EXTENSIONS OF PERIODIC LINEAR GROUPS WITH FINITE UNIPOTENT RADICAL

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Dedication

This article is published posthumously for the author Richard Phillips. The other authors dedicate this paper to his memory.

Abstract

A group is called p-linear if it is isomorphic to a subgroup of $\mathrm{GL}(n, K)$ for some field K of characteristic p and some integer n. Let H be a normal subgroup of G and assume that both H and G/H are periodic and p-linear. In addition, assume that both H and G/H have finite unipotent radicals and that the finite residual of G/H has finite index in G/H. The main result of this article is a proof that under these assumptions G is p-linear. An example is provided showing the result is false if the assumption regarding the finite residual is removed.

1. Results

For finite n, the group G is said to be K-linear of degree n if it is isomorphic to a subgroup of $GL(n, K)$ and is p-linear if it is K-linear for some field K of characteristic p.

The following theorem was proven as [5, Theorem A].

(1.1) THEOREM. *Let G be a periodic subgroup of $GL(n, K)$ with trivial unipotent radical. Then for every normal subgroup H of G, the image G/H is K-linear of degree bounded by a function of n.*

Here the statement on bounded degree indicates that there is a positive, non-decreasing function f such that the degree of G/H is always less than or equal to $f(n)$.

Wehrfritz [9] gave a second proof of the theorem and was able to weaken the hypotheses by assuming only that H has trivial unipotent radical. A slightly different proof of the theorem is given below (one that in fact includes groups G with finite unipotent radical, as remarked below).

The main concern here is proving a converse result to this theorem. It is not in general the case that an extension of a linear group by a linear group is linear. It is also not clear what shape the best converse to the theorem would take.

Consider the following:

HYPOTHESIS $(*)$

G is a locally finite group and M a normal subgroup of G satisfying:

(i) M is a central product $A \curlyvee L_1 \curlyvee \cdots \curlyvee L_i \curlyvee \cdots \curlyvee L_s$, for some s;

(ii) each L_i is a quasisimple group of Lie type in characteristic p;

(iii) A is a periodic abelian p'-group of finite rank r;

(iv) G/M is finite.

The rank of a periodic abelian group B is the largest m for which there is a prime q and an elementary abelian q-subgroup C of B having order q^m. If $p = 0$, then $s = 0$ in (i) and A is periodic abelian of finite rank in (iii).

Each of the groups L_i in (i) has, naturally defined, a Lie rank r_i and a field of definition $K_i \leq \overline{\mathbb{F}}_p$. (See [5, Lemma 9(iv)] or Theorem 2.6 below.) (In characteristic $p = 0$ we set $\mathbb{F}_p = \mathbb{Q}$ hence $\overline{\mathbb{F}}_p = \overline{\mathbb{Q}}$.) We define the M-rank of G with $(*)$ to be $r + \sum_{i=1}^{s} r_i$. We define the field K^M to be the smallest subfield of $\overline{\mathbb{F}}_p$ that contains each K_i and all q^kth roots of unity, for every k and all primes q for which A contains a subgroup $\mathbb{Z}_{q^\infty}$. Thus M is K^M-linear.

(1.2) THEOREM. *If G is a periodic K-linear group of degree n with finite unipotent radical $U(G)$, then G has a characteristic subgroup M for which (G, M) satisfies $(*)$, where $p = Char(K)$. In this case, M can be chosen so that the M-rank of G is bounded by a function of n and the index $|G/M|$ is bounded by functions of n and $|U(G)|$.*

(1.3) THEOREM. *A group G with $(*)$ is K^M-linear of degree bounded by a function of its M-rank and the index $|G/M|$.*

As a corollary, we get Winter's theorem [10]. The usual proofs are more elementary than this; see [3, Theorem 1.L.2] and [7, Theorem 9.5].

(1.4) COROLLARY. *If G is a periodic K-linear group with trivial unipotent radical, then G is $\overline{\mathbb{F}}_p$-linear, where p is the characteristic of K. In particular, G is countable.*

Since uncountable unipotent linear groups of exponent p exist, Winter's theorem is false without some restriction on the radical. Nevertheless, because of it we can restrict (most) discussion to the question of whether or not a given group is p-linear (that is, linear over some field of characteristic p).

We have a partial converse to Theorem 1.1. For a group G let $Res(G)$ be the finite residual of G, that is the intersection of the subgroups of finite index in G.

(1.5) THEOREM. *Let H be a normal subgroup of G and assume that*

(a) H is a periodic p-linear group with finite unipotent radical;

(b) G/H is a periodic p-linear group with finite unipotent radical;

(c) $Res(G/H)$ has finite index in G/H.

Then G is p-linear of degree bounded by a function of $\deg H, \deg G/H$ and $|G/H\colon Res(G/H)|$.

We claim that assumption (c) of the previous theorem is fulfilled if the Hirsch-Plotkin radical of G/H is Černikov. Indeed, let M be as in Theorem 1.2 applied to G/H in place of G. Let E_∞ be the product of all the infinite $L_i's$. Since A has finite rank, $Res(A)$ is the divisible part of A. Then $(*)$ implies that $Res(G/H) = Res(A)E_\infty$ and so $Res(G/H)$ has finite index in G/H if and only if $Res(A)$ has finite index in A. Note that this fulfilled if the Hirsch Plotkin radical of G/H is Černikov.

(1.6) THEOREM. *Let G be a periodic linear group with finite unipotent radical. Let $Res(G)^*$ be the image of $Res(G)$ in $Aut(G)$. Then $Out(G)$ is residually finite and $Res(Aut(G)) = Res(G)^*$.*

The next section contains proofs of Theorems 1.1-1.6. The final section contains two examples. The first example shows that Theorem 1.5 is false under (a) and (b) alone. The other example demonstrates the impossibility in Theorem 1.5 of bounding the representing degree of G in terms of $\deg G/H$ and $\deg H$, indeed in terms of the isomorphism class of H and the degree of G/H. The theorem shows that the degree is bounded in terms of the degree of H and the isomorphism class of G/H.

The problem discussed in this article was one that Richard Phillips was working on during the last years of his life. He wrote an initial draft of this article in conjunction with Julianne Rainbolt. After Richard Phillips' death,

Jonathan Hall, Ulrich Meierfrankenfeld, and Julianne Rainbolt completed the revisions of the article. The authors thank Felix Leinen and the referees for helpful remarks on earlier drafts of this article.

We also would like to thank Bert Wehrfritz for noticing that Theorem 1.5 in the published version of this article is false.

2. Proofs

Let G be a locally finite group and π be any set of primes. Then $O_\pi(G)$ denotes the largest normal subgroup of G all of whose elements are π-elements. If G is periodic and linear in characteristic p, then $O_p(G)$ is the unipotent radical of G. In the following, $E(G)$ denotes the subgroup of G generated by the components of G, where a component of G is a subnormal quasisimple subgroup.

(2.1) Theorem. (Schur, [3, Theorem 1.L.1]) *Periodic linear groups are locally finite.*

The next two lemmas are elementary.

(2.2) Lemma. *If G is K_0-linear of degree n and K is a subfield of K_0 such that $|K_0 : K|$ is finite, then G is K-linear of degree $n|K_0 \colon K|$.*

(2.3) Lemma. *If M is K-linear of degree m and has finite index in G, then G is K-linear of degree $m|G \colon M|$ via the induced representation.*

(2.4) Lemma. *Let H be K-linear and Z a finite normal subgroup of the center $Z(H)$. Then H/Z is K-linear of degree bounded by a function of the degree of H and $|Z|$.*

In particular, if the factors H_i are K-linear, then a central product $H_1 \curlyvee H_2 = (H_1 \times H_2)/Z$ over a finite central subgroup Z is K-linear of degree bounded by a function of the degrees of the H_i and $|Z|$.

Proof. As the direct product of two K-linear groups is K-linear with degree equal to the sum of the two degrees, this an immediate consequence of [5, Proposition 3(ii)].

(2.5) Lemma. *Let B be a periodic abelian group and let A be the divisible hull of B. Suppose K contains an n-root of unity whenever A contains an element of order n. Let V be faithful finite dimensional KB-module. If the unipotent radical of B is trivial, then V can be extended to a faithful KA-module with $End_{KA}(V) = End_{KB}(V)$.*

Proof. As V is finite dimensioal and B has trivial unipotent radical, B has finite rank. Note that every non-trivial subgroup of A intersects B and so any extension of V to a KA is faithful. Any direct summand of B is contained in a direct summand of A and so by induction on the rank of B we may assume that $B = \langle b \rangle$ is cylic and hence A is locally cyclic. Let λ be an eigenvalue for b on V.

Then there exists an homomorphism $\phi_\lambda : A \to (K^\sharp, \cdot)$ with $\phi(b) = \lambda$. From the assumption, V is the direct sum of the eigenspaces for b. Define $v^a = \phi_\lambda(a)v$ whenever $a \in A$ and v is in the eigenspace corresponding to λ.

A proof of the following theorem can be found at [4, Lemmas 15.6, 15.10, and Theorem 15.12]. It is very similar to [5, Proposition A] and [8, 1.2].

(2.6) THEOREM. *Let G be a periodic linear group of degree n over a field in characteristic p and having trivial unipotent radical.*

Let $\{ L_i \mid 1 \leq i \leq t \}$ consist of all components of G that have Lie type in characteristic p. The central product $E_{L,p}(G) = L_1 \mathsf{Y} \cdots \mathsf{Y} L_t$ is characteristic in G, and $t \leq n/2$. Furthermore, G has a characteristic abelian subgroup A such that the subgroup $M(G) = A \mathsf{Y} E_{L,p}(G)$ is characteristic in G and has finite index bounded by a function of n.

PROOF OF THEOREM 1.2.

G is locally finite by Theorem 2.1.

As the unipotent radical $U = U(G)$ is finite, $C_G(U)$ is a characterisitic subgroup of G of finite index bounded by a function of $|U|$. Therefore we may assume that $U \leq Z(G)$.

By Lemma 2.4, $\bar{G} = G/U$ is K-linear with trivial unipotent radical of degree bounded by a function of $|U|$ and n.

Let $\bar{N} = M(\bar{G})$ be the subgroup $\bar{A} \mathsf{Y} \bar{L}_1 \mathsf{Y} \cdots \mathsf{Y} \bar{L}_s$ of Theorem 2.6. Let L_i be the derived group of the preimage of $\bar{L}_i$ in G, A the p'-part of the preimage of $\bar{A}$, and $M = A \mathsf{Y} L_1 \mathsf{Y} \cdots \mathsf{Y} L_s$. Then M is characteristic in G, and the pair (G, M) satisfies $(*)$.

The various $\mathrm{rank}(L_i)$ are bounded by a function of n [5, Lemma 9(b)(i)]. Since A is p' and periodic, it has finite rank bounded by a function of n by Maschke's Theorem.

PROOF OF THEOREM 1.3.

By (iv) and Lemma 2.3, we may assume $G = M$. Each L_i is K^M-linear by assumption and has a center that is finite of order bounded by a function of the Lie rank of L_i ([5, Lemma 10(viii)] or Lemma 2.8). Thus $E(G) = L_1 \mathsf{Y} \cdots \mathsf{Y} L_s$ is K^M-linear by Lemma 2.4 and has finite center of order bounded in terms of the Lie ranks of the L_i. By definition, $(K^M)^\times$ contains a copy of $\mathbb{Z}_{q^\infty}$ whenever A has infinite q-part, so A is K^M-linear of degree equal to its rank r. A second application of Lemma 2.4 then proves that $G = M = A \mathsf{Y} E(G)$ is K^M-linear, as desired. Furthermore, its degree is controlled as described.

PROOF OF COROLLARY 1.4.

As G is periodic and K-linear, it has $(*)$ by Theorem 1.2 and so is $\overline{\mathbb{F}}_p$-linear by Theorem 1.3.

PROOF OF THEOREM 1.1.

By Theorem 1.2, G has $(*)$ for a normal subgroup M, and the M-rank of G and $|G/M|$ are both bounded by functions of n. In $\bar{G} = G/H$, set $\bar{M} = MH/H$. Then the pair $(\bar{G}, \bar{M})$ inherits $(*)$, so by Theorem 1.3 $\bar{G}$ is linear of degree bounded in terms of the $\bar{M}$-rank of $\bar{G}$ (at most the M-rank of G) and $|\bar{G}{:}\,\bar{M}|$ $(\leq |G{:}\,M|)$. Thus the degree of G/H is bounded by a function of n, as desired.

Remark. The same proof actually gives something slightly stronger than Theorem 1.1. We need only require that the unipotent radical of G be finite, in which case the representation degree of G/H is bounded by a function of n and of the order of the radical.

(2.7) Lemma. *Let B be a class 2 nilpotent group with $B' \leq H \leq Z(B)$. Assume that B/H is divisible and periodic. Then B is abelian.*

Proof. Let $b \in B$ and choose an integer n with $b^n \in H$. Then $B/C_B(b) \cong [B, b]$ has exponent dividing n. But $H \leq C_B(b)$ and so $B/C_B(b)$ is divisible. Thus $B = C_B(b)$ and B is abelian.

(2.8) Lemma. *Let periodic N have a central subgroup H such that $\bar{N} = N/H$ is a central product $\bar{M}_1 \curlyvee \cdots \curlyvee \bar{M}_t$ of finitely many infinite quasisimple groups $\bar{M}_i$ of Lie type in characteristic p. Then $N = H \curlyvee E(N)$, where $E(N) = N_1 \curlyvee \cdots \curlyvee N_t$ has finite center of $\deg \bar{N}$-bounded order, and is a central product of quasisimple groups N_i of Lie type with $\bar{N}_i = \bar{M}_i$. Moreover, $E(N)$ is p-linear of $\deg \bar{N}$-bounded degree.*

Proof. By the Three Subgroups Lemma, $Z_2(N) = Z(N)$. Also $N' = [N'H, N'H] \leq N''$; so N' is perfect, and $N = H \curlyvee N'$. Indeed $N/H = N'H/H \simeq N'/N' \cap H$ has image $N'/Z(N')$. Therefore $N' = E(N)$, the central product of the components $N_i = (M_i)'$, the derived subgroups of the preimages M_i of the various $\bar{M}_i$.

Each simple periodic infinite Lie type group L is a direct limit of finite simple groups of the same Lie type, and so any element of its multiplier occurs already within the multiplier of some finite subgroup of Lie type. Exceptional multipliers for finite Lie type groups occur only over small fields, and the canonicial multipliers come from the natural or spin representations of fixed degree bounded in terms of the Lie rank [2]. Thus the multiplier of L is finite of order bounded by the rank of L and comes from a representation of degree also bounded by the rank. (See also [5, Lemma 10].)

Proof of Theorem 1.6. By 1.2 there exists a characteristic subgroup M of G fulfilling $(*)$. Let E be the product of the infinite L_i. Using Lemma 2.8 we may choose A to be a characteristic subgroup of M. Then EA is a characterisic subgroup of finite index. Let $H = Res(G)$ and note that $H = Res(A)E$ and $Res(A)$ is the largest divisible subgroup of A. For $g \in G$ let $g^* \in Aut(G), h \to h^g$.

We first treat the case where $E = 1$. Let π be the set of prime divisors of $|G/A|$ and let D be the largest π-divisible subgroups of A. Note that A has finite rank, $O_{\pi'}(A) \leq D$ and $H = Res(A) \leq D$. Therefore A/D and G/D are finite π-groups. Let $n = |G/D|$ and for a positive integer i let $D_i = \{ d \in D \mid d^i = 1 \}$. Then D_i is finite. The following cohomological argument is taken from [1, Propostions 3.7.5]. Since D is n divisible, the following is a short exact sequence:

$$1 \longrightarrow D_n \longrightarrow D \overset{a \to a^n}{\longrightarrow} D \longrightarrow 1$$

Hence we also obtain the long exact sequence

$$\ldots \longrightarrow H^m(G/D, D_n) \longrightarrow H^m(G/D, D) \overset{\alpha \to n\alpha}{\longrightarrow} H^m(G/D, D) \longrightarrow \ldots$$

Suppose $m > 0$. By [1, Proposition 3.6.17] the map $\alpha \to n\alpha$ factors through the restriction map to the trivial subgroup of G/D and so is 0. Thus

$$H^m(G/D, D_n) \to H^m(G/D, D)$$

is onto. That is every cocyle for G/D in D arises from a cocycle of G/D in D_n.

For $m = 2$ we conclude that there exists $T \leq G$ with $G = DT$ and $D \cap T = D_n$. For $m = 1$ and G replaced by G/D_n we see that for all $\tilde{T} \leq G$ with $G = D\tilde{T}$ and $G \cap \tilde{T} = D_n$, there exists $d \in D$ with $\tilde{T} \leq T^d D_{n^2}$. Put $R = TD_{n^2}$. Let $\alpha \in Aut(G)$ and choose $d \in D$ with $T^\alpha \leq T^d D_{n^2}$. Then $R^\alpha = R^d$ and so $Aut(G) = N_{Aut(G)}(R)D^*$.

Put $F = C_{Aut(G)}(R)$. Since R is finite we get that FD^* has finite index in $Aut(G)$. Note that $C_F(D) \leq C_F(G) = 1$ and so F is isomorphic to a subgroup of $Aut(D)$. Since D is generated by its finite characteristic subgroup, $Aut(D)$ is residually finite. Thus F is residually finite. Note that $F \cap D^* \leq C_F(D) = 1$. Hence FD^*/D^* is residually finite and since this group has finite index in $Aut(G)/D^*$, $Aut(G)/D^*$ is residually finite. Moreover, G^*/D^* is finite and so $Aut(G)/G^*$ is residually finite.

Let p be a prime. Then $O_{p'}(D)H$ has finite index in D and so $FO_{p'}(D)^*H^*$ has finite index in $Aut(G)$. Also $FO_{p'}(D)^*H^*/O_{p'}(D)^*H^* \cong F$ is residually finite and so $Res(Aut(G)) \leq O_{p'}(D)^*H^*$. Since this holds for all primes, $Res(Aut(G)) \leq H^*$. But H^* has no proper subgroups of finite index and so $Res(Aut(G)) = H^*$.

Hence 1.6 holds if $E = 1$. In the general case, we have $E = L_1 L_2 \ldots L_k$ where L_i is a group of Lie type ${}^{\sigma_i}\Phi_i$ over an infinite field K_i. Let F_i be a finite subfield of K_i such that if $\sigma_i \neq 1$, σ_i acts nontrivially on F_i. Let $L_i(F_i)$ be the group of Lie type ${}^{\sigma_i}\Phi_i$ over the field F_i naturally embedded into L_i. We choose the F_i such that the Schur multipliers of L_i and $L_i(F_i)$ are identical. Let $X = \prod_{i=1}^k L_i(F_i) \leq E$. Since $G/C_G(E)E$ is finite we we can choose the F_i's such that $C_G(X) \leq C_G(E)$. Note that $L_i(F_i)$ is normalized by any field or graph automorphism of L_i. Moreover, our condition on the Schur multiplier ensures that every diagonal automorphism of L_i can be written as a product of an inner automorphism and an automorphism normalizing $L_i(F_i)$.

Put $B = N_G(X)$ and $F = N_{Aut(G)}(X)$. By the preceeding discussion $G = EB$ and $Aut(G) = FE^*$. Since B has no infinite component, we can apply the $E = 1$-case to B. We conclude that both $F/Res(B)^* C_F(B)$ and $F/B^* C_F(B)$ are residually finite. Note that

$$Aut(G)/G^* = FG^*/G^* \cong F/F \cap G^* = F/B^*$$

and so $Res(Aut(G)/G^*) \leq C_F(B)G^*/G^*$. Also $C_{G^*}(B) \leq C_{G^*}(X) \leq C_{G^*}(E)$. Since $BE = G$ we conclude that $C_{G^*}(B) = 1$. In particular, $G^* \cap C_F(B) = 1$. Thus

$$\bigcap C_F(B)G^* = (\bigcap C_F(B))G^* = G^*$$

where the intersection is taken over all the eligible F_i. Thus $Res(Aut(G)/G^*) = 1$.

Since $B/C_B(X)$, $C_G(X)/C_G(E)$ and $C_G(E)/A$ all are finite we conclude that $Res(B) = Res(A) \leq F$ and $H = Res(B)E$. So $C_F(B)Res(B)^* \leq F$,

$$F \cap (C_F(B)Res(B)^* E^*) = C_F(B)Res(B)^*(F \cap E^*) = C_F(B)Res(B)^*(B \cap E)^*$$

and

$$Aut(G)/C_F(B)H^* \cong FE^*/C_F(B)Res(B)^* E^* \cong F/C_F(B)Res(B)^*(B \cap E)^*.$$

Since $B \cap E$ is finite and $F/C_F(B)Res(B)^*$ is residually finite, $Res(Aut(G)) \leq C_F(B)H^*$. Intersecting over the various choices for F_i gives $Res(Aut(G)) \leq H^*$. Also H has no proper subgroup of finite index and so $Res(Aut(G)) = H^*$.

Proof of Theorem 1.5.

By 2.3 we may assume that $G/H = Res(G/H)$ and so G/H has no non-trivially residually finite quotient. By Theorem 1.6 $G/C_G(H)H$ is residually finite and so $G = C_G(H)H$. Thus $C_G(H)/Z(H) \cong G/H$. Since $Res(G)/H = G/H$ we conclude that $C_G(H) = A/Z(H) \times L/Z(H)$ where $A/Z(H)$ is divisible abelian and $L/Z(H)$ is the central product of infinite groups of Lie type. By 2.8, $L = E(L)Z(H)$, $E(L)$ is p-linear of $\deg G/H$-bounded degree and $|Z(L')|$ is $\deg G/H$-bounded. So by 2.4 there exists a faithful $\overline{\mathbb{F}}_p E(L)H$ module V of dimension bounded by a function of $\deg H$ and $\deg G/H$.

By 2.7 A is abelian. Let $A_0 \leq Res(A)$ with $Res(A) = Res(Z(H)) \times A_0$. Let A_1 be a divisible hull for $Z(H) \cap A_0$ in A_0. Then there exists $A_2 \leq A_0$ with $A_0 = A_2 \times A_1$. Moreover, $G = A_2 \times (A_1 E(L)H$. According to 2.5 we can extend the $\overline{\mathbb{F}}_p(Z(H) \cap A_0)$-module W to a module for A_1. So W becomes an $\overline{\mathbb{F}}_p G$ module with A_2 acting trivially. Take the direct sum of W with a faithful G/H module we obtain a faithful $\overline{\mathbb{F}}_p G$-module of dimension bounded in terms of $\deg H$ and $\deg G/H$.

3. Examples

(3.1) LEMMA. *There is an infinite sequence of primes*

$$q_1 < p_1 < \cdots < q_i < p_i < \cdots$$

for $i \in \mathbb{Z}^+$, *with* $q_i | p_i - 1$.

PROOF. The proof is by induction on i. Let q_1 be any prime number. A theorem of Dirichlet [6, p. 250] asserts that the sequence $\{1 + kq_1 \mid k = 1, 2, \ldots\}$ contains an infinite number of primes. Let p_1 be any such prime. Then $q_1|(p_1 - 1)$. Now let q_2 be a prime greater than p_1.

Suppose that we have chosen $q_1, \ldots, q_{s+1}$ and $p_1, \ldots, p_s$ such that

$$q_1 < p_1 < \cdots < q_s < p_s < q_{s+1}$$

Now choose the prime p_{s+1} in the sequence $\{1 + kq_{s+1} \mid k = 1, 2, \ldots\}$ and the prime q_{s+2} with $q_{s+2} > p_{s+1}$.

(3.2) EXAMPLE.

We first show that Theorem 1.5 is false when only (a) and (b) are assumed (even when (a) and (b) are strengthened to require that the unipotent radicals be trivial).

For p_i, q_i $(i \in \mathbb{Z}^+)$ as in Lemma 3.1, let S_i be cyclic of order p_i and R_i cyclic of order q_i. Let $F_i = S_i.R_i$ be the Frobenius group of order $p_i q_i$. Then $F^k = \oplus \sum_{i=1}^k F_i$ has representation degree at least $2k$ in each characteristic, even though it is the extension $S^k.R^k$ of cyclic $S^k = \oplus \sum_{i=1}^k S_i$ of degree 1 over any prime not in the sequence (and degree at most 2 in general) by cyclic $R^k = \oplus \sum_{i=1}^k R_i$ also of degree 1 over any prime not in the sequence (and degree at most 2 in general).

Thus $F^\infty = \oplus \sum_i F_i = \lim_k F^k$ is not linear in any characteristic even though it is the extension of rank 1 linear $S^\infty = \oplus \sum_i S_i = \lim_k S^k$ by rank 1 linear $R^\infty = \oplus \sum_i R_i = \lim_k R^k$. The residual core of F^∞ / S^∞ is trivial and has infinite index in F^∞ / S^∞.

(3.3) EXAMPLE.

The groups F^k of the first example show that the degree of linear G can not be bounded in terms of the degree, $\deg H$, of the normal subgroup H and $\deg G/H$. The next example will show that the degree of linear G can not in general be bounded in terms of $\deg G/H$ and the isomorphism type of H.

Fix a characteristic p and a prime $q \neq p$. Let, for $n \in \mathbb{Z}^+$,

$$E(n) = \langle x, y, z \,|\, x^{q^n} = y^{q^n} = z^{q^n} = 1,\ [x, y] = z, [x, z] = [y, z] = 1 \,\rangle\,.$$

Let $E^*(n)$ be the central product of $E(n)$ and $\mathbb{Z}_{q^\infty}$ with z identified with an element of order q^n of $\mathbb{Z}_{q^\infty}$. Then $H = \mathbb{Z}_{q^\infty}$ is p-linear of degree 1 and $E^*(n)/H \simeq Z_{q^n} \times Z_{q^n}$ is p-linear of degree 2.

We claim that the degree of $E(n)$ (and hence of $E^*(n)$) in characteristic p is q^n. Abelian $W = \langle x, z\rangle$ has index q^n, so this is an upper bound on the degree. Consider the restriction of a faithful representation to W. There is some degree 1 constituent ζ with $W = \langle w, z\rangle = C_{E(n)}(w)$, where $\langle w\rangle = \ker(\zeta)$. Then by Clifford's Theorem the ζ^{y^i}, for $i = 0, \ldots, q^n - 1$, are distinct constituents of the restriction.

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