Groups of local characteristic p

Barbara Baumeister

Andy Chermak

Andreas Hirn

Mario Mainardis

Ulrich Meierfrankenfeld

Gemma Parmeggiani

Chris Parker

Peter Rowley

Bernd Stellmacher

Gernot Stroth

An alternative title: An important step in a third generation proof for the classification of finite simple groups.

CFSG: Each finite simple group is isomorphic to one of the following:

a group of prime order.

an alternating group.

a group of Lie-type.

one of the 26 sporadics.

One of our goals: Uniform approach to the groups Lie-type.

Given a finite (simple) group G. How can one show that G is isomorphic to a group of Lie-type G^*.

G^* is (essentially) the automorphism group of a building $\mathcal{B}^*$.

Buildings are fairly easy to identify up to isomorphism:

Theorem Let $\mathcal{B}$ and $\mathcal{B}^*$ spherical buildings with the same set of types I and rank at least four. Let C and C^* be chambers of $\mathcal{B}$ and $\mathcal{B}^*$ respectively. Suppose that for all $J \subset I$, $\text{Res}_J(C) \cong \text{Res}_J(C^*)$. Then $\mathcal{B} \cong \mathcal{B}^*$.

This theorem follows from Tits' classification of the spherical bulidings, but also has fairly elementary proof (assuming (4.1.2) from Tits' notes) .

But how can we show that G acts on a building? That is how can we describe the building in group theoretical terms?

Building $\longleftrightarrow$ BN-pair $\longleftrightarrow$ parabolic subgroups.

parabolis subgroups = overgroups of the Borel subgroup,

Borel subgroup = (Normalizer of a) maximal unipotent subgroups

Maximal unipotent subgroups = Sylow p-subgroup, where p is the characteristic of the underlying field.

This raises two more questions: How to determine the correct prime p? How to find the parabolic subgroups, that is the overgroups of the Sylow p-subgroup?

An example: Let $G = GL_n(q)$. Then G acts on the building consisting of the (flags of) proper subspaces of $\mathbb{F}_q^n$.

The maximal parabolics are the normalizers of the subspaces:

$$M = \begin{pmatrix} * & 0 \\ * & * \end{pmatrix}$$

$$\text{Put} \qquad Q = \begin{pmatrix} I_k & 0 \\ * & I_{n-k} \end{pmatrix}$$

Then Q is a p-group, $M = N_G(Q)$ and $C_G(Q) \leq Q$, $(\mod Z(G))$

G is a finite $\mathcal{K}_p$-group, and p a fixed prime.

G has **characteristic** p if $C_G(O_p(G)) \leq O_p(G)$.

p-**local subgroup**: Normalizer of a non-trivial p-subgroup.

G has **local characteristic** p if all p-local subgroups of G have characteristic p.

G has **parabolic characteristic** p if all the p-local, parabolic subgroups of G have characteristic p.

Goal: Understand and classify the finite groups of local characteristic p with $O_p(G) = 1$.

Disclaimer: For p odd we do not expect to be able to achieve a complete classification. Some groups with a relatively small p-local structure will remain unclassified. In particular, we currently have no idea how to treat the case where G has a strongly p-embedded subgroup.

Characteristics of the known simple groups

1. Groups of Lie-Type

Let G be a finite simple group of Lie type defined over a field of characteristic r.

If $p = r$, then G is of local characteristic p.

If $p \neq r$ and a Sylow p-subgroup of G is not cyclic, then G is usually not of parabolic characteristic p.

Some exceptions:

$U_3(3) \cong G_2(2)'$, $Sp_4(2)' \cong L_2(9)$, $P\Omega_5(3) \cong \Omega_6^-(2)$, $L_3(4)$ and $U_4(3)$ all have local characteristics 2 and 3.

$L_4(3)$ has parabolic characteristics 2 and 3.

2. Alternating groups

The alternating groups usually have no local characteristic. But Alt$(p^n+\epsilon)$,$\epsilon \leq 2$ has parabolic characteristic p.

3. Characteristics of the sporadics

Group	local char.	parabolic char.
M_{11}	3	3
M_{12}		2, 3
J_1		
M_{22}	2	2
J_2		2
M_{23}	2	2
HS		2
J_3	2	2
M_{24}	2	2
McL	3	3
He		2
Ru		2, 5
Suz		2
ON	7	7
Co_3		3, 5
Co_2	2	3, 5
Fi_{22}	2	2
HN		2, 3
Ly	5	5
Th	2, 5	2, 3, 5
Fi_{23}		3
Co_1		2, 3, 5
J_4	2, 11	2, 11
Fi'_{24}		2, 3, 7
B		2, 3, 5
M		2, 3, 5, 7, 13

Notation

G is a group of local characteristic p with $O_p(G) = 1$.

$\mathcal{M}$ is the set of maximal p-local subgroups of G.

If $\mathcal{T}$ is a set of subgroups of G and $A \leq G$, then

$$\mathcal{T}(A) = \{T \in \mathcal{T} \mid A \leq T\}.$$

S is a fixed Sylow p-subgroup of G.

$\mathcal{M}(S)$ is the set of maximal p-local subgroups containg S.

Back to the example $G = GL_n(\mathbb{F}_q)$. The corresponding building is A_{n-1}. The residue corresponding to M has type $A_{k-1} \times A_{n-k-1}$. So the residue can be read of from the structure of $M/Q \cong GL_k(q) \times GL_{n-k-1}(q)$.

This suggest that our main task is to indentify $L/O_p(L)$ for various members L of $\mathcal{L}$.

Our favorite method for this is to study the action of L on p-reduced normal subgroups, i.e. elementary abelian normal p-subgroups Y of L with

$$O_p(L/C_L(Y)) = 1.$$

Y_L is the largest p-reduced normal subgroup of L.

Unfortunately the action of L on Y_L does not yield any information about $C_L(Y_L)$. An elementary argument shows that $\Omega_1 Z(S) \leq Y_L$ and so $C_L(Y_L) \leq C_G(\Omega_1 Z(S))$.

So to make up for this misfortune we also study the group $N_G(Z)$. For this we pick

$$\widetilde{C} \in \mathcal{M} \text{ with } N_G(Z) \leq \widetilde{C}.$$

For a group H, define $F_p^*(H)$ by $F_p^*(H)/O_p(H) = F^*(H/O_p(H))$.

Put $E := O^p(F_p^*(C_{\widetilde{C}}(Y_{\widetilde{C}})))$.

We now distinguish two cases:

E-**uniqueness** ($E!$): $\mathcal{M}(E) = \{\widetilde{C}\}$

and

non E-**uniqueness** ($\neg E!$): $|\mathcal{M}(E)| \geq 2$.

For the rest of this talk we assume that $E!$ holds.

The Structure Theorem

Let $M \in \mathcal{M}(S)$. Suppose $E!$ and that $M \neq \tilde{C}$. Then there exists $K \trianglelefteq M/C_M(Y_M)$ and $V \leq Y_M$ with $V \trianglelefteq M$ such K and V are as in the following table:

K	p	V	example
$SL_n(q)$	p	natural	$SL_{n+1}(q)$
$SL_n(q)$	p	ext. square	$\Omega_{2n}(q)$
$SL_n(q)$	p	sym. square	$Sp_{2n}(q)$
$SL_n(q^2)$	p	nat. $\otimes$ nat.$^\sigma$	$SU_{2n}(q)$
$SL_n(q) \circ SL_m(q)$	p	nat. $\otimes$ nat.	$SL_{n+m}(q)$
$Sp_{2n}(q)$	p	natural	B
A_6	2	natural	Suz
$3A_6$	2	6-dim	M_{24}
$\Omega^\pm_n(q)$	p	natural	$\Omega^\pm_{n+2}(q)$
$\Omega^\pm_{10}(q)$	p	half spin	$E_6(q)$
$E_6(q)$	p	$V(\lambda_1)$	$E_7(q)$
M_{11}	3	5-dim	Co_3
$2M_{12}$	3	6-dim	Co_2
M_{22}	2	10-dim	$M(22)$
M_{24}	2	11-dim	$M(24)$
$SL_2(q)^m$	$p = 2$	natural^m	-

A finite group L is p-**minimal** if a Sylow p-subgroup of L is contained in a unique maximal subgroup of L but is not normal in L.

$P \leq G$ is a **minimal parabolic subgroup** if $O_p(P) \neq 1$ and P is parabolic and p-minimal.

$\mathcal{P}$ denotes the set of minimal parabolics of G.

Define the **rank** of G to be the minimal n such that there exist $P_i \in \mathcal{P}(S)$,$1 \leq i \leq n$, with

$$O_p(\langle P_1, P_2, \ldots, P_n \rangle) = 1.$$

If no such n we define the rank to be 1.

Fix $P \in \mathcal{P}(S)$ with $P \not\leq \widetilde{C}$.

We say that $gp(P) = 1$ if there exists $M \in \mathcal{M}(P)$ with $Y_M \not\leq O_p(\widetilde{C})$. We say $gb(P) = 2$ if $gb(P) \neq 1$ and $\langle (Y_P)^{\widetilde{C}} \rangle$ is not abelian.

The Small World Theorem:

Suppose $E!$. Then one of the following holds:

1. G has rank 1 or 2.

2. $gb(P) = 1$ or $gb(P) = 2$.

Rank 2 Theorem, I

Suppose $E!$, $gp(P) \neq 1$ and that G has rank 2. Then there exists $\widetilde{P} \in \mathcal{P}(S)$ such that one of the following holds:

1. $(P, \widetilde{P})$ is a weak BN-pair. (slightly simplified statement)

2. The structure of P and $\widetilde{P}$ is as in one of the following groups.

    1. For $p = 2$: $U_4(3).2^e$, $G_2(3).2^e$, $D_4(3).2^e$, $HS.2^e$, F_3, $F_5.2^e$ or Ru.

    2. For $p = 3$: $D_4(3^n).3^e$, Fi_{23}, F_2.

    3. For $p = 5$: F_2.

    4. For $p = 7$: F_1.

gb(P)=1: The H-Structure Theorem

Supppose $E!$, and $gp(P) = 1$. Let $M \in \mathcal{M}(P)$ with $M^\circ := \langle O_p(\widetilde{C})^M \rangle$ maximal and $Y_M \not\leq O_p(\widetilde{C})$. Then either $M^\circ S$ is p-minimal or there exists $M^\circ S \leq H \leq G$ with $O_p(H) = 1$ such that H has same residuell type has one of the following groups:

1. A group of Lie-type in characteristic p.

2. For $p = 2$: M_{24}, He, Co_2, Fi_{22}, Co_1, J_4, Fi_{24}, Suz, B, M, $U_4(3)$ or $G_2(3)$.

3. For $p = 3$: Fi_{24}, Co_3, Co_1 or M.