ON THE CENTER OF MAXIMAL SUBGROUPS OF LOCALLY FINITE SIMPLE GROUPS OF ALTERNATING TYPE

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Abstract

Let G be a locally finite simple group of alternating type, p a prime, and $Z \leq G$ be elementary abelian of order p^2. We prove that there exists $1 \neq z \in Z$ with $C_G(z) \neq C_G(Z)$.

1 Introduction

In [LS] it was shown that the centers of maximal subgroups of finite simple groups are always cyclic. In this paper we extend this result to locally finite simple groups of alternating type. Note here that by [Me, 4.1] every countable locally finite simple group has maximal subgroups. While our result is of interest in itself, our paper is also meant as an illustration how the structure theory of locally finite simple groups in [Me] can be used. The techniques developed in this paper already have been used in [DM] to shed light on the structure of finite subgroups in locally finite simple groups of 1-type. We also expect the techniques to help answer some of the questions on centralizers in locally finite simple groups posted by B. Hartley in [Har].

A group G is called locally finite if every finite subset of G lies in a finite subgroup of G. A Kegel cover $\mathcal{K}$ of a locally finite group G is a set of pairs (H, M) such that H is a finite subgroup of G, M is a maximal normal subgroup of H, and for each finite subgroup K of G there exists $(H, M) \in \mathcal{K}$ with $K \leq H$ and $K \cap M = 1$. The groups H/M are called the factors of $\mathcal{K}$. We say that $\mathcal{K}$ is alternating if all of its factors are alternating groups. For a finite subgroup X of G, $\mathcal{K}(X)$ is the set of pairs $(H, N) \in \mathcal{K}$ with $X \leq H$ and $X \cap N = 1$. A group G is finitary if there exists a field F and a faithful FG-module V such that $\dim_F[V, g] < \infty$ for all $g \in G$. A locally finite simple group is of alternating type if it is not finitary and has an alternating Kegel cover. We are now able to state our main theorem

Theorem 1.1 *Let G be a locally finite simple group of alternating type, p a prime, and let Z be elementary abelian subgroup of order p^2. Then there exists $1 \neq z \in Z$ such that $C_G(z) \neq C_G(Z)$. In particular, if M is a maximal subgroup of G then $Z(M)$ is locally cyclic.*

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2 Preliminaries

Lemma 2.1 *Let H be a finite group, p a prime, S_0 a p-subgroup of H, $N \trianglelefteq H$ and $T \leq H$ minimal with respect to $S_0 \leq T$ and $H = TN$. Let $S \in Syl_p(T)$ with $S_0 \leq S$. Then*

(a) $S \cap N = O_p(T \cap N) \trianglelefteq T$.

(b) $T \cap N/O_p(T \cap N) \leq \Phi(T/O_p(T \cap N))$.

(c) If $O^p(H)N/N$ is a q-group for some prime $q \neq p$, then $O^p(T)O_p(T)/O_p(T)$ is a q-group.

Proof Note that $S_0 \leq N_T(N \cap S)$ and, by the Frattini argument, $T = N_T(N \cap S)(N \cap T)$. Thus by the minimality of T we have $T = N_T(S \cap N)$ and so $S \cap N = O_p(N \cap T)$. So (a) holds. Without loss we may assume $T = H$ and $O_p(T) = 1$. Then $N \cap S = 1$ and N is a p'-group. Suppose $N \not\leq \Phi(T)$. Then there exists a maximal subgroup M of T with $N \not\leq M$. Then $T = MN$ and, since N is a p'-group, M contains a Sylow p-subgroup of H. Thus $S_0 \leq M^k$ for some $k \in T$. Now as $T = NM^k$, we have $T = M^k = M$ by the minimality of T, a contradiction. Therefore (b) holds. For (c), since N is a p'-group, $O^p(T)$ is a p'-group. From [Gor, 6.2.2] we conclude that there exists an S-invariant $Q \in \mathrm{Syl}_q(O^p(T))$. Since $O^p(T)/N$ is a q-group we get $O^p(T) = QN$ and so $T = SQN$. Hence, by the minimality of T we get $T = QS$ and $O^p(T) = Q$ is a q-group.

□

Before we can state our next lemma we need to recall some definitions from [Me]. Suppose G is a group and that $G/M \cong \mathrm{Alt}(\Sigma)$ for some set Σ and some normal subgroup M of G. Let t be a positive integer with $t \leq |\Sigma|/2$. Then G acts t-pseudo naturally on Ω with respect to M if G acts transitively on Ω and if there exists a G-invariant partition Δ for G on Ω such that $C_G(\Delta) = M$ and the action of G on Δ is isomorphic to the action of G on subsets of size t of Σ. G acts pseudo naturally on Ω with respect to M, if G acts 1-pseudo naturally on Ω. G acts essentially on Ω with respect to M if $C_G(\Omega) \leq M$.

Lemma 2.2 *Let p be a prime. Then there exists an integer n (depending only on p) with the following property:*

If

(a) *H is a finite group,*

(b) *Z is an elementary abelian subgroup of H of order p^2,*

(c) *$M \trianglelefteq H$ with $H/M \cong Alt(\Omega)$,*

(d) *H acts transitively on a set Σ such that $C_H(\Sigma) \leq M$,*

(e) *Z has no regular orbits on Σ,*

and

(f) *$deg_\Omega(z) \geq n$ for all $1 \neq z \in Z$,*

then Σ is pseudo natural for H with respect to M.

Proof By [Me, 2.14] there exists an integer n such that under the above assumptions Σ is t-pseudo natural for some $t \leq p^2 - 2$. We do assume without loss that $n > 3p^2$. Then there exists an H-invariant partition Δ of Σ such that the action of H on Δ is isomorphic to the action of H on the set of subsets of size t of Ω. If $t = 1$ we are done. Suppose that $t > 1$. As Z has no regular orbits on Σ, Z has no regular orbits on Δ. Hence, every subset of size t of Ω is invariant under some nontrivial element of Z. Let Ω_1 and Ω_2 be two nontrivial orbits of Z on Ω such that Z acts faithfully on $\Omega_1 \cup \Omega_2$. Choose $T \subseteq \Omega$ such that $|T| = t$ and $|T \cap \Omega_i| = 1$ for $i = 1, 2$ (this is possible since $n \geq 3p^2$). Then there exists $1 \neq z \in Z$ leaving T invariant. But then z fixes $T \cap \Omega_i$ for $i = 1, 2$. Since Z is abelian and acts transitively on Ω_i we conclude that z fixes all elements of $\Omega_1 \cup \Omega_2$ contradicting the faithful choice of $\Omega_1 \cup \Omega_2$.

□

3 On a Lemma of Alperin

In this section we partially generalize a result of J.L. Alperin [Al] to odd primes. Let B be a group acting on a group A and i a non-negative integer. Define $[A,B,0]=A$ and inductively, $[A,B,i+1]=[[A,B,i],B]$.

Theorem 3.1 *Let H be a finite group, p and q distinct primes with $q\neq 2$, V a faithful $GF(p)H$-module, $Z\leq H$ a noncyclic elementary abelian p-group, $Q=[O_q(H),Z]$, and $\mathcal{X}$ be the set of all subgroups of index p in Z. For $X\in\mathcal{X}$, set $Q_X=[C_Q(X),Z]$ and suppose that*

$$[V,Z,p]=0.$$

Then

- *(a)* $[V,X,Q_X]=[V,Q_X,X]=0$*,*
- *(b)* $Q=\bigoplus_{X\in\mathcal{X}}Q_X$*,*
- *(c)* $[V,Q]=\bigoplus_{X\in\mathcal{X}}[V,Q_X]$*.*

Proof For (a), let $X\in\mathcal{X}$ and $z\in Z\setminus X$. Since $[V,Z,p]=0$, we have $[V,X,z,p-1]=0$. Put $H=Q_X\langle z\rangle$ and observe that H normalizes $[V,X]$. Let F be a composition factor for H on $[V,X]$. Then $O_p(H/C_H(F))=1$ and so, by the Hall-Higman Theorem B [Gor, 11.1.1], z centralizes F. Since

$$Q_X=[Q_X,Z]=[Q_X,X\langle z\rangle]=[Q_X,z],$$

also Q_X centralizes F. As Q_X is a p'-group, we get $[V,X,Q_X]=0$. Since Q_X centralizes X, $[V,[X,Q_X]]=0$ and so by the Three Subgroup Lemma, $[V,Q_X,X]=0$ and (a) holds.

For (b), let $D=N_Q([V,Q_X])$. Then $C_Q(X)\leq D$ and by (a) $\langle X^D\rangle$ centralizes $[V,Q_X]$. Therefore

$$[V,Q_X,[D,X]]=0.$$

Now by co-prime action we have $V=C_V([D,X])\oplus[V,[D,X]]$ and so since Q_X normalizes $[D,X]$ we get

$$[V,[D,X],Q_X]\leq[V,[D,X]]\cap[V,Q_X]\leq[V,[D,X]]\cap C_V([D,X])=0.$$

Thus by the Three Subgroup Lemma, $[V,[[D,X],Q_X]]=0$ and as H acts faithfully on V,

$$[[D,X],Q_X]=1.$$

By co-prime action $D=C_D(X)[D,X]$ and so since $C_D(X)\le C_Q(X)\le N_Q(Q_X)$, we have

$$Q_X\trianglelefteq D.$$

Suppose $D\neq Q$. Then since Q is a q-group, $D<N_Q(D)$. Since Q is Z-invariant, each of Q_X, D, and $N_Q(D)$ is Z-invariant. Let E be minimal Z-invariant with $D<E\le N_Q(D)$. Then Z acts irreducibly on the q-group E/D. Since Z is noncyclic, Z does not act faithfully on E/D. Thus, there exists $Y\in\mathcal{X}$ such that $[E/D,Y]=1$ and therefore $[E,Y]\le D$. By co-prime action $E=C_E(Y)[E,Y]=C_E(Y)D$ and so since $C_Q(X)\le D$, we have $Y\neq X$. Let $y\in Y\setminus X$ and put $W=[V,Q_X]$ and $U=[W,y,p-1]$. As we saw above $Q_X=[Q_X,y]$ and so y normalizes Q_X and $Q_X\langle y\rangle$ acts on $W/\langle U^{Q_X}\rangle$. By the Hall-Higman Theorem B, y acts trivially on the composition factors of $Q_X\langle y\rangle$ on $W/\langle U^{Q_X}\rangle$ and so Q_X acts trivially on $W/\langle U^{Q_X}\rangle$. Hence $W=[W,Q_X]\le\langle U^{Q_X}\rangle\le W$, and therefore $W=\langle U^{Q_X}\rangle$. In particular, as D normalizes W and $Q_X\le D$, we have

$$W=\langle W^D\rangle=\langle U^{Q_XD}\rangle=\langle U^D\rangle.$$

Let $e\in C_E(y)$ and $z\in Z$. Then

$$U^e=[W^e,y,p-1]\le[V,Z,p-1]\le C_V(Z)\le C_V(z).$$

It follows that $U^e\le W^e\cap W^{ez}$. As E normalizes D and W^{ez} is D-invariant, we compute

$$W^e=\langle U^{De}\rangle=\langle U^{eD}\rangle\le\langle W^{ezD}\rangle=W^{ez}$$

and so $W^e=W^{ez}$. Therefore Z acts trivially on the set $S=\{W^e|e\in C_E(y)\}$. Moreover, since E normalizes D and D normalizes W, D acts trivially on S as well. By co-prime action $E=C_E(y)[E,y]=C_E(y)D$ and so E acts on S.

Now $D[E,Z]$ is Z-invariant and so $D[E,Z]=D$ or $E=D[E,Z]$ by the minimality of E. In the first case, $[E,Z]\le D$ and so by co-prime action we get $E=C_E(Z)[E,Z]=C_E(Z)D\le N_Q(W)=D$, a contradiction. On the other hand, $E=D[E,Z]$ implies E acts trivially on S and again we get $E\le N_Q(W)=D$, a contradiction.

We conclude that $D=Q$ and so Q normalizes $[V,Q_X]$. By (a), X centralizes $[V,Q_X]$. Hence also $[Q,X]$ centralizes $[V,Q_X]$ and

$$[V, Q_X] \leq C_V([Q, X]).$$

Now by co-prime action $V = C_V([Q, X]) \times [V, [Q, X]]$ and so

$$[V, Q_X] \cap [V, [Q, X]] \leq C_V([Q, X]) \cap [V, [Q, X]] = 0.$$

In particular,

$$Q_X \cap [Q, X] = 1.$$

Let $Y \in \mathcal{X} \setminus \{X\}$. Then $Q_Y = [Q_Y, Z] = [Q_Y, XY] = [Q_Y, X] \leq [Q, X]$. Thus

$$Q_X \cap \langle Q_Y \mid Y \in \mathcal{X} \setminus \{X\}\rangle \leq Q_X \cap [Q, X] = 1.$$

Since $Q_X \trianglelefteq Q$, we conclude that

$$Q = [Q, Z] = \langle Q_X | X \in \mathcal{X}\rangle = \bigoplus_{X \in \mathcal{X}} Q_X.$$

Therefore by the commutator laws

$$[V, Q] = \prod_{X \in \mathcal{X}} [V, Q_X].$$

Moreover, since

$$[V, Q_X] \cap \prod_{Y \neq X, Y \in \mathcal{X}} [V, Q_Y] \leq [V, Q_X] \cap [V, [Q, X]] = 0,$$

we get

$$[V, Q] = \bigoplus_{X \in \mathcal{X}} [V, Q_X].$$

□

Lemma 3.2 *Let P be a p-group acting on an abelian p-group A. Suppose that $C_A(Z) = C_A(P)$ for some $Z \leq Z(P)$.*

(a) For a subset $X \subseteq P$ and non-negative integer i define $C_A(X, i)$ inductively by $C_A(X, 0) = 1$ and $C_A(X, i+1)/C_A(X, i) = C_{A/C_A(X,i)}(X)$. Then for all non-negative integers i,

$$C_A(Z, i) = C_A(P, i)$$

(b) If A is elementary abelian and $|Z| = p$, then $[A, P, p] = 1$.

Proof For (a) we use induction on i. The result holds for $i = 0$ and $i = 1$ by definition and by assumption respectively. Let $i \geq 2$ and suppose the result holds for all positive integers less than i. Then since $Z \leq P$, $C_A(P, i) \leq C_A(Z, i)$. Let $W = C_A(Z, i)/C_A(Z, i - 2)$. Then $[C_A(Z, i), Z] \leq C_A(Z, i - 1) = C_A(P, i - 1)$ and so $[W, Z, P] = 1$. Also, $[Z, P, W] = 1$ and so by the Three Subgroup Lemma $[W, P, Z] = 1$. We conclude that

$$[W, P] \leq C_{A/C_A(Z,i-2)}(Z) = C_A(Z, i - 1)/C_A(Z, i - 2).$$

Hence $[C_A(Z, i), P] \leq C_A(Z, i-1)$. By induction $C_A(Z, i-1) = C_A(P, i-1)$ and so $C_A(Z, i) \leq C_A(P, i)$. For (b), let $1 \neq z \in Z$. Then in $\mathrm{End}_{\mathbb{Z}}(A)$, $(z-1)^p \equiv z^p - 1 (\mod p)$, and so $[A, z, p] = A(z-1)^p = 1$. Thus $C_A(z, p) = A$. Hence, by (a) $A = C_A(P, p)$ and $[A, P, p] = 1$.

□

4 An Abelian Normal Subgroup

Lemma 4.1 *Let H be a finite group, p a prime, and $R = O^p(H)$. Then one of the following holds*

(a) $\Phi(O_p(R)) \leq Z(R)$.

(b) There exists an elementary abelian normal p-subgroup A of H with $[A, R] \neq 1$.

Proof Let $Q = O_p(R)$ and suppose $[\Phi(Q), R] \neq 1$. Let $D \trianglelefteq H$ minimal with respect to $D \leq \Phi(Q)$ and $[D, R] \neq 1$. Put $E = C_D(R)$. Then by the minimality of D, D/E is elementary abelian and H acts irreducibly on D/E. Since $R \trianglelefteq H$, $Q \trianglelefteq H$ and so $[D, Q] \trianglelefteq H$. Since Q is nilpotent, $[D, Q] < D$. Thus by the minimality of D, $[D, Q] \leq E$. Since $E \leq Z(R)$, $[D, Q, Q] = 1$ and therefore $[D, Q'] = 1$ by the Three Subgroup Lemma. Let $d \in D$ and $q \in Q$. Since D/E is elementary, $d^p \in E$. Hence, since $[D, Q] \leq E \cap Q \leq Z(Q)$, we get

$$1 = [d^p, q] = [d, q^p].$$

Therefore $[D, q^p] = 1$ and so $[D, \Phi(Q)] = 1$. Now since $D \leq \Phi(Q)$, D is abelian. Let $A = \Omega_1(D)$. Then A is an elementary abelian p-group and A char $D \trianglelefteq H$ implies $A \trianglelefteq H$. Also as $[D, R] \neq 1$ and $R = O^p(H)$, [Gor, 5.2.4] implies $[A, R] \neq 1$. □

5 Over-groups for Z in Alt(n)

Lemma 5.1 *Let p be a prime, Ω a finite set and $m > 1$. Suppose $Z \leq Alt(\Omega)$ is elementary abelian of order p^2 such that Z has at least $m \cdot p^3$ regular orbits on Ω. Then there exists a subgroup $H \leq Alt(\Omega)$ such that*

(a) $Z \leq H$ and $[\Phi(O_p(O^p(H))), O^p(H)] \neq 1$.

(b) $H/O_p(H) \cong Alt(mp^2)$ and Z acts semi-regularly on $\{1, 2, 3, \ldots, mp^2\}$.

Proof By assumption there exists a Z-invariant subset $\Sigma \subseteq \Omega$ such that $|\Sigma| = mp^5$ and Z acts semi-regularly on Σ. Let $n = mp^2$ and let P_1 be an extra special group of order p^3. By letting P_1 act on itself regularly and $\mathrm{Alt}(n)$ act naturally, we see that $\mathrm{Alt}(\Sigma)$ contains a subgroup $H^* = P_1 \wr \mathrm{Alt}(n)$. Now since $p^2 \mid n$, $\mathrm{Alt}(n)$ contains a subgroup Z^* isomorphic to Z with Z^* acting semi-regularly on $\{1, 2, \ldots, n\}$. Then Z^* is also semi-regular on Σ. Finally we can choose H^* and Z^* so that Z^* is the image of Z under the restriction map from $N_{\mathrm{Alt}(\Omega)}(\Sigma)$ to $\mathrm{Sym}(\Sigma)$. Letting H^* fix all the points in $\Omega \setminus \Sigma$ elementwise, we may view H^* as a subgroup of $\mathrm{Alt}(\Omega)$. Then since $Z^* \leq H^*$, Z normalizes H^*. Let $H = H^*Z$. Then $Z \leq H$ and $H = H^*C_H(\Sigma)$. Since H^* centralizes $\Omega \backslash \Sigma$, we conclude that $H/C_H(\Omega \backslash \Sigma) \cong Z/C_Z(\Omega \backslash \Sigma)$ is a p-group. Thus $C_H(\Sigma)$ is a p-group, $O_p(H) = O_p(H^*)C_H(\Sigma)$ and $H/O_p(H) \cong H^*/O_p(H^*) \cong H^*/\langle P_1^{H^*} \rangle \cong \mathrm{Alt}(n)$. Thus (b) holds. Let $P = \langle P_1^H \rangle$. Then P is a normal p-subgroup of H. We leave it as an exercise to the reader to verify that $P_1' \leq \Phi(O_p(O^p(H)))$. Also $O^p(H)$ does not normalize P_1' and (a) holds.

□

Theorem 5.2 *Let $m \geq 5^3$ and let $Z \leq Alt(mp^2)$ be an elementary abelian p-group of order p^2 acting semi-regularly. Then there exists a prime q with $2 \neq q \neq p$ and a Z-invariant q-subgroup Q of $Alt(mp^2)$ such that*

$$[Q_X, Q_Y] \neq 1$$

for all X and Y distinct proper subgroups of Z, where $Q_X = [C_Q(X), Z]$ and $Q_Y = [C_Q(Y), Z]$.

Proof. Let $q = 3$ if $p \neq 3$, $q = 5$ if $p = 3$, and Q_1 be an extra special q-group of order q^3. Since $m \geq 5^3 \geq q^3$, there exists a Z invariant subset $\Sigma \subseteq \{1, 2, \ldots, mp^2\}$ with $|\Sigma| = q^3p^2$. Moreover, as in Lemma 5.1, $\mathrm{Alt}(\Sigma)$ contains a subgroup $H^* = Q_1 \wr \mathrm{Alt}(p^2)$, such that Z normalizes H^* and H^* contains the image of Z in $\mathrm{Alt}(\Sigma)$.

Let $Q = \langle Q_1^Z \rangle$, X and Y distinct be proper subgroups of Z, $1 \neq x \in X$, and $1 \neq y \in Y$. Then Q is a Z-invariant q-subgroup of $\mathrm{Alt}(mp^2)$. Let π be the projection map from Q to Q_1. We claim $(Q_X)\pi = Q_1$. Clearly $(Q_X)\pi \leq Q_1$. Let $q_1 \in Q_1$. Then $q = q_1 q_1^x \ldots q_1^{x^{p-1}} \in C_Q(X)$ and

$$[q, y] = q^{-1} q^y = q_1^{-x^{p-1}} \ldots q_1^{-x} q_1^{-1} q_1^y \ldots q_1^{x^{p-1}y}$$

Now as Z acts semi-regularly on $\{1, 2, \ldots, mp^2\}$, it acts semi-regularly on $\langle Q_1^Z \rangle$. Therefore each of the $2p$ factors of $[q, y]$ lie in different conjugates of Q_1. In particular, $([q, y])\pi = q_1^{-1}$. Since $[q, y] \in Q_X$ and $q_1 \in Q_1$ was chosen arbitrarily, we have $Q_1 \leq (Q_X)\pi$ and so $(Q_X)\pi = Q_1$. By symmetry $(Q_Y)\pi = Q_1$ and hence $([Q_X, Q_Y])\pi = Q_1'$. Since Q_1 is non-abelian, we obtain $[Q_X, Q_Y] \neq 1$ and the theorem is proven.

□

6 The Regular Case

Theorem 6.1 *Let F be a finite group, Z an elementary abelian subgroup of order p^2, and $N \trianglelefteq F$ with $F/N \cong Alt(\Omega)$. Suppose that Z has at least $5^3 \cdot p^3$ regular orbits on Ω. Then $C_F(z) \neq C_F(Z)$ for some $1 \neq z \in Z$.*

Proof Let $H \leq F/N$ be given by 5.1 with $m = 5^3$. Let H^* and N^* be the pre-images of H and $O_p(H)$ in F, and $S \in \mathrm{Syl}_p(H^*)$ with $Z \leq S$. Also let $T \leq H^*$ be minimal with respect to $S \leq T$ and $H^* = N^*T$. Set $P = S \cap N^* = S \cap N^* \cap T$. By Lemma 2.1(a), $P \trianglelefteq T$. Since P is a Sylow p-subgroup of N^* and N^*/N is a p-group, $N^* = PN$. Thus $H^* = TN^* = TPN = TN$. Note that $O_p(O^p(H)) = (P \cap O^p(T))N/N$. As $[\Phi(O_p(O^p(H)), O^p(H)] \neq 1$ we conclude that $[\Phi(O_p(O^p(T))), O^p(T)] \neq 1$. Hence, by Theorem 4.1 there exists $A \trianglelefteq T$ elementary abelian with $[A, O^p(T)] \neq 1$. It follows that there exists a composition factor D for T on A with $[D, T] \neq 1$. Let $\overline{T} = T/C_T(D)$. Then D is a faithful irreducible $\mathrm{GF}(p)\overline{T}$-module and $O_p(\overline{T}) = 1$. Suppose that $C_T(D) \not\leq N^*$. Then since H^*/N^* is simple, $H^* = N^* C_T(D)$. Hence, by the minimality of T, we have $T = C_T(D)S$ and so $\overline{T}$ is a p-group. It follows that $\overline{T} = 1$, contradicting the choice of D. Since $O_p(\overline{T}) = 1$, $\Phi(\overline{T})$ is a p' group. By Lemma 2.1, $\overline{N^*} \cap \overline{T} \leq \Phi(\overline{T})$. Since $\overline{T}/\overline{N^*} \cap \overline{T}$ is simple we conclude that $\overline{N^*} \cap \overline{T} = \Phi(\overline{T})$ and so $\overline{T}/\Phi(\overline{T}) \cong \mathrm{Alt}(mp^2)$. Let Q be the Z-invariant q-subgroup of $\overline{T}/\Phi(\overline{T})$ given by Theorem 5.2. Since the pre-image of Q in $\overline{T}$ is a p'-group, the pre-image of Q contains a Z-invariant Sylow q-subgroup Q^*. As $[Q_X, Q_Y] \neq 1$ for all distinct proper subgroups X and Y of Z, $[Q_X^*, Q_Y^*] \neq 1$. Suppose that $C_A(z) = C_A(Z)$ for all $1 \neq z \in Z$.

Then by Lemma 3.2 $[A, Z, p] = 1$ and also $[D, Z, p] = 0$, where we view D as a vector space over GF(p). Thus Theorem 3.1 (b) implies $[Q_X^*, Q_Y^*] = 1$, a contradiction. Therefore $C_A(z) \neq C_A(Z)$ for some $1 \neq z \in Z$ and the result of the theorem follows.

□

7 A Bifurcation Lemma

Theorem 7.1 *Let G be a non-finitary, locally finite simple group with alternating Kegel cover $\mathcal{K}$ and let $E \leq G$ be a finite subgroup of G. Then one of the following holds*

(a) For all positive integers t

$$\mathcal{K}_{reg}^t(E) = \{(H_K, M_K) \in \mathcal{K}(E) |\ E \text{ has at least } t \text{ regular orbits on } \Omega_K\}$$

is a Kegel cover for G.

(b)

$$\mathcal{K}_{nat}(E) = \{(H_K, M_K) \in \mathcal{K}(E) |\ E \text{ has no regular orbits on } \Omega_K\}$$

is a Kegel cover for G.

Proof Suppose (a) does not hold. Then there exists a positive integer t such that $\mathcal{K}_{\mathrm{reg}}^t(E)$ is not a Kegel cover. Then $\mathcal{K}(E) \setminus \mathcal{K}_{\mathrm{reg}}^t(E)$ is a Kegel cover and we may assume that for all $(H_K, M_K) \in \mathcal{K}$, $E \leq H_K$ and E has less than t regular orbits on Ω_K. Let $F \leq G$ be a finite subgroup with $E \leq F$ and $|F/E| = r > t$. Suppose there exists some $(H_K, M_K) \in \mathcal{K}(F)$ such that F has a regular orbit s^F on Ω_K, for some $s \in \Omega_K$. Then each $s^{x_i E}$, where $\{x_i\}_{i=1}^r$ is a transversal for E in F, is a regular orbit for E on Ω_K, contrary to our assumptions. Therefore F has no regular orbits on Ω_K for all $(H_K, M_K) \in \mathcal{K}$. From [Me, 3.4] we conclude that there exists a Kegel cover $\mathcal{J} \subseteq \mathcal{K}$ for G such that for all (H_J, M_J), $(H_K, M_K) \in \mathcal{J}$ with $H_J \leq H_K$, all essential orbits for H_J on Ω_K are pseudo natural with respect to M_J. Without loss we may assume $\mathcal{K} = \mathcal{J}$.

Pick $(H_K, M_K) \in \mathcal{K}$ so that the number d of regular orbits of E on Ω_K is maximal. If $d = 0$ then (b) holds. Suppose $d > 0$. Let R be the unique minimal subnormal supplement to M_K in H_K and $1 \neq r \in R$. Then by Hall's Finitary Lemma ([Ha, 3.13]) there exists $(H_J, M_J) \in \mathcal{K}(H_K)$ such that $\deg_{\Omega_J}(r) > |H_K|$, where $\deg_{\Omega_J}(r)$ is the number of elements of Ω_J

moved by r. Hence there exists at least two orbits Ω_1 and Ω_2 for H_K on Ω_J which are not fixed elementwise by r. In particular, Ω_i is an essential orbit for H_K and therefore pseudo natural for $i = 1, 2$. So there exists an H_K-invariant partition Δ_i of Ω_i such that Δ_i and Ω_K are isomorphic as H_K-sets. As E has d regular orbits on Ω_K, it has d regular orbits on Δ_i for $i = 1, 2$. But then E has at least $2d$ regular orbits on Ω_J, contradicting the maximal choice of d.

□

8 The Proof of the Main Theorem

Let G be a locally finite simple group of alternating type and suppose Z is an elementary abelian subgroup of order p^2 such that $C_G(Z) = C_G(z)$ for all $1 \neq z \in Z$. Let $\mathcal{K}$ be an alternating Kegel cover for G and let $(H_K, M_K) \in \mathcal{K}$ with $Z \leq H_K$. Then by Theorem 6.1 Z has less than 5^3p^3 regular orbits on Ω_K. Therefore, by Theorem 7.1 we may assume that $\mathcal{K} = \mathcal{K}_{\text{nat}}(Z)$, that is, for all $(H_K, M_K) \in \mathcal{K}$, $Z \leq H_K$ and Z has no regular orbits on Ω_K.

Let n be chosen as in Lemma 2.2. By Hall's Finitary Lemma ([Ha, 3.13]) there exists $(H_K, M_K) \in \mathcal{K}$ such that $\deg_{\Omega_K}(z) \geq (p+1)(n+2)$ for all $1 \neq z \in Z$. Also let Δ be the union of the nontrivial orbits of Z on Ω_K and let $\mathcal{X}$ be the set of nontrivial proper subgroups of Z. As none of the orbits are regular, we have

$$\Delta = \bigcup_{X \in \mathcal{X}} \text{Fix}_\Delta(X)$$

where $\text{Fix}_\Delta(X)$ is the subset of Δ fixed elementwise by X. Since $|\Delta| \geq (p+1)(n+2)$ and Z has $p+1$ proper subgroups, $\Delta_2 = \text{Fix}_\Delta(X_1)$ has at least $n+2$ elements for some $X_1 \in \mathcal{X}$. Let $1 \neq x_1 \in X_1$. Then $|\Delta \setminus \Delta_2| \geq \deg_{\Omega_K}(x_1) \geq (p+1)(n+2)$. As above there exists $X_2 \in \mathcal{X} \setminus \{X_1\}$ so that $\Delta_1 = \text{Fix}_\Delta(X_2)$ has at least $n+2$ elements. Note that an element in $\Delta_1 \cap \Delta_2$ is fixed by $X_1X_2 = Z$. But Z has no fixed points on Δ and so Δ_1 and Δ_2 are disjoint.

If p is odd, let $\Omega_i = \Delta_i$ for $i = 1, 2$; if $p = 2$, let Ω_i be a Z-invariant subset of Δ_i such that $|\Delta_i \setminus \Omega_i| \leq 2$ and 4 divides $|\Omega_i|$. Then $|\Omega_i| \geq n$ and X_i has no fixed points on Ω_i for each i.

Let $\Omega = \Omega_1 \cup \Omega_2$, $H^* = \{h \in N_{H_K}(\Omega) | h \text{ is even on } \Omega\}$ and $M^* = C_{H^*}(\Omega)$. Then $Z = X_1X_2 \leq H^*$. Choose $H \leq H^*$ minimal so that $Z \leq H$ and $H^* = HM^*$, and let $M = M^* \cap H$. Then $H/M \cong \text{Alt}(\Omega)$. Let $R = H'$. Suppose $N \trianglelefteq H$ with $N \not\leq M$. Then, since H/M is simple, $H = MN$. Now by the minimality of H, $H = NZ$ and so $R \leq N$. Let $A_i = C_H(\Omega_{3-i})$ for $i = 1, 2$. Then $X_i \leq A_i$ and $A_i/M \cong \mathrm{Alt}(\Omega_i)$. Let $B_i = \langle X_i^{A_1A_2} \rangle$ for each i. Then as $X_i \not\leq M$ and A_i/M is simple, $A_i = B_iM$ for each i. Now $R \trianglelefteq H$ and $B_i \trianglelefteq A_i$ implies that $B_i \cap R \trianglelefteq A_i$. If $B_i \cap R \leq M$, then, as $B_i' \leq R$, $B_i' \leq M$. Hence, we get that A_i/M is abelian, a contradiction. Therefore $B_i \cap R \not\leq M$ and so $A_i = (B_i \cap R)M$.

Suppose $[B_1, B_2 \cap R] = 1$. Then $[X_1, B_2 \cap R] = 1$ and, as $C_G(X_1) = C_G(Z) = C_G(X_2)$, $[X_2, B_2 \cap R] = 1$. Also since $B_2 \cap R \trianglelefteq A_1A_2$ and $B_2 = \langle X_2^{A_1A_2} \rangle$, it follows that $[B_2, B_2 \cap R] = 1$. Thus $[A_2, A_2] = [B_2M, (B_2 \cap R)M] \leq M$, a contradiction since A_2/M is not abelian. Therefore $[B_1, B_2 \cap R] \neq 1$.

Let $D = [B_1, B_2 \cap R]$ and $(H_J, M_J) \in \mathcal{K}$ with $H \leq H_J$ and $H \cap M_J = 1$. Since $D \neq 1$, there exists an orbit Σ for H on Ω_J on which D acts nontrivially. Suppose $C_H(\Sigma) \not\leq M$. Then as seen above $D \leq R \leq C_H(\Sigma)$, a contradiction. Therefore $C_H(\Sigma) \leq M$. Note that for all $1 \neq z \in Z$, $\deg_\Omega(z) \geq \min(|\Omega_1|, |\Omega_2|) \geq n$. Also Z has no regular orbits on Ω_J and so Lemma 2.2 implies that Σ is pseudo natural for H with respect to M. That is, there exists a H-invariant partition Γ of Σ such that Γ and Ω are isomorphic H-sets. Let Γ_i be the images of Ω_i under the above isomorphism for $i = 1, 2$. Also let

$$\Sigma_i = \bigcup_{\gamma \in \Gamma_i} \gamma$$

for each i. Then $\Sigma = \Sigma_1 \cup \Sigma_2$. Let $\sigma \in \Sigma_{3-i}$. Since Z has no regular orbits on Σ, $C_Z(\sigma) \neq 1$. Let $\gamma \in \Gamma_{3-i}$ with $\sigma \in \gamma$. Then $C_Z(\sigma) \leq N_Z(\gamma) = X_i$. Thus $C_Z(\sigma) = X_i$ and X_i fixes Σ_{3-i} elementwise. Therefore, as A_1A_2 normalizes Σ_i for each i, $B_i = \langle X_i^{A_1A_2} \rangle$ fixes Σ_{3-i} elementwise for each i. Hence $B_1 \cap B_2$ acts trivially on $\Sigma = \Sigma_1 \cup \Sigma_2$. Since $D \leq [B_1, B_2] \leq B_1 \cap B_2$, we get D acts trivially on Σ, contradicting the choice of Σ.

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