Perfect Frobenius Complements

U. Meierfrankenfeld

May 17, 2002

Abstract

Let H be a finite Frobenius group with a perfect Frobenius complement G. Two new proofs that G is isomorphic to $SL_2(5)$ are given.

1 Introduction

A Frobenius group is a transitive but not regular permutation group on a set Ω such that every non-trivial element has at most one fixed-point. Let H be a finite Frobenius group with kernel N and complement G. That is, N consists of the identity and all the fixed-point free elements; and G is the stabilizer of some element in Ω. In [Fr, V] Frobenius proved that N is a normal subgroup of H. In [Za1, Satz 16] (and in revised form in [Za2]) Zassenhaus showed that if G is perfect then $G \cong SL_2(5)$. Both of these proofs involve character theory. A further proof of Zassenhaus' theorem based on the elementary theory of exceptional characters can be found in [Be]. In this note we will give two new proofs of Zassenhaus' theorem without using character theory (except that we assume Frobenius' theorem).

Theorem A *Let H be a finite Frobenius group with complement G. If G is perfect, then $G \cong SL_2(5)$.*

The first proof is based purely on standard group theory text book material. The second proof is slightly shorter but relies on a couple of simple facts about modular representations of finite groups (see the end of the introduction for the details).

Note that the action of G on Ω and the action of G on N by conjugation are isomorphic. Hence $C_N(g) = 1$ for all $g \in G^{\#}$. Let p be a prime dividing the order of N and put $V = \Omega_1(Z(O_p(N)))$. By [Th], N is nilpotent and so $V \neq 1$. (For perfect G a more elementary argument is possible: Let q the smallest prime dividing the order of G and S a Sylow q-subgroup of G. By [Go, 10.3.10], S is cylic or $q = 2$ and S is generalized quaternion. As G is perfect, Burnside's p-complement theorem [Go, 7.4.5] implies that S is not cyclic. Thus

$q = 2$ and G contains an involution t. But then t inverts N and so N is abelian and again $V \neq 1$.) Now V is a $GF(p)G$-module and all non-trivial elements of G act fixed-point freely on V. Hence Theorem A follows at once from (and is in fact equivalent to) Theorem B, which is also a theorem of Zassenhaus:

Theorem B *Let G be a non-trivial, finite perfect group, K a field and V a faithful KG-module so that all non-trivial elements of G act fixed-point freely on V. Then $G \cong SL_2(5)$.*

In section 3 we establish some basic facts about G which will be used in both of our proofs of Theorem B. Section 4 contains the first proof of Theorem B, while in section 5 we prove:

Theorem C *Let G be a non-trivial finite perfect group with cyclic or dihedral Sylow 2-subgroups, K a field of characteristic 2 and W a faithful KG-module so that all non-trivial elements of odd order of G act fixed-point freely on W. Then $G \cong Alt(5)$.*

We finish the introduction by showing how Theorem C implies Theorem B, and thus obtain our second proof of Theorem B.

The starting point are the following simple facts from the theory of modular representations of finite groups (which can be extracted for example from [CR]):

1. Let p be a prime and G a finite group so that p does not divide the order of G. Then every $GF(p)G$ module is isomorphic to the reduction modulo p of some module for G in characteristic 0.
2. Let G be a finite group, p a prime, V a G-module in characteristic 0 and W a reduction of V modulo p. Then a p'-element in G acts fixed-point freely on V if and only if it acts fixed-point freely on W.

Let G and V be as in Theorem B. Without loss K is a ground field. Since $|G|$ is co-prime to the characteristic of K, V is the reduction of some G-module X in characteristic zero. Then all non-trivial elements of G act fixed-point freely on X. Let W be a reduction of X modulo 2. Then all non-trivial $2'$-elements of G still act fixed-point freely on W.

Assume t is an involution in G and let $v \in V$. Then $v + v^t$ is fixed by t, $v^t = -v$ and so t is the unique involution in G. Hence $t \in Z(G)$. Moreover, $w = w^t$ for all $w \in W$. By [Go, 10.3.1] the Sylow 2-subgroups of S are cyclic or quaternion and so the Sylow 2-subgroups of $G/\langle t\rangle$ are cyclic or dihedral. By Lemma 3.6 below, $O_2(G) = \langle t\rangle$ and so $G/\langle t\rangle$ acts faithfully on W.

Hence we can apply Theorem C, to G if $|G|$ is odd, and to $G/\langle t\rangle$ if $|G|$ is even. We conclude that $|G|$ is even and $G/\langle t\rangle \cong Alt(5)$. Thus by [Hu, V25.7], $G \cong SL_2(5)$.

Acknowledgement: I would like to thank the Mathematisches Seminar der Universität Kiel for its hospitality. I also would like to thank B. Doerr, P. Hewitt, H. Schnabel, B. Stellmacher and most of all H. Bender for useful comments on an earlier draft of this paper.

2 $SL_2(3)$

In this section we proof a lemma on $SL_2(3)$ with fixed-point freely acting elements of order three. This lemma is at the heart of both of our proofs of theorem B.

Proposition 2.1 *Let $H \cong SL_2(3)$, $A = O_2(H)$, K a field and V a KH-module. Let d be an element of order three in H and $a \in A \setminus Z(H)$ so that also ad has order three. Let $b = a^{d^2}$ and $1 \neq z \in Z(H)$. Suppose that*

(i) *The elements of order three in H act fixed-point freely on V.*

(ii) *$vz = -v$ for all $v \in V$.*

Then

(a) *The following relations hold in the ring $\mathrm{End}_K(V)$:*

$$(1+a)d = b - 1 \quad \textit{and} \quad -2d = (a-1)(b-1).$$

(b) *If $\mathrm{char}K = 2$, then $C_V(a) = C_V(b) = C_V(A)$.*

(c) *If $\mathrm{char}K \neq 2$, then any A-invariant subspace of V is also H-invariant.*

Proof: Since d has order three, $(1+d+d^2)(d-1) = 0$. As d is fixed-point free we conclude

$$1 + d + d^2 = 0. \tag{1}$$

As ad also as order three, $1+ad+adad = 0$. Multiplying this equation with d from the right we get $d+ad^2+adad^2 = 0$. Hence by (1), $d-a-ad+adad^2 = 0$. Furthermore, $dad^2 = a^{d^2} = b$ and so $d - a - ad + ab = 0$. Thus $d - ad = a - ab$ and $(1-a)d = a(1-b)$. As $a^2 = z = -1$ we can multiply the last equation with a from the left to obtain

$$(a+1)d = b - 1. \tag{2}$$

Since $(a-1)(a+1) = a^2 - 1 = -2$ we get

$$-2d = (a-1)(b-1). \tag{3}$$

In particular, (a) holds. Suppose now that $\mathrm{char}K = 2$ and let $v \in C_V(a)$. Then $v = va$, $v + va = 0$ and so $v(1+a)d = 0$. Hence by (2) $v(b-1) = 0$ and $v \in C_V(b)$. As $A = \langle a, b\rangle$, (b) holds.

So suppose that $\mathrm{char}K \neq 2$. Then by (3), $d = -\frac{1}{2}(a-1)(b-1)$ and so every subspace invariant under A is also invariant under $H = A\langle d\rangle$. □

3 $Alt(4)$

In this section we assume that G is a non-trivial finite group with the following three properties:

(i) G is perfect.

(ii) The Sylow 2-subgroups of G are dihedral or cylic.

(iii) Every subgroup of G of order pq, p and q odd primes, is cyclic.

The main result of this section is Propostion 3.9 which establishes a subgroup isomorphic to $Alt(4)$ in G.

Let p be an odd prime dividing the order of G, T a Sylow p-subgroup and S a Sylow 2-subgroup of G.

Lemma 3.1 *All p-subgroups of G are cylic.*

Proof: By (iii) applied to the case $p = q$, all abelian p-subgroups of G are cyclic. Hence the lemma follows from [Go, 5.4.10i]. □

The following observations will be useful later on.

Lemma 3.2 (a) *If A is a p'-group acting on a cyclic p-group B, then either $[B, A] = 1$ or $C_B(A) = 1$.*

(b) *If α is an automorphism of order 2 of the cyclic p-group B, then α inverts B.* □

Lemma 3.3 *$N_G(T)/C_G(T)$ is a 2-group and each p-subgroup of G is inverted by some element in G.*

Proof: By induction on p. Suppose first that q is an odd prime dividing the order of $N_G(T)/C_G(T)$. Then $q \neq p$ and as T is cyclic, q divides $p-1$ and so $q < p$. Let R be a Sylow q-subgroup of $N_G(T)$ and $E = C_R(T)$. If $E = 1$ then $\Omega_1(R)\Omega_1(T)$ is not cylic, a contradiction to (iii). Thus $E \neq 1$. By induction there exists $y \in N_G(E)$ which inverts E. Note that T is a Sylow p -subgroup of $C_G(E)$. Thus by the Frattini argument [Go, 1.3.7] we may assume that y normalizes T. Now R is a Sylow q-subgroup of $N_G(T) \cap N_G(E)$ and so by another application of the Frattini argument we may assume that y also normalizes R. Since y does not centralize E, it does not centralize R. Thus by 3.2 y inverts R and so $R = [R, y]$. As the autmorphism group of T is abelian we conclude $R = [R, y] \leq C_G(T)$, a contradiction.

Thus $N_G(T)/C_G(T)$ is a 2-group. By Burnside's p-complement theorem [Go, 7.4.5], $N_G(T) \neq C_G(T)$. Hence T is inverted by some element of G and as any p-subgroup is conjugate to a subgroup of T, 3.3 is proved. □

Lemma 3.4 *$C_G(S) \leq S$.*

Proof: Suppose $C_G(S)$ contains an element x of order p. Then S is a Sylow 2-subgroup of $N_G(\langle x \rangle)$ and centralizes x. But this contradicts 3.3. $\square$

Lemma 3.5 *All involutions in G are conjugate.*

Proof: Since S is dihedral or cylic, S has a cyclic subgroup of index two. Since G has no subgroup of index two, Thompson transfer [Su, 5.1.8] implies that all the involutions in G are conjugate to the unique involution in this cyclic subgroup. $\square$

Lemma 3.6 *S is dihedral of order at least four and $Z(G) = O_2(G) = 1$.*

Proof: By 3.3, G has even order and no element of odd order is in the center of G. Thus $Z(G) \leq Z(S)$. By Burnside's p-complement theorem, S is not cylic and so is S is dihedral of order at least four. Hence G has more then one involution and so by 3.5 $S \cap Z(G) = 1$ and so $Z(G) = 1$. Since $O_2(G)$ is cyclic or dihedral, $Aut(O_2(G))$ is solvable and as G is perfect, $O_2(G) \leq Z(G)$. $\square$

Lemma 3.7 *Let A be a fours group in S. Then 3 divides $|N_G(A)/C_G(A)|$.*

Proof: It suffices to show that $N_G(A)$ acts transitively on $A^{\#}$. If $A = S$ this follows from 3.5 and a theorem of Burnside [Go, 7.1.1]. So suppose $A \neq S$ and let a, b be any two distinct involutions in A. Let c be the third involution in A. By 3.5, $c \in Z(S)^g$ for some $g \in G$. Then S^g is Sylow 2-subgroup of $C_G(c)$ and so we may assume that $A \leq S^g$. As S^g is dihedral, $A = C_{S^g}(A) < N_{S^g}(A)$. Since $N_{S^g}(A)$ fixes c it must permute a and b. $\square$

Let $F \leq Z(S)$ with $|F| = 2$.

Lemma 3.8 *$C_G(F) = O(C_G(F))S$.*

Proof: Put $R = O^2(C_G(F))$. If R has odd order we are done. So suppose that R has even order. Since R has no subgroup of index two we get as in 3.5 and 3.6 that all involutions in R are conjugate and $R \cap S$ is dihedral of order at least four. But then $F \leq R \cap S$ and we get a contradiction as F is normal in R. $\square$

Proposition 3.9 *G has a subgroup isomorphic to $Alt(4)$.*

Proof: By [Go, 6.2.2i] there exists an S-invariant Sylow 3-subgroup L of $O(C_G(F))$. Let A be a fours group in S. We consider the cases that $C_G(A)$ is a $3'$-group and that 3 divides $|C_G(A)|$ seperately.

3.9.1 *If $C_G(A)$ is a $3'$-group, then A is contained in a subgroup of G isomorphic to $Alt(4)$.*

Indeed, let D be a Sylow 3-subgroup of $N_G(A)$. By assumption $C_D(A) = 1$ and by 3.7, D does not centralizes A. Thus $D \cong C_3$ and $DA \cong Alt(4)$.

3.9.2 *If $C_G(A)$ is not a $3'$-group, then $1 \neq L$ is a Sylow 3-subgroup of $C_G(A)$, $S \neq A$ and if B is a fours group in S not conjugate to A in S, then B inverts L.*

Indeed by 3.4 we first conclude that $S \neq A$. Let $L^* \in \mathrm{Syl}_3(C_G(A))$. As S is dihedral, $F \leq A$ and so $L^* \leq C_G(F)$. By 3.8 $L^* \leq O(C_G(F))$. Since L^* is A-invariant we conclude from [Go, 6.2.2ii,iii] that some conjugate of L^* under $C_G(A)$ is contained in L. Hence we may assume without loss that $L^* \leq L$. Thus by 3.2, A centralizes L and so $L = L^*$. Hence $\langle A^S \rangle$ centralizes L. Note that S is a Sylow 2-subgroup of $N_G(L)$ and so by 3.3 S inverts L. As S is dihedral, $S = \langle A^S \rangle B$ and so B inverts L.

We are now able to prove 3.9. In case 3.9.1 we are done. So assume 3.9.2 holds. Then B does not centralize L. If $C_G(B)$ is not a $3'$-group, then 3.9.2 applied to B gives the contradiction $L \leq C_G(B)$. Thus $C_G(B)$ is a $3'$-group and by 3.9.1 B is contained in a subgroup isomorphic to $Alt(4)$. □

The next lemma is well known. For completeness we provide a simple (and also well known) counting argument.

Lemma 3.10 *If the centralizer of some involution in G has order four, then $G \cong Alt(5)$.*

Proof: Recall that by 3.5 G has a unique conjugacy class of involutions. Moreover, $C_G(F) = S$ and all elements in G have order either odd or two.

3.10.1 *Let M and M^* be a maximal abelian subgroup of G of odd order with $M \neq M^*$. Then $|N_G(M)/M| = 2$ and $M \cap M^* = 1$.*

Let b be an element of prime order in M and $C = C_G(b)$. Then C has odd order and by 3.4 there exists an involution z in G which inverts b. Then $C_C(z) = 1$ and so C is abelian and $C = M$. In particular, $b \notin M^*$ and $M \cap M^* = 1$. As any involution normalizing M has to invert M, M can not be normalized by a fours group. Thus $N_G(M) \cap C_G(z) = \langle z \rangle$ and by a Frattini argument applied to $M\langle z \rangle \trianglelefteq N_G(M)$,

$$N_G(M) = M(N_G(M) \cap C_G(z)) = M\langle z \rangle.$$

Thus 3.10.1 holds.

Let $M_1, M_2, \ldots, M_k$ be representatives for the conjugacy classes of maximal abelian subgroups of odd order in G, $n = |G|$ and $m_i = |M_i|$. By 3.10.1 each non-trivial element of odd order in G lies in exactly one conjugate of the M_i's. Moreover, there are $\frac{n}{4}$ involutions and so

$$n = 1 + \frac{n}{4} + \sum_{i=1}^{k} \frac{n}{2m_i} \cdot (m_i - 1).$$

Multipliying by $\frac{2}{n}$ we obtain

$$\frac{3}{2} > \sum_{i=1}^{k} \frac{m_i - 1}{m_i}.$$

Since $\frac{2}{3} + \frac{6}{7} = \frac{32}{21} > \frac{3}{2}$, we conclude $k = 2$, $m_1 = 3$, and $m_2 = 5$. Hence $n = 60$. In particular, the subgroup of G isomorphic to $Alt(4)$ has index five in G and so $G \cong Alt(5)$. □

4 The first proof of Theorem B

Let G and V be as in Theorem B. Morever, we assume without loss that K is algebraicly closed. Let S be a Sylow 2-subgroup of G. By [Go, 10.3.1] we have

Lemma 4.1 (a) *S is cylic or generalized quaternion.*

(b) *Every subgroup of G of order pq, p and q primes, is cyclic.* □

If G has odd order then G fullfils the assumptions but not the conclusion of section 3. Thus G contains an involution t. Then t inverts V, t is unique and $t \in Z(G)$. Put $\bar{G} = G/\langle t \rangle$. Then $\bar{S}$ is cyclic or dihedral and so we can apply the results of section 3 to $\bar{G}$. In particular, there exists $H \leq G$ with $\overline{H} \cong Alt(4)$. Let $A = O_2(H)$ and $D \in Syl_3(H)$. Then $A \cong Q_8$ and $H \cong SL_2(3)$. Without loss $A \leq S$. Let $\bar{F}$ be a subgroup of order two of $Z(\bar{S})$. Then $F \cong C_4$ and $F \leq A$. By 2.1c we have

Lemma 4.2 *All A-invariant subspaces of V are also invariant under H.* □

Lemma 4.3 *Let $H \leq R \leq G$ so that R normalizes a 2-dimensional subspace of V. Put $E = \langle H^R \rangle$. Then $E = H$ or $E \cong SL_2(5)$. Moreover, $C_R(E) = Z(R) = O(Z(R))Z(H)$ and $R/Z(R) \cong Alt(4), Sym(4)$ or $Alt(5)$.*

Proof: Let W be a 2-dimensional subspace of V normalized by R. By the fixed-point free action R acts faithfully on W and we may view R as a subgroup of $GL_K(W)$. Let M be a maximal abelian subgroup of R. As K is algebraicly closed, W is the direct sum of two 1-dimensional M-submodules. As M is maximal, $M \not\leq Z(R)$ and so these submodules are non-isomorphic and uniquely determined by each $m \in M \setminus Z(R)$. Hence $M \cap M^* = Z(R)$ for any two distinct maximal abelian subgroups M and M^* of R. Moreover $|N_R(M)/M| \leq 2$. Let $M_1, M_2, \dots M_k$ representatives for the classes of maximal abelian subgroups of R, $m_i = |M_i/Z(R)|$, $n = |R/Z(R)|$ and $\epsilon_i = |N_{N_R(M_i)}/M_i|$. Then

$$n = 1 + \sum_{i=1}^{k} \frac{n}{\epsilon_i m_i}(m_i - 1). \tag{1}$$

If $k = 1$ we conclude that $R = M_1$ is abelian, a contradiction to $H \leq R$. Hence we may assume from now on that $k \geq 2$. By (1)

$$1 = \frac{1}{n} + \sum_{i=1}^{k} \frac{m_i - 1}{\epsilon_i m_i}. \tag{2}$$

Since $\frac{m_i-1}{\epsilon_i m_i} \geq \frac{1}{4}$ we get $k \leq 3$.

Suppose first that $\epsilon_1 = 1$. Then $\frac{m_1-1}{m_1} < 1 - \frac{1}{4}(k-1)$ and so $k = 2$ and $m_1 \leq 3$. If $m_1 = 2$ we compute from (2) that $n = 2m_2$ and so M_2 is of index two in R. Then as H has no subgroup of index two, $H \leq M_2$, a contradiction. If $m_1 = 3$ we get $n = \frac{6m_2}{3-m_2}$. Thus $m_2 = 2$, $n = 12$ and $R = HZ(R)$.

Suppose next that $\epsilon_i = 2$ for all i. Then by (1), $k > 2$. Thus $k = 3$ and so by (2) $\frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3} = \frac{2}{n} + 1$. In particular, at least one of the $m_i's$ has to be 2. Say $m_1 = 2$. Then $\frac{1}{m_2} + \frac{1}{m_3} = \frac{2}{n} + \frac{1}{2}$. Then at least one of m_2 and m_3 has to be at most 3. Say $m_2 \leq 3$ and $m_2 \leq m_3$.

If $m_2 = 2$, then $n = 2m_3$, a contradiction as above.

If $m_2 = 3$, then $n = \frac{12m_3}{6-m_3}$. Thus m_3 is $3, 4$, or 5 and $n = 12, 24$ or 60 respectively. Hence $HZ(R)$ has index $1, 2$ or 5 in R. As $m_2 = 3$ and $e_2 = 2$ elements of order three in H are inverted by some element in R. So the case of index 1 is impossible while in the remaining two cases it is easy to see that $R/Z(R) \cong Sym(4)$ and $Alt(5)$, respectively.

Furthermore, as S is generalized quaternion, $O_2(Z(R)) = Z(H)$ and $Z(R) = O(Z(R))Z(H)$. □

Lemma 4.4 *$A \leq O_2(N_G(F))$. In particular, $O^2(N_G(F)) \leq C_G(A)$.*

Proof: Let $g \in N_G(F)$, $a \in A \setminus F$, $E = \langle aa^g \rangle$ and $D = \langle a, a^g \rangle$. Then $\bar{D}$ is dihedral, and EF as index at most 2 in DF. Since E centralizes F, EF is abelian. Since K is algebraically closed, EF normalizes a 1-dimensional subspace in V. Hence DF normalizes a 2-dimensional subspace W in V. Since $A = \langle a \rangle F \leq DF$, we conclude from 4.2 that also H normalizes W. Let $R = \langle D, H \rangle = \langle a^g, H \rangle$ and $E = \langle H^R \rangle$. Then $R = \langle a^g \rangle E$ and we conclude from 4.3 that $\bar{R} \cong Alt(4), Sym(4)$ or $Alt(5)$. Hence $DF \leq N_R(F) \cong Q_8$ or Q_{16}. In particular, A and A^g commute modulo F. Thus $\langle A^{N_G(F)} \rangle$ is a 2-group and so $A \leq Q := O_2(N_G(F))$. Clearly each element of odd order in $N_G(F)$ centralizes F and, as Q is quaternion, also Q. □

Lemma 4.5 *$S = A$.*

Suppose $S \neq A$ and let B be a quaternion group of order eight in S not conjugate to A in S.

Suppose that B is contained in a subgroup $H^* \cong SL_2(3)$. Put $R = \langle H, H^*, S\rangle$. As S has a cyclic subgroup of index two, there exists a 2-dimensional KS-submodule W in V. Then by 4.2 applied to H and H^*, R normalizes W. As $|S| \geq 2^4$ we conclude from 4.3 that $R/Z(R) \cong Sym(4)$. But then $A = O_2(H) = O_2(R) = O_2(H^*) = B$, a contradiction.

Thus B is not contained in an $SL_2(3)$. From 3.9.1 applied to B in place of A, we conclude that 3 divides $|C_G(B)|$. As $F \leq B$, 3 divides $|C_G(F)|$ and so by 4.4, 3 also divides $|C_G(A)|$. As the Sylow 3-subgroups of $N_G(A)$ are cyclic this implies that all elements of order three in $N_G(A)$ are already in $C_G(A)$, a contradiction to $H \leq N_G(A)$. □

We are now able to complete our first proof of Theorem B. Since $A = S$, 3.4 implies that $C_{\bar{G}}(\bar{A}) = \bar{A}$ is a 2-group. Hence by 4.4 also $C_{\bar{G}}(\bar{F})$ is a 2-group and so $C_{\bar{G}}(\bar{F}) = \bar{A}$. Thus by 3.10, $\bar{G} \cong Alt(5)$ and by [Hu, V25.7], $G \cong SL_2(5)$.

5 Theorem C

Let G and W be as in Theorem C. As in [Go, 10.3.1] we have that subgroups of order pq, p and q odd primes, are cyclic. Thus we can apply the results of section 3. In particular by 3.9 there exists $H \leq G$ with $H \cong Alt(4)$. Put $A = O_2(H)$ and let S be a Sylow 2-subgroup of G containing A. Let $1 \neq a \in Z(S) \leq A$.

Lemma 5.1 *$A = C_G(a) \cap C_G(C_W(a))$ and in particular, A is normal in $C_G(a)$.*

Let $B = C_G(a) \cap C_G(C_W(a))$. By definition, B centralizes $C_W(a)$. Since $[W, a] \leq C_W(a)$ and $[W, a]$ is isomorphic to $W/C_W(a)$ as $C_G(a)$-module, B also centralizes $W/C_W(a)$. It follows that $[W, B, B] = 0$. Hence B is elementary abelian. By 2.1b, $A \leq B$. Since S is a dihedral group, S has no elementary abelian subgroup of order larger than four, and so $B = A$.

Lemma 5.2 *$S = A$*

Suppose that $S \neq A$ and let B be a fours group in S distinct fom A. If B is not contained in an $Alt(4)$ then by 3.9.2 (with the roles of A and B interchanged), A inverts an element of order three in $C_G(B) \leq C_G(a)$, a contradiction since by 5.1, A is normal in $C_G(a)$. Thus B is contained in an $Alt(4)$ and hence 2.1b (applied to B in place of A) yields $B \leq C_G(a) \cap C_G(C_W(a))$. Thus by 5.1 $B \leq A$, a contradiction. □

Lemma 5.3 *$C_G(a) = S$.*

By 5.2 $S = A$ is a fours group. By 5.1, $C_G(a)$ normalizes A and so stabilizes the series $1 \leq \langle a\rangle \leq A$. Thus $O^2(C_G(a))$ centralizes A and so by 3.4 $C_G(a)$ is a 2-group. Thus $C_G(a) = S$. □

Theorem C now follows from 3.10.

References

[Be] H. Bender, Endliche Fastkörper und Zassenhausgruppen, in *Group Theory, Algebra and Number Theory-Colloquium in Memory of Hans Zassenhaus* ,(1996).

[CR] C.W. Curtis and I. Reiner, *Representation theory of finite groups and associative algebras*, Pure and Applied Mathematics **XI**, John Wiley & Sons, New York, (1962).

[Fr] G. Frobenius, Über auflösbare Gruppen IV, Sitz. Wiss. Akad. Berlin (1901)1216-1230.

[Go] D. Gorenstein,*Finite Groups*, Harper, New York (1980).

[Hu] B. Huppert, *Endliche Gruppen I*, Springer-Verlag, Berlin (1983).

[Su] M. Suzuki, *Group Theory II*, Grundlehren der Mathematik **248**, Springer Verlag,(1986).

[Th] J. Thompson, Finite groups with fixed-point free automorphisms of prime order, Proc.Nat.Acad.Sci.,**45**(1959), 578-581.

[Za1] H. Zassenhaus, Über endliche Fastkörper, Abhand.Math.Sem.Hamburg **11**(1936),187-220.

[Za2] H. Zassenhaus, On Frobenius groups I, Results in Math. **8**(1985),132-145.

Ulrich Meierfrankenfeld
Department of Mathematics
Michigan State University
East Lansing, Mi 48824
meier@math.msu.edu
http://www.math.msu.edu/~meier/