume that $\Lambda^a \cap \Lambda_1(L) = \Lambda^a \cap \Lambda_1(K)$ has size 3. Set $\Phi = \Lambda^a \cap \Lambda_1(L)$ and recall that, by 4.9 (b), $\Lambda_1(K)$ is the set of edges of a Hoffman-Singleton graph $\mathcal{H}$ with two edges adjacent if they are the ends of a path of length three in $\mathcal{H}$. Moreover, the edges of $\mathcal{H}$ which lie in $\mathcal{L}^1 \cup \mathcal{L}^2$ correspond to the vertices of $\Lambda_1(L)$ and the vertices of Ξ correspond to edges which join $\mathcal{L}^1$ to $\mathcal{L}^2$. As, by 4.5 (a), Φ is a maximal clique in $\Lambda_1(K)$, 4.8 (b) implies that Φ is every second edge on a cycle Π of length six in $\mathcal{H}$. Let Ψ consist of the three edges on Π which are not contained in Φ. Then Ψ is a clique of size 3 in Λ. Let $b \in \Psi$. Then b is not adjacent to any of the elements in Φ and so by 4.5 (b) is adjacent to both of the elements in $\Lambda^a \setminus \Phi$. Hence $\Psi \cup (\Lambda^a \setminus \Phi)$ is a clique of size 5 and thus, by 4.5 (a), $\Psi \cup (\Lambda^a \setminus \Phi) = \Lambda^c$ for some $c \in \Gamma_2$. Since Π is a closed path, an even number of edges of Π join $\mathcal{L}^1$ to $\mathcal{L}^2$. Thus, as $\Phi \cap \Xi = \emptyset$, $|\Xi \cap \Psi| = |\Xi \cap \Lambda^c| = 2$. Therefore, without loss of generality we may assume that $\Psi \cap \mathcal{L}^1 = \{e\}$ and $e \in X$ where $X \subseteq \mathcal{L}^1$ is an F-orbit. Since $\Phi \cap \Xi = \emptyset$, we deduce from 4.7 (c) that the two edges of Φ which are incident as edges in $\mathcal{H}$ to e also lie in X. Let $Y \subseteq \mathcal{L}^2$ be the F-orbit on $\mathcal{H}$ which contains the remaining element of Φ and $\mathcal{C}$ be the clique corresponding to the edges joining X to Y in the Petersen graph $X \cup Y$. Then $\mathcal{C}$ is an F-orbit on Ξ and so $\mathcal{C} = \Lambda^b$ for some $b \in \Gamma_2^F$. But then $|\Lambda^b \cap \Lambda^c| \geq 2 \leq |\Lambda^c \cap \Lambda^a|$ which, by 4.5 (c), implies that $c \in X_1(a) \cap X_1(b)$. Finally, 4.4, implies that $b \notin X_4(a)$ and we conclude that $\Gamma_2^F \not\subseteq X_4(a)$.

Lemma 4.11 (a) *Let Φ be a maximal clique in Λ and $c \in \Lambda \setminus \Phi$. Then $M'_{c\Phi}$ induces* $\mathrm{Sym}(3) \times \mathrm{Sym}(2)$ *on Φ.*

(b) *There exists a unique class of tuples (Φ, a, b, c), where Φ is a maximal clique in Λ, $a \neq b \in \Phi$ and c is an element of Λ neither adjacent to a nor to b.*

(c) *Let (Φ, a, b, c) be any tuple as in (b) and let $\{d, e\} = \Phi(c)$. Then $M'_{abc} \cong$* $\mathrm{Alt}(6)$ *and $O_3(M_{de}) \cap M_{abc}$ is a Sylow 3-subgroup of M_{abc}.*

Proof: By 4.5 (b), c is incident to exactly two elements of Φ. Let $\{e, d\} = \Phi(c)$. Put $Q = O_3(M_{de})$ and let Ψ be the unique maximal clique in Λ containing d,e and c. Then $O_3(M_\Phi)$ acts transitively on the nine maximal cliques containing $\{a, b\}$ but different from Φ. Also $Q = Q \cap O_3(M_\Phi)(Q \cap O_3(M_\Psi))$ and so, as Q preserves every clique containing $\{d, e\}$, $O_3(M_\Phi)$ acts transitively on $\Psi \setminus \{d, e\}$. Hence we see that the pair (Φ, c) is unique up to conjugation under M' and $M_{\Phi c} = M_{\Phi\{de\}} O_3(M_\Phi)$. Thus (a) and (b) hold. Moreover, $|M'_{abc} \cap Q| = 3^2$ and $M'_{abc\Phi} \sim 3^2.4$. By (b), M'_{abc} acts transitively on the ten maximal cliques containing a and b and so $|M'_{abc}/M'_{abc\Phi}| = 10$. Thus M'_{abc} is a subgroup of M'_{ab} of order 360 and so $M'_{abc} \cong \mathrm{Alt}(6)$.

5 $3^{\cdot}\mathrm{Aut}(\mathrm{McL})$ and its 90-dimensional representation over $\mathrm{GF}(5)$

In this section we determine the Schur multiplier of McL, construct a unique non-split group $3^{\cdot}\mathrm{Aut}(G)$ and prove that it has a unique 90-dimensional representation over GF(5).

Lemma 5.1 *The following hold:*

(i) The Schur multiplier of McL *has order* 3.

(ii) There exists a group $G = 3^{\cdot}\mathrm{Aut}(\mathrm{McL})$ and $O_3(G)$ is inverted by G.

Proof: Let $M = \mathrm{Aut}(\mathrm{McL})$, $G = M'$ and $m(G) = |\mathrm{H}^2(G, \mathbb{C}^*)|$ be the order of the Schur multiplier of G. Then, since $|G| = 2^7 \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11$, $m(G)$ is prime to 7 and 11 [Asch, 33.14]. Moreover, as the Schur multiplier of $2^{\cdot}\mathrm{Alt}(8)$ is trivial and the Schur multiplier of $\mathrm{U}_3(5)$ is a 3-group, $m(G)$ is a power of 3. Suppose that $\widetilde{G}$ is the universal covering group of G and let $A = Z(\widetilde{G})$. We shall show first that $|A| \leq 3$. Recall that G and hence $\widetilde{G}$ acts on the graph Λ of Section 4. Let Φ, a, b, c, d, e be as in 4.11 and set $Q_{ab} = O_3(G_{ab})$. Since both G_a and G_{ab} contain Sylow 3-subgroups of G, Gaschütz' Theorem implies that $\widetilde{G_a}$ and $\widetilde{G_{ab}}$ are also perfect central extensions. Since G_{ab} does not fix a symplectic form on Q_{ab}, we conclude that $\widetilde{Q_{ab}}$ is abelian. In particular, 4.11 (c) implies that $\widetilde{G_{abc}}$ has abelian Sylow 3-subgroups. Therefore, $\widetilde{G_{abc}}$ splits over A and is thus a complement to $\widetilde{Q_{ab}}$ in $\widetilde{G_{ab}}$. Hence, as $\widetilde{G_{ab}}$ is perfect, we must have $A \leq [\widetilde{Q_{ab}}, \widetilde{G_{ab}}]$. Since, by 2.5 (f), $|\mathrm{H}^1(G_{abc}, Q_{ab})| = 3$, we conclude that $|A| \leq 3$ as claimed. To complete the lemma it only remains to prove that $m(G) \geq 3$. To achieve this we employ what Griess calls the stable cocycle method [Gr]. Let B be a cyclic group of order 3 which is centralized by G and inverted by M. We recall from [CE, p. 257] that a cocycle in $\mathrm{H}^2(M_a, B)$ is M-stable if the conjugation map $c_g : \mathrm{H}^2(M_a \cap M_a^g, B) \to \mathrm{H}^2(M_a^{g^{-1}} \cap M_a, B)$ makes the triangular diagram made from the restriction maps $\mathrm{H}^2(M_a, B) \to \mathrm{H}^2(M_a \cap M_a^g, B)$ and $\mathrm{H}^2(M_a, B) \to \mathrm{H}^2(M_a^{g^{-1}} \cap M_a, B)$ commute. Clearly, it suffices to check this for representatives of the double cosets of M_a in M. In our special case we can choose x_0, x_1, x_2 double M_a-coset representatives so that $x_0 = 1$, $x_1 \in Z(M_{\{a,b\}c})^{\#}$ and $a^{x_2} = c$. Because M_a inverts B, we have $|\mathrm{H}^2(M_a, B)|_3 = 3$ from [ATLAS]. Let r be a non-trivial cocycle in $|\mathrm{H}^2(M_a, B)|$. To show that r is M-stable it suffices to show that, for $i \in \{0, 1, 2\}$, the restricted cocycle z_i in $\mathrm{H}^2(M_a \cap M_a^{x_i}, B)$ is $(M_a \cap M_a^{x_i})\langle x_i \rangle$-stable. We continue to use the notation of the first part of the lemma. Since $M_{abc} \cong \mathrm{Mat}_{10}$ and M_{abc} inverts B, $\mathrm{H}^2(M_{abc}, B) = 0$ [ATLAS]. Thus the restriction of r to M_{abc} is trivial. As M_{abc} contains a Sylow 3-subgroup of M_{ac}, z_2 is trivial and so clearly $M_{\{a,c\}}$-stable. Next we recall that $M_{ab} = M_{abc}Q_{ab}$, x_1 centralizes M_{abc} and x_1 inverts Q_{ab}. But this clearly implies z_1 is $M_{ab}\langle x_1 \rangle$-stable. We conclude that r is indeed M-stable and so the lemma follows from [CE, Proposition 9.4].

Our next objective is

Theorem 5.2 *Suppose that $G \cong 3{}^{\cdot}\mathrm{McL}$. Then G has a faithful irreducible representation of degree 45 in characteristic 5.*

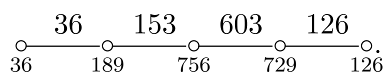
For the proof of 5.2 we set $H = 3_2{}^{\cdot}\mathrm{U}_4(3)$ ([ATLAS] notation), $G = 3{}^{\cdot}\mathrm{McL}$ and let k denote the algebraic closure of the field with 5 elements. We follow the Atlas [ATLAS] for conjugacy class names.

In Table 1 we give the faithful part of the ordinary character table of $H = 3_2{}^{\cdot}\mathrm{U}_4(3)$. The elements from classes 3B, 3C, 3D, 9A, 9B, 9C and 9D are not listed as their character values are all zero. We also recall that $b7 = \frac{-1+\sqrt{-7}}{2}$ and $b7^{**} = \frac{1+\sqrt{-7}}{2}$.

1A	2A	3A	4A	4B	5A	6A	6B	6C	7A	B**	8A	12A
36	4	9	4	0	1	1	-2	-2	1	1	0	1
45_1	-3	-9	1	1	0	3	0	0	b7	**	-1	1
45_2	-3	-9	1	1	0	3	0	0	**	b7	-1	1
126	14	-9	2	2	1	-1	2	2	0	0	0	-1
189	-3	27	5	1	-1	3	0	0	0	0	1	-1
315_1	11	18	-1	-1	0	2	2	2	0	0	1	2
315_2	-5	18	3	-1	0	-2	-2	4	0	0	-1	0
315_3	-5	18	3	-1	0	-2	4	-2	0	0	-1	0
630	6	-45	2	-2	0	3	0	0	0	0	0	-1
720	16	18	0	0	0	-2	-2	-2	-1	-1	0	0
729	9	0	-3	1	-1	0	0	0	1	1	-1	0
756	-12	27	-4	0	1	3	0	0	0	0	0	-1
945	-15	-27	1	1	0	-3	0	0	0	0	1	1

Table 1: Ordinary faithful irreducible characters of $3_2{\cdot}U_4(3)$

Lemma 5.3 *The faithful non-projective kH-modules form a unique block with Brauer tree as follows:*



Proof: All the character values in Table 1 which are divisible by 5 remain irreducible when reduced modulo 5 and are projective. So they form blocks of size 1. This leaves the five characters of degrees 36, 126, 189, 729 and 756. By [HL, 2.1.5], these characters form a single block $\mathcal{B}$. Furthermore, [HL, 2.1.22] tells us that the node of highest degree is not an end node. Thus the node labelled 756 is internal.

Since, by [HL, 2.1.5], two consequetive nodes correspond to a 5-modular projective indecomposable module, the sum of the degrees at those nodes has to be a multiple of 5. It follows that we have four possible Brauer trees for $\mathcal{B}$. On the other hand, from the ordinary character table of $U_4(3)$ [ATLAS], we see that H has two non-faithful, irreducible, projective 5-modular representation of degree 35, 35_1 and 35_2. Thus we may form the faithful, projective kH-module $35_1 \otimes 36$. Decomposing the corresponding complex character we have that

$$35_1 \otimes 36 = 36 + 189 + 315_3 + 720$$

and this establishes the existence of a projective 5-modular character of dimension 225. This now leaves two possible Brauer trees for $\mathcal{B}$. The false tree indicates that there should be a faithful irreducible kH-module of dimension

1A	2A	3A	4A	4B	6A	6B	6C	7A	B**	8A	12A
36	4	9	4	0	1	-2	-2	1	1	0	1
45_1	-3	-9	1	1	3	0	0	b7	**	-1	1
45_2	-3	-9	1	1	3	0	0	**	b7	-1	1
126	14	-9	2	2	-1	2	2	0	0	0	-1
153	-7	18	1	1	2	2	2	-1	-1	1	-2
315_1	11	18	-1	-1	2	2	2	0	0	1	2
315_2	-5	18	3	-1	-2	-2	4	0	0	-1	0
315_3	-5	18	3	-1	-2	4	-2	0	0	-1	0
603	-5	9	-5	-1	1	-2	-2	1	1	-1	1
630	6	-45	2	-2	3	0	0	0	0	0	-1
720	16	18	0	0	-2	-2	-2	-1	-1	0	0
945	-15	-27	1	1	-3	0	0	0	0	1	1

Table 2: 5-modular faithful irreducible characters of $3_2{\cdot}\mathrm{U}_4(3)$

27. This is immediately shown to be impossible by considering the subgroup of shape $H_3 \sim 3^{1+4}.\mathrm{Alt}(6)$ of H. Indeed, this subgroup has orbits of length 36 and 45 on the maximal subgroups of $O_3(H_3)$ which do not contain the centre of H. This shows that the minimal conceivable dimension for a faithful kH-module is 36. Hence the correct Brauer tree is the one described in the lemma. Alternatively, consult [MATLAS].

As a consequence of 5.3 we obtain the faithful part of the 5-modular character table of $3_2{\cdot}\mathrm{U}_4(3)$. This is recorded in Table 2.

Lemma 5.4 (a) $\mathrm{U}_4(3)$ *has a unique faithful irreducible ordinary character of degree less than* 22*. Partial information for this character is as follows:*

1A	*2A*	*3A*	*4A*	*4B*	*5A*	*6A*	*6B*	*6C*	*7A*	*B***	*8A*	*12A*
21	*5*	*-6*	*1*	*1*	*1*	*2*	*-1*	*-1*	*0*	*0*	*-1*	*-2*

(b) McL *has a unique irreducible ordinary character of degree* 22*. Partial character information is as follows:*

1A	2A	3A	3B	4A	5A	5B	6A	6B	7A	B**	8A	12A
22	6	-5	4	2	-3	2	3	0	1	1	0	-1

(c) *The character of degree* 21 *for* $\mathrm{U}_4(3)$ *remains irreducible after reduction modulo* 5*.*

(d) *The character of degree* 22 *for* McL *has composition factors* $1 + 21$ *when reduced modulo* 5*.*

Proof: (a) and (b) follow from the ordinary character table of $U_4(3)$ and McL respectively (see [ATLAS]).

Let T be the subgroup of $U_4(3)$ of shape $2^4.\mathrm{Alt}(6)$. Then, by considering the eigenspaces of $O_2(T)$ and noting that T acts transitively on the maximal subgroups of $O_2(T)$, we see that any faithful module for $U_4(3)$ in odd characteristic has dimension at least 15. Hence either the 21-dimensional complex representation remains irreducible when reduced modulo 5 or it has a trivial composition factor. However, by [ATLAS], the multiplicity of the trivial module for the $5'$-subgroup $C_3 \times Q_8$ (contained in $2^{1+4}2{\cdot}\mathrm{Sym}(4)$) on the 21-space is

$$\frac{1}{24}(21 + 5 + 2 \cdot -6 + 6 \cdot 1 + 2 \cdot 2 + 12 \cdot -2) = 0$$

and hence (c) holds. Alternatively consult [MATLAS].

For (d) we note that McL is the elementwise stabilizer of a triangle of type 223 in the Leech lattice, [CS, Table 10.4]. Hence McL stabilizes vectors a, b such that $(a, a) = (b, b) = 4$ and $(a + b, a + b) = 6$. Thus $(a, b) = -1$, and $(a, a - b) = 5$ and $(b, a - b) = -5$. Hence in $\bar{\Lambda} = \Lambda/5\Lambda$ we see that McL acts on the 21-space $M = \langle \bar{a}, \bar{b}\rangle^{\perp}/\langle \overline{a - b}\rangle$. By (c), $U_4(3)$ acts irreducibly on M and so (d) holds.

Lemma 5.5 *Partial information on the fusion of conjugacy classes for H in G is as follows:*

H	1A	2A	3A	4A	4B	5A	6A	6B	6C	7A	B**	8A	12A
G	1A	2A	3A	4A	4A	5B	6A	6B	6B	7A	B**	8A	12A

Proof: By 5.4 (a), McL has a character of degree 22. By 5.4 (b) this character decomposes as $1 + 21$ when restricted to $U_4(3)$. Comparing character values we see that the fusion must be as claimed.

¿From the character table of 3·McL [ATLAS] we have:

Lemma 5.6 3·McL *has (among others) the following faithful irreducible ordinary characters:*

1A	2A	3A	4A	5B	6A	6B	7A	B**	8A	12A
1980	-36	9	4	0	9	0	-1	-1	0	1
2376_1	-24	-56	0	1	6	0	b7	**	0	0
2376_2	-24	-56	0	1	6	0	**	b7	0	0
8019_2	-45	0	3	-1	0	0	-b7	**	-1	0
8019_1	-45	0	3	-1	0	0	**	-b7	-1	0

Lemma 5.7 *Restricted to* $3_2{\cdot}U_4(3)$ *we have*

(a) $1980 = 45_1 + 45_2 + 189 + 756 + 945$.

(b) *For* $i = 1, 2$, $2376_i = 45_i + 630 + 756 + 945$.

(c) *For* $i = 1, 2$, 45_i *is a summand of* 8019_i.

Proof: For (a) and (b) one verifies that character values on the right hand side add up to the character values on the left. For (c) we claim that $(45_i, 8019_i)$ is positive. Indeed for all x not in $7A$ or $7B**$, $45_i(x) \cdot 8019_i(x^{-1})$ is a non-negative integer. Furthermore $b7 = \frac{1}{2}(-1+\sqrt{-7})$ and so $b7 \cdot (-b7)$ has 3/2 as it real part. Thus $|3_2{}^{\cdot}\mathrm{U}_4(3)| \cdot (45_i, 8019_i)$ is the sum of positive integers.

We are now ready to begin our proof of 5.2. For $i = 1, 2$ we let λ_i be the $\mathbb{C}G$-module induced from the $\mathbb{C}H$-module corresponding to the character 45_i. Similarly, let $\widehat{\lambda_i}$ be the kG-module induced from the kH-module corresponding to the 5-modular character $\widehat{45_i}$. More generally we will use "hats" to indicate modules over k.

Lemma 5.8 *For* $i = 1, 2$ *we have* $\lambda_i = 1980 + 2376_i + 8019_i$.

Proof: By 5.7, we know that 1980, 2376_i and 8019_i are summands of λ_i. Since λ_i has degree $45 \cdot 275 = 12375 = 1980 + 2376 + 8019$, the lemma holds.

Lemma 5.9 *For* $i = 1, 2$ *we have* $\widehat{\lambda_i}$ *is a projective indecomposable* kG*-module.*

Proof: Since $\widehat{45_i}$ is a projective kH-module, $\widehat{\lambda_i}$ is a projective kG-module. Suppose that $\widehat{\lambda_i}$ is the sum of two proper projective kG-submodules $\widehat{\rho_1}$ and $\widehat{\rho_2}$. As, for $j = 1, 2$, $\widehat{\rho_j}$ is projective, $\widehat{\rho_j}$ is the reduction of some $\mathbb{C}G$-module ρ_j. By 5.8, $\rho_1 + \rho_2$ is the sum of three irreducibles, and so either ρ_1 or ρ_2 is irreducible. But none of 1980, 2376 and 8019 is divisible by 125, which is a contradiction. It follows that $\widehat{\lambda_i}$ is indecomposable.

Lemma 5.10 *Restricted to* $3_2{}^{\cdot}\mathrm{U}_4(3)$,

(a) $\widehat{1980}$ *has the following composition factors:*

$$\{\widehat{45}_1, \widehat{45}_2, \widehat{36}, \widehat{153}, \widehat{153}, \widehat{603}, \widehat{945}\}; \text{ and}$$

(b) *for* $i = 1, 2$, $\widehat{2376}_i$ *has the following composition factors:*

$$\{\widehat{45}_i, \widehat{153}, \widehat{603}, \widehat{630}, \widehat{945}\}.$$

Proof: Use 5.7 and 5.3.

Let $\widehat{\mu_i}$ be the unique irreducible quotient of $\widehat{\lambda_i}$ [Alp, Theorem 5.3]. Then, as $\widehat{45}_i$ is a summand of $\widehat{1980}\,|_H$, $\widehat{1980}$ is a quotient of $\widehat{\lambda_i}$ and consequently $\widehat{\mu_i}$ is a quotient of $\widehat{1980}$. Hence every composition factor of $\widehat{\mu_i}\mid_H$ is also a composition factor of $\widehat{1980}\mid_H$ and, using exactly the same argument, is a composition factor of $\widehat{2376_i}\mid_H$. From 5.10 we conclude that $\widehat{\mu_i}\mid_H = \widehat{45_i} + a_i \cdot \widehat{153} + b_i \cdot \widehat{603} + c_i \cdot \widehat{945}$, for some $\{a_i, b_i, c_i\} \in \{0, 1\}$. Since, by 5.5, $4A$ and $4B$ fuse in 3·McL the values of $\widehat{\mu_i}\mid_H$ on $4A$ and $4B$ elements must agree. As the character values on $4A$ and $4B$ elements agree on $\widehat{45_i}$, $\widehat{153}$ and $\widehat{945}$ and do not agree on $\widehat{603}$, we have $b_i = 0$. Moreover, $\widehat{\mu_1}$ and $\widehat{\mu_2}$ are also both composition factors of $\widehat{1980}$ and so $c_1 + c_2 \leq 1$. So we may assume without loss of generality that $c_1 = 0$. Put

$$\widehat{\mu} = \widehat{\mu_1}.$$

Then we have demonstrated that either

$$\widehat{\mu}\mid_H = \widehat{45_1}$$

or

$$\widehat{\mu}\mid_H = \widehat{45_1} \oplus \widehat{153}.$$

In the first case $\widehat{\mu}$ has degree 45 and 5.2 is proven. Therefore, we assume from now on that

$$\widehat{\mu}\mid_H = \widehat{45_1} \oplus \widehat{153}$$

and seek a contradiction.

Because we will now only be considering 5-modular modules, we drop the "hats" which were used to distinguish between ordinary and 5-modular representations. By 5.4 (c), kG has a (non-faithful) 21-dimensional representation. Set

$$\sigma = 21 \otimes \mu.$$

Then σ is a faithful kG-module. We have

Lemma 5.11 $\sigma\mid_H = (153 \setminus (36 \oplus 603) \setminus 153) \oplus 45_1 \oplus 45_2 \oplus 945 \oplus 945 \oplus 603$. *In particular, as a kH-module, $21 \otimes 45_1$ is the projective cover of the* 153.

Proof: Since 45_1 is projective, $\mu\mid_H = 45_1 \oplus 153$. Hence we need to compute the kH-modules $21 \otimes 45_1$ and $21 \otimes 153$. Their Brauer characters are:

	1A	2A	3A	4A	4B	6A	6B	6C	7A	B**	8A	12A
$21 \otimes 45_1$	945	-15	54	1	1	6	0	0	0	0	1	-2
$21 \otimes 153$	3213	-35	-108	1	1	4	-2	-2	0	0	-1	4

Now it is easy to verify that $21 \otimes 45_1$ has composition factors 36, 603 and 153 twice. Since 45_1 is projective, $21 \otimes 45_1$ is projective. Since 36 and 603 occur only once in $21 \otimes 45_1$, their projective cover cannot be a summand of $21 \otimes 45_1$. By 5.3 the projective cover of 153 is $153 \setminus (36 \oplus 603) \setminus 153$. Therefore $21 \otimes 45_1$ is the projective cover of 153. Finally we note that $21 \otimes 153$ has composition factors 45_1, 45_2, 945 twice and 603. All of these modules except 603 are projective kH-modules and so $21 \otimes 153$ is completely reducible and 5.11 holds.

Let α be an outer automorphism of G normalizing H. Then for each kG-module δ, we define δ^{α} to be the kG-module with G-action given by $v \cdot g = v^{\alpha(g)}$. We will be concerned with the kG-modules $\delta^{*\alpha}$ where $*$ as usual denotes duality. Since there is a unique kG-module of dimension 21 we have $21^{*\alpha} \cong 21$. Moreover, the $\mathbb{C}H$-module $\lambda_1 = 1980 + 2376_1 + 8019_1$ is also invariant under $*\alpha$. Thus $*\alpha$ fixes both μ and $\sigma = 21 \otimes \mu$. In particular, there exists a k-linear isomorphism

$$\beta : \sigma \to \sigma^*$$

such that

$$\beta(m^g) = \beta(m)^{\alpha(g)}$$

for all $m \in \sigma$ and all $g \in G$. For $\gamma \leq \sigma$, set

$$\tilde{\gamma} = \{m \in \sigma | \phi(m) = 0 \text{ for all } \phi \in \beta(\gamma)\}.$$

Then, if γ is a kG-submodule of σ, we have $\tilde{\gamma}$ is a kG-submodule of σ and $\sigma/\tilde{\gamma} \cong \gamma^{*\alpha}$.

Let ρ be the unique kH-submodule of σ isomorphic to 45_1 and let δ be the kG-submodule of σ spanned by ρ. Notice that, since ρ is unique in σ, any kG-submodule of σ that contains a composition factor isomorphic to 45_1 contains δ. Let ϵ be a maximal kG-submodule of δ. We claim that ϵ is unique and $\delta/\epsilon \cong \mu$. First note that by the definition of δ, ρ does not lie in any proper kG-submodule of δ. In particular, $\rho \not\leq \epsilon$ and so δ/ϵ is an irreducible kG-module which restricted to H has $(\rho+\epsilon)/\epsilon \cong \rho \cong 45_1$ as a submodule. Therefore, $\delta/\epsilon \cong \mu$. Furthermore, if τ is any proper kG-submodule of δ, then $\tau \leq \epsilon$. For otherwise, since δ/ϵ is irreducible, $\delta = \tau + \epsilon$ and so $\tau/(\tau \cap \epsilon) \cong (\epsilon + \tau)/\epsilon = \delta/\epsilon \cong \mu$. This implies that τ has a kH-composition factor isomorphic to 45_1. But then $\rho \leq \tau$, which is a contradiction. Hence $\tau \leq \epsilon$ and ϵ is unique.

Since μ is $*\alpha$-invariant and $\delta/\epsilon \cong \mu$, we have $\tilde{\epsilon}/\tilde{\delta} \cong \mu^{*\alpha} \cong \mu$. Therefore, $(\tilde{\epsilon}/\tilde{\delta})|_H$ has 45_1 as a composition factor. Hence $\rho \leq \tilde{\epsilon}$, $\rho \not\leq \tilde{\delta}$ and $\delta \leq \tilde{\epsilon}$. As $\tilde{\epsilon}/\tilde{\delta}$ is irreducible, we conclude that $\tilde{\epsilon} = \delta + \tilde{\delta}$ and $\delta/(\delta \cap \tilde{\delta}) \cong \tilde{\epsilon}/\tilde{\delta} \cong \mu$. Thus $\delta \cap \tilde{\delta}$ is a maximal submodule of δ and so $\delta \cap \tilde{\delta} = \epsilon$. So

$$\tilde{\epsilon}/\epsilon \cong \delta/\epsilon \oplus \tilde{\delta}/\epsilon = (\delta + \tilde{\delta})/\epsilon \cong \mu \oplus \tilde{\delta}/\epsilon.$$

Recall that, by hypothesis, $\mu|_H = 45_1 \oplus 153$. So, as 153 is $*\alpha$-invariant, we have 153 is a composition factor of $\epsilon|_H$ if and only if it is a composition factor of $(\sigma/\tilde{\epsilon})|_H$. Hence as, by 5.11, $\sigma|_H$ only has two composition factors isomorphic to 153, we see that the composition factors isomorphic to 153 only appear in $\tilde{\epsilon}/\epsilon$. But then 153 must occur as a composition factor of $(\tilde{\delta}/\epsilon)|_H$. It follows that $\sigma|_H$ has a section isomorphic to the direct sum of two copies of 153. This is the final straw, as by 5.11, $\sigma|_H$ contains no composition factor isomorphic to $153 \oplus 153$. This contradiction draws to a close the proof of 5.2.

Corollary 5.12 $3^{\cdot}\mathrm{Aut}(\mathrm{McL})$ *has a faithful irreducible representation of degree* 90 *over* GF(5).

Suppose now that $G \cong 3^{\cdot}\mathrm{Aut}(\mathrm{McL})$ and let

$$G_2 \sim 3^{1+1+4}.4.\mathrm{Sym}(5)$$

and

$$G_3 \sim 3^{1+4}.(\mathrm{C}_2 \times \mathrm{Mat}_{10})$$

be subgroups of G with

$$G_{23} := G_2 \cap G_3 \sim 3^{1+1+2+1}(\mathrm{C}_2 \times 3^2.\mathrm{Q}_8) = 3^{1+1+2+1}(\mathrm{C}_2 \times \mathrm{Mat}_9)$$

Set $Z = O_3(G)$, $Q_2 = O_3(G_2)$ and $Q_3 = O_3(G_3)$. Now let (G_2, G_3) be the abstract amalgam isomorphic to the amalgam just described in G.

Lemma 5.13 *Any two faithful irreducible* 90*-dimensional* GF(5)*-modules for the amalgam* (G_2, G_3) *are isomorphic. In particular, if* ϕ *is a faithful irreducible representation of the amalgam* (G_2, G_3) *into* $\mathrm{GL}_{90}(5)$*, then* $\langle G_2^\phi, G_3^\phi \rangle \cong 3^{\cdot}\mathrm{Aut}(\mathrm{McL})$.

Proof: Let V be any faithful irreducible 90-dimensional GF(5)-module for the amalgam (G_2, G_3), and view V as a 45-dimensional semilinear representation over GF(25). Note that both the normalizer of a Sylow 5-subgroup and a Sylow 2-subgroup are maximal, $3'$-subgroups of Mat_{10}. It follows that G_3 has two orbits $\mathcal{L}$ and $\mathcal{H}$ of lengths 36 and 45 on the maximal subgroups of Q_3 not containing Z. Since $[V, Q_3]$ is a direct sum of the centralizers in V of the maximal subgroups of Q_3 and Z fixes only 0 in V, we deduce that the Wedderburn components for Q_3 on $V = [V, Q_3] = [V, Z]$ are 1-dimensional and are transitively permuted by G_3. Moreover, if $C_V(H) \neq 0$ for H maximal in Q_3, then $H \in \mathcal{H}$ and $N_{G_3}(H) \sim 3^5.(\mathrm{C}_2 \times \mathrm{SD}_{16})$.

Let X_2 be the group of order three normal in G_2 and different from Z and A, B be the other two maximal subgroups of $\langle X_2, Z \rangle$. Note that $O^2(G_2)/Y \sim 3^{1+4}.2^{\cdot}\mathrm{Alt}(5)$ and Q_2/Y is extraspecial for each $Y \in \{Z, X_2, A, B\}$. Furthermore, we know that V has an $O^2(G_2)$-invariant decomposition

$$V = C_V(X_2) \oplus C_V(A) \oplus C_V(B).$$

Since G_{23} has orbits of length 9 and 36 on $\mathcal{H}$, V decomposes as a G_{23}-module into a direct sum of irreducible modules Z_4 and Z_5 of dimensions 9 and 36 respectively. Since $C_V(A)$ has dimension at least 9, we have $Z_4 = C_V(X_2)$ and $Z_5 = [V, X_2]$. In particular, Z_4 and Z_5 are also invariant under G_2. Thus, by 2.6 and 2.7 (a), Z_4 and Z_5 are uniquely determined when restricted to $O^2(G_2)$.

Suppose that $H \in \mathcal{H}$ and define $V^H = C_V(H)$. So V^H is 1-dimensional. To complete the determination of V as a G_3-module, we need to determine the action of $N_{G_3}(H)$ on V^H. Let S^* be a Sylow 2-subgroup of $N_{G_3}(H)$ and $S = C_{S^*}(Z)$. Then $S \cong \mathrm{SD}_{16}$ and it suffices to determine $C_S(V^H)$. For $i = 2, 3$, let $K_i = S \cap G_i'$ and set $K_0 = \Omega_1(K_2)$. Then $K_3 \cong \mathrm{Q}_8$, $K_2 \cong \mathrm{C}_4$ and $K_0 \cong \mathrm{C}_2$. We clearly have $S' \leq C_S(V^H)$. First, select $H \in \mathcal{H}$ such that $X_2 \leq H$. Then $V^H \leq C_{Z_4}(H \cap Q_2)$ and so, by 2.7 (d), K_2 centralizes V^H. Thus $S'K_2$ centralizes V^H and we know that $S'K_2 \cong \mathrm{Q}_8$. Thus $C_S(V^H) \cong \mathrm{Q}_8$ or SD_{16}. Next choose $H \in \mathcal{H}$ so that $X_2 \not\leq H$. Then $V^H \leq C_{Z_5}(H \cap Q_2)$ and, using 2.7 (d) again, K_0 inverts V^H. Thus $C_S(V^H) \cong \mathrm{Q}_8$. Therefore, V is uniquely determined as a G_3-module. It particular, Z_4 and Z_5 are uniquely determined as G_{23}-modules and then, by 2.3 and 2.7, Z_4 and Z_5 are also uniquely determined as G_2-modules. Since $\mathrm{End}_{\mathrm{GF}(5)G_2}(V) = \mathrm{End}_{\mathrm{GF}(5)G_{23}}(V)$, V is also uniquely determined as module for the amalgam (G_2, G_3). Finally, as by 5.12, $3^{\cdot}\mathrm{Aut}(\mathrm{McL})$ has an irreducible representation of degree 90 over GF(5), the lemma holds.

6 Existence and Uniqueness of Ly-amalgams

In this section we establish the existence and uniqueness of Ly-amalgams. The proof of our next lemma is taken from [IM, Lemma 6.3].

Lemma 6.1 *Let (M_1, M_2, M_3) and (M_1^*, M_2^*, M_3^*) be amalgams. Suppose that*

1. (i) *There exists $L \leq M_{123}$ such that $L \trianglelefteq M_2$, $C_{M_2}(L) = 1$ and for $i \in \{1, 3\}$, $M_{i2} = N_{M_i}(L)$.*
   (ii) *There exists $L^* \leq M_{123}^*$ such that $L^* \trianglelefteq M_2^*$, $C_{M_2^*}(L^*) = 1$ and for $i \in \{1, 3\}$, $M_{i2}^* = N_{M_i^*}(L^*)$.*
2. *For $i \in \{1, 3\}$ there exist isomorphisms $\phi_i : M_i \to M_i^*$ such that $\phi_1 |_{M_{13}} = \phi_3 |_{M_{13}}$, $M_{13}^{\phi_i} = M_{13}^*$ and $L^{\phi_i} = L^*$.*
3. *$M_2 = \langle M_{12}, M_{23} \rangle$ and $M_2^* = \langle M_{12}^*, M_{23}^* \rangle$.*

Then there exists an isomorphism $\phi_2 : M_2 \to M_2^*$ such that for $i \in \{1, 3\}$, $\phi_2 |_{M_{2i}} = \phi_i |_{M_{2i}}$ and $M_{2i}^{\phi_2} = M_{2i}^*$. In particular, (M_1, M_2, M_3) and (M_1^*, M_2^*, M_3^*) are isomorphic.

Proof: Let ϕ be the restriction of ϕ_1 or ϕ_3 to L. Then by assumption $L^\phi = L^*$. Define $\lambda : M_2 \to \mathrm{Aut}(L^*)$ by $m^\lambda : l^\phi \to (l^m)^\phi$ for all $m \in M_2$ and $l \in L$. Furthermore, define $\lambda^* : M_2^* \to \mathrm{Aut}(L^*)$ by $m^{\lambda^*} : l \to l^m$ for all $m \in M_2^*, l \in L^*$. Since $C_{M_2}(L) = 1 = C_{M_2^*}(L^*)$, λ and λ^* are monomorphisms. Let $i \in \{1,3\}$. Then $M_{i2}^{\phi_i} = N_{M_i}(L)^{\phi_i} = N_{M_i^*}(L^*) = M_{i2}^*$. We claim that λ and $\phi_i\lambda^*$ agree on M_{i2}. Indeed let $l \in L$ and $m \in M_{i2}$. Then

$$m^{\phi_i\lambda^*} : l^\phi \to (l^\phi)^{m^{\phi_i}} = (l^{\phi_i})^{m^{\phi_i}} = (l^m)^{\phi_i} = (l^m)^\phi,$$

proving the claim. In particular,

$$M_2^\lambda = \langle M_{12}^\lambda, M_{23}^\lambda \rangle = \langle M_{12}^{\phi_1\lambda^*}, M_{23}^{\phi_3\lambda^*} \rangle = \langle M_{12}^{*\lambda^*}, M_{23}^{*\lambda^*} \rangle = M_2^{*\lambda^*}.$$

Set $\phi_2 = \lambda\lambda^{*-1}$. Then ϕ_2 is an isomorphism between M_2 and M_2^* which agrees on M_{12} with ϕ_1 and on M_{23} with ϕ_3, completing the proof of the lemma.

We now recall the following definition from the introduction:

Definition 6.2 *An amalgam of groups* (M_1, M_2, M_3) *is called a* Ly**-amalgam** *provided that*

1. $M_1 \sim 3.\mathrm{McL}.2$, $M_2 \sim 3^6.2^3.\mathrm{Sym}(5)$ *and* $M_3 \sim 3^5.2.\mathrm{Mat}_{11}$.
2. $|M_2 : M_{12}| = 2$, $|M_2 : M_{23}| = 10$ *and* $|M_3 : M_{13}| = 11$.
3. $|M_{23} : M_{123}| = 2$.
4. *No non-trivial subgroup of* M_{123} *is normal in* M_1, M_2 *and* M_3.

The next proposition establishes the existence of a Ly-amalgam of groups.

Proposition 6.3 *There exists a* Ly*-amalgam of groups* (M_1, M_2, M_3) *such that*

(a) $M_1 \sim 3{}^\cdot\mathrm{Aut}(\mathrm{McL})$, $M_2 \sim 3^{2+4}.8.\mathrm{Sym}(5)$ *and* $M_3 \sim 3^5 : (\mathrm{C}_2 \times \mathrm{Mat}_{11})$.

(b) $M_{12} \sim 3^{1+1+4}.4.\mathrm{Sym}(5)$, $M_{13} \sim 3^{1+4}.(\mathrm{C}_2 \times \mathrm{Mat}_{10})$ *and* $M_{23} \sim 3^{2+2+1}.(\mathrm{C}_2 \times 3^2.\mathrm{SD}_{16})$.

(c) $M_{123} \sim 3^{1+1+2+1}.(\mathrm{C}_2 \times 3^2.\mathrm{Q}_8)$.

(d) $M_2 = M_{23}M_{21}$.

Proof: By 5.1 there exists a unique group of shape $3{}^\cdot\mathrm{Aut}(\mathrm{McL})$. Denote this group by M_1. By 2.5 (a), there is a unique group X of shape $3^5 : (\mathrm{C}_2 \times \mathrm{Mat}_{11})$ such that $C_X(O_3(X)) = O_3(X)$, $O_3(X)$ is an irreducible GF(3)Mat_{11}-module and $\mathrm{C}_2 \times \mathrm{Mat}_{10}$ inverts a cyclic subgroup in $O_3(X)$. Denote this group by M_3. Then both M_1 and M_3 contain subgroups of shape $X \sim 3^{(1+4)}{:}(\mathrm{C}_2 \times \mathrm{Mat}_{10})$ with $O_3(X)$ an indecomposable GF(3)Mat_{10}-module (see 2.5 (d) and the proof of Theorem 5.1). By 2.5 (f), a subgroup which has the shape of X is unique up to isomorphism. Hence we may arrange for M_1 and M_3 to intersect in a common subgroup M_{13} of shape $3^{(1+4)} : (\mathrm{C}_2 \times \mathrm{Mat}_{10})$. Let S be a Sylow 3-subgroup of M_{13}, Z_2 be the centre of S and set $Z_1 = O_3(M_1)$. So Z_2 has order 9 and Z_1 order 3. Let $T \in \mathrm{Syl}_2(N_{M_{13}}(S))$. Then $T \cong \mathrm{Q}_8 \times \mathrm{C}_2$. Write $T = A \times B$ where $|A| = 2$ and $AO_3(M_3)/O_3(M_3)$ centralizes $M_3/O_3(M_3)$ and, without loss of generality, $B \cong \mathrm{Q}_8$ is a subgroup of $O^2(M_3)$. Then, as $M_3/O_3(M_3)$ operates irreducibly on $O_3(M_3)$, A inverts every non-trivial element of Q_3 and hence of Z_2. As $Z(B) = B' \leq M_{13}' \leq M_1' = C_{M_1}(Z_1)$ and Z_2 admits B, $C_{Z_2}(Z(B)) = Z_2$ and B operates on Z_2 as a Klein four group. Thus, by 2.5 (g), there are distinct cyclic subgroups, B_1 and B_2, of order 4 in B such that $Z_2 = C_{Z_2}(B_1) \times C_{Z_2}(B_2)$. In particular, the third cyclic subgroup of order 4, B_3, in B inverts every element of Z_2. Let b be a generator of B_3 and a be an element of order 2 in A. Then $C = \langle ab \rangle$ is of order 4 and centralizes Z_2. Now $\langle A, B_3 \rangle \cong \mathrm{C}_4 \times \mathrm{C}_2$ inverts every element of Z_2 while the other two over-groups of C are quaternion and each centralizes a cyclic subgroup of Z_2 . As there are only two cyclic subgroups of order 4 in $\mathrm{C}_4 \times \mathrm{C}_2$ we conclude that there is a unique cyclic subgroup of order 4 in T which inverts Z_2, namely B_3.

We define

$$L = \langle B_3^{N_{M_{13}}(S)} \rangle = B_3[B_3, S] \leq M_{13}.$$

Then, $L \sim 3^{2+4}.4$ and $C_{M_1}(L) = C_{M_3}(L) = 1$. For $i = 1, 3$, and 13, define $M_{i2} = N_{M_i}(L)$. Then $T \leq M_{12}$ and $T \leq M_{23}$. Let $X{=}N_{M_1}(Z_2)$. As $C_{O_3(L)}(b^2) = Z_2$, $M_{12} \leq X \sim 3^{2+4}.4.\mathrm{Sym}(5)$. Now since $O_{2,3}(X)$ inverts Z_1, $O_{2,3}(X) \not\leq M_1'$. Hence, from the structure of McL, $O_{2,3}(X)$ also inverts Z_2/Z_1. Since $\mathrm{Sp}_4(3)$ does not contain a subgroup 4.Alt(5), we see that $O_{2,3}(X)$ inverts Z_2. Thus $O_{2,3}(X)$ and L are equal. Hence $M_{12} = X$ and $M_{12} \sim 3^{1+1+4}.4.\mathrm{Sym}(5)$. Because $M_{23} \geq N_{M_{13}}(S)$ and $[N_{M_3}(S) : N_{M_{13}}(S)] = 2$ the maximality of $N_{M_3}(S)$ in M_3 implies that

$$M_{23} = N_{M_3}(S) \sim 3^{2+2+1}.(\mathrm{C}_2 \times 3^2.\mathrm{SD}_{16}) \sim 3^{2+4}.8.(\mathrm{Sym}(3) \times \mathrm{Sym}(2)).$$

Finally we get

$$M_{123} = N_{M_{13}}(S) \sim 3^{2+2+1}.(\mathrm{C}_2 \times 3^2.\mathrm{Q}_8) \sim 3^{2+4}.4.(\mathrm{Sym}(3) \times \mathrm{Sym}(2)).$$

Since, for $i \in \{1, 3\}$, $C_{M_i}(L) = 1$, we identify M_{i2} with their images in $\mathrm{Aut}(L)$. Then $M_{12} \cap M_{23} = M_{123}$. Let M_2 be the subgroup of $\mathrm{Aut}(L)$ generated by M_{12} and M_{23}. We claim that $M_2 \sim 3^{2+4}.8.\mathrm{Sym}(5)$.

Let $T \in \mathrm{Syl}_2(L)$ and $D = C_{M_2}(T)$. Then, as $|T| = 4$, we have $|M_2/DL| = 2$, by the Frattini Argument, and, as D centralizes the element of order 2 in T, the Three Subgroup Lemma implies that D acts faithfully on $O_3(L)/Z_2$. Thus, since D centralizes T, D is isomorphic to a subgroup of $\mathrm{GL}_2(9)$. Let $D_i = M_{i2} \cap D$ for $i = 1, 3, 13$. Then $D_1/T \cong \mathrm{Alt}(5)$, $D_3/T \cong \mathrm{C}_2 \times \mathrm{Sym}(3)$ and $D_{13}/T \cong \mathrm{Sym}(3)$. Since all the 2-elements centralizing an element of order three in $\mathrm{GL}_2(9)$ are contained in $Z(\mathrm{GL}_2(9))$ we conclude that $D_3 = D_{13}X$ where X centralizes D_1. Thus $D = \langle D_1, D_2\rangle = \langle D_1, X\rangle \sim 8.\mathrm{Alt}(5)$ and so $M_2 \sim 3^{2+4}.8.\mathrm{Sym}(5)$ as claimed. Hence the amalgam (M_1, M_2, M_3) is a Ly-amalgam of groups and statements (a), (b) and (c) hold.

Theorem 6.4 *Up to isomorphism, there is a unique* Ly-*amalgam of groups* (M_1, M_2, M_3).

Proof: Let (M_1, M_2, M_3) be any Ly-amalgam of groups and, for $i = 1, 2, 3$, set $Q_i = O_3(M_i)$. Because $|M_3/M_{13}| = 11$, we have $M_{13} \sim 3^5.(2 \times \mathrm{Mat}_{10})$. Also, since elements of order five in McL are not centralized by a group of order 3^4, M_{13}'/Q_3, and hence also M_3'/Q_3, acts faithfully on Q_3. Thus, as M_3 operates non-trivially on Q_3 and M_3 possesses no 4-dimensional irreducible GF(3)-modules, M_3 operates irreducibly on Q_3. Hence, by 2.5 (a), Q_3 is the unique irreducible $\mathrm{GF}(3)M_3'$-module in which $M_{13} \cap M_3'$ stabilizes a 1-space. In particular, 2.5 (c) implies that Q_3 does not split over Q_1 as an M_{13}-module and, by 2.5 (d), $M_{13} \cap M_3'$ inverts Q_1. Thus $M_1 \sim 3^{\cdot}\mathrm{Aut}(\mathrm{McL})$. It follows that $C_{M_{13}}(Q_3) = Q_3$ and therefore, $M_3/Q_3 \cong \mathrm{C}_2 \times \mathrm{Mat}_{11}$ acts faithfully on Q_3. Thus M_1 and M_3 are unique up to isomorphism. Since the outer automorphism group of M_{13} is trivial (the group 3^4 does not admit $\mathrm{PGL}_2(9)$), the pair of groups (M_1, M_3) is unique up to isomorphism. As $M_{12} \sim 3^6.2^x.\mathrm{Alt}(5).2^y$ with $x + y = 3$, the structure of M_1 implies $M_{12} \sim 3^{1+1+4}.4.\mathrm{Sym}(5)$. Let $L = O_{3,2}(M_{12})$. As M_{12} is normal in M_2, L is normal in M_2. Clearly $M_{12} = N_{M_1}(L)$. Since $|M_{13}/M_{123}| = 10$, M_{123} has a unique Sylow 3-subgroup S and $M_{123} = N_{M_1}(S)$. In particular, $L \leq M_{123} \leq M_3$. Moreover, $|M_{23}/M_{123}| = 2$ implies that $S \trianglelefteq M_{23}$ and so $M_{23} \sim 3^{2+2+1}.(\mathrm{C}_2 \times 3^2.\mathrm{SD}_{16})$, $M_{23} = N_{M_3}(L)$ and $M_{123} \sim 3^{1+1+2+1}.(\mathrm{C}_2 \times 3^2.\mathrm{Q}_8)$. The argument concluding the proof of 6.3 shows that $M_2 \sim 3^{2+4}.8.\mathrm{Sym}(5)$ and $M_2 = M_{12}M_{23}$. Finally we apply 6.1 to get that (M_1, M_2, M_3) is unique up to isomorphism. This in conjunction with 6.3 gives the theorem.

We finish this section with a corollary to the proof of 6.3.

Corollary 6.5 *Suppose that* (M_1, M_2, M_3) *is a* Ly-*amalgam of groups and let* $K^* \in \mathrm{Syl}_2(M_{23})$. *Then*

(a) $C_{M_2}(Z(Q_2)) = O^2(M_2) = O^2(M_{12})$.

(b) $M_2/O^2(M_2) \cong \mathrm{D}_8$.

(c) $C_{K^*Q_2/Q_2}(O^2(M_2)/Q_2) \cong \mathrm{C}_8$.

(d) $O_{2,3}(M_2) \sim 3^{1+1+2+2}.\mathrm{C}_8$.

7 Completions of Ly-amalgams

Throughout this section M is assumed to be a faithful completion of a Ly-amalgam of groups (M_1, M_2, M_3). We identify M_1, M_2 and M_3 with their images in M. For the moment let $\Gamma = \Gamma(M; M_1, M_2, M_3)$ be the coset graph of M_1, M_2 and M_3 in M with two cosets incident if and only if they have non-trivial intersection. Then M acts on Γ by right multiplication. For $i = 1$, 2 and 3 set $\Gamma_i = M/M_i$. Our goal is to determine all the orbits of M_1 on Γ_1. For $a, b \in \Gamma_1$ we write $a \bowtie b$ if a and b are adjacent to a common vertex of type 2. We denote the distance between $a, b \in \Gamma_1$ in the graph $(\Gamma_1, \bowtie)$ by $d(a, b)$. For $a \in \Gamma$ set

$$Q_a = O_3(M_a)$$

and

$$Z_a = Z(Q_a).$$

We now assume that $a \in \Gamma_1$ is a fixed vertex and change notation so that $M_a = M_1$. Then we set

$$X_0(a) = \{a\}$$

and

$$X_1(a) = \{b \in \Gamma_1 \mid a \bowtie b\}.$$

A path $a, b, c, \ldots$ in Γ is said to be of type $i - j - k \ldots$ if $a \in \Gamma_i$, $b \in \Gamma_j$, $c \in \Gamma_k, \ldots$. We freely use the information about the structure of M_1, M_2 and M_3 guaranteed by 6.4 and given in 6.3.

Notice that the for $a \in \Gamma_1$, the results of Section 4 apply to describe the action of M_a on $\Gamma_i(a)$ for $2 \le i \le 4$. Also note that if $a \bowtie b$ and $c \in \Gamma_2(a) \cap \Gamma_2(b)$, then $\{c\} = \Gamma_2(a) \cap \Gamma_2(b)$ and $\{a, b\} = \Gamma_1(c)$. So, if $a \bowtie b$, then $M_{ab} = M_{abc} \sim 3^{2+4}.4.\mathrm{Sym}(5)$ has index 2 in M_c.

Lemma 7.1 *There exist unique M-classes of paths of type $1-2-1$ and $1-3-1$ in Γ.*

Proof: Because $|M_2 : M_{12}| = 2$, paths of type $1 - 2 - 1$ are in one-to-one correspondence with paths of type $1 - 2$. Thus there is a unique class of these paths. That there is a unique class of paths of type $1 - 3 - 1$ follows from the 2-transitivity of M_3 on the cosets of M_{13}.

Lemma 7.2 *Suppose that $b \in \Gamma_2$ and $c \in \Gamma_3(b)$. Then*

(a) $\Gamma_1(b) \subseteq \Gamma_1(c)$.

(b) if $f \in \Gamma_1(c)$ and $Z_f \le Q_c \cap Q_b$, then $f \in \Gamma_1(b)$ and $Z_f \le Z_b$.

Proof: By 6.3 (a) and (b), $|\Gamma_1(b)| = 2$. Let $\{a, e\} = \Gamma_1(b)$. We may assume that $a \in \Gamma_1(c)$. Since $M_b = M_{ba}M_{bc}$, by 6.3 (d), M_{bc} acts transitively on $\Gamma_1(b)$. Therefore, $e \in \Gamma_1(c)$ and so (a) holds.

Now note that M_c acts three-transitively on $\Gamma_1(c)$ and M_{cb} is the stabiliser of a set of two points under this action. Hence M_{cb} has an orbit of length 9 and an orbit of length 2 on $\Gamma_1(c)$, the latter is $\{a, e\}$. Suppose that $f \in \Gamma_1(c)$ and $Z_f \in Q_c \cap Q_b$. Since $\langle Z_a, Z_e\rangle = Z_b \leq Q_c \cap Q_f$, M_{cf} normalizes $Q_c \cap Q_f$ and $\langle Z_f^{M_c}\rangle = Q_c$, the action of M_{cb} now shows us that $Z_f \leq Z_b$ and $f \in \{a, e\}$ as claimed.

Lemma 7.3 *There exist four M-classes of paths a, b, c, d, e of type $1-2-1-2-1$ in Γ. These classes are distinguished by the isomorphism type of the subgroup of M_c generated by Z_a and Z_e as follows:*

$$\langle Z_a, Z_e\rangle \in \{3^2, 2^{\cdot}\mathrm{Alt}(4), 2^{\cdot}\mathrm{Alt}(5), 5^{1+2}.3\}.$$

Moreover,

(a) If $[Z_a, Z_e] = 1$, then $e \in X_1(a)$ and there exists a unique $f \in \Gamma_3(a) \cap \Gamma_3(c) \cap \Gamma_3(e)$.

(b) If $\langle Z_a, Z_e\rangle \sim 2^{\cdot}\mathrm{Alt}(4)$, then $M_{ae} \sim 2^{\cdot}\mathrm{Sym}(7)$.

(c) If $\langle Z_a, Z_e\rangle \sim 2^{\cdot}\mathrm{Alt}(5)$, then $M_{ae} \sim 2^{\cdot}(\mathrm{C}_2 \times \mathrm{Sym}(6))$.

(d) If $\langle Z_a, Z_e\rangle \sim 5^{1+2}.3$, then $M_{ace} \cong \mathrm{D}_{30}$.

Proof: By 7.1 we may assume that the path $a - b - c$ is fixed. We work inside M_c. By 4.4, M_{bc} has four orbits on $\Gamma_2(c) \setminus \{b\}$ and since a vertex of type 2 has only two neighbours of type 1, there exist exactly four paths of type $1-2-1-2-1$. Moreover, 4.4 (b), (c), (d) and (e) together show that the four classes are distinguished by the isomorphism type of $\langle Z_a, Z_e\rangle Z_c/Z_c$ as described.

Suppose that $[Z_a, Z_e] \leq Z_c$. Then, by 4.4 (b), there exists a unique $f \in \Gamma_3(b) \cap \Gamma_3(c) \cap \Gamma_3(d)$. Then $a, e \in \Gamma_1(f)$ and the two transitivity of M_f on $\Gamma_1(f)$ indicates that there is a $h \in \Gamma_2(f)$ with $\{a, e\} = \Gamma_1(h)$. So, by definition, $e \in X_1(a)$ and (a) holds.

Notice that if $\langle Z_a, Z_e\rangle$ is non-abelian, then, Z_a and Z_e are conjugate in $\langle Z_a, Z_e\rangle$.

Suppose that $\langle Z_a, Z_e\rangle Z_c/Z_c \sim 2^{\cdot}\mathrm{Alt}(i)$, $i = 4, 5$. Then 4.4 (c) and (d) (allowing for the central 3 in M_c which is inverted) give $M_{ace} \sim 2.(\mathrm{Sym}(3) \times \mathrm{Sym}(4))$ when $i = 4$ and $M_{ace} \cong 2.(\mathrm{C}_2 \times \mathrm{Sym}(3) \times \mathrm{Sym}(3))$ when $i = 5$. Let t be the involution in $\langle Z_a, Z_e\rangle$. Then clearly $M_{ae} \leq C_{M_a}(t) \sim 6.\mathrm{Sym}(8)$. Therefore as $Z_a \cap M_{ae} = 1$, $M_{ae}/\langle t\rangle$ is isomorphic to a subgroup of $\mathrm{Sym}(8)$.

We shall show that all the 3-cycles of $M_{ae}/\langle t\rangle$ are conjugate. So let $r\langle t\rangle/\langle t\rangle$ be such 3-cycle with $|\langle r\rangle| = 3$. If $\langle r, Z_c\rangle Z_a/Z_a \cong 2^{\cdot}\mathrm{Alt}(4)$ or $2^{\cdot}\mathrm{Alt}(5)$, $\langle r\rangle$ and Z_c

are conjugate in $\langle r, Z_c\rangle \le M_{ae}$. So suppose that $[Z_c, r] = 1$ and pick $y \in X_1(c)$ with $r \in Z_y$. Then y has distance at most two from a in $(\Gamma_1, \bowtie)$ and so, by (a), $a \bowtie y$ and as $[Z_y, Z_e] = 1$, the same argument yields $y \bowtie e$. Moreover, the path (a, y, e) in $(\Gamma_1, \bowtie)$ is unique up to conjugacy in M and so conjugate to (a, c, e). Hence y and c are conjugate in M_{ae}. Thus $M_{ae}/\langle t\rangle$ contains a unique class of 3-cycles and the normalizer in $M_{ae}/\langle t\rangle$ of such 3-cycles is conjugate to $M_{ace}/\langle t\rangle$ and so is isomorphic to $\mathrm{Sym}(3) \times \mathrm{Sym}(4)$ if $i = 4$ and $\mathrm{C}_2 \times \mathrm{Sym}(3) \times \mathrm{Sym}(3)$ if $i = 5$. Furthermore, we note that $M_{ae} > M_{ace}$ and $\langle Z_a, Z_e\rangle \cap Z_c = 1$, which gives the first parts of (b) and (c).

If $i = 4$, this implies $M_{ae} \sim 2{}^{\cdot}\mathrm{Sym}(7)$ or $M_{ae} \sim 2.(\mathrm{Sym}(4) \times \mathrm{Sym}(4)).2$. Let $x \in \Gamma_2(c)$ be chosen so that $\langle Z_a, Z_e\rangle$ normalizes Z_x. Then $N_{M_x}(\langle Z_a, Z_e\rangle) \sim 3^2.8.\mathrm{Sym}(4)$ and so $M_{xae} \sim 3^2.8.2$. Moreover, the elements of order eight in M_{xae} centralizes Z_a while M_{xae} inverts Z_a. Therefore, we conclude that $M_{axe}Z_a/Z_a\langle t\rangle$ is a subgroup of $\mathrm{Sym}(8) = N_{M_a}(t)/Z_a\langle t\rangle$ of shape $3^2.\mathrm{D}_8$ which intersects $\mathrm{Alt}(8)$ in $3^2.\mathrm{C}_4$. Now a subgroup of shape $(\mathrm{Sym}(4) \times \mathrm{Sym}(4)).2$ has a unique conjugacy class of subgroups of shape $3^2.\mathrm{D}_8$ and this class of subgroups intersects $\mathrm{Alt}(8)$ in a subgroup of shape $3^2.(\mathrm{C}_2 \times \mathrm{C}_2)$. Thus when $i = 4$ we cannot have $M_{ae} \sim 2.(\mathrm{Sym}(4) \times \mathrm{Sym}(4)).2$. Hence when $i = 4$ we have $M_{ae} \sim 2{}^{\cdot}\mathrm{Sym}(7)$ and so (b) holds.

If $i = 5$, we calculate that $M_{ae} \sim 2.(\mathrm{C}_2 \times \mathrm{Sym}(6))$ or $M_{ae} \sim 2.(\mathrm{C}_2 \times 3^2.\mathrm{D}_8)$. Let $f \in X_1(c)$ with $Z_f \le \langle Z_a, Z_e\rangle$ and $\langle Z_a, Z_f\rangle \sim 2.\mathrm{Alt}(4) \sim \langle Z_e, Z_f\rangle$. Then, by (b), $M_{af} \sim 2.\mathrm{Sym}(7) \sim M_{fe}$. Both of these group are in $C_{M_f}(t) \sim 6.\mathrm{Sym}(8)$ and thus M_{afe} contains a subgroup $2{}^{\cdot}\mathrm{Alt}(6)$. Therefore, $M_{ae} \sim 2.(\mathrm{C}_2 \times \mathrm{Sym}(6))$ and so (c) holds.

Finally, if $\langle Z_a, Z_e\rangle Z_c/Z_c \sim 5^{1+2}.3$, then, 4.4 gives $M_{ace} \cong \mathrm{D}_{30}$ and, as Z_e and Z_c are conjugate in $\langle Z_a, Z_e\rangle$, (d) also holds.

Notice that 7.3 completely determines those vertices in $(\Gamma_1, \bowtie)$ at distance 2 from a. Moreover, in all but the case of $a - b - c - d - e$ as in 7.3 (d), we know M_{ae} and thus the action of M_a on such paths. Define

$$X_2(a) = \{e \in \Gamma_1 \mid d(a, e) = 2 \text{ and } \langle Z_a, Z_e\rangle \sim 2.\mathrm{Alt}(4)\},$$

$$X_3(a) = \{e \in \Gamma_1 \mid d(a, e) = 2 \text{ and } \langle Z_a, Z_e\rangle \sim 2.\mathrm{Alt}(5)\}$$

and

$$X_4(a) = \{e \in \Gamma_1 \mid d(a, e) = 2 \text{ and } \langle Z_a, Z_e\rangle \sim 5^{1+2}.3\}.$$

Suppose that $e \in X_3(a)$, $a - b - c - d - e$ is a path as in 7.3 joining a to e and set $Z = Z(\langle Z_a, Z_e\rangle)$. Note that $Z \le M_a$. We define

$$H = \langle C_{M_a}(Z), Z_e\rangle.$$

Notice that, by 7.3, $C_{M_a}(Z)$ operates transitively on the set

$$\{f \in X_3(a) \mid Z = Z(\langle Z_a, Z_f\rangle)\}.$$

Since we claim that any faithful completion of a Ly-amalgam of groups is a group of Ly-type, the relevance of the next lemma is clear.

Lemma 7.4 *We have $H \cong 2{}^{\cdot}\mathrm{Alt}(11)$.*

Proof: Let $A = \langle Z_a, Z_e\rangle$. Then $A/Z \cong \mathrm{Alt}(5)$. Furthermore, we have $C_{M_a}(Z)/Z \cong \mathrm{Sym}(3) \wr \mathrm{Sym}(8)$. Since $A \le M_c$, we have

$$N_A(Z_a) \le N_{M_c}(Z_b) = M_{cb} = M_{ab} \le M_a.$$

Therefore, as $A \not\le M_a$,

$$C_{M_a}(Z) \cap A = N_A(Z_a) \sim 3.4.$$

Suppose that $f \in X_1(c)$ with $Z_f \in \mathrm{Syl}_3(A)$, then from 7.3 we have

$$C_{M_a}(Z) \cap C_{M_f}(Z) = \begin{cases} 2.\mathrm{Sym}(7) & \langle Z_a, Z_f\rangle \sim 2.\mathrm{Alt}(4) \\ 2.(\mathrm{C}_2 \times \mathrm{Sym}(6)) & \langle Z_a, Z_f\rangle = A \end{cases}.$$

We now apply 3.3 with H/Z, A/Z, $C_{M_a}(Z)/Z$ and 8, in place of M, A, L_D and n, to get $H/Z \cong \mathrm{Alt}(11)$. Finally, because $H \ge C_{M_a}(Z) \cong (2{}^{\cdot}\mathrm{Sym}(8)) \wr \mathrm{Sym}(3)$, we conclude that $H \cong 2{}^{\cdot}\mathrm{Alt}(11)$.

We now adjust our notation, if necessary, so that $Z_2 \le H$ and $|Z_3 \cap H| = 3^3$. So with this choice of H we have

$$C_{M_1}(Z(H)) \cong \mathrm{Sym}(3) \wr 2{}^{\cdot}\mathrm{Sym}(8),$$

$$C_{M_2}(Z(H)) \sim 3^2.8.\mathrm{Sym}(5)$$

and

$$C_{M_3}(Z(H)) \sim 3^3.(\mathrm{C}_2 \times \mathrm{GL}_2(3)) \sim 3^3(\mathrm{C}_2 \times 2{}^{\cdot}\mathrm{Sym}(4)).$$

We now extend Γ by adding vertices of type 4 corresponding to the cosets of H in M. Thus for the rest of this section

$$\Gamma = \Gamma(M; M_1, M_2, M_3, H),$$

$\Gamma_4 = M/H$ and once again two vertices in Γ are incident if and only if the cosets intersect non-trivially. Often we will denote H by M_4. Observe that at this stage we do not know the structure of $M_j \cap H$ for $j \in \{2, 3\}$, though we do know that $H \cap M_1 = C_{M_1}(Z)$. For $t \in \Gamma_4$, set $Z_t = Z(M_t)$ and let $\Omega(t)$ be a set of size 11 admitting M_t/Z_t naturally. Lastly, given $d, e \in \Gamma_4$ we write $d \star e$ provided that $\Gamma_1(d) \cap \Gamma_1(e) \ne \emptyset$ and $[Z_d, Z_e] = 1$.

Lemma 7.5 *Let b, a, c, d be a path of type $4-1-2-1$ in Γ such that $Z_b \leq M_{abcd}$. Then c and d are adjacent to b in Γ.*

Proof: Let t be an involution in M_a'. Then, as M_a' has a unique class of involutions, there exists $x \in \Gamma_4(a)$ with $t \in Z_x$. Since $M_{ax} = C_{M_a}(t)$, x is uniquely determined by a and t. As, by 4.1(d), $M_a' \cap M_c$ also has a unique class of involutions, we conclude that the path (b, a, c) and so also (b, a, c, d) is unique up to conjugation. Since (M_4, M_1, M_2) is a path like (b, a, c), we have that c is adjacent to b.

Let $y \in X_3(a)$ with $Z_b = Z(\langle Z_a, Z_y\rangle)$. Then, by the construction of H, $\langle Z_a, Z_y\rangle \leq M_b$ and so y is adjacent to b.

We note, by calculating in the centralizer of t in M_a, that there exists $u, v, w \in X_1(a)$ fixed by t such that $[Z_u, Z_w] = 1$ and $\langle Z_u, Z_v\rangle \sim \langle Z_v, Z_w\rangle \sim 2{}^{\cdot}\mathrm{Alt}(5)$. Select $r \in \Gamma_4(u)$ with $t \in Z_r$. Then by the previous paragraph applied to (u, v, r) in place of (a, y, b), r is adjacent to v. Applying the same trick once more this time to (v, w, r) yields that r is adjacent to w. As (b, a, d) is conjugate to (r, u, w), we conclude that d is adjacent to b.

We now clarify the structure of the edge stabilisers in Γ.

Lemma 7.6 *Suppose that $d \in \Gamma_4$ and $\{a, b, c, d\}$ is a clique in Γ. Then, for $\gamma \in \{a, b, c\}$, $M_{\gamma d} = C_{M_\gamma}(Z_d)$. In particular, for $i \in \{1, 2, 3\}$, we have $M_i \cap H = C_{M_i}(Z(H))$.*

Proof: Since $\{a, b, c\}$ is a clique in $\Gamma \setminus \Gamma_4$, we may assume that $a \in \Gamma_1$, $b \in \Gamma_2$ and $c \in \Gamma_3$. By the construction of H and 7.4 we have, $C_{M_a}(Z_d) = N_{M_d}(Z_a)$ and so $M_{da} = C_{M_a}(Z_d)$.

Let $\{a, x\} = \Gamma_1(b)$. By 7.5, x is incident to d and so there exists $h \in M_d$ with $a^h = x$ and $x^h = a$. Thus $h \in M_{db}$. Since $[C_{M_b}(Z_d) : C_{M_{ab}}(Z_d)] = 2$ and $h \notin C_{M_{ab}}(Z_d)$, we have $C_{M_b}(Z_d) = C_{M_{ab}}(Z_d)\langle h\rangle \leq M_d$ and hence $C_{M_b}(Z_d) = M_{bd}$. This in turn implies that $C_{M_c}(Z_d) = \langle C_{M_{ac}}(Z_d), C_{M_{bc}}(Z_d)\rangle = \langle M_{acd}, M_{bcd}\rangle = M_{cd}$. Finally, as M_1, M_2, M_3, H is a clique, we have the last statement of the lemma as well.

Let $d \in \Gamma_4$. Then $\Gamma_1(d)$ corresponds to subsets of $\Omega(d)$ of size 3, $\Gamma_2(d)$ corresponds to partitions of subsets of $\Omega(d)$ of size 6 into two parts of size 3 and the elements of $\Gamma_3(e)$ to partitions of subsets of $\Omega(d)$ of size 9 into three parts each of size 3. Incidence between elements of $\Gamma(d)$ is then recognised by symmetrized containment. The set $\{e \in \Gamma_4 \mid d * e\}$ corresponds to partitions of subsets of Ω of size 8 into four parts of size 2.

Proposition 7.7 *Let $d \in \Gamma_4$ and let Γ^d be the connected component of Γ^{Z_d}, the fixed points of Z_d on Γ, containing d. Then $M_d = \mathrm{Stab}_M(\Gamma^d)$. Moreover, for every $x \in \Gamma^d$, $M_{xd} = C_{M_x}(Z_d)$.*

Proof: Let $\Lambda = \Gamma^{Z_d}$. We follow the algorithm laid out in 2.2 so that we can apply 2.1. Let $\{a, b, c, d\}$ be a clique in Γ containing d with $a \in \Gamma_1$, $b \in \Gamma_2$ and $c \in \Gamma_3$. Then a, b, c are representatives for the orbits of M_d on $\Gamma(d)$. Moreover, by 7.6, $C_{M_{\gamma d}}(Z_d) = M_{\gamma d}$ for $\gamma \in \{a, b, c\}$. Let $e \in \Lambda_4(a) \setminus \{d\}$. Since, by assumption, Z_d fixes e, $[Z_e, Z_d] = 1$ and so

$$Z_e \leq C_{M_a}(Z_d) = M_{ad} \leq M_d.$$

In particular, by 4.2 (c) and (d), we have b, c, d, e is a complete set of representatives for the orbits of M_{ad} on $\Lambda(a)$. Now, Z_d is an involution in $M_e \sim 2{\cdot}\mathrm{Alt}(11)$ and so we know that the non-trivial element of Z_e has cycle shape $(2, 2, 2, 2)$ when it acts on $\Omega(e)$. Thus Z_a corresponds to the unique 3-cycle in M_e/Z_e centralized by Z_d. Hence $C_{M_e}(Z_d)$ normalizes Z_a, and therefore

$$C_{M_e}(Z_d) = C_{M_{ae}}(Z_d) = M_{ade} = M_{de} \sim 2.(\mathrm{Sym}(3) \wr 2^3).\mathrm{Sym}(4).$$

Observe that $M_{bd} \sim 3^2.8.\mathrm{Sym}(5)$ acts transitively on $\Lambda_1(b)$ with representative a and on $\Lambda_3(b)(= \Gamma_3(b))$ with representative c and that $\Lambda_4(b) = \emptyset$. Next we consider the action of M_{cd} on $\Lambda(c)$. Now $M_{cd} = C_{M_c}(Z_d) \sim (3^3.(2 \times 2{\cdot}\mathrm{Sym}(4)))$ and $O_3(M_{cd})$ fixes every element of $\Gamma_i(c)$ for $i = 1, 2$. Thus, for $i = 1, 2$, M_{cd} operates as $2 \times 2{\cdot}\mathrm{Sym}(4)$ on $\Lambda_i(c)$. Since Z_d acts with exactly 3 fixed points and 4 pairs of interchanged elements on the 11 elements in $\Gamma_1(c)$, we infer that M_{cd} is transitive on $\Lambda_1(c)$ (which has 3 elements and representative a) and has two orbits on $\Lambda_2(c)$ one of length 3 and one of length 4. Furthermore, M_{cd} has two orbits on $\Lambda_4(c)$ with representatives e and d. Let b and f be the respective representatives for these last two orbits. Notice that Z_d interchanges the two elements of $\Gamma_1(f)$ and so $\Lambda_1(f) = \emptyset$. Also, as Z_d centralizes Z_b and, by 7.2 (b), $Z_b \not\leq Q_c \cap Q_f$, $Z_bQ_f/Q_f = Q_cQ_f/Q_f$ is centralized by Z_d. It follows that $C_{M_f}(Z_d)$ normalizes Q_cQ_f and so $C_{M_f}(Z_d) \leq M_{fc}$. Therefore,

$$C_{M_f}(Z_d) \leq C_{M_c}(Z_d) \leq M_d$$

and so

$$M_{df} = M_{dcf} \sim 3^3.(\mathrm{C}_2 \times \mathrm{C}_2 \times \mathrm{Sym}(3)).$$

We will now investigate $\Lambda(e)$. Choose notation such that on $\Omega(e)$, a corresponds to $[1, 2, 3]$ and Z_d acts as $\langle(4, 5)(6, 7)(8, 9)(10, 11)\rangle$. Then we can identify the image of $C_{M_e}(Z_d)$ in M_e/Z_e acting on $\Omega(e)$ with

$$\langle(1, 2, 3), (1, 2)(5, 7)(4, 6)(8, 9), (6, 8, 10)(7, 9, 11), (4, 6)(5, 7)(8, 10)(9, 11)\rangle.$$

It is now straightforward to calculate that $M_{de}(= C_{M_e}(Z_d))$ has four orbits on $\Lambda_3(e)$ with representatives

$$\begin{array}{lcl} \tilde{c} & \leftrightarrow & [1, 2, 3 \mid 4, 6, 8 \mid 5, 7, 9] \\ g & \leftrightarrow & [1, 10, 11 \mid 4, 6, 8 \mid 5, 7, 9] \\ h & \leftrightarrow & [1, 10, 11 \mid 2, 4, 5 \mid 3, 6, 7] \\ i & \leftrightarrow & [1, 10, 11 \mid 2, 4, 5 \mid 3, 8, 9] \end{array}$$

where we remark that the last two orbits are fused under $N_{M_e}(\langle Z_d, Z_e\rangle)$. We also calculate that M_{de} has one orbit on $\Lambda_1(e) \setminus \{a\}$ with representative

$$j \leftrightarrow [1, 10, 11]$$

(note that in this case Z_d normalizes, but does not centralize, Z_j) and two orbits on $\Lambda_2(e)$ with representatives

$$\begin{array}{lcl} \tilde{f} & \leftrightarrow & [4,6,8 \mid 5,7,9] \\ k & \leftrightarrow & [1,10,11 \mid 2,4,5] \end{array}.$$

Recall that incidence of $x, y \in \Gamma(e)$ is recognized in $\Omega(e)$ by one the objects being a refinement of the other. Since $\tilde{c}$ is adjacent to a we may assume that $c = \tilde{c}$ and noting that $\tilde{f}$ is adjacent to $\tilde{c} = c$ we may also assume that $f = \tilde{f}$. Because Z_d inverts Z_j, 4.3 gives us that

$$C_{M_j}(Z_d) \cong \mathrm{C}_2 \times \mathrm{Mat}_{11}.$$

By determining the stabilizer of j (which represents an orbit of length 12 under M_{de}) in M_{de} we calculate

$$M_{dej} \cong \mathrm{C}_2 \times \mathrm{Q}_8 : \mathrm{Sym}(3) \cong \mathrm{C}_2 \times \mathrm{GL}_2(3).$$

As Z_d inverts Z_j but not Q_g, Z_dQ_g/Q_g is a diagonal involution in $M_g/Q_g \sim \mathrm{Mat}_{11} \times \mathrm{C}_2$ and so we deduce that

$$C_{M_g}(Z_d) \sim 3^2.(\mathrm{C}_2 \times \mathrm{GL}_2(3)).$$

Since we already know $C_{M_f}(Z_d) = M_{df} \leq M_d$ and $g \in \Lambda_3(f)$ is in an orbit of length 3, we conclude that

$$M_{dfg} = C_{M_g}(Z_d) \cap M_f \sim \mathrm{C}_2 \times 3^2.(\mathrm{C}_2 \times \mathrm{Sym}(3)).$$

Furthermore, as (using M_3 is 3-transitive on its neighbours of type 1) $M_{fgj} \sim 3^5(\mathrm{SD}_{16} \times \mathrm{C}_2)$,

$$M_{dfgj} \sim \mathrm{C}_2 \times 3^2.2^2.$$

In particular, we note $M_{dfgj} \not\leq M_{dej}$ and so, as M_{dej} is a maximal subgroup of $C_{M_j}(Z_d)$,

$$C_{M_j}(Z_d) = \langle M_{dej}, M_{dfgj}\rangle = M_{dj} \sim \mathrm{C}_2 \times \mathrm{Mat}_{11}.$$

Since Z_d inverts Z_k, 6.5 (a) and (b) indicate that the non-trivial element z of Z_d is a square in M_k/Q_k. It follows that z acts as an Alt(5)-type involution on $\Gamma_3(k)$. Hence, $|\Lambda_3(k)| = 2$ and so $\Lambda_3(k) = \{h, i\}$ (each adjacent to e). Now 2.5 (d) indicates that Z_d inverts exactly one of Q_h and Q_i. We choose notation so that Z_d inverts Q_i. Thus $C_{M_i}(Z_d) \cong \mathrm{C}_2 \times \mathrm{Mat}_{11}$, $M_{dei} = C_{M_i}(Z_d) \cap C_{M_i}(Z_e) \cong \mathrm{C}_2 \times \mathrm{GL}_2(3)$, and $M_{dji} = C_{M_j}(Z_d) \cap M_i \cong \mathrm{C}_2 \times \mathrm{Mat}_{10}$ by 4.3 (a). Hence,

$$C_{M_i}(Z_d) = \langle M_{dei}, M_{dji} \rangle = M_{di} \sim \mathrm{C}_2 \times \mathrm{Mat}_{11}.$$

Furthermore, by 4.3 (a), $M_{djg} = C_{M_j}(Z_d) \cap M_g = C_{M_{jg}}(Z_d) \sim \mathrm{C}_2 \times 3^2.\mathrm{SD}_{16}$ and so

$$C_{M_g}(Z_d) = \langle M_{dfg}, M_{djg} \rangle = M_{dg} \sim \mathrm{C}_2 \times 3^2.\mathrm{GL}_2(3).$$

Also, because $C_{M_k}(Z_d) \leq M_i$,

$$C_{M_k}(Z_d) = M_{dik} = M_{dk} \sim \mathrm{C}_2 \times 3^2.\mathrm{SD}_{16}.$$

Finally, we claim that $h^r = g$ for some $r \in M_d$. Indeed, by 4.3, such an r can be found in M_{dj}.

Put $\mathcal{F} = \{a, b, c, d, e, f, g, i, j, k\}$. We have demonstrated

(7.7.1) If $x \in \mathcal{F}$, then $C_{M_x}(Z_d) = M_{dx}$.

Also note that for $x \neq y \in \mathcal{F}$ having the same type, x, y are distinguished by the action of Z_d on Z_x and Z_y. Indeed, Z_d centralizes Z_a but inverts Z_j; Z_d centralizes Z_b, inverts Z_k and neither centralizes nor inverts Z_f; Z_d centralizes a 3^3 in Z_c, a 3^2 in Z_g and inverts Z_i; and Z_d equals Z_d but not Z_e. Thus $\mathcal{F}$ is a set of representatives of distinct orbits of M_d on Γ^d.

We next show that if $x \in \mathcal{F}$ and $y \in \Lambda(x)$, then y is M_d-conjugate to an element of $\mathcal{F}$. We do this by using (7.7.1) to show that, for $x \in \mathcal{F}$, there is a set Ξ of representatives for the orbits of M_{xd} on $\Lambda(x)$ with $\Xi \subset \mathcal{F} \cup \{h, k^r\}$ where $r \in M_{dj}$ is the element that achieves $h^r = g$. We find

$$\Xi = \begin{cases} \{b, c, d, e\} & \text{when } x = a \\ \{a, c, d\} & \text{when } x = b \\ \{a, b, d, e, f\} & \text{when } x = c \\ \{a, b, c\} & \text{when } x = d \\ \{a, c, f, g, h, i, j, k\} & \text{when } x = e \\ \{c, e, g\} & \text{when } x = f \\ \{e, f, j, k^r\} & \text{when } x = g \\ \{e, j, k\} & \text{when } x = i \\ \{e, g, k, i\} & \text{when } x = j \\ \{e, h, i, j\} & \text{when } x = k \end{cases}.$$

Indeed, for $x \in \{a, b, c, d, e\}$, the claim is transparent from our previous discussion. Suppose that $x = f$. Then $M_{df} \sim 3^2.(\mathrm{C}_2 \times \mathrm{C}_2 \times \mathrm{Sym}(3))$ which acts transitively on $\Gamma_1(f)$, so that $\Lambda_1(f) = \emptyset$. Now on $\Lambda_3(f)$, M_{df} has orbits of length 1, 3 and 6, the orbits of lengths 1 and 3 corresponding to the fixed points of Z_d with representatives c and g, respectively. Finally, we also have that M_{df}

acts transitively on $\Lambda_4(f)$ this time with representative e. Next assume that $x = g$, then $M_{dg} \sim \mathrm{C}_2 \times 3^2.\mathrm{GL}_2(3)$. Since $\mathrm{GL}_2(3)$ has one class of non-central involutions, M_{dg} operates transitively on $\Lambda_4(g)$ with representative e. Also $\Lambda_1(g)$ has three members and M_{dg} acts transitively on these, the representative is j. Finally $\Lambda_2(g)$ has two orbits under M_{dg}; one has representative f, the other k^r. Now assume that $x = i$. Then $M_{di} \sim \mathrm{C}_2 \times \mathrm{Mat}_{11}$, a complement to $O_3(M_i)$ with Z_d inverting $O_3(M_i)$. It follows that $\Lambda_1(i) = \Gamma_1(i)$ and $\Lambda_2(i) = \Gamma_2(i)$ both of which admit M_{di} transitively with representatives k and j. Furthermore, as Mat_{11} has a unique class of involutions, M_{di} is also transitive on $\Lambda_4(i)$. For $x = j$ we refer the reader to 4.3 to obtain representatives $\{e, j, k, i\}$. Finally we assume that $x = k$. Then $M_{dk} \sim \mathrm{C}_2 \times 3^2.\mathrm{SD}_{16}$. Since the elements of $\Lambda_4(k)$ correspond to involutions which invert $O_3(M_{dk})$, we have a single class here with representative e. The elements of $\Lambda_3(k)$ are h and k as discussed earlier and, as Z_d inverts Z_k, $\Lambda_1(k)$ has order 2 and representative j.

Hence, we have shown that the hypothesis of 2.1 holds. Therefore, $\Gamma^d = \mathcal{F}^{M_d}$ and $M_d = \mathrm{Stab}_M(\Gamma_D) \cap N_M(Z_d)$, as claimed.

Notice that one consequence of 7.7 is that for $d \in \Gamma_4$, M_d has exactly two orbits on $\Gamma_1 \cap \Gamma^d$. One orbit consists of those elements $a \in \Gamma^d$ of type 1 with $[Z_d, Z_a] = 1$, the other those $b \in \Gamma^d$ of type 1 with $[Z_d, Z_b] = Z_b$ and $M_{db} \sim \mathrm{C}_2 \times \mathrm{Mat}_{11}$. We now incorporate this fact into our notation. Let $d \in \Gamma_4$ and $x \in \Gamma_1^d$ we write $x \dagger d$ if Z_d inverts Z_x. Thus we have $d \dagger x$ if and only if $x \in \Gamma^d \setminus \Gamma_1(d)$. Another consequence of the proof of 7.7 is

Corollary 7.8 *Suppose that $d \in \Gamma_4$ and $f \in \Gamma_2 \cap \Gamma^d$. If Z_d neither inverts nor centralizes Z_f. Then there exists a unique $c \in \Gamma_3(d) \cap \Gamma_3(f)$.*

Proof: This follows as $M_{df} = M_{dcf}$ for any $c \in \Gamma_3(d) \cap \Gamma_3(f)$.

Lemma 7.9 *Let $d \in \Gamma_4$ and $x, y \in \Gamma_1^d$. Then $d(x, y) \leq 2$.*

Proof: If $x, y \in \Gamma_1(d)$, then, since Z_x and Z_y are generated by 3-cycles in M_d/Z_d, there is a $u \in \Gamma_1(d)$ with $x \bowtie u \bowtie y$. Thus $d(x, y) \leq 2$. Suppose that $x \notin \Gamma_1(d)$ and $y \in \Gamma_1(d)$. Then $M_{xd} \cong \mathrm{C}_2 \times \mathrm{Mat}_{11}$, acts 3-transitively on $\Omega(d)$ and, in particular, M_{xd} acts transitively on $\Gamma_1(d)$. Hence we only need to show that $d(x, y) \leq 2$ for some x, y fulfilling the assumptions of the current case. Pick $e \in \Gamma_4(y)$ with $e \star d$. We may suppose that the action of y and d on $\Omega(e)$ are described as follows: $y \leftrightarrow [1, 2, 3]$ and $d \leftrightarrow [4, 5][6, 7][8, 9][10, 11]$. Select $\tilde{x} \in \Gamma_1(e)$ with $\tilde{x} \leftrightarrow [1, 4, 5]$. Then $\tilde{x} \in \Gamma_1^d \setminus \Gamma_1(d)$ and $d(\tilde{x}, y) = 2$ and we are done in this case.

Suppose finally that neither x nor y are adjacent to d. Then as there are only two orbits of M_d on Γ_1^d, $y \in \{x^{M_d}\}$. Recall that permutation rank for M_d on $\{x^{M_d}\}$ is that of Alt(11) on Mat_{11} is five (see [ATLAS]) and so there are exactly 5 distinct classes of triples (x,y,d) with the property that neither x nor y is adjacent to d. Let $a,b \in \Gamma_1$ with $a \bowtie b$. We now work in M_b. For $0 \le i \le 4$, let $Y_i(a)$ be the five distinct orbits of M_{ab} on the neighbours of b in $(\Gamma_1, \bowtie)$(see 4.4). Then for $0 \le i \le 4$, select $c_i \in Y_i(a)$. By 4.4, there exists $d_i \in \Gamma_4$ with $d_i \dagger b$ and $Z_{d_i} \le M_{abc_i}$. Then $a \dagger d_i \dagger c_i$ and we have found representatives for all five classes of triples (x,y,d) with $x \dagger d \dagger y$ and in each case $d(x,y)=2$.

Lemma 7.10 *Let $b \in X_4(a)$. Then $M_{ab} \sim 5^{1+2}\mathrm{Sym}(3)$ and $|X_1(a) \cap X_1(b)| = 25$. In particular, $O_5(M_{ab})$ acts transitively on $X_1(a) \cap X_1(b)$ and $Z(O_5(M_{ab}))$ fixes $X_1(a) \cap X_1(c)$ pointwise.*

Proof: Let $c \in X_1(a) \cap X_1(b)$ and put $A = \langle Z_a, Z_b \rangle$. Then, by 7.3 (d), $A \sim 5^{1+2}.3$, $\mathrm{C}_5 \cong Z(A) \le M_{abc} \cong \mathrm{D}_{30}$ and $M_{ab} \le N_{M_a}(Z(A)) \sim 3.5^{1+2}.3.\mathrm{SD}_{16}$. Pick $e \in \Gamma_4$ with $e \dagger a$ and $Z_e \le M_{abc}$. Then $e \dagger b$, $M_{abce} = Z_e$ and $M_{ae} \sim \mathrm{C}_2 \times \mathrm{Mat}_{11} \sim M_{be}$. In particular, by orders, $M_{abe} = M_{ae} \cap M_{be} \ne Z_e = M_{abce}$ and thus $M_{ab} \ne M_{abc}$. Since $M_{abc} = N_{M_{ab}}(Z_c)$, the structure of M_{abc} and $N_{M_a}(Z(A))$ imply that $M_{ab} \sim 5^{1+2}.\mathrm{Sym}(3)$, as required.

Suppose that $b \in X_4(a)$, put $A = \langle Z_a, Z_b \rangle$ and define $F = Z(A)$. Let $c \in X_1(a) \cap X_1(b)$ and let $f = \Gamma_2(c) \cap \Gamma_2(a)$. Notice that M_{abc} inverts both Z_a and Z_b. Pick $d \in \Gamma_4(a)$ so that Z_d centralizes both Z_f and F (so we can choose $M_a = M_1$, $M_d = H$ and $M_f = M_2$). Then we have

$$N_{M_d}(F) \sim 2^{\cdot}(5.4 \curlywedge \mathrm{Sym}(6))$$

$$N_{M_a}(F) \sim 3.5^{1+2}.3.\mathrm{SD}_{16}$$

and

$$N_{M_{ad}}(F) = 6.(5 \times 3).4.$$

Define

$$R = \langle N_{M_d}(F), N_{M_a}(F) \rangle.$$

Also, just as in 7.10, fix $e \in \Gamma_4$ with $a \dagger e$, $b \dagger e$, $d \star e$ and so that Z_e inverts Z_f. (So Z_e is an involution in $M_{abc} \sim \mathrm{D}_{30}$). Notice that Z_e inverts F. Since $Z_d \le C_{M_a}(F)$ and acts fixed-point-freely on $O_5(C_{M_a}(F))/F$. Hence Z_d fixes a unique complement to $O_5(C_{M_a}(F))/F$ in $N_{M_a}(F)/F$ and so we infer that Z_d centralizes a unique group of order 3^2 in $C_{M_a}(F)$. In particular, c is the unique member of $X_1(a) \cap X_1(b)$ fixed by Z_d.

We exploit this notation in the following lemma where we uncover the structure of the subgroup R of M.

Lemma 7.11 *We have $R \sim 5^{1+4}.4.\mathrm{Sym}(6)$ and for $x \in \{a,f,d,e\}$, $N_{M_x}(F) \le R$.*

Proof: Set $\mathcal{A} = X_1(c)^F$, $A_0 = O_5(\langle Z_x \mid x \in \mathcal{A}\rangle)$, $\mathcal{B} = X_1(a)^F$, $B_0 = O_5(\langle Z_x \mid x \in \mathcal{B}\rangle)$ and $Y = \{a, f, d, e\}$.

From the structure of $N_{M_a}(F) \sim 3.5^{1+2}.3.\mathrm{SD}_{16}$, we have $|\mathcal{B}| = 25$. On the other hand, we have both $X_1(a) \cap X_1(b) \subseteq \mathcal{B}$ and $|X_a(a) \cap X_1(b)| = 25$ from 7.10. Thus $\mathcal{B} = X_1(a) \cap X_1(b)$ and $\mathcal{B}$ corresponds to the set of Sylow 3-subgroups of M_{ab}.

In particular, we have $B_0 = O_5(M_{ab})$ is extraspecial of order 5^3 with centre F and for $x \neq y \in \mathcal{B}$ we have $x \in X_4(y)$.

Similarly, as Z_a and Z_b are both subgroups of M_c and centralize F, $A \leq M_c$ and $\mathcal{A}$ corresponds to the set of Sylow 3-subgroups of A. Therefore, $A_0 = O_5(A)$ is also extraspecial of order 5^3 with centre F. Because A_0 centralizes B_0 and $A_0 \cap B_0 = Z(A_0) = Z(B_0) = F$, A_0B_0 is extraspecial of order 5^5. Notice also that, by 7.10, B_0 operates transitively on $\mathcal{B}$. Since $\mathcal{A}$ corresponds to the set of Sylow 3-subgroups of A and $\mathcal{B}$ corresponds to the set of Sylow 3-subgroups of M_{ab}, we have $\mathcal{A} \subseteq X_1(x)$ for all $x \in \mathcal{B}$.

For each $x \subseteq Y$ define $R_x = M_x \cap N_M(F)$. Then, by definition, $R = \langle R_d, R_a\rangle$. Since $F \in \mathrm{Syl}_5(M_{ac})$ and M_{ac} has index 2 in M_f the Frattini argument implies $M_f = R_f M_{ac}$ and so R_f acts non-trivially on $\Gamma_1(f) = \{a, c\}$. Therefore, the definitions of $\mathcal{A}$ and $\mathcal{B}$ imply that R_f interchanges A_0 and B_0 and so normalizes A_0B_0. Put $R_0 = A_0B_0R_f$. Then, from 6.5 (i) and (ii) we get

(7.11.1) $R_f \sim 3^2.5.4.\mathrm{D}_8$ and $R_0 \sim 5^{1+4}.4.3^2.\mathrm{D}_8$.

From the structure of $2{}^{\cdot}\mathrm{Alt}(11)$, we get

(7.11.2) $R_d \sim 2{}^{\cdot}(5.4 \wr \mathrm{Sym}(6))$.

Our goal is to show that R_d normalizes A_0B_0 and $R = A_0B_0R_d$. Using 7.7 and 7.11.1, we get

$$R_{fd} = M_{fd} \cap N_M(F) = C_{R_f}(Z_d) = R_f \sim 3^2.4.5.\mathrm{D}_8.$$

In particular, R_f is a maximal subgroup of R_d.

We now investigate R_e, so set $R_{0e} = C_{R_0}(Z_e)$. Then, as Z_e normalizes both A_0 and B_0 we have

$$R_{0e} = C_{A_0}(Z_e)C_{B_0}(Z_e)C_{R_f}(Z_e).$$

Now, recalling that Z_e inverts F, we have $C_{A_0}(Z_e) \sim \mathrm{C}_5 \sim C_{B_0}(Z_e)$ and, as A_0 and B_0 commute, $C_{A_0}(Z_e)C_{B_0}(Z_e)$ is an elementary abelian subgroup of order 25 which intersects F trivially. On the other hand, as $R_f \leq R_d$ and $Z_e \leq R_f$, $C_{R_f}(Z_e)$ can be calculated in R_d. We find that $C_{R_f}(Z_e) \sim 2^4$. Therefore,

$$R_{0e} \sim 5^2.2^4.$$

In particular, R_{0e} is a subgroup of the normalizer of a Sylow 5-subgroup of M_e; that is a subgroup of index 2 in a group of shape $2.(5:4 \wr 5:4).2$. Furthermore, in R_d we compute

$$R_{de}{=}C_{R_d}(Z_e) \sim 4.2^3 \text{ and } C_{R_{de}}(F) = C_{R_{def}} \cong \mathrm{Q}_8.$$

We conclude that $C_{R_{de}}(F)$ is a Sylow 2-subgroup of $C_{R_{e0}}(F)$. Notice also that as $N_M(F)/C_M(F)$ has order at most 4, we have $R_e = R_{de}C_{R_e}(F)$ and $R_e/C_{R_e}(F) = 4$. Furthermore, as Z_e inverts F and so $Z_e \not\leq C_M(F) \cap R_e$. Hence $C_{R_e}(F)$ splits over Z_e. Recalling that $C_{R_{0e}}(F) \sim 5^2 : \mathrm{Q}_8$, it follows from the structure of $2^{\cdot}\mathrm{Alt}(11)$ that the overgroups in M_e of $R_{de}R_{0e}$ are M_e, $2^{\cdot}\mathrm{Alt}(10)$, or $2^{\cdot}(\mathrm{Alt}(5) \times \mathrm{Alt}(5)) : 4$ (using $R_e/C_{R_e}(F)$ has order 4). Since $C_{R_e}(F)$ splits over F, we conclude that R_e is the normalizer of Sylow 5-subgroup in M_e and hence

(7.11.3) $R_e \sim 5^2.2^5$, R_{0e} is a normal subgroup of R_e of index 2 and $R_e = C_{R_{de}}(F)R0e$.

Let $t \in C_{R_{de}}(F) \setminus R_0$. Then $t^2 \in R_{0e}$. In R_d we calculate that $R_{fd} \cap R_{fd}^t \sim 5.4.\mathrm{D}_8$ and, by 7.11.3, $R_{0e} = R_{0e}^t$. In R_0 we see that $\langle R_{fd} \cap R_{fd}^t, R_{0e}\rangle \sim 5^{1+4}.4.\mathrm{D}_8$ and thus $A_0B_0 \leq R_0 \cap R_0^t$. It follows that $A_0B_0 = O_5(R_0 \cap R_0^t)$ and so t normalizes A_0B_0. As $R_d = \langle R_{df}, t\rangle$, R_d normalizes A_0B_0. Now for $x \in Y$ we have $R_x = (R_x \cap R_d)(R_x \cap A_0B_0)$. Hence $R_x \leq A_0B_0R_d$ and so, by 7.11.2, $R = A_0B_0R_d \sim 5^{1+4}.4.\mathrm{Sym}(6)$ as claimed.

We continue with the notation of 7.11 in the next lemma.

Lemma 7.12 *Let Γ^R be the connected component of Γ^F containing a. Then $R = \mathrm{Stab}_M(\Gamma^R)$.*

Proof: Let $X = \{a, d, f\}$. Then by 7.11 $N_{M_x}(F) \leq R$ for all $x \in X$. We claim that X is a complete set of representatives for the orbits of R on Γ^R. But this is clear, as for all $x \in X$, $X \setminus \{x\}$ is a complete set of representatives for the orbits of $R \cap M_x$ on $\Gamma(x)^F$. Indeed, if $x = a$, then, as we saw in the first few lines of 7.11, $X_1(a) \cap X_1(b)$ admits R_a transitively with representative c. Since each element of $X_1(a) \cap X_1(b)$ determines a unique element of $\Gamma_2(a)^F$, R_a is transitive on $\Gamma_2(a)^F$ with representative f. Now $\Gamma_3(a)^F$ is empty and $\Gamma_4(a)^F$ corresponds to the conjugates of Z_d in R_f and so we have a single orbit with representative d. As seen just before (7.11.1), R_f is transitive on $\Gamma_1(f)^F$ with representative a. Also $\Gamma_3(f)^F = \emptyset$ and $\Gamma_4(f)^F = \{d\}$. Finally, since the neighbours of type 1, 2 and 3 of d correspond to a subgroups or M_d/Z_d generated by a 3-cycle, two commuting 3-cycles and 3 commuting 3-cycles respectively, we see that $\Gamma_3(d)^F = \emptyset$ and $\Gamma_2(d)^F$ and $\Gamma_1(d)^F$ admit R_d transitively with representatives a and f respectively.

Hence we may apply 2.1 to obtain $R = \mathrm{Stab}(\Gamma^R) \cap N_M(F)$. Note that F is the elementwise stabilizer of Γ^R and so $\mathrm{Stab}_M(\Gamma^R) \leq N_M(F)$. Therefore, R and $\mathrm{Stab}_M(\Gamma^R)$ have the same orbits and stabilizers on Γ^R. Thus, by a Frattini argument, $R = \mathrm{Stab}_M(\Gamma^R)$.

Lemma 7.13 *Let $x \neq y \in \Gamma_1^R$ and put $B = \langle Z_x, Z_y \rangle$. Then either $Z_xO_5(R) = Z_yO_5(R)$ and $y \in X_4(x)$ or $Z_xO_5(R) \neq Z_yO_5(R)$ and $B \cong BO_5(R)/O_5(R)$. In particular, $d(x, y) \leq 2$.*

Proof: This follows from the structure of R as follows. Set $\bar{R} = R/F$ and let $x \in \Gamma_1^R$. Then Z_x maps to a 3-cycle in $R/O_{5,2}(R)$ and $N_{\bar{R}}(\bar{Z}_x) \sim 3 \times 5^2.3.\mathrm{SD}_{16}$. Thus in $\bar{R}$ there are 500 conjugates of $\bar{Z}_x$. Let C be a complement to $O_5(\bar{R})$ which contains $\bar{Z}_x$. Since $C_{O_5(\bar{R})}(\bar{Z}_x) \sim 5^2$, $\bar{Z}_x$ is contained in exactly 25 complements. Let C_1 be a further complement containing $\bar{Z}_x$ and let t and t_1 be the respective central involutions of C and C_1. Suppose that $y \in \Gamma_1^R$, $x \neq y$ and $\bar{Z}_y \leq C_1 \cap C$. Put $B = \langle \bar{Z}_x, \bar{Z}_y \rangle$. Since $B \leq C$, $Z_xO_5(R) \neq Z_yO_5(R)$. Thus $BO_5(R)/O_5(R) \cong \mathrm{C}_3 \times \mathrm{C}_3, 2^{\cdot}\mathrm{Alt}(4)$ or $2^{\cdot}\mathrm{Alt}(5)$. In any case B acts fixed-point freely on $O_5(\bar{R})$. On the other hand B centralizes $\langle t, t_1 \rangle \cap O_5(\bar{R}) \sim \mathrm{C}_5$, a contradiction. Therefore, any two complements that contain $\bar{Z}_x$ intersect in a subgroup which normalizes $\bar{Z}_x$. So the complements which contain $\bar{Z}_x$ account for $19.25 + 1$ conjugates of $\bar{Z}_x$. Moreover, in $\overline{Z_xO_5(R)}$ there are 25 conjugates of $\bar{Z}_x$. Thus we have accounted for all the conjugates of $\bar{Z}_x$ and any pair lie in $\overline{Z_xO_5(R)}$ and generate a group of shape $5^2.3$ or live in a complement and thus centralize an involution. In the latter case the distance between the corresponding vertices is at most 2 by 7.9 while in the former case there is a $u \in \Gamma_1^R$ for which $\bar{Z}_u$ centralizes $\langle \bar{Z}_x, \bar{Z}_y \rangle$ and so $y \in X_4(x)$ and this completes the proof.

Lemma 7.14 *Suppose that $e, f \in \Gamma_4$ with $e \star f$, $d \in \Gamma_1(f)$ and for $1 \neq t \in Z_e$ assume that $d^t \in X_2(d)$. Let $T = \langle h \in M_{def} \mid h^2 = 1 \rangle$. Then*

(a) $T \cong \mathrm{D}_{24}$.

(b) $|M_{def} : T| = 2$;

(c) *T has three orbits $\mathcal{O}_1$, $\mathcal{O}_2$ and $\mathcal{O}_3$ on $\Omega(b)$ of lengths 3, 4 and 4 respectively. T induces $\mathrm{Sym}(3)$ on $\mathcal{O}_1$, and D_8 on $\mathcal{O}_2$ and $\mathcal{O}_3$. The orbits $\mathcal{O}_2$ and $\mathcal{O}_3$ are not isomorphic as T-sets and $\mathcal{O}_2 \cup \mathcal{O}_3$ is an orbit for M_{def}.*

Proof: We first compute the structure of T and M_{def} inside M_f. Because $d^t \in X_2(d)$ we may choose notation such that in $\Omega(e)$,

$$d \leftrightarrow [1, 2, 3] \text{ and } e \leftrightarrow [1, 2][3, 4][5, 6][7, 8].$$

Then

$$M_{def}/Z_f \leftrightarrow \langle (9,10,11), (1,2)(5,7)(6,8)(9,10), (5,6)(7,8), (1,2)(5,6) \rangle$$

and so

$$T/Z_f \leftrightarrow \langle (9,10,11), (1,2)(5,7)(6,8)(9,10), (56)(78) \rangle.$$

In particular $T \cong \mathrm{D}_{24}$, $|M_{def}/T| = 2$ and M_{def} fuses the two classes of fours groups in T. Let $g \in \Gamma_1(e) \cap \Gamma_1(f)$. Then $Z_g = O_3(T)$. Note also that $Z_e \not\leq M_{def}$ and so M_{def} operates faithfully on $\Omega(e)$. The statements about the orbits of T on $\Omega(e)$ are now readily verified.

Proposition 7.15 *We have $d(x,y) \leq 2$ for all $x, y \in \Gamma_1$.*

Proof: Suppose that the statement is false. Then there is a pair $a, d \in \Gamma_1$ with $d(a,d) = 3$. Let a, b, c, d be a path in $(\Gamma_1, \bowtie)$ connecting a and d. Then $c \in X_i(a)$ and $b \in X_j(d)$ for some $i, j \in \{2,3,4\}$. We begin with

(7.15.1) Suppose that $c \in X_2(a)$. Let $e \in \Gamma_4(a) \cap \Gamma_4(c)$ and let $1 \neq t \in Z_e$, then $d^t \notin X_1(d)$.

If $d^t \in X_1(d)$, then, by 7.3 (a), there exists a unique f in $\Gamma_3(c) \cap \Gamma_3(d) \cap \Gamma_3(d^t)$. By the uniqueness of f, $e \in \Gamma_4(f)$ and so $|\Gamma_1(f) \cap \Gamma_1(e)| = 3$. As $c \in X_2(a)$ we see by calculating in $\Omega(e)$, that $a \bowtie g$ for at least one $g \in \Gamma_1(f) \cap \Gamma_1(e)$. But $g \bowtie d$ and so $d(a,d) \leq 2$, which is a contradiction.

(7.15.2) Suppose that $f \in \Gamma_4$ and $a \in \Gamma^f$. Then $\Gamma_2(d) \cap \Gamma^f = \emptyset$.

Suppose that there exists $g \in \Gamma_2(d)$ and $f \in \Gamma_4$ with $a, g \in \Gamma^f$. If Z_f centralizes or inverts Z_g, then $d \in \Gamma^f$ and $d(a,d) \leq 2$ by 7.9. So assume that Z_f neither inverts nor centralizes Z_g. Then, by 7.8, there exists a unique $h \in \Gamma_3(f) \cap \Gamma_3(g)$. So $|\Gamma_1(f) \cap \Gamma_1(h)| = 3$. Let $i \in \Gamma_1(f) \cap \Gamma_1(h)$. Then $i \bowtie d$. Since $a, i \in \Gamma^f$, $d(a,i) \leq 2$, by 7.9. Hence $d(a,i) = 2$. As Z_f centralizes Z_i, the the three possible structures for $\langle Z_a, Z_i \rangle$ indicate that $a \in \Gamma_1(f)$ and $a \in X_2(i) \cup X_3(i)$. Let $1 \neq t' \in Z_f$. Then $d^{t'} \in X_1(d)$. Hence 7.15.1 implies that $a \in X_3(i)$ for each $i \in \Gamma_1(f) \cap \Gamma_1(h)$. But then using the shape of these elements on $\Omega(f)$ we see that there is an involution t_1 in M_{fh} inverting Q_h and centralizing Z_a. Let $q \in \Gamma_4$ with $Z_q = \langle t_1 \rangle$. Then $a, d \in \Gamma^q$ and so 7.9 implies that $d(a,d) \leq 2$, a contradiction. Thus 7.15.2 holds.

(7.15.3) Up to changing the roles of a and d we may assume that $c \notin X_4(a)$.

Suppose that $c \in X_4(a)$ and set $F = Z(\langle Z_a, Z_c \rangle)$. Then, by 7.10, $X_1(a) \cap X_1(c)$ has order 25 and is fixed pointwise by F. Now considering F as a subgroup of M_c and applying 4.10 we see that $X_1(a) \cap X_1(c) \not\subset X_4(d)$. Hence there exists $b_1 \in X_1(a) \cap X_1(c)$ such that $b_1 \notin X_4(d)$ which is precisely what we required.

By 7.15.3 we may assume that $c \in X_2(a) \cup X_3(a)$. Let $e \in \Gamma_4(a) \cap \Gamma_4(c)$ and let $1 \neq t \in Z_e$. We focus our attention on the path d, c, d^t in $(\Gamma, \bowtie)$.

(7.15.4) $d \neq d^t$.

If $d = d^t$, then $a, d \in \Gamma^e$ and so $d(a, d) \leq 2$ by 7.9. Thus $d \neq d^t$.

(7.15.5) $d^t \notin X_1(d)$.

Suppose that $d^t \in X_1(d)$. Then there exists $g \in \Gamma_2(d) \cap \Gamma_2(d^t)$ with $g, a \in \Gamma^e$, this is against 7.15.2. Thus $d^t \notin X_1(d)$.

(7.15.6) $d^t \notin X_3(d)$.

Let $f \in \Gamma_4(c) \cap \Gamma_4(d) \cap \Gamma_4(d^t)$. Then $f * e$ and calculating in $\Omega(f)$ shows that it is impossible for $d^t \in X_3(d)$.

(7.15.7) $d^t \notin X_2(d)$.

Aiming for another contradiction we suppose that $d^t \in X_2(d)$. Let f be in $\Gamma_4(c) \cap \Gamma_4(d) \cap \Gamma_4(d^t)$. Then $f \star e$. Using 7.14 we compute in M_e that M_{edf} contains an involution s with $[Z_a, Z_a^s] = 1$. But this contradicts 7.15.2 applied with the roles of a and d interchanged.

By 7.15.4 through 7.15.7, we have $d^t \in X_4(d)$. Set $T = M_{cdd^t}$. Then, by 7.3 (d), $T \cong \mathrm{D}_{30}$. Put $F = O_5(T)$. Because $t \in M_c'$ and normalizes F, t centralizes both F and Z_c and hence also T. Therefore $T \leq M_e$. Since $a \in X_2(c) \cup X_3(c)$ we see in M_e acting on $\Omega(e)$, that either F fixes a or there exist involutions s in T with $[Z_a, Z_a^s] = 1$. We then obtain contradictions via 7.13 and 7.15.2 (applied with the roles of a and d interchanged) respectively.

We now prove the main theorem of this section.

Theorem 7.16 *Suppose that M is a faithful completion of a* Ly*-amalgam of groups (M_1, M_2, M_3). Then*

(a) *M_1 has five orbits on M/M_1. Moreover, the orbit stabilisers have shape* $3^{\cdot}\mathrm{Aut}(\mathrm{McL})$, $3^{2+4}.4.\mathrm{Sym}(5)$, $2.\mathrm{Sym}(7)$, $4.\mathrm{Sym}(6)$ *and* $5^{1+2}.\mathrm{Sym}(3)$.

(b) *M has order* $2^8 \cdot 3^7 \cdot 5^6 \cdot 7 \cdot 11 \cdot 31 \cdot 37 \cdot 67$.

(c) *M has a unique class of involutions and* $C_M(t) \cong 2^{\cdot}\mathrm{Alt}(11)$ *for any involution t in M.*

(d) *For* $i = 1, 2, 3$, M_i *is a maximal* 3*-local subgroup of* M.

(e) M *has a subgroup* F *of order five with* $N_M(F) \sim 5^{1+4}.4.\mathrm{Sym}(6)$.

(f) M *is a group of* Ly*-type.*

Proof: Part (a) follows from 7.3 (a), (b) and (c), 7.10 and 7.15 and part (b) is a direct consequence of (a). Let H be the subgroup of M found in 7.4. Then $H \cong 2{\cdot}\mathrm{Alt}(11)$ and so, by (b), $1 \neq t \in Z(H)$ is a 2-central involution. Since H has a unique class of involution different from t and these involutions are conjugate to t, M contains a unique class of involutions. If $g \in \Gamma_1^{Z(H)}$, then 7.15 implies that $d(M_1, g) \leq 2$ and so $\Gamma_1^{Z(H)}$ is connected. Thus $N_M(\Gamma_1^{Z(H)}) = C_M(t) = H$ by 7.7. Hence (c) holds. That M_1 is a maximal subgroup follows immediately from parts (a) and (b). As for M_i, $i = 2, 3$, M_i acts transitively on the M-conjugates of Z_1 in Q_i and so by a Frattini argument $N_M(Q_i) = M_i(N_M(Q_i) \cap N_M(Z_1)) = M_i N_{M_1}(Q_i) = M_i$. Hence M_2 and M_3 are also maximal 3-locals. Let F be the cyclic group of order 5 introduced just before 7.11. Then 7.15 shows that Γ^F is connected and so (e) follows from 7.12. Finally we come to part (f). If N is a normal subgroup of M and $N \cap M_i = 1$ for $i = 1, 2, 3$, then M/N is also a faithful completion of (M_1, M_2, M_3). Since, by (a), $|M| = |M/N|$ we get $N = 1$. So suppose that $N_i = M \cap M_i \neq 1$ for some $i \in \{1, 2, 3\}$ and fix i. It follows that $Z_1 \leq N_i \leq N$. Thus $Z_2 = \langle Z_1^{M_2} \rangle \leq N$ and $M_1' = \langle Z_2^{M_1} \rangle \leq N$. Then $M_3 = \langle (M_1' \cap M_3)^{M_3} \rangle \leq N$, $M_1 = M_1' M_{13} \leq N$ and $M_2 = M_{12} M_{23} \leq N$. Thus $N = M$ and M is simple. Finally (c) implies that M is of Ly-type.

8 A Ly-amalgam in $\mathrm{GL}_{111}(5)$

In this section we show that there exists a Ly-amalgam of groups, (M_1, M_2, M_3), in $\mathrm{GL}_{111}(5)$. This then establishes the existence of a faithful completion of a Ly-amalgam of groups. Our calcuations depend on the details of the structure of such an amalgam which is given in 6.3. Let F be the field with 5 elements. The following notation is used to describe the action of the groups M_1, M_2 and M_3 on the various modules as we build the 111-dimensional F-space. Let $K^* \in \mathrm{Syl}_2(M_{23})$ and define

$$\begin{aligned} B &= M_{123}, \\ S &= O_3(B) \quad \text{and} \\ K &= K^* \cap B. \end{aligned}$$

Also, for $i \in \{1, 2, 3, 12, 13, 23\}$, we define

$$\begin{aligned} K_i &= K \cap O^2(M_i), \\ Q_i &= O_3(M_i) \quad \text{and} \\ Z_i &= Z(Q_i). \end{aligned}$$

Recall that $S \in \mathrm{Syl}_3(B)$. Finally we let K_0 be the subgroup of K_2 of order two, for $i = 2, 3$, D_i be the maximal subgroup of K satisfying $[D_i, O^2(M_i)] \leq Q_i$, and D_2^* be the maximal subgroup of K^* with $[D_2^*, O^2(M_2)] \leq Q_2$.

In the following technical lemma we detail the structure of, and relationship between, many of the subgroups that we have just introduced.

Lemma 8.1 (a) *$B = SK$, $M_{23} = SK^*$ and $M_2 = O^2(M_2)K^*$.*

(b) *$K^* \cong \mathrm{C}_2 \times \mathrm{SD}_{16}$.*

(c) *$K \cong \mathrm{C}_2 \times \mathrm{Q}_8$.*

(d) *$K_1 \cong \mathrm{Q}_8$ and $K_1 = C_K(Q_1)$.*

(e) *$D_3 \cong \mathrm{C}_2$.*

(f) *$D_2 \cong \mathrm{C}_4$ and $D_2^* \cong \mathrm{C}_8$.*

(g) *$K_2 \cong \mathrm{C}_4$, $K_0 \cong \mathrm{C}_2$ and $K_2 = K_{12} = C_{K^*}(Z(Q_2)) \leq K_1$.*

(h) *$D_2K_2 \cong \mathrm{C}_4 \times \mathrm{C}_2$ and $D_3 \leq D_2K_2$.*

(i) *$K_3 \cong \mathrm{Q}_8$ and $K_2 \cap K_3 = K_0$.*

(j) *$D_2 \leq K_3$ and $D_2 = K^{*\prime}$.*

(k) *$D_2^* \not\leq M_3'$.*

(l) *$C_{K^* \cap M_3'}(S/Q_2) \cong \mathrm{D}_8$.*

Proof: Parts (a), (b), (c), (e) and $K_3 \cong \mathrm{Q}_8$ follow from 6.3 and (f) follows from 6.5. Since $O^2(M_{12}) = O^2(M_2) = C_{M_2}(Z_2)$, by 6.5 (a), and $O^2(M_{12})/Q_2 \cong 2{\cdot}\mathrm{Alt}(5)$, we have $K_2 = K_{12} = C_K(Z_2)$ and so (g) holds. As, by definition, D_2 centralizes K_2, D_2K_2 is the unique normal subgroup of K^* isomorphic to $\mathrm{C}_2 \times \mathrm{C}_4$. In particular, $K^{*\prime} = D_2K_2 \cap K_3 \cong \mathrm{C}_4$ and so, as D_2K_2 contains only two cyclic groups of order 4, $D_2K_2 \cap K_3 \in \{K_2, D_2\}$. Since, by 6.5 (b), M_2 and K^* induce D_8 on $Z(Q_2)$, $K^*/K_2 \cong \mathrm{D}_8$ and thus we infer that $K_2 \neq K^{*\prime}$. Hence $D_2 = D_2K_2 \cap K_3 = K^{*\prime}$ and $K_2 \cap K_3 = K_0$. So (h), (i) and (j) hold. It only remains to prove (k) and (l). Let X be the unique conjugate of Z_1 in Q_3 fixed by K^*. Then, by 2.5 (d), $C_{M_3'}(X) \sim 3^5.\mathrm{Alt}(6)$, Since $Q_3 = X(Q_3 \cap Q_2)$, we have $S = XQ_2$. Therefore, $C_{K^* \cap M_3'}(S/Q_2)$ is a subgroup of $3^5.\mathrm{Alt}(6)$ of order 8. This proves (l). Moreover, D_2^* also centralizes S/Q_2 and so X. But by (f), $D_2^* \cong \mathrm{C}_8$ and and so by (l) $D_2^* \not\leq M_3'$. Thus (k) is also proven and the lemma is complete.

Let $1 \leq i, j \leq 3$ with $i \neq j$. If X is an FM_i-module, then $R_{ij}(X)$ denotes the restriction of X to M_{ij}. If, on the other hand, W is an FM_{ij}-module, then $I^i(W) = W \otimes_{FM_{ij}} FM_i$ denotes the FM_i-module induced from W and $R_B(W)$ denotes the restriction of W to B. Finally, if U is an FB-module, then $I^{ij}(U) = U \otimes_{FB} FM_{ij}$.

In the ensuing construction $X_t(i)$ will always denote an FM_i-module, $W_t(ij)$ an FM_{ij}-module and U_t an FB-module. If G is a group, $H \leq G$, U is an FH-module and W is an FG-module we write $U \to W$ provided that U is isomorphic to an FH-submodule of W restricted to FH. (We remark that in all cases below the FH-submodule of W isomorphic to U will be unique).

Recall that Aut(McL) has two irreducible 21-dimensional representation over F arising from the Leech-lattice modulo 5; they differ by the -1 representation (see 5.4 (d)). For the moment let $X_1(1)$ be either of the irreducible 21-dimensional FM_1/Q_1-modules. So

(1) *$X_1(1)$ is irreducible of dimension 21.*

Put

$$W_1(13) = C_{X_1(1)}(Q_3) \text{ and } W_2(13) = [X_1(1), Q_3].$$

Then $X_1(1) = W_1(13) \oplus W_2(13)$. By 2.5 (e), M_{13} has two orbits one of length 10 and one of length 30 on the maximal subgroups of Q_3/Z_1. Set $Q_3^* = Q_3 D_3$. Then, since Q_3^* inverts Q_3, it follows that

(2) (a) *$W_1(13)$ is 1-dimensional and $W_1(13) \to X_1(1)$.*

(b) *$W_2(13)$ is irreducible of dimension 20 and $W_2(13) \to X_1(1)$.*

(c) *$W_2(13)$ is, as an FQ_3^*-module, the direct sum of ten 2-dimensional Wedderburn components.*

The two 21-dimensional FM_1-modules can be distinguished by the action of D_3 on $W_1(13)$. We choose $X_1(1)$ so that

(3) *D_3 inverts $W_1(13)$.*

Set

$$U_1 = R_B(W_1(13)).$$

Since B permutes the ten Wedderburn components for Q_3^* in $W_2(23)$ as Mat_9, it follows from (2) (c) that

(4) *Restricted to B, $W_2(13)$ is the direct sum of irreducible modules U_2 and U_3 of dimension 2 and 18 respectively.*

Put

$$W_1(12) = C_{X_1(1)}(Q_2) \text{ and } W_2(12) = [X_1(1), Q_2].$$

Then $W_1(12)$ is 3-dimensional and, by 2.7, $W_2(12)$ is 18-dimensional. Since $X_1(1)$ is irreducible, M_{12}' does not normalize $W_1(13)$. Hence, as $R_B(W_1(12)) = U_1 + U_2$, we conclude that $W_1(12)$ is irreducible as an M_{13}-module. Because Q_2Q_3 is contained in M_{13}', $C_{W_1(12)}(Q_2Q_3) = W_1(13)$ and so (using facts about the 3-dimensional $F(2{}^{\cdot}\mathrm{Alt}(5))$-module, which is in fact the irreducible constituent of the degree 5 permutation module), K_2 inverts $W_1(13)$. Now (3), 8.1 (h) and (j) imply that D_2 centralizes $W_1(13)$ and hence, as $W_1(12)$ is irreducible, D_2 centralizes $W_1(12)$. Therefore, $C_{M_{12}}(W_1(12)) = Q_2D_2$. Pulling these facts together we have

(5) (a) *Restricted to M_{12}, $X_1(1)$ is the direct sum of irreducible modules $W_1(12)$ and $W_2(12)$ of dimension* 3 *and* 18 *respectively.*

(b) $C_{M_{12}}(W_1(12)) = Q_2D_2$.

(c) $C_{M_{12}}(W_2(12)) = Q_1$.

(d) *Restricted to B, $W_1(12)$ is isomorphic to the direct sum U_1 and U_2.*

(e) *Restricted to B, $W_2(12)$ is isomorphic to U_3.*

Since $M_2/Q_2D_2 \cong \mathrm{Sym}(5) \times \mathrm{C}_2$, there exists a unique FM_2-module $X_1(2)$ such that

(6) (a) *$X_1(2)$ is irreducible of dimension* 3.

(b) *Restricted to M_{12}, $X_1(2)$ is isomorphic to $W_1(12)$.*

(c) *D_2^* inverts $X_1(2)$ and $C_{M_2}(X_1(2)) = Q_2D_2$.*

We have

$$X_1(2) = C_{X_1(2)}(S) \oplus [X_1(2), S].$$

Put

$$W_1(23) = C_{X_1(2)}(S) \text{ and } W_2(23) = [X_1(2), S].$$

Then it is straightforward to verify

(7) (a) *$W_2(23)$ is irreducible of dimension* 2.

(b) *Restricted to B, $W_2(23)$ is isomorphic to U_2.*

(c) *Restricted to M_{23}, $X_1(2)$ is isomorphic to the direct sum of $W_1(23)$ and $W_2(23)$.*

By 8.1 (j), $D_2 \le K_3 \le M_3'$. Thus $D_2 \le M_{13} \cap M_3'$. Since $D_2 \not\le M_1'$ whereas $M_{13}' \le M_1'$, we get $D_2M_{13}' = M_{13} \cap M_3'$. Hence, by (5) (b), $M_{13} \cap M_3'$ centralizes $W_1(13)$. Let $X_1(3)$ be the 1-dimensional FM_3-module $X_1(3)$ which is centralized by M_3' and inverted by M_3. Then the restriction of $X_1(3)$ to M_{13} is isomorphic to $W_1(13)$. Then as FB-modules $R_{23}(X_1(3))$ and $W_1(23)$ are both isomorphic to U_1. On the other hand as, by 8.1 (k), $D_2^* \not\le M_3'$, D_2^* inverts both $W_1(23)$ and $C_{X_1(2)}(S)$. Thus $R_{23}(X_1(3))$ and $W_1(23)$ are isomorphic as FM_{23}-modules. We have proved:

(8) (a) $X_1(3)$ *is* 1*-dimensional.*

(b) *Restricted to* M_{13}*,* $X_1(3)$ *is isomorphic to* $W_1(13)$*.*

(c) *Restricted to* M_{23}*,* $X_1(3)$ *is isomorphic to* $W_1(23)$*.*

(d) *Restricted to* B*,* $X_1(3)$ *is isomorphic to* U_1*.*

Put

$$X_2(2) = I^2(W_2(12)).$$

Then $X_2(2)$ is 36-dimensional and when restricted to M_{12}, $X_2(2)$ is the direct sum of $W_2(12)$ and $W_3(12)$, where $W_3(12)$ is irreducible of dimension 18. As Q_1 centralizes $W_2(12)$ but not $W_3(12)$, $W_2(12)$ and $W_3(12)$ are not isomorphic. Therefore, $X_2(2)$ is irreducible of dimension 36. Put

$$U_4 = R_B(W_3(12)) \text{ and } W_3(23) = R_{23}(X_2(2)).$$

We have

(9) (a) $W_3(12)$ *is irreducible of dimension* 18 *and is not isomorphic to* $W_2(12)$ *as an* FM_{12}*-module.*

(b) $X_2(2)$ *is irreducible of dimension* 36*.*

(c) *Restricted to* M_{12}*,* $X_2(2)$ *is isomorphic to the direct sum of* $W_2(12)$ *and* $W_3(12)$*.*

(d) U_4 *is irreducible of dimension* 18 *and not isomorphic to* U_3*.*

(e) *Restricted to* B*,* $W_3(12)$ *is isomorphic to* U_4*.*

(f) $W_3(23) \cong I^{23}(U_3)$ *is irreducible of dimension* 36*.*

(g) *Restricted to* B*,* $W_3(23)$ *is isomorphic to the direct sum of* U_3 *and* U_4*.*

Let

$$X_2(3) = I^3(W_2(23)).$$

Then $X_2(3)$ is 110-dimensional. Since M_3 operates on M_3/M_{23} as it does on duads in the S(4,5,11)-Steiner system, following back through (7) and (2) c) delivers

(10) (a) $W_2(23)$ *is a Wedderburn component for* Q_3 *on* $X_2(3)$*.*

(b) $X_2(3)$ *is irreducible of dimension* 110*.*

By (2) b) and (4), we have $W_2(13) \cong I^{13}(U_2)$ and $U_3 \to W_2(13)$. Thus $W_2(13) \to X_2(3)$ and so $U_3 \to X_2(3)$. Since, M_3 acts on M_3/M_{23} as it does on duads in the Steiner system and since M_3 permutes the Wedderburn components in $X_2(3)$ as is permutes the cosets M_3/M_{23}, we can use (10) to calculate

(11) (a) *Restricted to* M_{13}, $X_2(3)$ *is isomorphic to the direct sum of* $W_2(13)$ *and* $W_3(13)$ *where* $W_3(13)$ *is irreducible of dimension* $90 = 45 \cdot 2$.

(b) *Restricted to* M_{23}, $X_2(3)$ *is isomorphic to the direct sum of* $W_2(23)$, $W_3(23)$ *and* $W_4(23)$, *where* $W_4(23)$ *is irreducible of dimension* 72.

(c) *Set* $U_5 = R_B(W_4(23))$. *Then* U_5 *is irreducible of dimension* 72.

(d) $R_B(W_3(13))$ *is the direct sum of* U_4 *and* U_5.

(e) *Restricted to* B, $X_2(3)$ *is isomorphic to the direct sum of* U_2, U_3, U_4 *and* U_5.

We now claim that there is a 72-dimensional FM_2-module which when restricted to M_{23} is isomorphic to $W_4(23)$. Let E_1 and E_2 be the two subgroups of order three in Z_2 not conjugate to Q_1 in M_2 and set $R = O^2(M_2)$. Then, by 6.5, R centralizes Z_2 and so, in particular, normalizes both E_1 and E_2. Now for $i = 1, 2$, we let W_i be the unique FR/E_i-module of dimension 36 and $W = W_1 \oplus W_2$ (see 2.7). Then W is a faithful FR-module. Next let $R_3 = R \cap M_3$. We will show that W is isomorphic to $W_4(23)$ as an FR_3-module. Thus in $\mathrm{GL}_{72}(F)$ we may arrange that the images of M_{23} and R intersect in a subgroup isomorphic to R_3. Denote the images by M_{23}^*, R^* and R_3^* respectively. Following the same methods as in the end of the proof of 6.4 we see that $\langle R^*, M_{23}^* \rangle \cong M_2$. Therefore, to prove our claim it suffices to show that W and $W_4(23)$ are isomorphic as FR_3-modules. Set $Y = W_4(23)$ and, for $i = 1, 2$, set $Y_i = C_Y(E_i)$. Since $W_3(13)$ admits M_{13} faithfully, $0 = C_{W_3(13)}(Q_1) = C_{U_5}(Q_1) = C_{W_4(23)}(Q_1)$. Thus $Y = Y_1 \oplus Y_2$. Since both Y_1 and Y_2 are FR_3-modules, our claim will be proved when we show that, for $i = 1, 2$, the FR_3-modules Y_i and W_i are isomorphic. It clearly suffices to show Y_1 and W_1 are isomorphic.

Let E be the preimage of the complement to Z_2/E_1 in $(Q_2 \cap Q_3)/E_1$ which is normalized by K_2 (so $E = [Q_2 \cap Q_3, K_2]E_1/E_1$) and set $W_E = C_{W_1}(E)$ and $Y_E = C_{Y_1}(E)$ and consider them as $FN_{R_3}(E)$-modules. From 2.7 and 2.6 we have $\dim_F W_E = 4 = \dim_F Y_E$. Note that the FR_3-modules W_1 and Y_1 are induced from the $FN_{R_3}(E)$-modules W_E and Y_E respectively. Hence to show that W_1 is isomorphic to Y_1 as an FR_3-module it suffices to show that W_E and Y_E are isomorphic as $N_{R_3}(E)$-modules.

Note that $N_{R_3}(E) = Q_3K_2$ acts as $Q_3K_2/E \sim 3^2.\mathrm{C}_4$ (by 8.1 (e)) on both W_E and Y_E. Now K_2 normalizes Z_2E and so, as K_0 does not centralize Q_3/E, we see that K_2 has an orbit of length 2 and two orbits of length 1 on the maximal subgroups of Q_3 which contain E. Since Z_2 acts with out fixing any non-zero vectors on both W_E and Y_E, we conclude that

$$\begin{aligned} Y_E &= C_{Y_E}(A_1) \oplus C_{Y_E}(A_2) \\ W_E &= C_{W_E}(A_1) \oplus C_{W_E}(A_2) \end{aligned}$$

where A_1 and A_2 are the maximal subgroups of Q_3 which contain E and are interchanged by K_2. Thus to conclude the proof that Y_E and W_E are isomorphic as $FN_{R_3}(E)$-modules it only remains to show that they are isomorphic as FK_0-modules. We will show that K_0 in fact inverts both spaces. This follows directly from 2.7 (d) for W_E. Thus we concentrate on Y_E. Let $t \in C_{K^* \cap M_3'}(S/Q_2)$ be an involution (see 8.1 (l)). Then, as t centralizes S/Q_2, (4) and (7) (b) imply that t either centralizes or inverts $W_2(23)$. On the other hand, by (8) and (6) (c), t centralizes $X_1(3)$, but not $X_1(2)$. Thus we see that t inverts $W_2(23)$. Since t is conjugate into K_0 in M_3', we see that $C_{X_2(3)}(E)$ is not centralized by K_0. Now, as mentioned earlier, E is contained in exactly four maximal subgroups of Q_3; extend our notation above by denoting the other two maximal subgroups of Q_3 which contain E by A_3 and A_4 and assume that $A_3 = EZ_2$. Then

$$C_{X_2(3)}(E) = C_{Y_E}(A_1) \oplus C_{Y_E}(A_2) \oplus C_{X_2(3)}(A_3) \oplus C_{X_2(3)}(A_4),$$

where each summand is 2-dimensional. Now we plainly have $C_{X_2(3)}(A_3) \cong W_2(23)$ and $C_{X_2(3)}(A_4) \cong C_{X_2(2)}(E)$ and these are both centralized by K_0. Since A_1 and A_2 are interchanged by K_2 we know that K_0 operates in the same way on both $C_{Y_E}(A_1)$ and $C_{Y_E}(A_2)$. Therefore, K_0 inverts Y_E as claimed. Thus we have shown that there exists an FM_2-module, $X_3(2)$, which satisfies

(12) $X_3(2)$ *is irreducible of dimension* 72 *and is restricted to* M_{23} *isomorphic to* $W_4(23)$.

Let $W_4(12) = R_{12}(X_3(2))$. Note that both $W_3(13)$ and $W_3(12) \oplus W_4(12)$ are, when restricted to B, isomorphic to $U_4 \oplus U_5$. Thus taking $G_2 = M_{12}$, $G_3 = M_{13}$ and $G_{23} = M_{123}$, we can apply 5.13 to get an FM_1-module $X_2(1)$ so that

(13) (a) $X_2(1)$ *is irreducible of dimension* 90.

(b) *Restricted to* M_{13}, $X_2(1)$ *is isomorphic to* $W_3(13)$.

(c) *Restricted to* M_{12}, $X_2(1)$ *isomorphic to the direct sum of* $W_3(12)$ *and* $W_4(12)$.

Let X be a 111-dimensional vector space over F. We are now able to construct a completion of the Ly-amalgam in $\mathrm{GL}(X)$. Set $H = \mathrm{GL}(X)$ and $A = \mathrm{Inn}(\mathrm{GL}(X))$. Then, by (1)-(13), for $i = 1, 2, 3$ there exist monomorphism $\alpha_i : M_i \to H$ such that as an $FM_1^{\alpha_1}$-module $X = X(1) = X_1(1) \oplus X_2(1)$, as an $FM_2^{\alpha_2}$-module $X = X(2) = X_1(2) \oplus X_2(2) \oplus X_3(2)$ and as an $FM_3^{\alpha_3}$-module $X = X(3) = X_1(3) \oplus X_2(3)$. Let $\{i, j\} \subset \{1, 2, 3\}$. Then using (1)-(13), $X(i)$ and $X(j)$ are isomorphic as FM_{ij}-modules. Thus there exist $a_1, a_2, a_3 \in A$ such that

$$\alpha_1|_{M_{13}} a_2 = \alpha_3|_{M_{13}}, \alpha_2|_{M_{12}} a_3 = \alpha_1|_{M_{12}} \text{ and } \alpha_3|_{M_{12}} a_1 = \alpha_2|_{M_{23}}.$$

Therefore, the assumptions of 2.4 are fulfilled. Set $M^*_{12} = M^{\alpha_1}_{12}$, $M^*_{23} = M^{\alpha_3 a_2^{-1}}_{23}$, $M^*_{13} = M^{\alpha_1}_{13}$, and $B^* = B^{\alpha_1}$. If N is one of B^*, M^*_{23}, M^*_{13} and M^*_{12}, then X is the direct sum of pairwise non-isomorphic, absolutely irreducible FN-modules and so $C_H(N)$ consists of exactly those linear transformations which act as non-zero scalars on each of the irreducible FN-submodules. By 2.4 (b), $B^* \leq N$ and so for each choice of N, $C_A(N) \leq C_A(B^*)$. Since, on restriction to M_{ij} (respectively B), X is a direct sum of pairwise non-isomorphic FM_{ij}-modules, it is now easy to verify that $C_A(B^*)$ has order 4^4 and that $C_A(B^*) = C_A(M^*_{23})C_A(M^*_{13})C_A(M^*_{12})$. Thus by 2.4 (a), 2.4 (a1) holds. Put $M^{**}_i = M^{\alpha_i b_i}_i$. Then by 2.4 (c) we get

Theorem 8.2 *Suppose that X is a 111-dimensional vector space over F. Then there exists a* Ly*-amalgam in* GL(X).

Theorem 1.3 now follows by combining 6.4, 8.2 and 7.16.

References

[Alp] J.L. Alperin, Local Representation Theory, Cambridge University Press, Cambridge, 1986.

[Asch] M. Aschbacher, Finite Group Theory, Cambridge University Press, Cambridge, 1986.

[Asch1] M. Aschbacher, Sporadic Groups, Cambridge University Press, Cambridge, 1994.

[AschS] M. Aschbacher and Y. Segev, The uniqueness of groups of Lyons type, J. AMS, Vol 5, No. 1, 75-98 1992.

[Be] D. Benson, Modular Representation Theory: New Trends and Methods, Springer-Verlag, Berlin, New York, Tokyo, (1984).

[ATLAS] J.H. Conway, R.T. Curtis, S.N. Norton, R.A. Parker, R.A. Wilson, Atlas of Finite Groups, Clarendon Press, Oxford, 1985.

[CE] H. Cartan, S. Eilenberg, Homological Algebra, OUP, 1956.

[CS] J.H.Conway, N.J.A.Sloane, Sphere Packings, Lattices and Groups, Springer Verlag, New York, 1988.

[CFYT] G. Cooperman, L. Finkelstein, B. York and M. Tselman, Constructing Permutation Representations for Large Matrix Groups, Proc. Int. Symposium on Symbolic and Algebraic Manipulation (ISSAC-94), ACM Press, 1194.

[Fink] L. Finkelstein, The Maximal Subgroups of Conway's Group C_3 and McLaughlin's groups, J. Alg. 25 (1973) 58-89.

[Frob] G. Frobenius, Über die Charaktere der mehrfach transitiven Gruppen, Sitzung berichte Ak. Berlin, (1904), 558-571.

[Gol] H. W. Gollan, A New Existence Proof for Ly, The Sporadic Simple Group of R. Lyons, Habilitationsschrift, Universität GH Essen, FB Mathematik und Informatik, 1995.

[Go] D. Gorenstein, Finite Groups, Harper & Row, 1968.

[Gr] R.L. Griess, Schur Multipliers of Finite Simple Groups of Lie Type, Trans. A.M.S. Vol 183, 355- 421, 1973.

[HL] G. Hiss, K. Lux, Brauer Trees of Sporadic Groups, Clarendon Press, Oxford, 1989.

[HLP] G. Hiss, K. Lux, R. Parker, The 5-Modular Characters of the McLaughlin Group and its Covering Group, Manuscripta Math. 73, 91-114 (1991).

[IM] A.A. Ivanov, U. Meierfrankenfeld, A Computer Free Construction of J_4, J. Alg. 219 (1999), no. 1, 113–172.

[Iv] A.A. Ivanov, Geometry of Sporadic Groups I, Petersen and Tilde Geometries, Encyclopedia of Mathematics and its Applications, CUP, 1999.

[Jam] G.D. James, The Modular Characters of the Mathieu Groups, J. Alg. 27, 57-111 (1973.)

[MATLAS] Ch. Jansen, K. Lux, R.A. Parker and R.A. Wilson, An Atlas of Brauer Characters, Clarendon Press, Oxford, 1995.

[JW] Ch. Jansen and R.A. Wilson, The Minimal 3-Modular Representation for the Lyons Group, Comm. Alg. 24(3), 873-879 (1996).

[Ly] R. Lyons, Evidence for a New Finite Simple Group, J. Alg. 20 540-569, (1972); Errata, J. Alg. 34, 188-189, (1975).

[McL] McLaughlin, J, A Simple Group of Order 898, 128,000, *Theory of Finite Groups (Symposium, Harvard, Univ., Cambridge, Mass., 1968)*, Ed, R. Brauer and C. Sah, pp. 109-111, Benjamin, New York, 1969.

[MNP] W. Meyer, W. Neutsch, R. Parker, The Minimal 5-Representations of Lyons' Sporadic Group, Math. Ann. 272 (1985), no. 1, 29–39.

[Si] C. C. Sims, The Existence and Uniqueness of Lyons' Group, in Finite groups '72 (eds. T. Gagen, M. P. Hale and E. E. Shult), North Holland, 1973, pp.138-141.

[Wi] R.A. Wilson, A Representation for the Lyons Group in $\mathrm{GL}_{2480}(4)$, and a New Uniqueness Proof, Arch. Math. (Basel) 70 (1998), no. 1, 11–15.