FINITE GROUPS GENERATED BY A PAIR OF MINIMAL PARABOLIC SUBGROUPS: PART I

Andrew Chermak

Department of Mathematics
Kansas State University
Manhattan KS, 66502

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Section 0: Notation, terminology, and quadratic action

All groups considered here are understood to be finite.

Let X be a group, and let V be a group on which X acts. We write

$$\eta(X,V)$$

for the number of chief factors in any X-chief series through V on which the action of X is non-trivial. If V_0 is a subgroup of V, and Y is an element or a subgroup of X, we write

$$C_V(Y \ mod \ V_0)$$

for the set of elements v of V such that $[v,Y] \leq V_0$.

Suppose further that V is an $\mathbb{F}_pX$-module for some prime p. We say that V is an $F1$**-module** (resp. a **quadratic** $F2$**-module**, resp. a **quadratic** $F2^*$**- module**) for X if there exists a subgroup A of X such that $[V,A,A]=1 \neq [V,A]$, and such that $|A/C_A(V)||C_V(A)| \geq |V|$ (resp. $|A/C_A(V)|^2|C_V(A)| \geq |V|$, resp. $|A/C_A(V)|^2|C_V(A)| > |V|$). Such a subgroup A is then said to be a quadratic $F1$**-offender** (resp. $F2$**-offender**, resp. $F2^*$**-offender**).

When there is a need to be more precise, we employ the notation, due, as far as I know, and in one form or another, to Stellmacher, as follows. Assuming that there exists a non-identity subgroup of X which acts quadratically on V, we define $r(X,V)$ to be the smallest real number r such that there exists a non-identity subgroup A of X with $[V,A,A]=0$ and with $|A/C_A(V)|^r|C_V(A)|=|V|$. We then set

$$\mathcal{A}(X,V)=\{A \leq X \ : \ A \neq 1, \quad [V,A,A]=0, \quad and \quad |A/C_A(V)|^{r(X,V)}|C_V(A)|=|V|\}.$$

Lemma 0.1. *Let X be a finite group, p a prime, and V a faithful $\mathbb{F}_pX$-module. Let A and K be subgroups of X, such that $1 \neq K = O^p(K) = [K, A]$, and such that $[V, A, A] = 0$. Assume also that $A \in \mathcal{A}(A, V)$, and set $W = [V, K]$. Then $C_A(K) = C_A(W)$, and for any complement A_0 to $C_A(K)$ in A we have $r(A_0, W) \leq r(A, V)$.*

Proof. By quadratic action, we have $[V, \langle A^K \rangle, C_A(K)] = 0$, and since $K = [K, A] \leq \langle A^K \rangle$ we then have $[W, C_A(K)] = 0$. Now $[V, C_A(W), K] = 0$, and the Three Subgroups Lemma then yields $C_A(W) = C_A(K)$.

Choose a complement A_0 to $C_A(K)$ in A, and set $r = r(A, V)$. Suppose that $|A_0|^r|C_W(A_0)| < |W|$. We then have

$$|V| = |A|^r|C_V(A)| = |A_0|^r|C_A(W)|^r|C_W(A)||C_V(A)/C_W(A)|$$

$$< |C_A(W)|^r|C_V(A)/C_W(A)||W| = |C_A(W)|^r|W + C_V(A)|$$

$$\leq |C_A(W)|^r|C_V(C_A(W))|.$$

this is contrary to the assumption that A lies in $\mathcal{A}(A, V)$, so in fact $|A_0|^r|C_W(A_0)| \geq |W|$, and the lemma is proved. □

Section 1: P and H

These notes are intended as a preliminary draft of a more extensive work. We assume familiarity with some of the notation and basic concepts associated with the project initiated by Meierfrankenfeld, on groups of characteristic p-type. We begin with a finite group G, a prime p, and a Sylow p-subgroup S of G. The following condition is assumed, throughout.

1.0. *For any subgroup L of G containing S, with $O_p(L) \neq 1$, we have $C_L(O_p(L)) \leq O_p(L)$.*

For any subgroup L of G, we denote by Y_L the *p-reduced core* of L. That is, Y_L is the (uniquely determined) normal p-subgroup Y of L such that $O_p(L/C_L(Y_L)) = 1$, and which is largest for this condition. Here are the general properties of the p-reduced core that we will require.

Lemma 1.1. *Let X be a group with $C_X(O_p(X)) \leq O_p(X)$.*

(a) *We have $Y_X \leq \Omega_1(Z(O_p(X)))$, and if $X/O_p(X)$ acts faithfully on $\Omega_1(Z(O_p(X)))$ then $Y_X = \Omega_1(Z(O_p(X)))$.*
(b) *If $X \leq X^*$, where $C_X(O_p(X^*)) \leq O_p(X^*)$, then $Y_X \leq Y_{X^*}$.*

We set $Z = \Omega_1(Z(S))$, we set $C = C_G(Z)$, and we choose a maximal p-local subgroup $\widetilde{C}$ of G containing C. Further, we set $E = O^p(F_p^*(C_G(Y_{\widetilde{C}})))$.

For X a group and R a Sylow p-subgroup of X, we denote by $\mathcal{P}_X^*(R)$ the set of all subgroups P of X such that R is contained in a unique maximal subgroup of P, and satisfying also the condition that R is not a normal subgroup of P. We set $\mathcal{P}_X(R) = \{P \in \mathcal{P}_X^*(R) \ : \ O_p(P) \neq 1\}$. We shall write $\mathcal{P}$ for $\mathcal{P}_G(S)$.

Lemma 1.2. *For any finite group X and Sylow subgroup R of X, we have $X = \langle \mathcal{P}_X^*(R) \rangle N_X(R)$, and if $O_p(X) \neq 1$ then $X = \langle \mathcal{P}_X(R) \rangle N_X(R)$.*

Henceforth, all groups under consideration will be subgroups of our fixed finite group G. For any subgroup X of G and p-subgroup R of X we denote by $\mathcal{L}_X(R)$ the set of subgroups L of X containing R, with $O_p(L) \neq 1$. We write $\mathcal{L}$ for $\mathcal{L}_G(S)$.

At this point we may state the hypotheses under which we will be working. One of these is that whenever it is inconvenient to do otherwise, we assume that the non-abelian simple sections of elements of $\mathcal{L}$ are alternating groups, groups of Lie type, or may be found among the twenty-six "sporadic" groups. We shall make use of this "$\mathcal{K}$-group" hypothesis when we have quadratic action by a group of order bigger than 2 or, at one point (see 3.3, below), to identify simple groups with dihedral or semi-dihedral Sylow 2-subgroups.

1.3 Main Hypothesis.

(1) *$C_L(O_p(L)) \leq O_p(L)$ for any $L \in \mathcal{L}$.*

(2) *$\widetilde{C}$ is the unique maximal p-local subgroup of G containing E.*

(3) *There is a unique member P of $\mathcal{P}$ such that $P \not\leq \widetilde{C}$. Moreover, setting $P^0 = \langle O_p(\widetilde{C})^P \rangle$, we have the following.*

  (a) *$P^0 S = P$.*

  (b) *$P^0/O_p(P^0) \cong SL(2,q)$ for some power q of p, and $Y_p = Y_{P^0}$ is a natural $SL(2,q)$-module for P^0.*

  (c) *$C_{Y_P}(S \cap P^0) \trianglelefteq \widetilde{C}$.*

(4) *There is at most one member P_1 of $\mathcal{P}$ such that $\langle P, P_1 \rangle \in \mathcal{L}$ and such that Y_P is not normal in P_1. Moreover, if such a P_1 exists then, setting $L = \langle P, P_1 \rangle$ and $L^0 = \langle O_p(\widetilde{C})^L \rangle$, we have the following.*

  (a) *$L = L^0 S$.*

  (b) *$L^0/C_{L^0}(Y_L) \cong SL(3,q)$ or $Sp(4,q)$, and $Y_L = Y_{L^0}$ is a natural module for $L^0/C_{L^0}(Y_L)$.*

(5) *There exists $\widetilde{P} \in \mathcal{P}_{ES}(S)$ such that $\langle P, \widetilde{P} \rangle \notin \mathcal{L}$.*

(6) *We have $Y_P \leq O_p(\widetilde{C})$, and if L is as in (4) then also $Y_L \leq O_p(\widetilde{C})$.*

From the outset, we fix $\widetilde{P}$ as in 1.3(5) so that, first, the group $H := \langle P \cap \widetilde{C}, \widetilde{P} \rangle$ is as small as possible, and then, subject to this condition, so that $\widetilde{P}$ is as small as possible. Then $\langle P, H \rangle \notin \mathcal{L}$.

Our goal is to determine all groups G satisfying the above Hypothesis. In this preliminary set of notes we will fall far short of this goal. For one thing, we will only be dealing, at present, with the case "$b \geq 3$" (where b is a parameter to be defined later), and we leave the case $b \leq 2$ (which, by 1.3(6) amounts to $b = 2$) for a second installment. For another thing, we will be content in these notes (so far) to obtain suitable p-local information about G, from which the actual identification of G will follow at a later stage. We mention that, in the case of $p = 2$, we expect to be able to use (perhaps "steal" would be a better word) large portions of the relevant parts of the "Quasithin"

Project of Aschbacher and Smith, in determining the group G from the local data. In fact, Michael and Steve should really by writing up the stuff I need as a separate paper, as far as I'm concerned. In these notes as they are, we only aim at the following.

Theorem A. *Assume $b \geq 3$. Then either $(N_H(P)P, H)$ is a weak BN-pair, or there are two maximal 2-local subgroups of G having the "same general structure" as maximal 2-local subgroups in the Rudvalis simple group. In particular, in the latter case, we have $|S| = 2^{14}$.*

In fact, in the case where $(N_H(P)P, H)$ is not a weak BN-pair we can show that $|G| = |Ru|$. But, as mentioned, we expect to follow Aschbacher and Smith in their approach to the construction of Ru from the materials at their hands.

In the remainder of this section we will develop basic properties of P and H, and of the coset graph which they determine.

Lemma 1.4. *Let K be a subgroup of H containing S. Then one of the following holds.*

(i) $\langle K, P \cap H\rangle = H$.
(ii) $K \leq N_G(Y_P)$.
(iii) $K = \langle N_K(Y_P), P_1\rangle$, *where $P_1 \in \mathcal{P}_K(S)$ and P_1 is the unique element of $\mathcal{P}$ satisfying the conditions in 1.3(4).*

Proof. Immediate from 1.3(4) and from the minimality of H. $\square$

Lemma 1.5. *We have*

$$N_P((S \cap P^0)Q_P)) = N_P(S \cap P^0) = P \cap \widetilde{C} = P \cap H,$$

and $P \cap H$ is the unique maximal subgroup of P containing S.

Proof. We have $N_P(S) \leq P \cap \widetilde{C}$ by the E-uniqueness condition 1.3(2). The P-uniqueness condition in 1.3(3), together with 1.1, shows that $P \cap \widetilde{C}$ is a maximal subgroup of P containing S. But 1.3(3b) shows that $N_P(S \cap P^0)$ is maximal in P. As $P \in \mathcal{P}$, 1.5 follows. $\square$

In all that follows, set $H_0 = \langle O^p(\widetilde{P})^H\rangle$. Let us also set

$$Q_P = O_p(P), \quad \text{and} \quad Q_H = O_p(H).$$

Further, set

$$V = \langle (Y_P)^H\rangle, \quad \widetilde{V} = V/Y_H, \quad \text{and} \quad \overline{H} = H/C_H(\widetilde{V}).$$

Lemma 1.6. *We have $Y_P = \Omega_1(Z(Q_P))$ and $Y_H = C_{Y_P}(S \cap P^0) = C_{Y_P}(O_p(P \cap H))$.*

Proof. We have $Y_P \leq \Omega_1(Z(Q_P))$, by 1.1(a). But 1.3(3) shows that $C_P(\Omega_1(Q_P)) = Q_P$, so the first part of 1.6 follows from the maximality of Y_P among the p-reduced subgroups of P.

We have $H = H_0(P \cap \widetilde{P})$, and 1.3(5) shows that $H_0 \leq E$. Set $Q = O_p(\widetilde{C})$. Then $Q \leq Q_H$, and so $Y_H \leq \Omega_1(Z(Q)))$ by 1.1. As $[Y_{\widetilde{C}}, E] = 1$, the Thompson $P \times Q$

Lemma shows that also $[\Omega_1(Z(Q)), E] = 1$, and so $[Y_H, H_0] = 1$. Thus $H/C_H(Y_H) \cong (P \cap \widetilde{C})/C_{P\cap\widetilde{C}}(Y_H)$. It now follows (from the definition of Y_H) that $Y_H \leq Z(O_p(P\cap\widetilde{C}))$. Setting $Z_0 = C_{Y_P}(S \cap P^0)$, we have $Z_0 = \Omega_1(Z(O_p(P \cap \widetilde{C})))$, and we conclude from 1.1(a) that $Y_H \leq Z_0$. On the other hand, we observe that $Y_{P\cap\widetilde{C}} = Z_0$, and so $Z_0 \leq Y_H$ by 1.1(b). This yields the second part of 1.6. $\square$

Lemma 1.7. *We have $N_G(Y_P) \leq N_G(O^2(P))$.*

Proof. Set $N = N_G(Y_P)$. As P is transitive on $(Y_P)^\sharp$, we have $N = C_N(z)P$ for any $z \in Z^\sharp$. Then $\langle (O_2(C_G(z)))^N \rangle = \langle (O_2(C_G(z)))^P \rangle \geq \langle (O_2(\widetilde{C}))^P \rangle = O^2(P)O_2(\widetilde{C})$. It follows that $O^2(P)$ is normal in N. $\square$

Lemma 1.8. *Let $g \in P - H$, and set $R = (Q_H)^g$. Then $C_V(R) = (Y_H)^g$.*

Proof. Set $U = C_V(R)Y_P$. Then U is Q_P-invariant, and we have $[U, R][U, Q_H] = Y_P$. Thus $U \trianglelefteq P$ and $[U, O^p(P)] = Y_P$. Assume that $U \neq Y_P$ and let U_0 be a normal subgroup P contained in U, with $|U_0/Y_P| = p$. As $Y_P = \Omega_1(Z(Q_P))$ we have $[U_0, Q_P] = Y_P$, and then $Q_P/C_{Q_P}(U_0)$ is isomorphic to Y_P as modules for P. As $[R, Q_P] \leq R \cap Q_P \leq C_{Q_P}(U_0)$, it follows that R acts trivially on Y_P, so $R \leq Q_P$. As $Q_H \not\leq Q_P$, by 1.3(3), we have a contradiction. $\square$

Lemma 1.9. *Let H_1 be a subgroup of H containing $\widetilde{P}$, and set $R = Q_P \cap O_p(H_1)$. Then R is normal in neither P nor H_1.*

Proof. Set $U = \Omega_1(Z(O_p(\widetilde{C})))$. It is then a fundamental fact, concerning E, that $[U, E] = 1$. (See Lemma — in the P-Uniqueness paper, for example.)

Set $W = \Omega_1(Z(R))$ and set $W_1 = \Omega_1(Z(O_p(H_1)))$. As $O^p(\widetilde{P}) \leq E$ we have $H_0 \leq E$, and so $O^p(H_1) \leq E$, and $[U, O^p(H_1)] = 1$. But $W_1 \leq U$, so $[W_1, O^p(H_1)] = 1$. Suppose that R is normal in H_1. The Thompson $P \times Q$ Lemma then yields $[W, O^p(H_1)] = 1$. As $Y_P \leq W$, by 1.3(6), we obtain $Y_P \trianglelefteq \langle P, \widetilde{P} \rangle$, whereas $\widetilde{P}\rangle \notin \mathcal{L}$ by assumption. Thus R is not normal in H_1. In particular, $O_p(H_1) \not\leq Q_P$.

Suppose next that R is normal in P, and set $P_0 = \langle O_p(H_1)^P \rangle$. We have $O_p(\widetilde{C}) \leq O_p(H_1)$, and we have $O_p(\widetilde{C}) \leq O_p(P\cap H)$ and $O_p(\widetilde{C}) \not\leq O_p(P)$ by 1.3(3a). It follows that $O_p(H_1) \cap O_p(P \cap H)$ is invariant under $P \cap H$, and that $O_p(H_1)$ is a Sylow p-subgroup of P_0. Then $P = P_0S$. As $Y_P \trianglelefteq \langle P, \widetilde{P} \rangle \notin \mathcal{L}$, no non-identity characteristic subgroup of $O_p(H_1)$ is normal in P_0. Further, as $Z(P) = 1$ we have also $Z(P_0) = 1$, and then the Pushing Up Theorem of Niles [Ni] or of Stellmacher [St1] implies that $Q_P = Y_P$. As $Y_P \leq Q_H$ by 1.3(6), and since $P \cap H$ acts irreducibly on $O_p(P \cap H)/Q_P$, it follows that $Q_H = O_p(P \cap H)$, and that Q_H is the product of any two H-conjugates of Y_P in Q_H. If $p = 2$ then all of the involutions in Q_H lie in two conjugates of Y_P, so that $|H : N_H(Y_P)| \leq 2$. But $Y_P \trianglelefteq S$, so $Y_P \trianglelefteq H$ in this case, contrary to $\langle P, H \rangle \notin \mathcal{L}$. Thus p is odd.

Recall the standard notation: $V = \langle (Y_P)^H \rangle$ and $\widetilde{V} = V/Y_H$. Here we have found that $V = Q_H$. Set $\Lambda = Aut(V)/C_{Aut(V)}(\widetilde{V})$. It is well known [REFERENCE ?] that, in odd characteristic, $\widetilde{V}$ admits the structure of a 2-dimensional vector space over $\mathbb{F}_q$, in such a way that Λ is isomorphic to the associated group of all semilinear transformations. Denote by $\overline{\Lambda}$ the quotient of Λ by the scalar transformations, and let $\overline{\Lambda}_0$ be the linear subgroup of $\overline{\Lambda}$. Thus $\overline{\Lambda}_0 \cong PGL(2,q)$.

Let D be a Hall p'-subgroup of $P \cap H$, and denote by $\overline{D}$ the image of D in $\overline{\Lambda}$. Thus D is cyclic of order $q-1$, and since $D \cap Z(O^p(P) \; mod \; Q_P) \neq 1$ the irreducible D-modules Y_P/Y_H and V/Y_P are non-isomorphic. Thus, the image of D in Λ intersects the set of scalar transformations trivially, and so D is isomorphic to $\overline{D}$. Further, as D acts regularly on Y_P/Y_H, $\overline{D}$ acts linearly on $\widetilde{V}$, and so $\overline{D}$ is a cyclic subgroup of $\overline{\Lambda}_0$ of order $q-1$.

As $\widetilde{P} \in \mathcal{P}$, we have $Q_H \neq S$. As $Q_H = C_S(Y_H)$ it follows that H_0Q_H/Q_H is a p'-group and that S induces a non-identity group of Galois automorphisms on $\overline{\Lambda}_0$ and on $\widetilde{V}$. In particular we have $q-1 > 5$, and then the only p'-subgroups of $\overline{\Lambda}_0$ containing $\overline{D}$ are contained in the normalizer of $\overline{D}$. As D leaves invariant only two 1-dimensional $\mathbb{F}_q$-subspaces of $\widetilde{V}$ we again arrive at $|H : N_H(Y_P)| \leq 2$. As p is odd, $\widetilde{P}$ has no subgroups of index 2, and thus $\widetilde{P} \leq N_H(Y_P)$. As $\langle P, \widetilde{P} \rangle \notin \mathcal{L}$, we have a contradiction at this point. Thus R is not normal in P.

□

Corollary 1.10. *We have* $[Q_P, O^p(P)] \nleq O_p(\widetilde{P})$. □

Lemma 1.11. *We have*

(a) $[V, Q_H] = Y_H$.
(b) $[C_H(\widetilde{V}), S \cap O^p(P)] \leq Q_H$.
(c) $O^p(H)Q_H/Q_H$ *is a central extension of* $O^p(\overline{H}))$ *by the* p'- *group* $C_H(\widetilde{V})/Q_H$.

Lemma 1.12. *Suppose that some element of* Q_H *induces a transvection on a non-central chief factor for* P *in* Q_P. *Then* $q = p$.

Proof. □

We now form the **coset graph**

$$\Gamma = \Gamma(P, H).$$

Thus, Γ is the bipartite graph whose **vertices** are the right cosets of P and the right cosets of H, with adjacency of vertices given by non-empty intersection. The stabilizer in G of a vertex Pg, for $g \in G$, is then the conjugate P^g of P (and similarly for vertices Hg). In this way we speak of a vertex α of Γ being of **type** P (resp. H) if α is of the form Pg (resp. Hg) for some $g \in G$. An **edge** of Γ consists of an un-ordered pair $\{\alpha, \beta\}$ of adjacent vertices. If $\alpha = Pg$ and $\beta = Hg$ then the stabilizer in G of the edge $\{\alpha, \beta\}$ is $(P \cap H)^g$.

For any vertex δ of Γ, we denote by $\Delta(\delta)$ the set of vertices of Γ which are adjacent to δ. For any integer $n \geq 1$, we denote by $\Delta^{(n)}(\delta)$ the set of all vertices γ such that $d(\delta, \gamma) \leq n$, where d is the natural metric on Γ. We also set $G_\delta^{(n)} = G_{\Delta^{(n)}(\delta)}$; the point-wise stabilizer in G of $\Delta^{(n)}(\delta)$.

For any vertex δ of Γ we set

$$Y_\delta = Y_{G_\delta}, \qquad and \qquad Q_\delta = O_p(G_\delta).$$

For β a vertex of type H we set

$$V_\beta = \langle Y_\beta \ : \ \delta \in \Delta(\beta) \rangle \quad and \quad \overline{G_\beta} = G_\beta / C_{G_\beta}(V_\beta / Y_\beta).$$

and for α a vertex of type P we set

$$W_\alpha = \langle V_\beta \ : \ \beta \in \Delta(\alpha) \rangle.$$

Further, for α of type P we set

$$D_\alpha = \cap_{\beta \in \Delta(\alpha)} V_\beta.$$

We assume that the reader of these notes has a good deal of familiarity and, indeed, facility, with the so-called **amalgam method**. In particular, we do not wish to develop the notion of **critical pair**, for the uninitiated, at this time. For a critical pair (α, α'), the distance $b = d(\alpha, \alpha')$ is an invariant of Γ, which we denote by $b(\Gamma)$ or by $b(P, H)$. A path of length b from α to α' will be called a **critical path**.

The following result is so basic to what follows that it will most often be used without explicit reference. Recall that a group X is said to be *p-closed* if $O_p(X)$ is a Sylow subgroup of X.

Lemma 1.13. *Let δ be a vertex of Γ.*

(a) *If α is of type P then $G_\alpha^{(1)}$ is p-closed, and $O_p(G_\alpha^{(1)} = Q_\alpha$.*
(b) *If β is of type H then $C_{G_\beta^{(1)}}(Y_\beta)$ is p-closed, and $O_p(C_{G_\beta^{(1)}}(Y_\beta)) = Q_\beta$.*

Proof. Set $R_P = \cap\{(P \cap H)^g\}_{g \in P}$ and $R_H = \cap\{(P \cap H)^g\}_{g \in H}$. The Frattini argument yields $P = N_P(S \cap R_P)R_P$, and since $P \in \mathcal{P}$ we then have $P = N_P(S \cap R_P)$. Thus $S \cap R_P = Q_P$, and we have (a).

Set $T = S \cap C_{R_H}(Y_H)$. Then $H = N_H(T)C_{R_H}(Y_H)$. But $C_{P \cap H}(Y_H) = O_p(P \cap H)$, as follows from 1.3(3) and from the characterization of Y_H given in 1.6. Thus $T \trianglelefteq H$, so $T = Q_H$, and we have (b). $\square$

Lemma 1.14. *Let (κ, λ, μ) be a a path of length 2 in Γ. Then $Q_\kappa \cap Q_\lambda \leq [Q_\lambda, O^p(G_\lambda)]Q_\mu$.*

Proof. Set $R = [Q_\lambda, O^p(G_\lambda](Q_\lambda \cap Q_\mu)$. Then $R \trianglelefteq G_\lambda$. As G_λ is transitive on $\Delta(\lambda)$ we then have $Q_\kappa \cap Q_\lambda \leq R$. $\square$

Lemma 1.15. *Let $(\alpha, \beta, \cdots, \alpha')$ be a critical path in Γ.*

(a) *If b is even then $b \geq 2$ and $[Y_\alpha, Y_{\alpha'}] = Y_\beta = Y_{\alpha'-1}$.*
(b) *If b is odd then $b \geq 3$ and $Y_\beta \cap Y_{\alpha'} = 1$.*

Proof. $\square$

Lemma 1.16. *Let (β, γ, δ) be a path of length 2 in Γ, with δ of type P. Then $Q_\beta(G_\beta \cap G_\delta) = G_\beta \cap G_\gamma$.*

Proof. Set $P = \delta$ and $H = \beta$, and let $g \in P$ so that $\gamma = \beta^g$. Notice that the induced action of P on $\Delta(\delta)$ is equivalent to the natural doubly transitive permutation representation of P on the cosets of $P \cap H$. The two- point stabilizer $D := G_\beta \cap G_\gamma$ therefore has the property that $O_p(P \cap H)D = P \cap H$. But $O_p(P \cap H) = Q_P Q_H$, as D acts irreducibly on $O_p(P \cap H)/Q_H$. As $Q_P \leq D$, the lemma follows. □

Recall that for α of type P we have set $D_\alpha = \cap_{\beta \in \Delta(\alpha)} V_\beta$.

Lemma 1.17. *Let α be a vertex of Γ of type P, let σ and λ be distinct neighbors of α, and set $Q^*_\alpha = (Q_\sigma \cap Q_\alpha)(Q_\alpha \cap Q_\lambda)$. Then the following hold.*

(a) $V_\sigma \cap V_\lambda = D_\alpha$.
(b) $[D_\alpha, O^p(G_\alpha)] = Y_\alpha$.
(c) $Q^*_\alpha \trianglelefteq G_\alpha$, *and* $[Q_\alpha, O^p(G_\alpha)] \leq Q^*_\alpha$.

Lemma 1.18. *Suppose that b is odd, and let $(\alpha, \beta, \cdots, \alpha')$ be a critical path in Γ. Suppose that $V_{\alpha'} \not\leq Q_\beta$, and suppose that $[V_\beta \cap Q_{\alpha'}, V_{\alpha'}] = [V_\beta, V_{\alpha'} \cap Q_\beta] = 1$. Then either $V_{\alpha'}$ is an $F1$-offender on V_β/Y_β, or V_β is an $F1$-offender on $V_{\alpha'}/Y_{\alpha'}$.*

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Section 2: Stellmacher's Basic Amalgam Lemmas

For the remainder of this paper we shall assume:

Hypothesis 2.0. *We have $b \geq 3$.*

Lemma 2.1. *Assume that b is even, $b \geq 4$, and let $(\alpha, \beta, \cdots, \alpha')$ be a critical path in Γ. Let $\beta' \in \Delta(\alpha') - \{\alpha' - 1\}$. Then $V_{\beta'}$ fixes the path $(\beta, \cdots, \alpha')$, $V_{\beta'} \not\leq G_\alpha$, and $V_{\beta'}$ is a quadratic F-offender on $\widetilde{V_\beta}$.*

Proof. As b is even, 1.15(a) yields:

(1) $[Y_\alpha, Y_{\alpha'}] = Y_\beta = Y_{\alpha'-1}$.

Then $Y_{\alpha'} = Y_{\alpha'-1}Y_{\beta'}$, and so $[Y_\alpha, Y_{\beta'}] \neq 1$. In particular, we have $Y_{\beta'} \neq Y_{\alpha+3}$. As $V_{\alpha+3} \leq Q_{\beta'}$ and $V_{\beta'} \leq Q_{\alpha+3}$, it now follows from 1.11(a) that

(2) $[V_{\alpha+3}, V_{\beta'}] = 1$.

Observe that $Y_{\beta+1} = Y_{\alpha'-1}Y_{\alpha+3}$ by (1). Then $V_{\beta'} \leq C_{G_{\beta+1}}(Y_{\beta+1})$ and so

(3) $V_{\beta'} \leq Q_{\beta+1} \leq G_\beta$.

Notice that $V'_\beta \cap Q_\beta \trianglelefteq \langle Y_\alpha, Q_{\beta'} \rangle \geq O^p(G_{\alpha'})$. Then $V_{\beta'} \cap Q_\beta \leq D_\alpha$, and so

(4) $|V_{\beta'}/D_{\alpha'}| \leq |V_{\beta'}Q_\beta/Q_\beta|$.

Notice also that, by (2), we have $[D_{\alpha+2}, V_{\beta'}] = 1$, and so (4) yields

$$|V_\beta / C_{V_\beta}(V_{\beta'})| \leq |V_\beta / D_{\alpha+2}| = |V_{\beta'}/D_{\alpha'}| \leq |V_{\beta'} Q_\beta / Q_\beta|.$$

But $|\widetilde{V_\beta}/C_{\widetilde{V_\beta}}(V_{\beta'})| \leq |V_\beta/C_{V_\beta}(V_{\beta'})|$, so we have shown that either $V_{\beta'}$ is an F-offender on $\widetilde{V_\beta}$ or $V_{\beta'} \leq Q_\beta$. Thus, it remains to show that $V_{\beta'}$ acts quadratically on $\widetilde{V_\beta}$ and that $V_{\beta'} \not\leq G_\alpha$. The required quadratic action follows from the observation that $[V_\beta, V_{\beta'}, V_{\beta'}] \leq [W_{\alpha'}, V_{\beta'}] \leq Y_{\alpha'-1} = Y_\beta$. So assume that we have $V_{\beta'} \leq G_\alpha$. Then $[Y_\alpha, V_{\beta'}] \leq Y_\beta = Y_{\alpha'-1} \leq V_{\beta'}$, whence

$$V_{\beta'} \trianglelefteq \langle Y_\alpha, G_{\beta'} \rangle = \langle Y_\alpha, G_{\alpha'} \cap G_{\beta'}, G_{\beta'} \rangle = \langle G_{\alpha'}, G_{\beta'} \rangle.$$

As $O_p(\langle P, H \rangle) = 1$, we have a contradiction, and the lemma is proved. $\square$

Here is a "QRC"-argument that will be used in the proof of lemma 2.3. It was shown to me by Ulrich M.

Lemma 2.2. *Let V be a p-group, and let $\{Z_i\}_{1 \leq i \leq n}$ be a collection of normal subgroups of V which together generate V. Let $s > 0$ be a positive real number, and let B be a p-group acting on V. Assume that each Z_i is B-invariant, and assume that*

$$|Z_i / C_{Z_i}(D)| \geq |D / C_D(Z_i)|^s$$

for every subgroup D of B. Then $|V/C_V(B)| \geq |B/C_V(B)|^s$.

Proof. Set $B_1 = B$ and, inductively, set $B_{i+1} = C_{B_i}(Z_i)$, $1 \leq i \leq n$. Then $B_{n+1} = C_B(V)$, and

$$|B_i/B_{i+1}|^s = |B_i/C_{B_i}(Z_i)|^s \leq |Z_i/C_{Z_i}(B_i)|.$$

Then

$$|B/C_B(V)|^s = \prod_{i=1}^{n} |B_i/C_{B_i}(Z_i)|^s \leq \prod_{i=1}^{n} |Z_i/C_{Z_i}(B_i)|.$$

But also

$$|Z_i/C_{Z_i}(B_i)| = |Z_i C_V(B_i)/C_V(B_i)| \leq |C_V(B_{i+1})/C_V(B_i)|$$

and so we get

$$\prod_{i=1}^{n} |Z_i/C_{Z_i}(B_i)| \leq \prod_{i=1}^{n} |C_V(B_{i+1})/C_V(B_i)| = |V/C_V(B)|.$$

Thus $|V/C_V(B)| \geq |B/C_V(B)|^s$. $\square$

Lemma 2.3. *Assume that we have $b \geq 3$, and assume also that $\widetilde{V}$ is not an $F1$-module for H via an $F1$-offender contained in $O_p(O^p(P))O_p(H)$. Then b is odd, and there exists a critical path $\pi = (\alpha, \beta, \cdots, \alpha')$ such that $V_{\alpha'} \not\leq Q_\beta$, and such that one of the following holds.*

(i) *V_β is a quadratic $F2^*$-offender on $\widetilde{V_{\alpha'}}$ and $Y_{\alpha'} \not\leq V_\beta$,*

(ii) *V_β is a quadratic $F2^*$-offender on $\widetilde{V_{\alpha'}}$ and $b = 3$, or*

(iii) *V_β is a quadratic $F2$-offender on $\widetilde{V_{\alpha'}}$, $V_{\alpha'}$ is a quadratic $F2$-offender on $\widetilde{V_\beta}$, and $b = 3$.*

Proof. By 2.1, b is odd. Fix a critical path $\pi = (\alpha, \beta, \cdots, \alpha')$ and suppose first that $V_{\alpha'} \leq Q_\beta$. Then $|V_{\alpha'}/C_{V_{\alpha'}}(Y_\alpha)| \leq q$. Here $Y_\alpha \leq O_p(O^p(G_{\alpha'-1}))Q_{\alpha'}$, so $|Y_\alpha/C_{Y_\alpha}(\widetilde{V_{\alpha'}})| < |\widetilde{V_{\alpha'}}/C_{\widetilde{V_{\alpha'}}}(Y_\alpha)|$, by hypothesis. Thus, we have $Y_\alpha/(Y_\alpha \cap Q_{\alpha'})| < q$. $[Y_\alpha \cap Q_{\alpha'}, V_{\alpha'}] \leq Y_\beta \cap Y_{\alpha'} = 1$, by 1.15(b), and then $[Y_\alpha, V_{\alpha'}] = 1$ as Y_α is a natural module for G_α. But $Y_\alpha \not\leq Q_{\alpha'}$, so we have a contradiction. Thus:

(1) $V_{\alpha'} \not\leq Q_\beta$.

Notice that the proof of (1) involved also a proof of:

(2) $Y_\alpha \cap Q_{\alpha'} = Y_\beta$.

Further, it follows from 1.11(a) that we have

(3) $[V_\beta \cap Q_{\alpha'}, V_{\alpha'} \cap Q_\beta] = 1$.

Since V_β and $V_{\alpha'}$ act quadratically on each other, V_β is not an F-offender on $\widetilde{V_{\alpha'}}$, and $V_{\alpha'}$ is not an F-offender on $\widetilde{V_\beta}$. Therefore

(4) $[V_\beta \cap Q_{\alpha'}, V_{\alpha'}] \neq 1$ or $[V_{\alpha'} \cap Q_\beta, V_\beta] \neq 1$.

As $(\alpha', \alpha' - 1, \cdots, \beta)$ is a pre-critical path, by (1), we may assume, by (4) and symmetry, that $[V_\beta, V_{\alpha'} \cap Q_\beta] \neq 1$,. Thus, we have $[V_\beta, V_{\alpha'} \cap Q_\beta] = Y_\beta$, and $Y_\beta \leq V_{\alpha'}$.

Let $\delta \in \Delta(\beta)$ with $[Y_\delta, V_{\alpha'} \cap Q_\beta] \neq 1$. Then (3) shows that $Y_\delta \not\leq Q_{\alpha'}$, and we may therefore take $\delta = \alpha$. Choose $t \in V_{\alpha'} \cap Q_\beta$ with $[Y_\alpha, t] \neq 1$, and choose $\alpha - 1 \in \Delta(\alpha) - \{\beta\}$. Then:

(5) $\langle Q_{\alpha-1}, t\rangle \geq O^p(G_\alpha)$.

Suppose next that $[V_{\alpha-1}, V_{\alpha'-2}] = 1$. Then $V_{\alpha-1} \leq Q_{\alpha'-2} \cap Q_{\alpha'-1}$. Set $A = V_{\alpha-1} \cap V_\beta Q_{\alpha'}$. Then

$$A \leq V_{\alpha-1}V_\beta \cap V_\beta Q_{\alpha'} = V_\beta(V_{\alpha-1}V_\beta \cap Q_{\alpha'}) \leq V_\beta(Q_{\alpha'-1} \cap Q_{\alpha'})$$

and so $[A, t] \leq [V_\beta, t][Q_{\alpha'-1} \cap Q_{\alpha'}, t] \leq Y_\beta Y_{\alpha'}$.

Set $X = C_A(t \ mod \ Y_\beta)$. Then $|A/X| \leq q$, and $X \trianglelefteq \langle Q_{\alpha-1}, t\rangle$ as $Y_\beta Y_{\alpha-1} \leq X$. As $O^p(G_\alpha)$ is transitive on $\Delta(\alpha)$, (5) then implies that $X \leq D_\alpha$. But also $[D_\alpha, t] \leq [V_\beta, t] \leq Y_\beta$, so in fact

(6) $X = D_\alpha$.

We now have

$$|V_{\alpha'}/C_{V_{\alpha'}}(V_{\alpha-1}V_\beta)| \leq |V_{\alpha'}/D_{\alpha'-1}|$$
$$= |V_{\alpha-1}/D_\alpha| = |V_{\alpha-1}/A||A/X| \leq |V_{\alpha-1}V_\beta Q_{\alpha'}/V_\beta Q_{\alpha'}|q.$$

But also

$$|V_{\alpha-1}V_\beta Q_{\alpha'}/Q_{\alpha'}| = |V_{\alpha-1}V_\beta Q_{\alpha'}/V_\beta Q_{\alpha'}||V_\beta/(V_\beta \cap Q_{\alpha'})| \geq |Y_\alpha/Y_\alpha \cap Q_{\alpha'}| = q,$$

and so we obtain $|V_{\alpha-1}V_\beta Q_{\alpha'}/Q_{\alpha'}| \geq |V_{\alpha'}/C_{V_{\alpha'}}(V_{\alpha-1}V_\beta)|$. Thus $V_{\alpha-1}V_\beta$ is an F-offender on $\widetilde{V_{\alpha'}}$, contrary to hypothesis. Thus

(7) $[V_{\alpha-1}, V_{\alpha'-2}] \neq 1$.

If now $V_{\alpha-1} \leq Q_{\alpha'-2}$ then we may apply the result (1) to the pre-critical path $(\alpha'-2, \cdots, \alpha-1)$ and obtain $V_{\alpha'-2} \leq Q_{\alpha-1}$. Then (7) yields $Y_{\alpha-1} = Y_{\alpha'-2}$, and then $Y_\alpha = Y_{\alpha'-2}Y_\beta \leq C_G(V_{\alpha'})$, which is contrary to the case. Thus:

(8) $V_{\alpha-1} \nleq Q_{\alpha'-2}$.

Suppose now that we have $b > 3$. Then $[V_{\alpha'-2}, V_{\alpha'}] = 1$, and so $Y_{\alpha-1} \nleq V_{\alpha'-2}$. Since both $(\alpha-1, \cdots, \alpha'-2)$ and $(\alpha'-2, \cdots, \alpha-1)$ are pre-critical paths, we are free to replace $(\beta, \cdots, \alpha'-2)$ by one of these, obtaining either $Y_\beta \nleq V_{\alpha'}$ or $Y_{\alpha'} \nleq V_\beta$. But in fact $Y_\beta \leq V_{\alpha'}$ as $[V_\beta, V_{\alpha'} \cap Q_\beta] \neq 1$, so we have shown:

(9) If $b > 3$ then $Y_{\alpha'} \nleq V_\beta$ and $[V_{\alpha'}, V_\beta \cap Q_{\alpha'}] = 1$.

Suppose that $|V_\beta Q_{\alpha'}/Q_{\alpha'}| \leq |V_{\alpha'}Q_\beta/Q_\beta|$. Then (9) yields

$$|\widetilde{V_\beta}/C_{\widetilde{V_\beta}}(V_{\alpha'})| \leq |V_\beta/C_{V_\beta}(V_{\alpha'})| = |V_\beta/(V_\beta \cap Q_{\alpha'})| \leq |V_{\alpha'}Q_\beta/Q_\beta|$$

and $V_{\alpha'}$ is an F-offender on $\widetilde{V_\beta}$, contrary to assumption. Thus:

(10) If $b > 3$ then $|V_\beta Q_{\alpha'}/\alpha'| > |V_{\alpha'}Q_\beta/qb|$.

Notice that 2.2 applies to our situation, with $s = 1$, $V = V_\beta$, $B = V_{\alpha'} \cap Q_\beta$, and with $\{Z_i\}_{1 \leq i \leq n} = \{Y_\delta\}_{\delta \in \Delta(\beta)}$. We therefore have

(11) $|V_\beta/C_{V_\beta}(V_{\alpha'} \cap Q_\beta)| \geq |(V_{\alpha'} \cap Q_\beta)/C_{V_{\alpha'}}(V_\beta)|$.

If now $|V_\beta Q_{\alpha'}/Q_{\alpha'}| > |V_{\alpha'}Q_\beta/Q_\beta|$ then

$$|V_{\alpha'}/C_{V_{\alpha'}}(V_\beta)| = |V_{\alpha'}Q_\beta/Q_\beta||(V_{\alpha'} \cap Q_\beta)/C_{V_{\alpha'}}(V_\beta)| <$$
$$|V_\beta Q_{\alpha'}/Q_{\alpha'}||V_\beta/C_{V_\beta}(V_{\alpha'} \cap Q_\beta)| \leq |V_\beta Q_{\alpha'}/Q_{\alpha'}||V_\beta/(V_\beta \cap Q_{\alpha'})| = |V_\beta Q_{\alpha'}/Q_{\alpha'}|^2.$$

This shows:

(12) If $b > 3$ then V_β is a quadratic $F2^*$-offender on $\widetilde{V_{\alpha'}}$.

Suppose next that we have $b = 3$. If also $|V_\beta Q_{\alpha'}/Q_{\alpha'}| \geq |V_{\alpha'}Q_\beta/Q_\beta|$ then the argument for (12) shows that V_β is a quadratic $F2$-offender on $\widetilde{V_{\alpha'}}$, while if $|V_\beta Q_{\alpha'}/Q_{\alpha'}| \leq |V_{\alpha'}Q_\beta/Q_\beta|$ then $V_{\alpha'}$ is a quadratic $F2$-offender on $\widetilde{V_\beta}$. Alternatively, if $|V_\beta Q_{\alpha'}/Q_{\alpha'}| < |V_{\alpha'}Q_\beta/Q_\beta|$ then $V_{\alpha'}$ is a quadratic $F2^*$-offender on $\widetilde{V_\beta}$. In this way we obtain the lemma. □

Lemma 2.4. *In outcome (iii) of 2.3 we have $Q_\beta V_{\alpha'} \trianglelefteq G_\beta \cap G_{\beta+1}$.*

Proof. Immediate from 1.16. □

Section 3: Quadratic modules

This section contains detailed results on quadratic $F2$-modules and $F2^*$-modules. Some of these results are stated without proof.

Hypothesis 3.1. *X is a finite group with $O_p(X) = 1$, and such that every non-abelian simple section of X is an alternating group, a group of Lie type, or one of twenty-six "sporadic" groups. Further, we are given a faithful $\mathbb{F}_pX$-module V.*

Proposition 3.2. *Assume hypothesis 3.1, with $p = 2$, and let t be an involution in X. Assume that $F^*(X)$ is quasisimple, and that $X = \langle t^X \rangle$. Assume further that $|V/C_V(t)| \leq 4$. Then one of the following holds.*

(i) *There is a fours group E in X with $t \in E$ and with $[V, E, E] = 0$.*
(ii) *A Sylow 2-subgroup of X is dihedral or semi-dihedral.*

Proof. To be provided. □

Remark: The proof of 3.2 does not require that the simple sections of X be of "known" type. The "known" simple groups with dihedral or semi-dihedral Sylow 2-subgroups are the groups $L_2(r)$, r odd; $L_3(r)$, $r \equiv 3$ $(mod\ 4)$; $U_3(r)$, $r \equiv 1$ $(mod\ 4)$; $Alt(7)$; and M_{11}. Using this, we may obtain the following result.

Proposition 3.3. *Assume hypothesis 3.1. Let X, t, and V be as in 3.2, and set $K = F^*(X)$. Assume that there does not exist a fours group E in X, containing t, and such that $[V, E, E] = 0$. Then either $X \cong Alt(5)$ and V is the $O_4^-(2)$-module for X, or else $X \cong Sym(5)$ and V is the $\Gamma L(2, 4)$- module for X. In the latter case there exists a fours group in K acting quadratically on V.*

Proof. To be provided. □

Corollary 3.4. *Assume hypothesis 3.1. Let K be a component of X and let $A \leq X$ be a quadratic $F2$-offender on V, with $[K, A] \neq 1$, and set $L = \langle K^A \rangle$. Then one of the following holds.*

(i) *There exists a subgroup B of LA, acting quadratically on some non-trivial L-invariant section of V and acting faithfully on L, such that $|B| > 2$.*

(ii) $p = 2$, $K = L \cong SL(2, 4)$, *and* $|A/C_A(K)| = 2$.
(iii) $p = 2$, $K \neq L$, *and* $K \cong SL(2, 2^n)$ *for some* n.

Proof. Suppose that (i) is false, and let A_0 be a complement to $C_A(K)$ in A. Then $|A_0| = 2$. Suppose that K is A-invariant. Then 0.1 implies that A_0 induces either a transvection or a 2-transvection on $[V, K]$, and then 3.3 yields (ii). So assume that K is not A-invariant. Theorem 3 in [Chermak] then yields (iii). □

Proposition 3.5. *Assume hypothesis 3.1. Set* $K = F^*(X)$, *and assume that* K *is quasisimple. Then one of the following holds.*

(i) $K/Z(K)$ *is of Lie type in characteristic* p.
(ii) $p = 2$ *and* $K/Z(K)$ *is an alternating group.*
(iii) $p = 2$ *and* $K/Z(K)$ *is isomorphic to* $U_4(3)$.
(iv) $p = 2$ *and* $K/Z(K)$ *is isomorphic to a sporadic group*

Proof. For $p = 2$ this is a consequence of the work [MS1] of Meierfrankenfeld and Stroth on quadratic fours groups in groups of Lie type, with some help from 3.3. For $p > 3$ one may appeal to work on so-called quadratic pairs by Thompson, Betty Salzberg [Sa], or by Timmesfeld [Ti]. The case of quadratic pairs for $p = 3$ was investigated by Meierfrankenfeld in unpublished notes. The author of these notes will produce a paper on quadratic pairs for $p = 3$, more than sufficient for the purposes here, someday, soon. □

Proposition 3.6. *Assume hypothesis 3.1. Let* S *be a Sylow* p*-subgroup of* X, *and assume that* S *is contained in a unique maximal subgroup of* X. *Assume further that* V *is a quadratic* $F2$*-module for* X. *Let* K *be a component of* X, *set* $U = [V, K]$, *and set* $\widetilde{U} = U/C_U(K)$. *Then one of the following holds.*

(i) $K \cong SL(2, p^n)$ *and every constituent for* K *in* $\widetilde{U}$ *is a natural module for* K. *There are at most two such constituents.*
(ii) $K \cong SU(3, p^n)$ *and* U *is a natural module for* K.
(iii) $K \cong \Omega_4^-(p^n)$ *and* U *is a natural orthogonal module for* K.
(iv) $p = 2$, $K \cong Sz(2^n)$, *and* $\widetilde{U}$ *is a natural module for* K.
(v) $p = 2$, $K \cong SL(3, 2^n)$ *(resp.* $Sp(4, 2^n)$*), and* $\widetilde{U}$ *is the direct sum of a natural module and its dual (resp. a natural module and its contragredient), for* K. *Moreover, there exists an element of* $N_S(K)$ *which induces a polarity on the standard Coxeter diagram for* K *and which interchanges the two irreducible* K*-submodules of* $\widetilde{U}$.
(vi) $p = 2$, $K \cong Alt(2^n + 1)$, *and every non-trivial constituent for* K *in* U *is a natural module (derived from the permutation module) for* K. *There are at most two such constituents.*
(vii) $p = 2$, $K \cong Alt(9)$, *and* $\widetilde{U}$ *is isomorphic to a spin module for* K, *of dimension* 8.

Proposition 3.7. *In Proposition 3.6, suppose that* V *is a quadratic* $F2^*$*-module for* X. *Then one of the following holds.*

(i) $K \cong SL(2, p^n)$ *and* $\widetilde{V}$ *is a natural module for* K.

(ii) $p = 2$, $K \cong SL(2, 2^n)$, *and there exists a quadratic* $F2^*$*-offender* A *in* X *such that* $\langle K^A \rangle \cong {\Omega_4}^+(2^n)$, *and* U *is a natural orthogonal module for* $\langle K^A \rangle$.

(iii) $p = 2$, $K \cong {\Omega_4}^-(2^n)$, U *is a natural orthogonal module for* K, *and* K *contains no quadratic* $F2^*$*-offender on* U.

(iv) $p = 2$, $K \cong SL(3, 2^n)$, *and* $\widetilde{U}$ *is the direct sum of a natural module and its dual, for* K. *Moreover, there exists an element of* $N_S(K)$ *which induces a polarity on the standard Coxeter diagram for* K *and which interchanges the two irreducible* K*-submodules of* $\widetilde{U}$.

(v) $p = 2$, $K \cong Alt(2^n + 1)$, *and* $\widetilde{U}$ *is a natural module (derived from the permutation module) for* K.

Moreover, if V *is an* $F1$*-module for* X *then either (i) or (v) holds.*

Proposition 3.8. *In Proposition 3.5, suppose that there is a component* K_1 *of* X*, distinct from* K*, such that* $[V, K, K_1] \neq 0$. *Then* $KK_1 \cong {\Omega_4}^+(p^n)$*, and* $[V, K] = [V, K_1]$. *Moreover,* K_1 *is the unique such component of* X *distinct from* K.

Proposition 3.9. *Assume hypothesis 3.1, and assume that* $X = RA$*, where* $R = [R, A]$ *is a normal* p'*-subgroup of* X*, and where* $|A| = p$ *and* $|V/C_V(A)| \leq p^2$. *Assume also that* $V = [V, R]$. *Then one of the following holds.*

(i) $X \cong SL(2, 2)$ *and* $|V| = 4$ *or* 16.

(ii) R *is elementary abelian of order* 9, $|A| = 2$, *and* $|V| = 16$.

(iii) X *is isomorphic to the commutator subgroup of* $SU(3, 2)$*, and* V *is a natural* $SU(3, 2)$*-module for* X*, of order* 64.

(iv) X *is dihedral of order* 10*, and* $|V| = 16$.

(v) $X \cong SL(2, 3)$ *and* $|V| = 9$ *or* 81.

(vi) R *is a commuting product of two quaternion groups, and* $|V| = 81$.

Proposition 3.10. *Assume hypothesis 3.1. Let* $X = RA$*, with* A *a* p*-group and with* $R = [R, A]$ *a normal* p'*-subgroup of* X. *Assume that* A *is a quadratic* $F2$*-offender on* V. *Denote by* $\mathcal{D}$ *the set of non-identity subgroups* D *of* X *such that* $D = [C_R(B), A]$ *for some maximal subgroup* B *of* A. *Then* R *is the direct product of the elements of* $\mathcal{D}$*, and* $[V, R]$ *is the direct sum of the subspaces* $[V, D]$, $D \in \mathcal{D}$. *Further, for any* $D \in \mathcal{D}$ *and for any* $a \in A - C_A(D)$*, the triple* $(D, \langle a \rangle, [V, D])$ *satisfies the hypothesis of 3.8, in place of* (R, A, V). *Moreover,* A *is generated by its elements* a *such that* $|V/C_V(a)| \leq p^2$.

Proposition 3.11. *Assume hypothesis 3.1, with* X *solvable. Let* S *be a Sylow* p*-subgroup of* X*, and assume that* S *is contained in a unique maximal subgroup of* X. *Assume further that* V *is a quadratic* $F2$*-module* V *for* X. *Then* $p = 2$ *or* 3*, and there exists an* S*-transitive set* $\mathcal{K} = \{K_1 \cdots K_r\}$ *of subgroups of* $O_{p'}(X)$ *such that* $O_{p'}(X) = K_1 \times \cdots \times K_r$*, with* $[V, O_{p'}(X)] = [V, K_1] \oplus \cdots \oplus [V, K_r]$*, and such that one of the following holds for al* i.

(i) $K_i \cong O^p(SL(2, p))$ *and* $[V, K_i]$ *is a natural module or a direct sum of two natural modules for* K_i.

(ii) $K_i \cong O^p({O_4}^+(p))$ *and* $[V, K_i]$ *is a natural orthogonal module for* K_i.

(iii) $p = 2$, K_i *is cyclic of order* 5, *and* $|[V, K_i]| = 16$.
(iv) $p = 2$, $K_i \cong O^2(SU(3, 2))$, *and* $|[V, K_i]| = 64$.

The next two results come from joint work with Christopher Parker. The reader may wish to recall some of the notation from 1.2.

Lemma 3.12. *Assume hypothesis 3.1, and assume that* $X = DP_0$ *where* D *is a normal* r*-subgroup of* X *for some prime* r*, and where* $P_0 \in \mathcal{P}^*_X(S)$ *for some Sylow* p*-subgroup* S *of* X*. Assume further that* $O^p(P_0) \not\leq C_{P_0}(D)$ *and that there exists an element* t *of* S *such that* $[V, t, t] = 0$ *and such that* $|V/C_V(t)| \leq p^2$*. Assume also that* $V = \langle C_V(S)^X \rangle$*. Then* $p = 2$ *and the following hold.*

(a) $|V/C_V(t)| = 4$.
(b) $[D, t]$ *is a direct factor of* D *of order* 3, *and* $|[V, [D, t]]| = 16$.
(c) *Let* h *be an element of odd order in* $P_0 - D$*, with* $h^t = h^{-1}$*, and set* $D^* = D\langle h \rangle$*. Then* $\langle t^{D^*} \rangle \cong SU(3, 2)'$*, and* $|[V, [D^*, t]]| = 64$.

Proof. We have $[D, t] \neq 1$ as $[D, O^p(P_0)] \neq 1$. Then the quadratic action of t implies that $p = 2$ or 3. If $p = 3$ then there is a subgroup $K = \langle t^K \rangle$ of P with $K \cong SL(2, 3)$ and with $O_2(K) \not\leq \Phi_3(P)$. On the other hand, if $p = 2$ then the Baer-Suzuki Theorem implies that there is a subgroup $K = \langle t^K \rangle$ of P with K dihedral of twice odd order, and with $O_{2'}(K) \not\leq \Phi_2(P)$. Fix K as above, for the two cases, and put $J = O_{p'}(K)$, $R = [D, t]$, and $R^* = [DJ, t]$. Suppose first that t is a transvection on V, or that R^* is cyclic. Then 3.9 shows that $R^* = J$, contrary to $[D, t] \neq 1$. We conclude that t is a 2-transvection and that R^* is non-cyclic. Then 3.9 implies that $R^* \cong 3^2$ or 3^{1+2} if $p = 2$, or a commuting product of two copies of Q_8 if $p = 3$. Moreover, in the case that $R^* \cong 3^{1=2}$, we have $[Z(R^*), t] = 1$, and thus in any case we have $R\langle t \rangle \cong SL(2, p)$.

Now $C_D(t)$ centralizes R and normalizes R^*. Suppose $p = 3$. Then R and J are the unique normal quaternion subgroups of R^*, and so $C_D(t)$ normalizes J. But the centralizer in $Aut(J)$ of t is $\langle t \rangle$, so $C_D(t)$ centralizes J, and thus J centralizes $[D, t]C_D(t) = D$. As $\langle J^{P_0} \rangle$ contains $O^3(P_0)$, we have a contradiction at this point, and so $p = 2$. Now suppose that $R^* \cong 3^2$. There is then precisely one cyclic subgroup J_1 of R^* satisfying $R^* = R \times J_1$ and with $|[V, J_1] = |[V, R]|$. (Here it should be noted that $|[V, R]|$ may be either 4 or 16.) Then J_1 is D-invariant, so J_1 centralizes D, and then also $O^2(P_0)$ centralizes D, as in the case $p = 3$. So we conclude at this point that $R^* \cong 3^{1+2}$. Here $[V, R^*]$ is a natural $SU(3, 2)$-module for $R^*\langle t \rangle$, as is shown by 3.9, and the Lemma thereby follows. $\square$

Lemma 3.13. *Assume hypothesis 3.1, and assume that all of the following conditions hold.*

(1) $F^*(X)$ *is an* r*-group for some prime* r, $r \neq p$.
(2) $X \in \mathcal{P}^*_X(S)$ *for a Sylow* p*-subgroup* S *of* X.
(3) *We have* $V = \langle C_V(S)^X \rangle$.
(4) *There exists a non-identity element* t *of* S*, acting quadratically on* V*, such that* $|[V, t]| \leq p^2$.

Set $D = \langle [O_r(X), T]^X \rangle$, and denote by $\mathcal{B}$ the set of all maximal subgroups B of D such that $C_V(B) \neq 0$. Then $p = 2$, $r = 3$, D is elementary abelian, and X/D induces the full symmetric group on $\mathcal{B}$, with t inducing a transposition.

Proof. Set $\mathcal{T} = t^X$ and set $H = \langle \mathcal{T} \rangle$. Then 3.11 implies that $p = 2$, and 3.12(b) in particular implies that D is an elementary abelian 3-group, with $F(X) = D \times C_{F(X)}(H)$. As $X \in \mathcal{P}^*_X(S)$, by assumption, we have also $H \geq O^2(X)$.

Set $\overline{V} = V/C_V(D)$, and let R be a Sylow 2-subgroup of the pre-image in X of $O_2(X/C_X(\overline{V}))$. Suppose that $R \neq 1$. As $O_p(X) = 1$ and $X \in \mathcal{P}^*_X(S)$, the frattini argument then yields $O^2(X) \leq C_X(\overline{V})$, contrary to $D \neq 1$. Thus $R = 1$, and we may therefore replace V with $\overline{V}$ and replace X by $X/C_X(\overline{V})$ while retaining the hypotheses of the lemma. So, we may assume to begin with that $C_V(D) = 0$.

Write $[D, t] = \langle d \rangle$, and notice that 3.12(c) implies that $[V, d] = [V, d, C_D(t)]$, while 3.12(b) yields $|[V, d]| = 16$. It follows that there are hyperplanes I and J of D with $I^t = J$ and with $[V, d] = C_V(I) \times C_V(J)$. As t is a 2-transvection on V, it then follows that t acts trivially on $\mathcal{B} - \{I, J\}$. Thus:

(1) t acts as a transposition on $\mathcal{B}$.

We also record:

(2) We have $[V, d] = C_V(I) \times C_V(J)$, where I and J are the two elements of $\mathcal{B}$ which are interchanged by t.

Denote by $\mathcal{F}$ the set of fixed-points for H on $\mathcal{B}$, and set $W = \langle C_V(B) : B \in \mathcal{B} - \mathcal{F} \rangle$. Then W is X-invariant, and (2) shows that $W \geq [V, H] \geq [V, D_0] = V$. Then $\mathcal{B} = \mathcal{B} - \mathcal{F}$, and so

(3) H has no fixed-points on $\mathcal{B}$.

Now (1) and (3) together imply that X acts transitively on $\mathcal{B}$. As $X = SH$, it follows that S is transitive on the set of H-orbits in $\mathcal{B}$. As $\langle t \rangle$ is normal in S, by assumption, we conclude that H has just one such orbit. Thus

(4) H is transitive on $\mathcal{B}$.

It now follows from (1) and (4) that X induces the full symmetric group on $\mathcal{B}$. $\square$

Section 4: The structure of H, with $b \geq 3$

Recall from Hypothesis 2.0 that we have $b \geq 3$.

We begin this section by considering the case where H is solvable. As always, we set $\overline{H} = H/C_H(\widetilde{V})$, and we set $H_0 = \langle O^p(P)^H \rangle$.

Lemma 4.1. *Assume that H is solvable and that P is not. Then the following hold.*

(a) *$\overline{H}_0$ is an r-group, where $\{p, r\} = \{2, 3\}$.*

(b) *$[\Phi_p(H_0), S \cap O^p(P)] \leq O_p(H)$.*

(c) *H acts irreducibly on $H_0/\Phi_p(H_0)$.*
(d) *$(P \cap H)\Phi_p(H_0)$ is the unique maximal subgroup of H containing $P \cap H$.*
(e) *Let A be a subgroup of $S \cap O^p(P)$ such that $\overline{A}$ is elementary abelian, and let $\mathcal{X}_A$ denote the set of elements $g \in H_0$ such that $|A/C_A(g \bmod O_p(H))| = p$ and such that for any $a \in A$ with $[\overline{g}, \overline{a}] \neq 1$, we have $H = \langle a, (P \cap H)^g \rangle$. Then $H_0 = \langle \mathcal{X}_A \rangle C_{H_0}(A \bmod O_p(H))$.*

Proof. Let H_1 be a normal subgroup of H_0 such that $H_1(P \cap H)$ is a proper subgroup of H. By our minimal choice of H, we have $\langle P, P_0 \rangle \in \mathcal{L}$ for any $P_0 \in \mathcal{P}_{H_1}(S)$. For any such P_0, P_0 is solvable, by assumption, and since P is non-solvable it follows from 3.1(4) that $P_0 \leq N_G(O^p(P))$. As also $O^p(P)$ is invariant under $N_G(S)$, by P-Uniqueness (i.e. by 1.3(3)) we conclude from 1.2 that $H_1 \leq N_G(O^p(P))$. But $O_p(O^p(P) \cap H) = S \cap O^p(P)$, so $[H_1, S \cap O^p(P)] \leq O_p(H_1)$. In particular, by taking $H_1 = \Phi_p(H_0)(P \cap H)$ we obtain (b).

As $\widetilde{P} \in \mathcal{P}$ and $\widetilde{P}$ is solvable, S acts irreducibly on $O^p(\widetilde{P})/\Phi_p(O^p(\widetilde{P}))$. As $S \cap O^p(P) \not\leq O_p(\widetilde{P})$, by 1.10, it follows that $O^p(\widetilde{P}) = [O^p(\widetilde{P}), S \cap O^p(P)]$, and so also $H_0 = [H_0, \langle (S \cap O^p(P))^H \rangle]$. Thus, there does not exist a normal subgroup H_1 of H which lies properly between $\Phi_p(H_0)$ and H_0, and we obtain (c). Now (d) is an immediate consequence of (c), as is the fact that $\overline{H}_0$ is an r-group for some prime r. Then (a) follows from the quadratic action on $\widetilde{V}$ given by 2.3.

Let A be as in (e), and set $U = [\overline{H}_0/\Phi(\overline{H}_0), A]$. Assume that $U \neq 1$. As $\overline{A}$ is elementary abelian, we have $U = \langle u \in U : |A/C_A(u)| = p$. Let $g \in H_0$ such that g is incident with an element u of U with $|A/C_A(u)| = p$, and let $a \in A - C_A(u)$. Then $\langle a^g, P \cap H \rangle \geq \langle [a, g], P \cap H \rangle = H$, as a consequence of (d). Further, (b) implies that $[g, C_A(u)] \leq [C_H(\widetilde{V}), C_A(u)]$. Since $[C_H(\widetilde{V}), S \cap O^p(P)] \leq O_p(H)$, by 1.11(b), we see that $[g, C_A(u)] \leq O_p(H)$, and thus $g \in \mathcal{X}_A$. This yields (e). □

Lemma 4.2. *If H is solvable then so is P.*

Proof. Assume false, so that H is solvable and P is not. Then 4.1(a) implies that $q > p$.

Define r as in 4.1, so that $\overline{H}_0$ is an r-group. Let $(\alpha, \beta, \cdots, \alpha')$ be a critical path in Γ, and suppose first that b is odd. Then $Y_\beta \cap Y_{\alpha'} = 1$, by 1.15(b). Take $H = \alpha'$ and $P = \alpha' - 1$, and set $A = V_\beta$. We have $A \leq (S \cap O^p(P))Q_H$, by 1.14. For any $h \in H$ such that $[A, h] \leq Q_H$ we have $A \leq (\alpha' - 1)^h$. As $A \not\leq Q_H$ it then follows from 4.1(e) that there exists $g \in H_0$ and a hyperplane A_0 of A such that $[A_0, g] \leq Q_H$ and such that, for any $a \in A - A_0$, we have $H = \langle a, (P \cap H)^g \rangle$. Setting $\mu = (\alpha' - 1)^g$, we then have:

(1) There exists $\mu \in \Delta(\alpha')$ such that $|A : A \cap G_\mu| = p$ and such that $\langle a, G_{\alpha'} \cap G_\mu \rangle = G_{\alpha'}$ for any $a \in A - G_\mu$.

In particular, (1) implies that for any $a \in A - G_\mu$ we have $[a, Y_\mu] \neq 1$. Write A_0 for $A \cap G_\mu$.

Suppose that $Y_\mu \leq Q_\beta$. Then $[A, Y_\mu] = Y_\beta$. As $[A_0, Y_\mu] \leq Y_{\alpha'}$, we then have $A_0 = C_A(Y_\mu)$, and thus $|A/C_A(Y_\mu)| = p$. But $p < q$, from which it now follows that $[Y_\delta, Y_\mu] = 1$ for all $\delta \in \Delta(\beta)$. This is contrary to $[A, Y_\mu] \neq 1$, so we conclude that $Y_\mu \not\leq Q_\beta$.

By analogy with (1), we may now choose $\delta \in \Delta(\beta)$ so that:

(2) $|Y_\mu : Y_\mu \cap G_\delta| = p$, and $G_\beta = \langle G_\delta \cap G_\beta, b\rangle$ for any $b \in Y_\mu - G_\delta$.

Now $[Y_\delta \cap G_\mu, Y_\mu \cap G_\delta] \leq Y_\beta \cap Y_{\alpha'} = 1$, and as $p < q$ we then have $[Y_\delta, Y_\mu \cap G_\delta] = 1$ and $[Y_\delta \cap G_\mu, Y_\mu] = 1$. But $[Y_\delta, Y_\mu] \neq 1$, as follows from (2), so $Y_\delta \nleq G_\mu$. Then (1) yields $G_{\alpha'} = \langle Y_\delta, G_{\alpha'} \cap G_\mu\rangle$, and we obtain $Q_{\alpha'} \cap Q_\mu = C_{Q_{\alpha'}}(Y_\mu \cap G_\delta) \trianglelefteq G_{\alpha'}$, contrary to 1.9. This contradiction shows that b is even.

We now have $[Y_\alpha, Y_{\alpha'}] = Y_\beta = Y_{\alpha'-1}$. Also, as $b > 2$ by assumption, we have $b \geq 4$. Let A now denote W_α. Then $[A, Y_\beta] = 1$. As $A \leq Q_{\alpha'-3} \leq G_{\alpha'-2}$, and as $Y_{\alpha'-2} = Y_{\alpha'-3}Y_{\alpha'-1} = Y_{\alpha'-3}Y_\beta$, we have $A \leq Q_{\alpha'-2}$, and so $A \leq G_{\alpha'-1}$. We observe also that $\Phi(A) \leq Y_\alpha \leq Q_{\alpha'-1}$. If $A \leq Q_{\alpha'-1}$ then $[A, Y_{\alpha'}] = [Y_\alpha, Y_{\alpha'}] \leq Y_\alpha$. But $\eta(G_\alpha, A) \geq 2$, as V_β is not normal in G_α. Thus $A \nleq Q_{\alpha'-1}$. By analogy with (1) and (2) we may then choose $\lambda \in \Delta(\alpha' - 1)$ so that:

(3) $|A : A \cap G_\lambda| = p$, and $G_{\alpha'-1} = \langle a, G_{\alpha'-1} \cap G_\lambda\rangle$ for any $a \in A - G_\lambda$.

Set $A_0 = A \cap G_\lambda$. Then $[A_0, Y_\lambda] \leq Y_{\alpha'-1} \leq Y_\alpha$. By 1.12, no element of $Q_\beta - Q_\alpha$ induces a transvection on A/Y_α, so we have $Y_\lambda \leq Q_\alpha$. We observe that, as a consequence of (3), Y_λ is not A-invariant, and so $[A, Y_\lambda] \nleq Y_\beta$. As $Y_\lambda \leq Q_\beta$ we have $[V_\beta, Y_\lambda] \leq Y_\beta$, so we may fix $\delta \in \Delta(\alpha) - \{\beta\}$ with $[V_\delta, Y_\lambda] \nleq Y_\beta$. In particular, we have $[V_\delta, Y_\lambda] \neq 1$, while $Y_\lambda \leq Q_\alpha \leq G_\delta$. As $Y_\delta Y_\beta = Y_\alpha$ we have $[Y_\delta, Y_{\alpha'}] \neq 1$, and so $Y_\delta \nleq V_{\alpha'-1}$. Then $Y_\delta \nleq [V_\delta, Y_\lambda]$, so $[V_\delta, Y_\lambda \cap leqQ_\delta] = 1$. In particular, we have $Y_\lambda \nleq Q_\delta$, and by analogy with (1) we may fix $\gamma \in \Delta(\delta)$ so that $|Y_\lambda : Y_\lambda \cap G_\gamma| = p$ and so that $G_\delta = \langle G_\gamma \cap G_\delta, Y_\lambda\rangle$. Then $[Y_\gamma, Y_\lambda] \neq 1$. But $[Y_\gamma, Y_\lambda \cap G_\gamma] = 1$ as $Y_\delta \nleq V_{\alpha'-1}$. As $p < q$ it follows that $Y_\gamma \nleq G_\lambda$, and then (3) yields $\langle Y_\gamma, G_{\alpha'-1} \cap G_\lambda\rangle = G_{\alpha'-1}$. Then $Q_{\alpha'-1} \cap Q_\lambda = C_{Q_{\alpha'-1}}(Y_\lambda) \trianglelefteq G_{\alpha'-1}$, again contrary to 1.9. This contradiction completes the proof of 4.2. $\square$

Lemma 4.3. *Assume that $b \geq 3$ and that $F_p^*(H) = F_p(H)$. Then H is solvable, P is solvable, and either $p = 2$ or $(N_H(P)P, H)$ is a weak BN-pair in characteristic* 3.

Proof. Set $D = F_p(H)$, $X = O^p(\widetilde{P})$, and $R = O_p(O^p(P))Q_H$. As always, set $\overline{H} = H/C_H(\widetilde{V})$. Suppose first that $X \leq D$. Then $H = (P \cap \widetilde{C})D$, H is solvable, and 4.2 implies that P is solvable. By 2.3 there exists a subgroup A of R such that $\overline{A}$ is a non-identity quadratic $F2$-offender on $\widetilde{V}$. Set $R_0 = \langle A^{P \cap H}\rangle$, and suppose that $R_0 \leq O_p(\widetilde{P})$. As $[O_p(\widetilde{P}), O^p(\widetilde{P})]$ is a p- subgroup of $F_p(H)$, we then have $\overline{R}_0 Q_H \trianglelefteq \langle O^p(\widetilde{P}), P \cap H\rangle = H$, contrary to $A \nleq Q_H$. Thus we may assume to begin with that $A \nleq O_p(\widetilde{P})$.

Suppose that $p \neq 2$. As $\overline{A}$ acts non-trivially on $\overline{D}$, $\overline{DA}$ involves $SL(2, p)$, so we obtain $p = 3$, $\widetilde{P}$ is a $\{2, 3\}$-group, and $\overline{D}$ is a 2-group. As P is solvable we get also $q = 3$, and $P \cap H = S\langle t\rangle$ where t is an involution satisfying $[S, t] \leq R$. Set $L = \langle A^H\rangle$, and set $L_0 = O^3(L)$. Then 3.10 shows that $\overline{L}_0$ can be written as $\overline{L}_0 = \overline{K}_1 \cdots \overline{K}_s$, where each $\overline{K}_i$ is a normal quaternion subgroup of $\overline{L}$, and where $|[\widetilde{V}, \overline{K}_i]| = 9$ or 81. Denote by L_1 the inverse image in L_0 of $Z(\overline{L}_0)$. Thus $\overline{L}_1$ is an elementary abelian 2-group, and it then follows from "P_1- uniqueness (i.e. from 1.2 and 1.3(4)) that $L_1 \leq N_G(Y_P)$.

Suppose $s = 1$. As $X \leq L_0$ we then have $\overline{H} \cong GL(2, 3)$ and $(N_H(P)P, H)$ is a weak

BN-pair. Thus, we may assume that $s > 1$. Notice that $C_{L_1}(\widetilde{Y}_P)$ centralizes $\widetilde{V}$, as $\widetilde{V} = \langle (\widetilde{Y}_P)^{L_0} \rangle$. Thus $\overline{L}_1$ acts faithfully on $\widetilde{Y}_P$, and since $|\widetilde{Y}_P| = 3$ we then have $s = 2$, $Z(\overline{K}_1) = Z(\overline{K}_2)$, and $\widetilde{V} = [\widetilde{V}, D]$ is a natural $O_4^+(3)$-module for $\overline{H}$. Here $|\overline{A}| = 3$ and $|\widetilde{V}/C_{\widetilde{V}}(\overline{A})| = 9$, so 2.3 shows that $b = 3$, and if $(\alpha, \beta, \gamma, \alpha')$ is a critical path in Γ then V_β is a quadratic $F2$-offender on $\widetilde{V_{\alpha'}}$. But $V_\beta Q_{\alpha'}$ is normal in $(G_\beta \cap G_{\alpha'})Q_{\alpha'}$, and one may observe that $(G_\beta \cap G_{\alpha'})Q_{\alpha'} = G_\gamma \cap G_{\alpha'} = P \cap H$. But $\overline{A}$ is not normal in $P \cap H$ since t interchanges $\overline{K}_1$ and $\overline{K}_2$. This contradiction proves that either the lemma holds holds or:

(1) We have $X \not\leq D$.

We may now assume that (1) is the case. If $D \leq N_H(R)$ then $[D, R] \leq Q_H$, and then 1.10 yields $[\overline{D}, \overline{O}^p(H)] = 1$. This implies that $O^p(H) \leq D$, contrary to (1), so we conclude that $D \not\leq N_H(R)$. In particular, $D \not\leq N_G(O^p(P))$. Minimality of H implies that $\langle P, P_1 \rangle \in \mathcal{L}$ for any $P_1 \in \mathcal{P}_{DS}(S)$, and then P_1-Uniqueness (1.2 and 1.3(4)) yields $q = 2$ or 3. Notice that, by 3.10, $\overline{A}$ is generated by elements $\overline{a}$ such that $|\widetilde{V}/C_{\widetilde{V}}(\overline{a})| \leq p^2$. Then 3.12 yields $p = 2$, whence also $q = 2$ and $P \cap H = S$. Thus $H = \widetilde{P}$, and $\overline{D}$ is a 3-group, with $|[\overline{D}, \overline{R}]| = 3$. Then $|\overline{R}| = 2$, so $\overline{R} = \overline{A}$ and $|[\widetilde{V}, R]| \leq 4$. We now appeal to 3.13, and find that H/D is a symmetric group of degree $2^n + 1$ for some n, and that RD/D is generated by a transposition. Then $N_H(R)D$ is not maximal in H. This is contrary to P_1-Uniqueness, and the lemma is thereby proved. □

Lemma 4.4. *Suppose that $F_p^*(H) \neq F_p(H)$. Then $H = (P \cap H)E_p(H)$, and $O^p(\widetilde{P}) \leq E_p(H)$. Moreover, H acts transitively on its set of p-components.*

Proof. Let K be a p-component of H, and set $L = \langle K^H \rangle$. Suppose that $O^p(\widetilde{P}) \not\leq L$. Then $S \cap L \trianglelefteq \widetilde{P}$, and since $P \cap H_0 = O_p(P \cap H)$ we have also $S \cap L \trianglelefteq P \cap H$. Thus $S \cap L \trianglelefteq H$, and $\overline{L}$ is a p'-group.

As $L \leq H_0$ we have $[L, H_0] \not\leq Q_H$, and so $[L, O^p(\widetilde{P})] \not\leq Q_H$. Set $R = O_p(O^p(P))$. As $R \not\leq O_p(\widetilde{P})$, by 1.10, $[L, R]$ is not a p-group. Then 1.4 shows that $L = \langle N_L(R) \rangle P_1$ where $P_1 \in \mathcal{P}_{LS}(S)$, and where P_1 involves $SL(2, q)$. As $\overline{L}$ is a p'-group we then have $q \leq 3$, $|P_1 : N_{P_1}(R)| \leq 4$, and $\overline{P}_1 \cap \overline{L}$ is an r-group, where $\{p, r\} = \{2, 3\}$. Further, a Frattini Argument implies that $\overline{S}$ normalizes a Sylow subgroup of $\overline{L}$ for every prime divisor of $|\overline{L}|$, so 1.4 now yields $|L : N_L(R)| \leq 4$. As no non-abelian simple group has a subgroup of index less than 5, we have a contradiction. Thus $O^p(\widetilde{P}) \leq L$, and then $L = H_0$. This yields the lemma. □

In the following lemma there is some non-standard notation. For a group X we write $Z_p(X)$ for the complete pre-image in X of $Z(X/O_p(X))$.

Lemma 4.5. *Suppose that H is non-solvable. Then the following hold.*

(a) *$O^p(\widetilde{P})$ is a product of p-components of H.*
(b) *If K is any p-component of H then $K/Z_p(K) \in \mathcal{L}ie(p)$ or $p = 2$ and $K/Z_2(K)$ is an alternating group $Alt(2^n + 1)$, $n \geq 3$.*

Proof. As H is non-solvable, 4.3 implies that $F^*_p(H) \neq F_p(H)$, and the preceding lemma then yields $H = (P \cap H)E_p(H)$, with $O^p(\widetilde{P}) \leq E_p(H)$.

Set $T = C_S(Y_H)$. Then T is a Sylow p-subgroup of TH_0, and $T \geq O_p(O^p(P))Q_H$. Recall the notation given in section 0, and set $r = r(T, \widetilde{V})$ and $\mathcal{A} = \mathcal{AF}^*(T, \widetilde{V})$. Further, set $T_0 = \langle \mathcal{A} \rangle$. Notice that, by 2.3, we have $r \leq 2$. As $T_0 \not\leq Q_H$, we have $T_0 \not\leq O_p(\widetilde{P})$, and therefore we may fix $A \in \mathcal{A}$ with $A \not\leq O_p(\widetilde{P})$. As $F^*_p(H_0) = E_p(H_0)$ we may choose a p-component K of H such that $[\overline{K}, \overline{A}] \neq 1$. Then $H_0 = \langle K^H \rangle$, by 4.4.

Set $L = \langle K, A \rangle$, and suppose first that $K \cong SL(2, p^n)$ for some n. Then (ii) holds, and we may assume that (i) is false. Then $S \cap E_p(H) \trianglelefteq \widetilde{P}$. As $C_{P \cap H}(Y_H) = O_p(P \cap H)$, a Frattini Argument then shows that $S \cap E_p(H) \trianglelefteq H$, which is absurd. Thus, the lemma holds in this case, and we may assume henceforth that K is not isomorphic to $SL(2, p^n)$. Then 3.4 implies that $L = KA$, K is invariant under $\langle A^H \rangle$, and there exists a subgroup B of L with $|\overline{B}| > 2$, such that $\overline{B}$ acts faithfully on $\overline{K}$ and quadratically on some non-trivial L-invariant section of $\widetilde{V}$. Then 3.5 shows yields:

(1) We have $\overline{K}/Z(\overline{K}) \in \mathcal{L}ie(p)$, or else $p = 2$ and $\overline{K}/Z(\bar{(K)})$ is isomorphic to $Alt(m)$ for some m, or to $U_4(3)$, or a sporadic group.

Set $X = O^p(\widetilde{P})(S \cap E_p(H))$. As $O^p(\widetilde{P}) \leq \langle A^H \rangle$, K is X- invariant, and indeed every p-component of H is X-invariant. We may therefore assume that K has been chosen so that $S \cap K \not\leq O_p(X)$. Then $O^p(\widetilde{P}) \leq [O^p(\widetilde{P}), S \cap K] \leq K$. Set $X_1 = (S \cap K)O^p(\widetilde{P})$ and set $N = N_{P \cap H}(K)$. As we have already seen, (ii) follows from (i) via 3.6, so we may assume that $K \neq X_1$. Denote by $\mathcal{M}$ the set of maximal subgroups of K containing X_1, and let $M \in \mathcal{M}$. As $H_0Q_H = E_p(H) = \langle X_1, P \cap H \rangle$ we have $\overline{K} = \langle (\overline{M})^N \rangle$. Thus, $\overline{K}$ has an automorphism ϕ which fixes a Sylow p-subgroup of K, and which moves a maximal subgroup of K containing that Sylow p-subgroup. In the case of the sporadic groups listed in (1), with $p = 2$, there is no such automorphism, as one may check using [Aschbacher].

Suppose that $p = 2$ and $\overline{K}/Z(\overline{K}) \cong U_4(3)$. Then $\overline{M}/Z(\overline{K})$ is of the form $2^4 : Alt(6)$, and $|N : N_N(\overline{M})| = 2$. The two maximal subgroups of $\overline{M}$ containing $Z(\overline{K})(\overline{S} \cap \overline{K})$ are invariant under all automorphisms of $\overline{K}$ which fix $\overline{S} \cap \overline{K}$, so we conclude in this case that X_1 is N-invariant. But $K = \langle (X_1)^N \rangle$, and so we have a contradiction in this case.

If $p = 2$ and $\overline{K}$ is an alternating group $Alt(m)$ then $N \leq S$, and we again have $K = \langle (X_1)^N \rangle = X_1$, for a contradiction. Thus, we are now reduced to the case where $\overline{K}$ is a group of Lie type in characteristic p. We shall say that a subgroup $\overline{P}_0$ of $\overline{K}$ is "parabolic" if $P_0 \geq Z(K \; mod \; C_H(\widetilde{V}))$ and $\overline{P}_0/Z(\overline{K})$ is a parabolic subgroup of $\overline{K}/Z(\overline{K})$ in the ordinary sense. Thus, $\overline{M}$ is a maximal parabolic subgroup of $\overline{K}$, and as $\overline{M}$ is not N-invariant it follows that some element of N induces a non-trivial automorphism on the Coxeter diagram associated with $\overline{K}$. Let M_0 be a parabolic subgroup of M which is minimal subject to containing $\overline{X}_1$. Then $K = \langle (M_0)^N \rangle$. As $\widetilde{P} \in \mathcal{P}$, it follows from inspection of the various Coxeter diagrams that $\overline{M}_0$ is in fact a minimal parabolic subgroup of $\overline{K}$, that $M_0 = M$ is uniquely determined, and that $\overline{K}/Z(\overline{K})$ is of Lie rank

2. Then S normalizes M_0, so p is odd, and $\overline{K}/Z(\overline{K})$ is isomorphic to $L_3(p^n)$ or $G_2(3^n)$ for some n. Moreover, we have $|N : N_N(M)| = 2$.

Set $D = N_H(S \cap E_p(H))$, and suppose next that $D \leq N_H(O^p(P))$. Let N_1 be a Hall p'-subgroup of N. Then $[D \cap K, N_1] \leq P \cap K \leq O_p(P \cap H) \cap K$, and thus $\overline{N}_1$ acts trivially on the group $\overline{I} = (\overline{D} \cap \overline{K})/(\overline{S} \cap \overline{K})$. But in fact, any automorphism of $\overline{K}$ which acts non-trivially on the diagram for $\overline{K}$ also acts non-trivially on $\overline{I}$, so we have a contradiction in this case. Thus $D \not\leq N_H(O^p(P))$. Then 1.4 shows that there exists $P_1 \in \mathcal{P}_{DS}(S)$ such that $\langle P, P_1 \rangle \in \mathcal{L}$, with $P_1 \not\leq N_H(O^p(P_1))$. Set $J = \langle P, P_1 \rangle$. By 1.3(4) we have $J/C_J(Y_J) \cong SL(3, q)$ or $Sp(4, q)$, and since D is solvable, by 1.11(b), we conclude that $q = 3$, $P_1/O_3(P_1) \cong SL(2, 3)$, and $|P \cap H : S| = 2$. As $O^3(\widetilde{P}) \leq K$, K is S-invariant, and we may now conclude that $N = P \cap H$ and that $Q_H K = E_p(H)$. Then $E(\overline{H})\overline{S}/E(\overline{H})$ may be identified with a subgroup of $Out(\overline{K})$ covered by field automorphisms. It follows that $[\overline{I}, \overline{S}] \leq \overline{S} \cap \overline{K}$. As $[\overline{P}_1, \overline{S}]$ is not a p-group, we have a final contradiction at this point, proving 4.5. $\square$

Lemma 4.6. *There is then a unique maximal subgroup of H containing $P \cap H$.*

Proof. Immediate from 4.5 and 4.3. $\square$

For the remainder of this section, denote by M the unique maximal subgroup of H containing $P \cap H$, and set $N = N_H(Y_P)$. Also set $R = O_p(O^p(P))Q_H$, set $r = r(R, \widetilde{V})$, and fix $A \in \mathcal{A}(R, \widetilde{V})$.

Lemma 4.7. *Let K be a p-component of H. Then $\overline{K}/Z(\overline{K})$ is of Lie type in characteristic p. Moreover, the Lie rank of K is at most 2, and if equal to 2 then $p = 2$ and $\overline{K}/Z(\overline{K}) \cong L_3(2^n)$ or $Sp(4, 2^n)$ for some n.*

Proof. Evidently $N \leq M$. Suppose first that this inclusion is proper. Set $M_0 = O^p(M \cap H_0)$. Then $M = (P \cap H)M_0$, and so $M_0 \not\leq N_G(Y_P)$. As $M_0 S$ is a proper subgroup of H, we have $\langle P, P_1 \rangle \in \mathcal{L}$ for every $P_1 \in \mathcal{P}_{M_0 S}(S)$, and the P_1!-Theorem then says that there is a unique $P_1 \in \mathcal{P}_{M_0 S}(S)$ such that $P_1 \not\leq N_G(Y_P)$, and moreover, we have $O^p(P_1/O_p(P_1)) \cong SL(2, q)'$. Suppose now that a p-component K of H is of Lie type in characteristic p. Then $M_0/O_p(M_0)$ is abelian, and hence $q = 2$, $P \cap H = S$, $H = \langle \widetilde{P}, P \cap H \rangle = \widetilde{P}$, and the lemma holds. In view of —- we may therefore assume that $p = 2$ and that $\overline{K} \cong Alt(2^n + 1)$ for any 2-component K of H. If $n \geq 4$ then $M_0 S$ is generated by the elements of $\mathcal{P}_{M0S}(S) - \{P_1\}$, by —-, and $M = N$ in that case. Thus, we have $n = 3$, and $\overline{M}_0 \cong Alt(8) \cong L_4(2)$. Again, P_1! implies that $q = 2$ and then that $H = \widetilde{P}$. Then S acts transitively on the set of 2-components of H, and since $P_1/O_2(P_1) \cong L_2(2)$ it follows that there is just one 2-component in H.

Suppose that $a^* < 2$. Then —- shows that $[\widetilde{V}, K]$ is the irreducible permutation module for $\overline{K}$. Then $C_{[\widetilde{V},K]}(S \cap K) = C_{[\widetilde{V},K]}(M_0)$, and this group is of order 2. But also $\widetilde{V} = \langle (\widetilde{Y}_P)^K \rangle$, where $|\widetilde{Y}_P| = 2$, so it follows that Y_P is M_0-invariant in this case. We conclude from —- that $a^* = 2$, $b = 3$, and A may be chosen so that $\overline{A}$ is normal in $\overline{N}$. Here $\overline{N} \cap \overline{M}_0$ is a maximal parabolic subgroup $\overline{M}_1$ of $\overline{M}_0$, and inspection of these then shows that $\overline{A} = O_2(\overline{M}_1)$ (of order 8 or 16). Now — shows that there is no $\mathbb{F}_2$-module for $\overline{K}$ on which $\overline{A}$ is a quadratic $F2$-offender. By this contradiction, we may assume henceforth that $M = N$.

Then —- shows that $p = 2$, $\overline{K}$ is isomorphic to $Alt(2^n + 1)$ for some n, $n \geq 3$, and we have $M = N$. But one may observe that $O_2(M \cap H_0) = O_2(H)$, and hence $C_S(Z_0) = O_2(H)$. But this means that $S \cap H_0 = O_2(H)$, which is evidently untrue. □

Lemma 4.8. *Assume that H is non-solvable. Then there are normal subgroups L_i of H_0 and subgroups V_i of V, $1 \leq i \leq s$, such that*

(1) $H_0 = L_1 \cdots L_s$,
(2) $[L_i, L_j] \leq Q_H$ *for all* $i \neq j$,
(3) $V_i = [V, L_i]$, *and* $[\widetilde{V}_i, L_j] = 1$ *for all* $i \neq j$,
(4) *Set* $L = L_1$ *and set* $W = V_1/C_{V_1}(L)$. *Assume that the indexing has been chosen so that* $[\overline{L}, \overline{A}] \neq 1$, *let* A_0 *be a complement in* A *to* $C_A(\overline{L})$, *and set* $r_0 = r(\overline{A}_0, W)$. *Then one of the following holds.*
  (a) $\overline{L}$ *is isomorphic to* $SL(2, p^n)$, *and* W *is a natural module for* $\overline{L}$.
  (b) $r_0 = 2$, $\overline{L} \cong SU(3, p^n)$ *or* $Sz(p^n)$, *and* W *is a natural module for* $\overline{L}$.
  (c) $r_0 = 2$, $\overline{L} \cong SL(2, p^n)$, *and* W *involves two non-trivial constituents for the action of* $\overline{L}$, *each of which is a natural module.*
  (d) $r_0 = p = 2$, $\overline{L} \cong Sp(4, 2^n)'$ *and* W *is the direct sum of a natural and a contragredient module for* $\overline{L}$.
  (e) $p = 2$, $\overline{L} \cong SL(3, 2^n)$ *and* W *is the direct sum of a natural and a dual module for* $\overline{L}$.
  (f) $\overline{L} \cong \Omega_4^\epsilon(p^n)$ *and* W *is a natural orthogonal module for* $\overline{L}$.
(5) $\widetilde{V} = \widetilde{V}_1 \cdots \widetilde{V}_r C_{\widetilde{V}}(H_0 A)$.

Proof. Choose K so that $[\overline{K}, \overline{A}] \neq 1$, and with $K \leq \widetilde{P}$. Set $L = [K, A]$ and $W = [V, L]$. Then 0.1 says that $C_A(\overline{L}) = C_A(\widetilde{W})$ and that $r(\overline{A}_0, \widetilde{W}) \leq r(\overline{A}, \widetilde{V})$ for any complement A_0 to $C_A(\overline{K})$ in A. In particular, we have $r(\overline{A}_0, \widetilde{W}) \leq 2$. Fix an irreducible K-submodule $\widehat{U}$ of $\widehat{W}$, and denote by X the product of all of the p-components of H which are not contained in L.

Suppose first that $L \neq K$, and let $a \in A_0$ with $K \neq K^a$. Then $\widehat{U} \cap (\widehat{U})^a = 1$, and so, in particular, we have $|\widetilde{W}/C_{\widetilde{W}}(a)| > 4$. Then $|\overline{A}_0| \neq 2$, and we may appeal to Theorem 3 of [$P_G(V)$-paper] for the structure of $\overline{LA_0}$ and of $\widetilde{W}$. Thus, $p = 2$, $\overline{L} \cong {\Omega_4}^+(2^n)$ for some n, and $\widetilde{W}$ is a direct sum of natural orthogonal modules for $\overline{L}$. As $r(\overline{A}_0, \widetilde{W}) \leq 2$, one easily determines that $\eta(L, \widetilde{W}) = 1$, and thus $[\widetilde{W}, X] = 1$.

Suppose next that $L = K$, and set $K^* = N_S(K)K$. Suppose that $\eta(K^*, \widetilde{W}) > 1$. Then 3.— says that $\overline{K} \cong SL(2, p^n)$, that $\eta(\overline{K}, \widetilde{W}) = 2$, and that the K-irreducible constituents of $\widehat{W}$ are natural $SL(2, p^n)$-modules for $\overline{K}$. It follows from —– that there exists at most one p-component K_1 of X such that $[\widetilde{W}, K_1] \neq 1$, and that if such a p-component K_1 exists, then $\overline{KK_1} \cong {\Omega_4}^+(p^n)$ and $\widetilde{W}$ is a natural orthogonal module for $\overline{KK_1}$.

If $\eta(K^*, \widetilde{W}) = 1$ then also $\eta(K, \widetilde{W}) = 1$ and $[\widetilde{W}, X] = 1$. With 3.— , the lemma now follows □

Lemma 4.9. *Suppose that H is non-solvable, and suppose that $M \neq N$. Set $U = \langle (Y_P)^M \rangle$. Then $q = 2$, $H = \widetilde{P}$, $|Y_P| = 4$, and $|U| = 8$.*

Proof. As $M \neq N$, and since $M = (M \cap H_0)(P \cap H)$, it follows from 1.4 that there exists $P_1 \in \mathcal{P}_{(McapH_0)S}(S)$ such that $\langle P, P_1 \rangle \in \mathcal{L}$, and such that $O^p(P_1/O_p(P_1))$ is isomorphic to the commutator subgroup of $SL(2,q)$. On the other hand, $(M \cap H_0)/O_p(M \cap H_0)$ is abelian, by inspection of the set of outcomes in 4.8(4). Thus $q = 2$, $P \cap H = S$, and $H = \widetilde{P}$. Set $J = \langle P, P_1 \rangle$. Then 1.3(4) says that $J/C_J(Y_J)$ is isomorphic to $L_3(2)$ or $Sp(4,2)$, and that Y_J is a natural module for $J/C_J(Y_J)$. Then $|U| = 8$. $\square$

Lemma 4.10. *Assume that H is non-solvable, let K be a p-component of H, and set $\widehat{V} = V/C_V(H_0)$. Then one of the following holds.*

(i) *K is the unique p-component of H, $M = N$, $\overline{K} \cong SL(2,q)$ or $Sz(q)$, and $\widehat{V}$ is a natural module for $\overline{K}$. Moreover, if $\overline{K} \cong Sz(q)$ then $b = 3$.*
(ii) *$q = 2$ and $\overline{K} \cong L_3(2)$ or $Alt(6)$.*
(iii) *$q = 2$, $M \neq N$, $\overline{H} \cong Sym(5)$, and $\widehat{V}$ is a natural $\Gamma L(2,4)$-module for $\overline{H}$.*
(iv) *$q = 2$, $M = N$, $b = 3$, $\overline{H} \cong Alt(5)$ or $Sym(5)$, and $\widehat{V}$ is an ${\Omega_4}^-(2)$-module for $\overline{H}$.*

Proof. Set $U = \langle (Y_P)^M \rangle$ and set $M_i = O^p(M \cap L_i)$, $1 \leq i \leq s$. If $M_1 = 1$ then it follows from 4.8(4) that $\overline{L} \cong L_3(2)$ or $Alt(6)$, so that outcome (ii) of the lemma holds in this case. We may therefore assume that $M_1 \neq 1$.

Suppose first that $[U, M_1] = 1$. Then $[U, O^p(M \cap H_0)] = 1$, and since $M = (M \cap H_0)(P \cap H)$ it follows that Y_P is M-invariant. Further, as $C_{\widetilde{V}}(M \cap H_0) \neq C_{\widetilde{V}}(H_0)$, and since $M_1 \neq 1$, it follows that 4.8(4)(f) holds with $p^n = 2$ and with $\epsilon = -1$. Here R is M-invariant, by 1.7, so we have $A \leq H_0 Q_H$, and then $r_0 = 2$. As $r_0 \leq r$, by 0.1, 2.3 now implies that $b = 3$. Thus (iv) holds, and we may assume henceforth that $[U, M_1] \neq 1$.

Suppose next that we have $s > 1$, and then suppose further that there exists an element z in $([U, M_1] \cap Y_P) - Y_H$. Then $Q_P \cap Q_H = C_{Q_H}(z)$ is invariant under $\langle P \cap H, L_2 \rangle = H$, contrary to 1.9. Thus $[\widetilde{U}, M_1] \cap \widetilde{Y}_P = 1$. This yields $U \neq Y_P$, and then 4.9 yields $|\widetilde{U}| = 4$. Since $[\widetilde{U}, M_1] \neq 1$ we obtain $\widetilde{U} \leq V_1$, contrary to $s > 1$. We have therefore shown that $s = 1$.

As $P \cap H$ acts irreducibly on $\widetilde{Y}_P$, and as Y_P is not normal in H, we have $C_{Y_P}(O^p(H)) = Y_H$. Thus $|\widehat{Y}_H| = |\widetilde{Y}_H| = q$. Further, we have $[Y_P, S \cap L] \leq Y_H$, since $S \cap L \leq O_p(P \cap H)$.

Suppose that $\overline{L} \cong SL(2,p^n)$ and that $\widehat{V}$ is a natural module for $\overline{L}$. Supposing that $M = N$, we obtain $q = p^n$, and so (i) holds. On the other hand, suppose that $M \neq N$. Then $q = 2$, $H = \widetilde{P}$, $|\widetilde{U}| = 4$ and $\overline{L} \cong SL(2,4)$, by 4.9. Further, there exists $P_1 \in \mathcal{P}_M(S)$, and therefore $\overline{H} \cong Sym(5)$. Thus (iii) holds in this case, and we may assume henceforth that the pair $(\overline{L}, \widehat{V})$ does not consist of $SL(2,p^n)$ and a natural module. In particular, b is then odd, as follows from 2.1 and 3.9.

Fix a critical path $(\alpha, \beta, \cdots, \alpha')$, and take $H = \beta$ and $P = \beta + 1$. As b is odd, 1.15(b) yields $Y_\beta \cap Y_{\alpha'} = 1$. Setting $V_0 = V \cap Q_{\alpha'}$, and taking $A = V_{\alpha'}$, we then have $[V_0, A \cap Q_H] = 1$.

Suppose next that $|Y_\alpha/(Y_\alpha \cap Q_{\alpha'})| < q$. Then $A \cap Q_\beta$ centralizes Y_α. As M is the unique maximal subgroup of H containing $P \cap H$, we may choose a vertex $\delta \in \Delta(\alpha')$ so that $\langle Y_\alpha, G_{\alpha'} \cap G_\delta \rangle = G_{\alpha'}$. Suppose further that $|Y_\delta/(Y_\delta \cap Q_\beta)| < q$. Then $C_{Q_{\alpha'}}(Y_\delta \cap Q_\beta) = Q_{\alpha'} \cap Q_\delta$ which is then normal in $G_{\alpha'}$, contrary to 1.9. By symmetry, we may then assume:

(1) $|Y_\alpha/(Y_\alpha \cap Q_{\alpha'})| \geq q$.

Suppose next that $r = 2$. Then 2.3 implies $b = 3$, and 1.16 implies that $\overline{A}$ is normal in $\overline{P} \cap \overline{H}$. Suppose that $\overline{L} \cong SU(3, p^n)$, $Sz(2^n)$, or $SL(2, p^n)$, as in 4.8(4b) or 4.8(4c). The quadratic action of V_β on $V_{\alpha'}/Y_{\alpha'}$ then yields $|V/V_0| \leq p^n$, and then $q \leq p^n$ by (1). Here $[V_0, A] \leq Y_{\alpha'} \leq Y_P$. The condition $q \leq p^n$ then eliminates the possibility that $\widehat{W}$ is a 3-dimensional unitary module over $\mathbb{F}_{p^{2n}}$, and shows that $|\widehat{Y}_P| = p^n$. If $\overline{H}_0 \cong Sz(2^n)$ we then have outcome (i) of the lemma. So assume that $\overline{L} \cong SL(2, p^n)$ and (as stated in 4.8(4c)) that $\widehat{V}$ involves two natural modules for $\overline{L}$. Then $|[\widehat{V}, \overline{A}]| \geq p^{2n}$, and thus $[V, A] \nleq Y_P$. Then also $V \cap A \nleq Y_P$. On the other hand, we have $[V \cap A, Q_P] = [V \cap A, (Q_P \cap Q_H)A] \leq [V, Q_H] = Y_H$, and by symmetry we have also $[V \cap A, Q_P] \leq Y_{\alpha'}$. As $Y_\beta \cap Y_{\alpha'} = 1$, we have thus shown that $V \cap A \leq \Omega_1(Z(Q_P))$, which is to say that $V \cap A \leq Y_P$. This contradiction eliminates the case 4.8(4c).

Suppose next that $\overline{L}$ is an orthogonal group $\Omega_4^\epsilon(p^n)$, and that $V/C_V(H_0)$ is a natural orthogonal module. Further, assume that $\overline{A} \leq \overline{L}$. Then $r = 2$ and, as we have seen, $b = 3$ and A is normal in $P \cap H$. Quadratic action yields $|\overline{A}| \leq p^n$. Further, we have $|C_{\widetilde{V}}(\overline{S} \cap \overline{L})| = p^n$, and so $q \leq p^n$. Set $W = Y_{\alpha'}/C_{Y_{\alpha'}}(O^p(G_{\alpha'}))$, and denote by W_0 the image of $A \cap Q_H$ in W. Then $|W/W_0| \leq p^n$ and $|[W_0, V]| \leq q$. It follows that $q = p^n$ and that $|\overline{A}| = q$. Suppose $q = 2$. As H is non-solvable we then have $\epsilon = -1$ and $\overline{L} \cong Alt(5)$. Here $\widetilde{Y}_P$ centralizes $\overline{M}$, so 1.7 yields $M = N$, and we have thus returned to outcome (iv) of the lemma. We may therefore assume that $q > 2$, and then $M = N$ by 4.9. Observe that there exists a subgroup X of $C_M(Y_P)$ with $|\overline{X}| = q - \epsilon$. It follows that $[O^p(P), X] \leq Q_P$. Since $A = V^g$ for some $g \in O^p(P)$, A is then X-invariant. It follows that $\epsilon = +1$.

As $\widetilde{P}$ contains a p-component of H, by 4.5(a), there exists an element f of $P \cap H$ such that f interchanges the two p-components of H. If p is odd then 1.16 shows that f may be chosen in $G_\beta \cap G_{\alpha'}$, so that f normalizes A. But for p odd the only quadratic subgroups of $\overline{L}$ are contained in components of $\overline{L}$, so we now conclude that $p = 2$. Then f may be chosen to induce an $\mathbb{F}_q$-transvection on $\widehat{V}$, centralizing $\widehat{Y}_P$ in particular. Then $f \in O_2(P \cap H) = Q_P Q_H$, so we may take $f \in Q_P$. Then f normalizes A, and indeed $[A, f] \leq [A, Q_P] \leq A \cap Q_H$, since ' $[W, Q_P] = [W, O_2(G_\gamma \cap G_{\alpha'})]$ is the unique 3-dimensional $\mathbb{F}_q$-subspace of W which is invariant under Q_P. Thus $[\overline{A}, \overline{f}] = 1$.

Let D be a subgroup of $P \cap H$ of order $q - 1$. Then D acts regularly on $\widehat{Y}_P$, and hence $\overline{D}$ acts as a group of inner automorphisms of $\overline{L}$. By 1.16 we may take D to normalize A. We recall however that $\overline{A}$ is also invariant under a subgroup $\overline{X}$ of $\overline{M}$ of order $q - 1$, such that $\overline{X}$ centralizes $\widehat{Y}_P$. It follows that $\overline{A}$ is normal in $\overline{M}$. But the only proper $\overline{M}$-invariant subgroups of $\overline{S} \cap \overline{L}$ are contained in components of $\overline{L}$, so we have a contradiction at this point.

We are reduced to the case where $\overline{A} \nleq \overline{L}$. Then $[\overline{M}, \overline{A}]$ is not a p-group. We recall however that $A \leq R$ and that N normalizes R. Thus $M \neq N$, and 4.9 yields $q = 2$ and $|U| = 8$. As Y_P is not normal in M we then have $\widehat{U} \neq \widehat{Y}_P$, and so $|\widehat{U}| = 4$. This implies that $2^n = 4$. Also, we note that $[Y_P, A] = 1$, $\overline{A}$ acts as a group of $\mathbb{F}_4$-linear automorphisms of $\widehat{V}$, and hence $|\overline{A} : \overline{A} \cap \overline{H}| = 2$. Then also $|\overline{A}| \leq 8$. By symmetry we have also $|V/V_0| \leq 8$

Suppose that $|\overline{A}| > 2$, and let $\overline{a}$ be a non-identity element of $\overline{A} \cap \overline{L}$. Then $|C_{\widehat{V}}(\overline{a})| = 16$, and since $[V_0, A] \leq Y_{\alpha'}$, of order 2, it follows that $|V/V_0| \geq 8$. Again by symmetry, we conclude that $|\overline{A}| = 8$ and that $|\overline{A} \cap \overline{L}| = 4$ But $C_{\widehat{V}}(\overline{a}) = C_{\widehat{V}}(\overline{b})$ for any non-identity element $\overline{b}$ of $\overline{A} \cap \overline{L}$, and this is inconsistent with $[\widehat{V}_0, \overline{A}]$ being of order at most 2, and with $|\widehat{V}/\widehat{V}_0| < 16$. We therefore conclude that $|\overline{A}| = 2$, and also, by symmetry, $|V/V_0| = 2$. Then $\overline{A}$ induces an $\mathbb{F}_4$-transvection on $\widehat{V}$, and similarly for V on W. As V_0 and $A \cap Q_H$ are index-2 subgroups of V and A, respectively, we obtain $[V_0, A] = Y_{\alpha'}$ and $[V, A \cap Q_H] = Y_H$. Now 2.3 implies that $b = 3$.

As $\overline{A}$ is $\mathbb{F}_4$-linear on $\widehat{V}$, and as $[V, A] \geq Y_H$, we have $|[V, A]| \geq 8$. Now $[\widehat{V}, \overline{Q}_P]$ is the unique Q_P-invariant 3-dimensional subspace of $\widehat{V}$, and is therefore equal to $C_{\widehat{V}}(\overline{A})$. Thus $[V, Q_P, A] \leq C_V(O^p(H))$. But also $[V, Q_P, A] \leq [V_0, A] \leq Y_{\alpha'}$, and since $Y_P = Y_H Y_{\alpha'}$ we have $[Y_{\alpha'}, O^p(H)] \neq 1$. Thus $[V, Q_P, A] = 1$, and symmetry yields also $[A, Q_P, V] = 1$. Then $[A, V] \leq \Omega_1(Z(Q_P))$, by the Three Subgroups Lemma, and so $[A, V] \leq Y_P$. As $|[A, V]| > 4$ we have a contradiction, and have thereby succeeded in ruling out the case given by 4.8(4f).

It remains now to consider the cases where $p = 2$ and $\overline{L} \cong SL(3, 2^n)$ or $Sp(4, 2^n)$, with $n > 1$. Here (1) implies that $|\widetilde{Y}_P| > 2$ and so $M = N$. Thus $\widetilde{Y}_P$ is invariant under a Borel subgroup of $\overline{L}$, as well as being invariant under an element of S which induces a diagram automorphism on $\overline{L}$. It follows that $\widetilde{Y}_P$ covers $C_{\widehat{V}}(\overline{S} \cap \overline{L})$, and so $|\widehat{Y}_P| = 2^{2n}$. But $O^2(P \cap H)$ acts transitively on $\widehat{Y}_P$ while preserving any non-trivial irreducible $\overline{L}$-invariant section of $\widehat{V}$. This is contrary to $\eta(\overline{L}, V) = 2$, so we have succeeded in eliminating cases (4d) and (4e) of 4.8. The proof of 4.10 is thereby complete. $\square$

Section 5: Two special cases

Our goal in this section is to show that outcomes (ii) and (iv) in lemma 4.10 can not occur.

Lemma 5.1. *Suppose that $q = 2$, and let K be a 2-component of H. Then $\overline{K}$ is not isomorphic to $L_3(2)$ or to $Alt(6)$.*

Proof. Suppose false. Choose a critical path $(\alpha, \beta, \cdots, \alpha')$ in Γ, and take $H = \beta$ and $P = \beta + 1$. Notice that $\widetilde{V}$ is not an $F1$-module for H, as follows from 3.7, and then 2.1 implies that b is odd. Note also that we have $G_\beta \cap G_{\beta+1} = P \cap H = S$, and hence $H = \widetilde{P}$.

Denote by $\mathcal{K}$ the set of 2-components of H, and write $\mathcal{K} = \{K_1, \cdots, K_s\}$. For any i, $1 \le i \le s$, set $V_i = [V, K_i]$. Then 4.8 shows that we have $\widetilde{V} = C_{\widetilde{V}}(H_0) \times \widetilde{V}_1 \times \cdots \times \widetilde{V}_r$, and $\widetilde{V}_i$ is the direct sum of two non-isomorphic modules for K_i, each of dimension 3 if $\overline{K} \cong L_3(2)$, or 4 if $\overline{K} \cong Alt(6)$. We recall also that $N_S(K_i)$ contains an element which induces a diagram automorphism on $\overline{K}_i$ (viewed as a group of Lie type in characteristic 2) and which interchanges the two irreducible submodules of $\widetilde{V}_i$.

Suppose first that we have $b \ge 5$. By 2.3 we may then assume that $V_{\alpha'}$ is a quadratic $F2^*$-offender on $\widetilde{V_\beta}$, and further that $Y_\beta \not\le V_{\alpha'}$, while $Y_{\alpha'} \le V_\beta$. In particular, 4.8 and 0.1 show that $\overline{K}_i \cong L_3(2)$ for all i.

Fix a 2-component K of H such that $[\overline{K}, V_{\alpha'}] \ne 1$, and set $V_K = [V, K]$. As $V_{\alpha'}$ acts quadratically on $\widetilde{V_\beta}$ we have $|V_{\alpha'}/C_{V_{\alpha'}}(\overline{K})| \le 4$. As $Y_\beta \not\le V_{\alpha'}$ we have also $[V_K, C_{V_{\alpha'}}(\overline{K})] = 1$. Thus $|\widetilde{V_{\alpha'}}/C_{\widetilde{V_{\alpha'}}}(V_K)| \le 4$. But also $|[\widetilde{V}_K, V_{\alpha'}] \ge 4$ as $\eta(K, V_K) = 2$, and so $V_K \not\le Q_{\alpha'}$. It now follows that $|V_K/(V_K \cap Q_{\alpha'})| = 2$ and that $|\widetilde{V_{\alpha'}}/C_{\widetilde{V_{\alpha'}}}(V_K)| = 4$. Further, we have $\overline{V_{\alpha'}} \le C_{\overline{H}}(\overline{K})\overline{K}$.

Let $\widetilde{U}_1$ and $\widetilde{U}_2$ be the two K-submodules of $\widetilde{V}_K$ of order 8, with the indexing chosen so that $V_{\alpha'}$ projects modulo $C_{\overline{H}}(\overline{K})$ to the unipotent radical of the stabilizer of a point in $\widetilde{U}_1$ and of a line in $\widetilde{U}_2$. Set $U_0 = V_K \cap Q_{\alpha'}$. As we have seen, $\widetilde{U}_0$ is a hyperplane of $\widetilde{V}_K$, and $[\widetilde{U}_0, V_{\alpha'}] = \widetilde{Y_{\alpha'}}$, of order 2. Then $\widetilde{U}_0 = \widetilde{U}_1 \times [\widetilde{U}_2, V_{\alpha'}]$, and $\widetilde{Y_{\alpha'}} \le U_1$. Now K is uniquely determined among the 2-components of H by the condition that $Y_{\alpha'}$ lie inside V_K. Thus $[\overline{H}_0, V_{\alpha'}] = \overline{K}$, and $\overline{K} \ge \overline{V_{\alpha'}}$. In particular, we now have $|V_{\alpha'}/C_{V_{\alpha'}}(V_\beta)| = 4$. Then also $|V_\beta/(V_\beta \cap Q_{\alpha'})| = 2$, and $[O^2(G_{\alpha'}), V_\beta]$ contains a unique 2-component L of $G_{\alpha'}$.

Set $X_\beta = \langle (W_\alpha)^{G_\beta} \rangle$. As $b \ge 5$ we have $[W_\alpha, X_\beta, W_\alpha] \le [W_\alpha, W_\alpha] = 1$. Also $[V_\beta, X_\beta] = 1$ and so, recalling that $Y_{\alpha'} \le V_\beta$, we have $[Y_{\alpha'}, X_\beta] = 1$. By the definition of b, we have $X_\beta \le G_{\alpha'-2}$, and X_β centralizes $Y_{\alpha'-2}Y_{\alpha'} = Y_{\alpha'-1}$. Since $P \cap H = S$ is maximal in H, it now follows that $X_\beta \le Q_{\alpha'-1}$, and so $X_\beta \le G_{\alpha'}$. Then L is X_β-invariant. Set $\overline{G_{\alpha'}} = G_{\alpha'}/C_{G_{\alpha'}}(\widetilde{V_{\alpha'}})$. As $[W_\alpha, X_\beta, W_\alpha] = 1$, where also $\Phi(W_\alpha) = 1$, it follows that either

(i) $|W_\alpha/C_{W_\alpha}(\overline{L})| = 2$, or

(ii) $|X_\beta/C_{X_\beta}(\overline{L})| \le 8$.

Set $V_L = [V_{\alpha'}, L]$, and suppose that (i) holds. Then $W_\alpha = C_{W_\alpha}(\overline{L})V_\beta$, and hence

$$[V_L, W_\alpha] \le Y_{\alpha'}[V_L, V_\beta] \le V_\beta \le W_\alpha.$$

and hence $W_\alpha \trianglelefteq \langle G_\alpha, V_L \rangle$. But $V_K = (V_K \cap Q_{\alpha'})Y_\alpha$, so $|[Y_\alpha, V_L]| \ge 4$, and thus $V_L \not\le G_\alpha$. Then $\langle G_\alpha, V_L \rangle = \langle G_\alpha, G_\beta \rangle$, contrary to $O_2(\langle P, H \rangle) = 1$. Thus (ii) holds. Then $|X_\beta/C_{X_\beta}(\overline{L})V_\beta| \le 4$. But $[C_{X_\beta}(\overline{L})V_\beta, V_L] \le V_\beta$, so X_β/V_β is an $F1$-module for G_β via the action of V_L. (Indeed, we have $|V_L/C_{VL}(V_\beta)| = 4$.) As $\overline{G_\beta}$ has no $F1$-modules, we conclude that $[X_\beta, O^2(G_\beta)] \le V_\beta$, so that $W_\alpha \trianglelefteq G_\beta$, again contrary to $O_2(\langle P, H \rangle) = 1$. This proves:

(1) $b = 3$.

Set $\gamma = \beta+1$. By 1.16 we have $Q_\beta V_{\alpha'} \trianglelefteq G_\beta \cap G_\gamma$, and quadratic $F2$-action then implies that $\overline{V_{\alpha'}} = Z(\overline{S} \cap \overline{H}_0)$. As $b = 3$, 1.17 implies that $[V_\beta, V_{\alpha'}] \leq D_\gamma$ and that $[D_\gamma, G_\gamma] \leq Y_\gamma$, where we recall from section 1 that D_γ denotes $\cap\{V_\delta\}_{\delta \in \Delta(\gamma)}$. Thus $[\widetilde{V}, Z(\overline{S} \cap \overline{H}_0), \overline{S}]$ is of order at most 2. As S acts transitively on the set $\mathcal{K}$ of 2-components of H, it follows that $|\mathcal{K}| = 1$.

Write V_0 for $V_\beta \cap Q_{\alpha'}$ and set $\widehat{V} = V/C_V(H_0)$. Suppose that $\overline{H}_0 \cong Alt(6)$. Then $|V/V_0| = 4$ and $|[V_0, V_{\alpha'}]| \leq 2$. Further, we have $V_0 \trianglelefteq Q_\beta(G_\beta \cap G'_\alpha) = S$, by 1.16, which implies that $\widehat{V}_0$ intersects each of the two irreducible H_0-submodules of $\widehat{V}$ in a hyperplane. But then $|[\widehat{V}_0, V_{\alpha'}]| > 2$, and we have a contradiction. As $\overline{V_{\alpha'}} \trianglelefteq \overline{S}$ we conclude:

(2) $\overline{H} \cong Aut(L_3(2))$ and, setting $A = V_{\alpha'}$, we have $\overline{A} = Z(\overline{S})$.

Observe that $[V, A] \geq Y_\beta Y_{\alpha'} = Y_\gamma = Y_P$. As $V = \langle (Y_P)^{H_0} \rangle$ and as $\overline{A} \leq \overline{H}_0$, it follows that $\widetilde{V} = [\widetilde{V}, H_0]$. That is:

(3) $C_V(H_0) = Y_H$.

Choose $\lambda \in \Delta(\alpha) - \{\beta\}$, and set $Q^*_\alpha = (Q_\lambda \cap Q_\alpha)(Q_\alpha \cap Q_\beta)$. Then $Q^*_\alpha \trianglelefteq G_\alpha$, and $[O_2(G_\alpha), O^2(G_\alpha)] \leq Q^*_\alpha$. As usual, set $D_\alpha = \cap\{V_\delta\}_{\delta \in \Delta(\alpha)}$. We then claim:

(4) $[D_\alpha, G_\alpha] = Y_\alpha$.

Indeed, setting $X = \langle (Q_\beta)^{G_\alpha} \rangle$, we have $[D_\alpha, X] = Y_\alpha$ and $Q^*_\alpha \leq X$. There is a natural embedding of $Q_\alpha/C_{Q_\alpha}(D_\alpha)$ into $Hom(D_\alpha/Y_\alpha, Y_\alpha)$, and so $Q_\alpha/C_{Q_\alpha}(D_\alpha)$ is isomorphic to a direct sum of natural modules for G_α/Q_α. Thus $Q_\alpha = C_{Q_\alpha}(D_\alpha)[Q_\alpha, O^2(G_\alpha)] = C_{Q_\alpha}(D_\alpha)Q^*_\alpha$, and this proves (4).

Notice that $\widetilde{Y}_P = C_{\widetilde{V}}(S)$. Notice further that $C_{\widetilde{V}/\widetilde{Y}_P}(S) = C_{\widetilde{V}}(S \cap H_0)/\widetilde{Y}_P$. As D_α/Y_P centralizes S, by (4), we then have:

(5) $|D_\alpha/Y_\alpha| \leq 2$.

Next, since Y_α does not induce a transvection on $\widetilde{V_{\alpha'}}$ we obtain $|V_{\alpha'}/C_{V_{\alpha'}}(Y_\alpha)| \geq 4$. We may then choose an element $t \in V_{\alpha'} \cap Q_\beta - Q_\alpha$. Now $C_{V_\lambda}(t)Y_\alpha \trianglelefteq \langle Q_\lambda, t \rangle$, and $\langle Q_\lambda, t \rangle \geq O^2(G_\alpha)$. Therefore $C_{V_\lambda}(t)Y_\alpha \leq V_\lambda \cap V_\beta$. Further, we have $V_\lambda \cap V_\beta \trianglelefteq O^2(G_\alpha)Q_\beta$, so that $V_\lambda \cap V_\beta = D_\alpha$. With (5) we then have:

(6) $|C_{V_\lambda}(t)Y_\alpha| \leq 8$, and $C_{V_\lambda}(t)Y_\alpha \leq D_\alpha$.

On the other hand we have $|V_\lambda/(V_\lambda \cap G_{\alpha'})| \leq 4$, so $V_\lambda/(V_\lambda \cap Q_{\alpha'})| \leq 16$, and so $|V_\lambda/C_{V_\lambda}(t)| \leq 32$. As $Y_\alpha \not\leq C_G(t)$ we then have $|V_\lambda/C_{V_\lambda}(t)Y_\alpha| \leq 16$, and as $|V_\lambda| = 2^7$ we conclude from (6) that $|C_{V_\lambda}(t)Y_\alpha| = 8$, that $C_{V_\lambda}(t)Y_\alpha = D_\alpha$, and that equality holds in each step in achieving this calculation. In particular, we have $[V_\lambda \cap Q_{\alpha'}, t] \neq 1$, and so

(7) $[V_\lambda \cap Q_{\alpha'}, t] = Y_{\alpha'}$.

Similarly, we have $V_\lambda \cap G_\gamma \not\leq Q_\gamma$. Choose $t' \in (V_\lambda \cap G_\gamma) - Q_\gamma$. By symmetry, we have $C_{V_{\alpha'}}(t')Y_\gamma = D_\gamma$, and $[V_{\alpha'} \cap Q_\lambda, t'] = Y_\lambda$. Now

$$V_{\alpha'} \cap Q_\lambda \leq V_{\alpha'} \cap G_\lambda \leq C_{V_{\alpha'}}(Y_\lambda) \leq [V_{\alpha'}, G_\gamma \cap G_{\alpha'}]$$

$$= [V_{\alpha'}, Q_\gamma] \leq [W_\gamma, Q_\gamma].$$

As $t' \in G_\gamma$ we then have $[t', V_{\alpha'} \cap Q_\lambda] \leq [W_\gamma, Q_\gamma]$, and so $Y_\lambda \leq [W_\gamma, Q_\gamma]$.

For any $\delta \in \Delta(\gamma)$ we have $V_\delta Q_{\alpha'}/Q_{\alpha'} \leq Z((G_\delta \cap G_{\alpha'})/Q_{\alpha'})$, and so $W_\gamma Q_{\alpha'}/Q_{\alpha'} \leq Z((G_\delta \cap G_{\alpha'})/Q_{\alpha'})$. Then $[W_\gamma, Q_\gamma] \leq Q_{\alpha'}$, and so $Y_\alpha \leq Q_{\alpha'}$. As (α, α') is a critical pair, we have a final contradiction, which then completes the proof of 5.1. $\square$

Lemma 5.2. *[Outcome (iv) of lemma 4.10 is out]*

Proof. To be provided. (It is in fact very easy, and very short. Maybe set it inside 4.10.) $\square$

Section 6: The Rudvalis Case

We begin by warning the reader that we shall often write Z for Y_H.

This section is concerned with outcome (iii) of lemma 4.10. We begin with two lemmas on modules for $L_3(2)$.

Lemma 6.1. *Let V be a module for L over $\mathbb{F}_2$, where $L \cong L_3(2)$. Assume that there exists a parabolic subgroup P of L and an element x of V such that $V = \langle x^L \rangle$ and $dim(\langle x^P \rangle) = 2$. Then $\eta(L, V) \leq 2$.*

Proof. Set $S = C_P(x)$. Then S is a Sylow 2-subgroup of L, and so S is contained in a (proper) parabolic subgroup P_1 of L, $P_1 \neq P$. Set $U = \langle x^P \rangle$ and set $U_1 = \langle x^{P_1} \rangle$. Let W be a maximal L-submodule of V.

Observe that $dim(U_1) \leq 3$. Suppose first that $U_1 = \langle x \rangle$. Then $|x^L| = 7$ and so $dim(V) \leq 7$. As any non-trivial L-module has dimension at least 3, we have $\eta(L, V) \leq 2$ in this case. Suppose next that $dim(U_1) > 1$, and let y be an element (possibly 0) of $C_{U_1}(P_1)$. If $y \notin W$ for any choice of W then $V = \langle y^L \rangle$ and, again, we obtain $dim(V) \leq 7$ and $\eta(L, V) \leq 2$. As $dim(C_{U_1}(P_1)) \leq 1$ we may therefore take $C_{U_1}(P_1) \leq W$. Setting $\overline{V} = V/W$, $barV$ is then an irreducible L-module, with $dim(\langle \overline{x}^P \rangle) = dim(\langle \overline{x}^{P_1} \rangle) = 2$. There are only four isomorphism classes of irreducible $\mathbb{F}_2 L$-modules, and the given conditions identify $\overline{V}$ as the adjoint module (which is also the Steinberg module) for L. In particular $\overline{V}$ is projective, so $W = 0$, and $\eta(L, V) = 1$. $\square$

Lemma 6.2. *Take $L = L_3(2)$ and let $W = sl(3, 2)$ be the $\mathbb{F}_2 L$-module consisting of three-by-three matrices of trace zero. Then L has exactly six orbits on the set of non-zero vectors of W. We may denote these orbits by $\mathcal{O}_n$, $n = 2, 3, 4, 6, 7, 8$, where n is the cardinality of $C_L(w_n)$, w_n a representative of $\mathcal{O}_n$. The elements w_n may be taken as* *follows.*

$$w_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad w_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \qquad w_4 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$w_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad w_7 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \qquad w_8 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Proof. To determine the cardinalities of the centralizers, notice that w_3 and w_7 are elements of $L_3(2)$ of order 3 and 7, respectively, and that $w_4 + I$ and $w_8 + I$ are elements of order 4 and 2, respectively. A slight knowledge of centralizers in $L_3(2)$ thus yields $|C_L(w_i)|$ for $i = 3, 4, 7, 8$. For w_2 and w_6, some straightforward matrix computation is called for. One observes that the summation of all the numbers $168/n$, $n = 2, 3, 4, 6, 7, 8$, is equal to $|W| - 1$. $\square$

Lemma 6.3. *Suppose that H is non-solvable and that $N_H(Y_P)$ is not a maximal subgroup of H. Then $q = 2$, $|S| = 2^{14}$, and the following hold.*

(a) *$H/O_2(H) \cong Sym(5)$, and $\widetilde{V}$ is the $\Gamma L(2,4)$ module for $H/O_2(H)$. Further, $C_H(V)/V$ is a trivial module for $O^2(H)$ of order 4, and $O_2(H)/C_H(V)$ is isomorphic to V/Z as modules for H.*

(b) *Let P_1 be the maximal subgroup of H containing S, and set $L = \langle P, P_1 \rangle$. Then $L/O_2(L) \cong L_3(2)$, $\Phi(O_2(L)) = Y_L$, Y_L is a natural $L_3(2)$-module for $L/O_2(L)$, and $O_2(L)/Y_L$ is the adjoint module for $L/O_2(L)$.*

Proof. By 4.10, we have $q = 2$, H_0 is a 2-component of H, $\overline{H}$ is isomorphic to $Sym(5)$, and $V/C_V(H_0)$ is a natural $\Gamma L(2,4)$-module for $\overline{H}$. Thus $N_H(Y_P)$ has index 3 in the maximal subgroup M of H containing $P \cap H$, and $|\langle (Y_P)^M \rangle| = 8$. Observe that $M \in \mathcal{P}$. For this reason we shall henceforth write P_1 for M.

Set $L = \langle P, P_1 \rangle$. Then 1.3(4) shows that $L/C_L(Y_L)$ is isomorphic to $L_3(2)$, or $Sp(4,2)$, with Y_L a natural $L_3(2)$-module or a natural $Sp(4,2)$-module for L. Form the coset graph $\Gamma^* = \Gamma(L, H)$, and set $b^* = b(L, H)$. Take $\alpha = L$ and $\beta = H$.

For the remainder of this section, take $V = V_\beta = \langle (Y_P)^H \rangle$ and set $W = W_\beta = \langle (Y_L)^H \rangle$. Then V_δ and W_δ are defined for any conjugate δ of β in Γ^*. Also, set $U = U\alpha, \beta = \langle (Y_P)^{P_1} \rangle$. Then conjugation defines $U\lambda, \mu$ for any edge $\{\lambda, \mu\}$ of Γ^*.

Notice that since $|V| > |Y_L|$, $Y_L \cap V$ is a proper P_1-invariant subspace of V. Then $Y_L \cap V = U$. Further, as $Y_P \not\leq Z(O_2(H))$ we have also $U \not\leq Z(O_2(H))$. Then $O_2(H)O_2(L)/O_2(L)$ is a non-trivial subgroup of $O_2(P_1)/O_2(L)$, and in the case that $L/C_L(Y_L)$ is isomorphic to $Sp(4,2)$ it follows that $O_2(H)O_2(L)/O_2(L)$ is not the transvection subgroup of order 2 in $O_2(P_1)/O_2(L)$. Thus, in any case we have:

(1) $[O_2(H), O^2(P_1)] \not\leq O_2(L)$.

We now assume that we have:

Case A. $L/C_L(Y_L) \cong Sp(4,2)$**.**

Notice that, in this case, (1) implies that $[Y_L, O_2(H)] = [Y_LV, O_2(H)] = U$. As U is not normal in H it then follows that Y_LV is not normal in H, and so $Y_LV \neq W$. Thus:

(2) $\eta(H, W) \geq 2$.

We also note the following.

(3) There is a unique element element $\bar{t}$ of $O_2(P_1)/O_2(L)$ such that $\bar{t}$ induces a transvection on Y_L. Moreover, we have $[Y_L, \bar{t}] = Y_H$.

Suppose next that $Y_lV \trianglelefteq L$. As $|Y_LV/Y_L| = |V/(Y_L \cap V)| = 4$, we then have $[V, O^2(L)] \leq Y_L$. But evidently $[V, O^2(P_1)] = V \not\leq Y_L$. Thus:

(4) Y_LV is not normal in L, and if $V \leq O_2(L)$ then $\eta(L, \langle V^L \rangle) \geq 2$.

Fix a critical path $(\alpha, \beta, \cdots, \alpha')$ in Γ^*, and suppose first that b^* is odd. Then $b^* \geq 3$, by 1.3(6). [OR MAYBE BY AN ARGUMENT TO BE SUPPLIED LATER].

Suppose that $V_{\alpha'} \leq Q_\beta$. Then $[V_\beta, V_{\alpha'}] \leq Y_\beta$, where Y_β is of order 2. As V/Z is a $\Gamma L(2,4)$-module for H, no element of H acts as a transvction on V/Z, and so $V_\beta \leq Q_{\alpha'}$. Then $[V_\beta, V_{\alpha'}] \leq Y_\beta \cap Y_{\alpha'}$, and so $[V_\beta, V_{\alpha'}] = 1$, by 1.15(b). Then $V_{\alpha'}$ centralizes the hyperplane $Y_\alpha \cap V_\beta$ of Y_α, and so $|V_{\alpha'}/C_{V_{\alpha'}}(Y_\alpha)| \leq 2$. This is contrary to $Y_\alpha \not\leq Q_{\alpha'}$, so we conclude that $V_{\alpha'} \not\leq Q_\beta$. In particular, $(\alpha', \cdots, \beta)$ is a pre-critical path, and so also $V_\beta \not\leq Q_{\alpha'}$, by symmetry.

Notice that $V_\beta \leq Q_{\alpha'-1} \leq O_2(G_{\alpha'-1} \cap G_{\alpha'})$. For any element x of $O_2(P_1) - O_2(H)$ we have $[V, x]Z = U$, so now $[V_\beta, V_{\alpha'}]Y_{\alpha'} = U_{\alpha'-1,\alpha'}$. For the same reason, we have also $[W_\beta, V_{\alpha'}]Y_{\alpha'} = U_{\alpha'-1,\alpha'}$. In view of (2), it follows that $V_{\alpha'}$ acts as a transvection group on W_β/V_β. But $vap \leq O_2(G_\beta \cap G_{\beta+1}) \leq O^2(H)O_2(H)$, and since no element of $SL(2,4)$ induces a transvection on any $SL(2,4)$-module over $\mathbb{F}_2$, we have a contradiction at this point. We conclude that b^* is even, $b^* \geq 2$.

Let $P_0 = (P_1)^g$ be an L-conjugate of P_1 such that $(P_0 \cap P_1)/O_2(L) \cong SL(2,2)$. (The existence of such a conjugate P_0 of P_1 can be read off easily from a picture of the B_2 root system.) One then has $\langle t, P_1 \rangle = L$ for any $t \in O_2(P_0) \vee_2 (L)$. By edge-transitivity on Γ^* we may then fix a vertex $\lambda \in \Delta(\alpha')$ such that:

(5) $\langle G_{\alpha'} \cap G_\lambda, t \rangle = G_{\alpha'}$, for any $t \in O_2(G_{\alpha'-1} \cap G_{\alpha'}) - Q_{\alpha'}$.

We next show:

(6) $Y_\beta = Y_{\alpha'-1}$.

In order to prove (6), suppose first that $[V_\beta, Y_{\alpha'}] \neq 1$. As $yap \leq Q_\beta$ we then have $[V_\beta, Y_{\alpha'}] = Y_\beta$, and so V_β induces a transvection group on $Y_{\alpha'}$. Then (2) yields $[V_\beta, Y_{\alpha'}] = Y_{\alpha'-1}$, and so (6) holds in this case. So assume that $[V_\beta, Y_{\alpha'}] = 1$. Then $C_{Y_\alpha}(Y_{\alpha'}) = Y_\alpha \cap V_\beta$ is a hyperplane of Y_α, whence also $C_{Y_{\alpha'}}(Y_\alpha)$ is a hyperplane of $Y_{\alpha'}$. Then (3) yields $[V_\beta, Y_{\alpha'}] = Y_\beta = Y_{\alpha'-1}$, and (6) holds in any case.

The next step will be to show:

(7) We have $b^* = 2$.

Assume that (7) is false, so that $b^* \geq 4$. Suppose that $[Y_{\beta+1}, W_\lambda] \neq 1$. There is then a critical pair $(\beta+1, \mu)$ with $\mu \in \Delta(\lambda)$. Applying (6) to this critical pair, we obtain $Y_{\beta+2} = Y_\lambda$, and then $C_G(Y_\lambda) \geq \langle Y_\alpha, G_\lambda \rangle$. But (5) then yields $C_G(Y_\lambda) \geq \langle G_{\alpha'}, G_\lambda \rangle$, contrary to $\langle L, H \rangle \notin \mathcal{L}$. We therefore conclude that $[Y_{\beta+1}, W_\lambda] = 1$.

Now $W_\lambda \leq Q_{\beta+1} \leq G_\beta$. Since $Y_{\alpha'} \leq Q_\beta$ we have $|[V_\beta, Y_{\alpha'}]| \leq 2$, and then $|V_\beta/(V_\beta \cap Q_{\alpha'})| \leq 2$. Suppose that $V_\lambda \not\leq Q_\beta$. Then $[V_\beta, V_\lambda]Y_\beta = U_{\beta,\beta+1} \geq [V_\beta, W_\lambda]$. Since $O_2(P_1)$ contains no transvections on W/V, it follows that $V_\beta \cap G_\lambda \leq Q_\lambda$. As $[Y_\alpha, Y_\lambda] \neq 1$, as we have seen, we have $Y_\lambda \not\leq V_\beta$, and then $[V_\beta \cap Q_\lambda, V_\lambda] = 1$. Thus V_λ centralizes a hyperplane of V_β, and so $V_\lambda \leq Q_\beta$.

We now have $[V_\beta, V_\lambda] \leq Y_\beta = Y_{\alpha'-1} \leq Y_{\alpha'}$. If $V_\beta \not\leq G_\lambda$ it then follows from (5) that $Y_{\alpha'}V_\lambda$ is normal in $G_{\alpha'}$, contrary to (4). Thus $V_\beta \leq G_\lambda$, and since $|[V_\beta, V_\lambda]| \leq 2$ we obtain $V_\beta \leq Q_\lambda$ and $[V_\beta, V_\lambda] \leq V_\beta \cap Y_\lambda$. We have already seen that $[Y_\alpha, Y_\lambda] \neq 1$, so in fact $[V_\beta, V_\lambda] = 1$. Then V_λ induces a transvection on Y_α, and $[ya, V_\lambda] = Y_\beta \leq Y_{\alpha'}$. This again yields $Y_{\alpha'}Y_\lambda \trianglelefteq G_{\alpha'}$, contrary to (4), and completing the proof of (7).

Since $[Y_L, O_2(P_1)] \leq V$, we now have $[Y_\alpha, Y_{\alpha'}] \leq Y_\alpha \cap V_\beta \cap Y_{\alpha'}$. But $(Y_\alpha \cap V_\beta)/Y_\beta$ and $(Y_{\alpha'} \cap V_\beta)/Y_\beta$ are two different 1-dimensional $\mathbb{F}_4$-subspaces of V/Z (different since $G_\alpha \cap G_\beta \neq G_\beta \cap G_{\alpha'}$, by maximality of P_1 in H). Therefore $[Y_\alpha, Y_{\alpha'}] = Y_\beta$. This yields:

(8) $[W_\beta, W_b] = Y_\beta$.

Further, $Y_\alpha \cap V_\beta$ is the hyperplane of Y_α which centralizes W_β. The normal closure of $Y_\alpha \cap V_\beta$ in G_β is V_β, so we have:

(9) $[V_\beta, W_\beta] = 1$.

Applying (9) at λ, we have $[Y_{\alpha'}, V_\lambda] = 1$, so $V_\lambda \leq Q_{\alpha'} \leq G_\beta$. Suppose $V_\lambda \leq Q_\beta$. Then $[V_\beta, V_\lambda] \leq Y_\beta$, and since $V_\beta \leq Q_{\alpha'} \leq G_\lambda$ we get $V_\beta \leq Q_\lambda$ and $[V_\beta, vl] \leq Y_\beta \cap Y_\lambda = 1$. As in the proof of (7), we then have $[Y_\alpha, V_\lambda] = Y_\beta$ and $Y_{\alpha'}V_\lambda \trianglelefteq G_{\alpha'}$, contrary to (4). We therefore conclude that $V_\lambda \not\leq Q_\beta$.

As $b^* = 2$ we have $W_\lambda \not\leq Q_{\alpha'}$. Let $t \in W_\lambda - Q_{\alpha'}$. Then $[V_\beta, (V_\beta)^t] \leq V_\beta \cap (V_\beta)^t \cap Y_{\alpha'} = (V_\beta \cap Y_{\alpha'}) \cap (V_\beta \cap Y_{\alpha'})^t$. By our choice of λ in (5), we have $O_2(G_{\alpha'} \cap G_\lambda) \cap G_\beta = Q_{\alpha'}$, so $t \notin G_\beta$. As t induces a transvection on $V_{\alpha'}$, by (8), we then have $Y_{\alpha'} = C_{Y_{\alpha'}}(t)Y_\beta$ and $(V_\beta \cap Y_{\alpha'}) \cap (V_\beta \cap Y_{\alpha'})^t = C_{V_\beta \cap Y_{\alpha'}}(t)$ is of order 4. Then $C_{V_\beta \cap Y_{\alpha'}}(t) = [V_\beta, V_\lambda] \leq V_\lambda$, and so we have shown that $[V_\beta, (V_\beta)^t] \leq V_\lambda$. But $[W_\lambda, V_\beta] = [(W_\lambda \cap G_\beta)\langle t \rangle, V_\beta] \leq V_\beta(V_\beta)^t$, and so $[W_\lambda, V_\beta, V_\beta] \leq [V_\beta, (V_\beta)^t] \leq V_\lambda$. As $|[V_\beta, V_\lambda]| > 2$ we have $V_\beta \not\leq Q_\lambda$, and we have shown that V_β acts quadratically on W_λ/V_λ.

Set $W_0 = W_\lambda \cap Q_{\alpha'}$. Then $|W_\lambda/W_0| = 2$ and we have $[V_\beta, W_0] \leq U_{\beta,\alpha'} = [V_\beta, V_\lambda]Y_\beta$. Setting $\widehat{W_\lambda} = W_\lambda/V_\lambda$, we then see that V_β acts on $\widehat{W_\lambda}$, with $|[\widehat{W}_0, V_\beta]| \leq 2$. It follows that $\widehat{W_\lambda}$ involves no natural $\Gamma L(2,4)$-module for $G_{\alpha'}$, and then quadratic action implies that $|V_\beta/C_{V_\beta}(\widehat{W_\lambda})| = 2$. Thus $V_\beta \cap Q_\lambda$ is a hyperplane of V_β. We have seen that $[V_\beta, V_\lambda] = V_\beta \cap Y_{\alpha'} \cap V_\lambda$ is of order 4, so $Y_\beta \not\leq [V_\beta, V_\lambda]$ and $Y_\lambda \not\leq [V_\beta, V_\lambda]$. Then

$[V_\beta \cap Q_\lambda, V_\lambda] = 1$, which yields $V_\lambda \leq Q_\beta$ and a final contradiction in Case A. Thus, we are reduced to

Case B: $L/C_L(Y_L) \cong L_3(2)$**.**

We may begin by observing that in this case we have $U = Y_L \leq V = W$. In particular, W is abelian, and so $b^* \geq 3$.

Suppose that b^* is even. As $V_\beta \leq Q_{\alpha'-1}$ and $V_{\alpha'-1} \leq Q_\beta$, we have $[V_\beta, V_{\alpha'-1}] = [Y_\alpha, Y_{\alpha'}] = Y_\beta = Y_{\alpha'-1}$. As $C_L(Z) = P_1 \leq H$ we have $Y_\beta \neq Y_{\beta+2}$, and so $b* \geq 6$.

Set $X = \langle (V_\beta)^{G_\alpha} \rangle$. As V is not normal in L we have $X \neq V_\beta$ and so $\eta(G_\alpha, X) > 1$. Now $X \leq Q_{\alpha'-3} \leq G_{\alpha'-2}$, and $[X, Y_{\alpha'-1}] = [X, Y_\beta] = 1$. It follows that $X \leq G_{\alpha'-1}$. If $X \leq Q_{\alpha'-1}$ then $[X, V_{\alpha'-1}] \leq Y_\beta \leq Y_\alpha$, contrary to $\eta(G_\alpha, X) > 1$. Thus $X \not\leq Q_{\alpha'-1}$. But $X \leq O_2(G_{\alpha'-1} \cap G_{\alpha'-1}) \leq O^2(G_{\alpha'-1})O_2(G_{\alpha'-1})$, and since Sylow 2-subgroups of $O^2(\overline{H})$ are TI-sets it follows that $X \not\leq G_{\alpha'}$.

Next, observe that $V_\lambda \leq Q_{\beta+2} \leq G_{\beta+1}$, and $[Y_\beta, V_\lambda] = [Y_{\alpha'-1}, V_\lambda] = 1$, so $V_\lambda \leq G_\beta$. If $V_\lambda \leq Q_\beta$ then $[V_\lambda, V_\beta] \leq Y_\beta \leq Y_{\alpha'} \leq V_\lambda$, and then $V_\lambda \trianglelefteq \langle V_\beta, G_{\alpha'} \cap G_\lambda \rangle = G_{\alpha'}$, whereas V is not normal in L. Thus $V_\lambda \not\leq Q_\beta$, and it follows that $C_{V_\beta}(V_\lambda) = Y_{\beta+1}$, of index 4 in V_β.

As $L/O_2(L) \cong L_3(2)$ we have $O_2(P_1) \cap (P_1)^g \not\leq O_2(L)$ for any $g \in L$. As $V_\beta \leq O_2(G_{\alpha'-1} \cap G_{\alpha'}$, and as $vb \not\leq G_\lambda$ by definition of λ, we then have $|V_\beta/(V_\beta \cap G_\lambda)| = 2$. Now $[V_\beta \cap Q_\lambda, V_\lambda] \leq V_\beta \cap Y_\lambda$. But yl is not centralized by $G_{\alpha'}$, so $[V_\beta, Y_\lambda] \neq 1$, and so $[V_\beta \cap Q_\lambda, V_\lambda] = 1$. Thus $V_\beta \cap Q_\lambda = C_{V_\beta}(V_\lambda)$ which, we have seen, has index 4 in V_β. We may then conclude that $V_\beta \cap G_\lambda \not\leq Q_\lambda$. This yields $[V_\beta \cap G_\lambda, V_\lambda]Y_\lambda = Y_{\alpha'}$. As $Y_{\alpha'-2} \cap Y_{\alpha'} = Y_{\alpha'-1}$, of order 2, we have $[V_\beta \cap G_\lambda, V_\lambda] \not\leq Y_{\alpha'-2}$. Now X centralizes $Y_{\alpha'-2}$ and, as $b^* \geq 6$, X centralizes V_β. As no element of H induces a transvection on V/Z we have the desired contradiction at this point. That is:

(10) b^* is odd.

[AS BEFORE, ASSUME FROM GLOBAL HYPOTHESIS or 1.3(6) THAT $b^* \neq 1$.]

As V_β does not induce a transvection on $V_{\alpha'}/Y_{\alpha'}$ we have $V_{\alpha'} \not\leq Q_\beta$, and since $V_{\alpha'} \leq O^2(G_\beta)O_2(G_\beta)$ we get also $V_{\alpha'} \not\leq G_\alpha$.

Suppose that $Y_{\alpha'} \leq V_\beta$. As $Y_{\alpha'-1} = [V_\beta, V_{\alpha'}]Y_{\alpha'}$, we then have $Y_{\alpha'-1} \leq V_\beta$. Set $X = X_\alpha = \langle (V_\beta)^{G_\alpha} \rangle$, and assume now that $b^* \geq 5$. Then $X \leq Q_{\alpha'-3} \leq G_{\alpha'-2}$, and X centralizes $Y_{\alpha'-3}Y_{\alpha'-1} = V_{\alpha'-2}$. We then have $X \leq Q_{\alpha'-1} \leq G_{\alpha'}$, and so $[X, V_{\alpha'}] \leq Y_{\alpha'-1}V_\beta \leq X$. Thus X is normal in $\langle G_\alpha, V_{\alpha'} \rangle$, and since $V_{\alpha'} \not\leq G_\alpha$ we get X normal in G_β, contrary to $\langle L, H \rangle \notin \mathcal{L}$. We may now conclude that $yap \not\leq V_\beta$. As also $(\alpha', \cdots, \beta)$ is a pre-critical path, we obtain also $Y_\beta \not\leq V_{\alpha'}$, by symmetry. Then $|V_{\alpha'}/(V_{\alpha'} \cap Q_\beta)| = |V_\beta/(V_\beta \cap Q_{\alpha'})| = 4$.

Set $D = \langle (X)^{G_\beta} \rangle$. Then $D \leq Q_\beta$ and indeed $[D, V_\beta] = 1$. Set $F = [V_\beta, V_{\alpha'}]$. Then F is a complement to $Y_{\alpha'}$ in $Y_{\alpha'-1}$, contained in $C_{V_{\alpha'-2}}(D)$. As D does not induce a transvection on $V_{\alpha'-2}/Y_{\alpha'-2}$ we get $D \leq Q_{\alpha'-2} \leq G_{\alpha'-1}$. As $D \leq O_2(G_{\alpha'-2} \cap G_{\alpha'-1})$, we have $|D/(D \cap G_{\alpha'})| \leq 2$. As X is not normal in G_β we have $\eta(G_\beta, X) \geq 2$, and then $|D/C_D(V_{\alpha'})| \geq 16$, and $|[D, V_\lambda]| \geq 16$. Then $D \not\leq G_{\alpha'}$. Suppose that D is abelian. Then

$Y_{\alpha'} \not\leq D$, so that $D \cap Q_{\alpha'} \leq C_D(V_{\alpha'})$, and then since a Sylow 2-subgroup of $G_{\alpha'}/Q_{\alpha'}$ is non-abelian of order 8 we get $|D/C_D(V_{\alpha'})| < 16$. This contradiction shows that D is non-abelian, and so $b^* = 5$. Then $F \leq Y_{\beta+1} \cap Y_{\beta+3} = Y_{\beta+2}$, which is contrary to $|F| = 4$. Thus, we have shown:

(11) $b^* = 3$.

Set $\gamma = \beta + 1$, so that also $\gamma = \alpha' - 1$. Here $V_{\alpha'} \not\leq Q_\beta$ since V_β is not a transvection group on $V_{\alpha'}/Y_{\alpha'}$.

We have

$$V_\beta \cap V_{\alpha'} \geq Y_\gamma[V_\beta, V_{\alpha'}] \trianglelefteq \langle Q_\beta, Q_{\alpha'} \rangle Q_\gamma = G_\gamma.$$

Suppose that $|V_\beta| > 32$. That is, suppose that there are non-trivial fixed points for H on V/Z. Since $V = \langle (Y_L)^H \rangle$ where $Y_L = [Y_L, P_1]$, it then follows that $|C_{V/Z}(H_0)| = 2$ or 4, and that $V = [V, H]$. Thus, V/Z is an indecomposable module for H_0, and we have $|V/[V, O_2(P_1)]| = |V/[V, x]Y_L| = 4$ for any element x of $O_2(P_1) - O_2(H)$. Thus $|V_\beta/[V_\beta, V_{\alpha'}]Y_\gamma| = 4$, and since $[V_\beta, V_{\alpha'}]Y_\gamma \leq V_{\alpha'}$ it follows that Y_γ is a proper subgroup of $V_\beta \cap V_\gamma$. [SHORTER ARGUMENT FOR THIS ?] Let $y \in V_\beta \cap V_{\alpha'}$ with $[y, O^2(G_\beta)] = 1$ and with $Y_\gamma\langle y \rangle$ normal in $G_\beta \cap G_\gamma$. Then $Y_\gamma\langle y \rangle$ is normal in $\langle Q_\beta, Q_{\alpha'} \langle Q_\gamma = G_\gamma$, and y centralizes $[Q_\gamma, O^2(G_\beta \cap G_\gamma)]$. But the commutator map defines a pairing of $\langle y \rangle \times O_2(H)/C_{O_2(H)}(y)$ onto Y_L, as $y \notin Y_L = \Omega_1(Z(O_2(L)))$, and so y does not centralize any conjugate of $O^2(P_1)$ in L. This contradiction shows that, in fact, we have $|V| = 32$, and V/Z is irreducible for H.

Observe that $|Q_\beta/(Q_\beta \cap Q_\gamma| \leq 4$, and that $[Q_\beta \cap Q_\gamma, V_{\alpha'}] \leq [Q_\gamma, V_{\alpha'}] \leq Y_\gamma \leq V_\beta$. Since no element of $Alt(5)$ induces a transvection on any $\mathbb{F}_2$-module for $Alt(5)$, it follows that $\eta(H, O_2(H)/V) \leq 1$. Set $Q = O_2(H)$ and set $R = C_Q(V)$. If $[Q, O^2(H)] \leq R$ then $V \leq Z(Q)$ by the Thompson $P \times Q$ Lemma, whereas $[V, O_2(H)] \geq [Y_L, O_2(H)] = Z$, by (1). Thus $\eta(H, O_2(H)/V) = 1$, and $[R, O^2(H)] = V$. The commutator map defines a pairing of $Q/R \times V/Z$ onto Z, so Q/R is isomorphic to the dual of V/Z as a module for H. We record the results so far as follows.

(12) Both V/Z and $O_2(H)/C_H(V)$ are natural $\Gamma L(2,4)$-modules for $H/O_2(H)$, and $[C_H(V), O^2(H)] = V$.

For any vertex δ of Γ and for any positive integer n, define $G_\delta^{(n)}$ to be the point-wise stabilizer in G_δ of all the vertices at distance at most n from δ. Observe that $Y_\alpha \cap G_\alpha^{(4)} = 1$ as $b^* = 3$. As $Y_\alpha = Y_L = \Omega_1(Z(O_2(L)))$, we then have $G_\alpha^{(4)} = 1$. In a similar vein, observe that since $[Q_\alpha, V_\beta, Q_\alpha] \leq [Y_\alpha, Q_\alpha] = 1$, we have $[Q_\alpha, Q_\alpha] \leq C_{Q_\alpha}(X_\alpha)$ by the Three Subgroups Lemma, and so $[Q_\alpha, Q_\alpha] \leq G_\alpha^{(3)}$. We note also that $C_{G_\beta}(V_\beta) = G_\beta^{(2)}$.

Let us write Q_L for $O_2(L)$, and Q_H for $O_2(H)$. Write $X = X_\alpha$. Then $Q_L = (Q_L \cap Q_H)X$. Now $(O_2(L) \cap O_2(H))/C_H(V)$ is a 1-dimensional $\mathbb{F}_4$-subspace of $Q_H/C_H(V)$ as seen in (12). As $Q_H X = O_2(P_1)$ it then follows that $Q_L \cap Q_H = (X \cap Q_H)C_H(V)$. Thus

$$Q_\delta = (Q_\alpha \cap Q_\delta)X = G_\delta^{(2)} X$$

for all $\delta \in \Delta(\alpha)$, and we have the following result.

(13) $Q_\alpha = \cap_{\delta\in\Delta(\alpha)} C_{Q_\alpha}(V_\delta)X$

We now have

$$[G_\alpha^{(3)}, Q_\alpha] = \bigcap_{\delta\in\Delta(\alpha)} [G_\alpha^{(3)}, C_{Q_\delta}(V_\delta)X] = \bigcap_{\delta\in\Delta(\alpha)} [G_\alpha^{(3)}, G_\delta^{(2)}]$$

$$\leq \bigcap_{\delta\in\Delta(\alpha)} [G_\delta^{(2)}, G_\delta^{(2)}] \leq \bigcap_{\delta\in\Delta(\alpha)} G_\delta^{(3)} \leq G_\alpha^{(4)} = 1.$$

This shows:

(14) We have $G_\alpha^{(3)} = Y_\alpha$.

Suppose next that there exists an element $h \in C_H(O^2(H)) - Z$. We may then choose h so that $[H, h] = Z$. By (12) we have $h \in C_H(V)$, and then $[h, X] \leq [O_2(L), X] \leq Y_L$. As $C_{Y_L}(O^2(H)) = Z$ we then have $[h, X] \leq Z$. For any $\mu \in \Delta(\alpha)$ we have $h \in G_\mu$, and since no element of G_μ induces a transvection on V_μ/Y_μ we further have $h \in Q_\mu$. But then $[V_\mu, h] \leq Y_\mu \neq Y_\alpha = Z$, and so $[V_\mu, h] = 1$. Thus $h \in G_\alpha^{(3)}$, and so $h \in Y_\alpha$ by (14). As $C_{Y_L}(O^2(H)) = Z$ we have a contradiction. This proves:

(15) $C_H(O^2(H)) = Z$.

Here is a further consequence of (13). Namely, we have

$$[Q_\alpha, Q_\beta] \leq [G_\beta^{(2)}X, Q_\beta] \leq [G_\beta^{(2)}, Q_\beta]X = [C_H(V), Q_H]X.$$

But (15) implies that $C_H(V)/Z$ is abelian, so $C_H(V)/V$ is abelian, and we then have $[C_H(V), Q_H] \leq V$, by (12). We then conclude that $[Q_\alpha, Q_\beta] \leq X$, and so $X \geq [Q_L, O^2(L)]$. As $X = \langle V^L \rangle$, where $V = [V, O^2(P_1)]$, we then have $X = [Q_L, O^2(L)]$. On the other hand, 6.1 shows that $\eta(L, X/Y_L) \leq 2$. If X involves no Steinberg module for $L/O_2(L)$ it follows that $\eta(P_1, X/Y_L) = 2$ and that $\eta(P_1, O_2(P_1)) = 4$. But it is evident from the structure of H that $\eta(P_1, O_2(P_1)) = 5$. Thus X has a constituent W which is a Steinberg module for $L/O_2(L)$. Then $\eta(P_1, W) = 3$ and it follows that W is the unique non-trivial constituent for L in X/Y_L. Projectivity of W now yields $W = X/Y_L$. As $X = [Q_L, O^2(L)]$ it follows that also $Q_L = C_{Q_L}(X)X$. But $C_{Q_L}(X) = G_\alpha^{(3)} \leq X$ by (14). Thus:

(16) Q_L/Y_L is a Steinberg module for $L/O_2(L)$, and $|S| = 2^{14}$.

Finally, we observe from $|S|$ that $|C_H(V)/V| = 4$. This completes the proof of lemma 6.3. $\square$

Lemma 6.4. *Set $W = Q_L/Y_L$, identify W with $sl(3,2)$, and identify L/Q_L with $L_3(2)$, with action on W given by matrix conjugation. Let the orbits for L on $W - \{0\}$ be given as in 6.2. For any $x \in Q_L$, write $\widehat{x}$ for the image of x in W. Then $|x| \leq 2$ if $x \in Y_L$ or if $\widehat{x} \in \mathcal{O}_7 \cup \mathcal{O}_8$, and otherwise $|x| = 4$.*

Proof. We have maps $\mathcal{S} : W \longrightarrow Y_L$ and $\mathcal{B} : W \times W \longrightarrow Y_L$, given by squaring and commutation, respectively. Then, for any x and y in Q_L, we have

$$\mathcal{S}(\widehat{x} + \widehat{y}) = \mathcal{S}(\widehat{x}) + \mathcal{S}(\widehat{y}) + \mathcal{B}(\widehat{x}, \widehat{y}). \tag{1}$$

We shall also identify Y_L with a natural $SL(3,2)$-module, written additively.

Let $t \in V - Y_L$ with $[t, S] \leq Y_L$. Then $\langle \widehat{t} \rangle = C_W(S)$. We take S to induce conjugation by upper triangular matrices on W. Adopting the notation of 6.2, we then have $\widehat{t} = w_8$. As $\Phi(V) = 1$, and as $\mathcal{S}$ is constant on L-orbits, we see that $\mathcal{S}$ is trivial on $\mathcal{O}_8$.

As $\mathcal{S}(w_7)$ is invariant under $C_L(w_7)$ (which is of order 7) it follows that $\mathcal{S}(w_7) = (0,0,0)$, and thus $\mathcal{S}$ is trivial on $\mathcal{O}_7$.

Denote by $\widehat{t}'$ the transpose of $\widehat{t}$. The set of all matrices in W whose lower-left entry is 0 is the unique maximal S-invariant subspace of W, and it therefore contains $C_{Q_L}(t)/Y_L$. It follows that $\mathcal{B}(\widehat{t}, \widehat{t}') \neq (0,0,0)$. Then $\mathcal{S}(w_2) = \mathcal{S}(\widehat{t} + \widehat{t}') \neq (0,0,0)$, by (1). Thus, $\mathcal{S}$ is non-trivial on $\mathcal{O}_2$.

Next, observe that $w_3 + \widehat{t} \in \mathcal{O}_7$. (To see this, either check that, as an invertible matrix, $w_3 + \widehat{t}$ is of order 7, or observe that conjugation of $w_3 + \widehat{t}$ by w_3 yields w_7.) Then $(0,0,0) = \mathcal{S}(w_3 + \widehat{t}) = \mathcal{S}(w_3) + \mathcal{B}(w_3, \widehat{t})$. But evidently w_3 does not lie in the unique maximal S-invariant subspace of W, so $\mathcal{B}(w_3, \widehat{t}) \neq (0,0,0)$, and so $\mathcal{S}$ is non-trivial on $\mathcal{O}_3$. Further, as $\mathcal{S}(w_3)$ is invariant under $C_L(w_3)$ we obtain $\mathcal{S}(w_3) = (1,1,1)$. For the same reason, we have $\mathcal{S}(w_3') = (1,1,1)$, where w_3' is the transpose of w_3.

We next check that $w_3' + w_6 \in \mathcal{O}_7$. (Indeed, one observes that $w_3' + w_6$ is the square of the matrix w_7.) Suppose that $\mathcal{S}(w_6) = (0,0,0)$. Then $(0,0,0) = \mathcal{S}(w_3' + w_6) = \mathcal{S}(w_3') + \mathcal{B}(w_3', w_6)$, and thus $\mathcal{B}(w_3', w_6) = (1,1,1)$. Thus $\mathcal{B}(w_3', w_6)$ is invariant under any element g of L whose matrix for the L-action on W is given by w_3'. The matrix of g^2 is then w_3, and setting $u = (w_6)^g$ and $v = (w_6)^{g^2}$, we have $w_6 = u + v$. Then

$$\begin{aligned} (1,1,1) &= \mathcal{B}(w_3', w_6) = \mathcal{B}(w_3', u + v) = \mathcal{B}(w_3', u) + \mathcal{B}(w_3', v) \\ &= \mathcal{B}(w_3', w_6)^g + \mathcal{B}(w_3', w_6)^{g^2} = (1,1,1) + (1,1,1) = (0,0,0). \end{aligned}$$

This contradiction shows that, in fact, $\mathcal{S}$ is non-trivial on $\mathcal{O}_6$.

It only remains to treat $\mathcal{O}_4$. For this, define matrices y and g as follows.

$$y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix} \qquad g = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Then $y^g = w_2'$, and we have $w_4 = y + w_7$. Further, we have

$$(w_7)^g = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

and hence $\mathcal{B}(y, w_7) \neq (0,0,0)$. Now $\mathcal{S}(w_4) = \mathcal{S}(y+w_7) = \mathcal{B}(y, w_7)$, and so $\mathcal{S}$ is non-trivial on $\mathcal{O}_4$. $\square$

Corollary 6.5. *Set $R = C_H(V)$, and let x be an element of order 3 in P_1. Then $R = C_R(x)V$, and $C_R(x)$ is a quaternion group.*

Proof. As V is a natural $\Gamma L(2,4)$-module for H/Q_H, we have $V = [V, x]Z$. By 6.3(a), R/V centralizes $O^2(H)$, so $R = C_R(x)V$ with $C_R(x) \cap V = Z$, and with $|C_R(x)| = 8$. We have $R \leq Q_L$ since $P_1/Q_L \cong Sym(4)$. As $C_R(x) - Y_L = C_R(x) - Z$, it now follows from 6.4 that $C_R(x) - Y_H$ consists entirely of elements of order 4, and hence that $C_R(x)$ is a quaternion group. $\square$

Proposition 6.6. *Both L and H are maximal 2-local subgroups of G.*

Proof. Let L^* be a 2-local subgroup of G containing L. Suppose first that Q_L properly contains $O_2(L^*)$. Then $O_2(L^*) = Y_L$ and thus $C_L(O_2(L^*)) \not\leq O_2(L^*)$, contrary to the Main Hypothesis concerning $\mathcal{L}(S)$. Thus $O_2(L^*) = Q_L$. But then $L^* = C_{L^*}(Y_L)L$, and $C_{L^*}(Y_L)/Q_L$ is of odd order, centralized by $\langle Q^L \rangle$ (where we recall that Q denotes $O_2(\widetilde{C})$). Thus, we have $L^*/Q_L = L/Q_L) \times O_{2,2'}(L)/Q_L$. As L acts irreducibly on Q_L/Y_L, it then follows that $L^* = L$.

Now let H^* be a 2-local subgroup of G containing H. Suppose first that $O_2(H^*)$ does not contain $[Q_H, O^2(H)]$. Then $O_2(H^*) \leq C_H(V)$, and since H^* is 2-constrained we have V properly contained in $O_2(H^*)$. The preceeding lemma then shows that $V = \Omega_1(Z(O_2(H^*)))$ and that $Z = \Phi(O_2(H^*))$. Then $H^* = C_{H^*}(V)H$, as $Sym(5)$ is maximal among subgroups of $L_4(2)$ having a Sylow 2-subgroup of order 8. Now $C_{H^*}(V) \leq C_G(Y_L) \leq L$, by what has already been proved, and then $C_{H^*}(V) \leq P_1 \leq H$. Thus, we may assume that $[Q_H, O^2(H)] \leq O_2(H^*)$. But this yields $V = Z(O_2(H^*))$, and $Z = [V, O_2(H^*)]$, with the result, as just argued, that $H = H^*$. $\square$

[NOW COMES THE HARD PART: IDENTIFYING Ru. I have some notes which yield $|Ru|$. Better yet, I have Michael's notes which do the same thing, with a clever idea for avoiding some of the computation required for a strict applicaion of the Thompson order formula. He also tells how to get an amalgam for ${}^2F_4(2)$ from (H, L), and with $|G|$ (and much of the local structure for odd primes) in hand, one can obtain ${}^2F_4(2)$ using a result of Bennett and Shpectorov. At that point, he produces Ru as a rank-3 permutation group.]

Section 7: The Solvable Case, $p = 2$, Part I

In this section we consider the case where $p = 2$ and where both P and H are solvable. In particular, we then have $q = 2$, so that $H = \widetilde{P}$. Our treatment may be viewed as a re-working of parts of Stellmacher's N-Group Paper [St2] (specifically, sections 9 and 10). In the case $b = 3$, we will stick closely to the original (section 10).

This section and the next will be devoted to the proof of the following result.

Theorem 7.0. *Assume that $b \geq 3$, $q = 2$, and $H = \widetilde{P}$ is solvable. Then $(P, \widetilde{P})$ is a weak BN-pair. Moreover, we have $b = 3$, and $\widetilde{P}/O_2(\widetilde{P})$ is isomorphic to either $SL(2,2)$ or $Sz(2)$.*

Remark: From the data given by the Theorem 7.1, we will recover the amalgam for M_{12} or $Aut(M_{12})$ (when $\widetilde{P}/O_2(\widetilde{P})$ is isomorphic to $SL(2,2)$), or the amalgam for the Tits group or for ${}^2F_4(2)$ (when $\widetilde{P}/O_2(\widetilde{P})$ is isomorphic to $Sz(2)$). The actual construction of these groups from the local data promises to be an interesting exercise. Or not: it being likely that one has only to copy the relevant portions of the Aschbacher-Smith Opus.

Lemma 7.1. *Denote by $\mathcal{T}$ the set of elements t of S such that $|\widetilde{V}/C_{\widetilde{V}}(t)| = 2$. Let $\mathcal{T}_0$ be a subset of $\mathcal{T}$, and put $T_0 = \langle \mathcal{T}_0 \rangle$. Suppose that $Y_P \leq [V, T_0]Y_H$. Then $C_V(O^2(H)) = Y_H$ and $T_0 Q_H = \langle \mathcal{T} \rangle Q_H$.*

Proof. By assumption we have $[V, T_0] \nleq Y_H$, so $T_0 \nleq Q_H$. Set $T = \langle \mathcal{T} \rangle$. Then 3.11 implies that $O^2(\overline{H})\overline{T}$ is a direct product of groups $\overline{K}_i$, $1 \leq i \leq r$, where $\overline{K}_i \cong SL(2,2)$ and where $|[\widetilde{V}, \overline{K}_i]| = 4$ for all i. It follows that $[\widetilde{V}, T] \leq [\widetilde{V}, O^2(H)]$, and thus $Y_P \leq [V, O^2(H)]Y_H$, by assumption. This shows that $C_V(O^2(H)) = Y_H$ and that $\widetilde{V}$ is the direct product of the fours groups $[\widetilde{V}, \overline{K}_i]$, $1 \leq i \leq r$. Suppose that $\overline{\mathcal{T}_0}$ is a proper subset of $\overline{\mathcal{T}}$. Then $[\widetilde{V}, \overline{T}_0]$ centralizes at least one of the groups $\overline{K}_i$. As $\overline{H} = \langle \overline{K}_i, \overline{S} \rangle$ for any i, we then have $Y_P \trianglelefteq H$, which one recognizes as a contradiction, and which proves the lemma. $\square$

Lemma 7.2. *Suppose that there exists an element t of $[Q_P, O^2(P)]O_2(H)$ such that $|\widetilde{V}/C_{\widetilde{V}}(t)| = 2$. Then $C_H(\widetilde{V}) = O_2(H)$.*

Proof. Set $N = C_H(\widetilde{V})$. As $H \in \mathcal{P}$ (or, more generally, by 1.11(b)) Q_H is a Sylow 2-subgroup of N. As N normalizes Y_P, 1.7 implies that $N \leq N_G(O^2(P))$. Then $[N, t]$ is a 2-group, and thus $[N, t] \leq Q_H$. Set $\widehat{H} = H/Q_H$.

Define subgroups $\overline{K}_i$ of $\overline{H}$, $1 \leq i \leq s$, as in the proof of Lemma 7.1. Then $\overline{K}_i = \langle \bar{t}_i, \bar{a}_i \rangle$, where $|\bar{a}_i| = 3$ and where $\bar{t}_i \in \overline{S}$. As $H \in \mathcal{P}$, $\overline{S}$ acts transitively on $\{\overline{K}_i\}_{1 \leq i \leq s}$. Then each $\bar{t}_i$ is the image in $\overline{H}$ of a conjugate t_i of t. Set $X_i = [O^2(H), t_i]$. As $[N, t_i] \leq Q_H$, it follows that $|\widehat{X}_i| = 3$. In particular, we have $[\widehat{X}_i, \widehat{X}_j] = 1$ for all i and j. Setting $T = \langle t_1, \cdots, t_s \rangle$, we now have $[O^2(\widehat{H}), \widehat{T}]$ elementary abelian of order 3^s, and where $1 \neq \widehat{T} \trianglelefteq \widehat{S}$. But $H \in \mathcal{P}$, so $[O^2(\widehat{H}), \widehat{T}] = O^2(\widehat{H})$. This completes the proof that $N = Q_H$. $\square$

Lemma 7.3. *Let (ρ, σ, τ) be a path in Γ with σ of type P. Set $D = V_\rho \cap V_\tau$, $T = Q_\rho \cap Q_\sigma$, and assume:*

(i) *$|V_\tau/D| = 2$, and*
(ii) *there exists $t \in T - Q_\tau$ with $[D, t] = 1$.*

Then $H/O_2(H) \cong SL(2,2)$, and $|V| = 8$.

Proof. We take $S = G_\sigma \cap G_\tau$, and set $\overline{G}_\tau = G_\tau/C_{G_\tau}(V_\tau/Y_\tau)$. Thus, we have $\overline{S} \cong S/Q_\tau$. As $S = Q_\sigma Q_\tau$ and $T \trianglelefteq Q_\sigma$, we then have $\overline{T} \trianglelefteq \overline{Q}_\sigma = \overline{S}$.

We have $[D,t]=1$, by assumption, and thus t induces a transvection on V_τ/Y_τ. Proposition 3.11 then yields the structure of V_τ/Y_τ as a module for the group $X=\langle t^{G_\tau}\rangle$. Thus, $\overline{X}$ is a direct product of s copies of $Sym(3)$ and, setting $\widetilde{V}=V_\tau/Y_\tau$, we have $\widetilde{V}=C_{\widetilde{V}}(O^2(X))\times V_1\times\cdots\times V_s$, where each V_i is an X-invariant fours group. As $t\in T$, $\overline{T}$ contains all the transvections in $\overline{S}$. As $[D,T]\le Y_\rho$, of order 2, it follows that $s\le 2$. But also $\widetilde{D}$ is $\overline{S}$-invariant, as $D\trianglelefteq Q_\sigma$. If $s=2$ (so that $\overline{S}$ is dihedral of order 8) this condition uniquely determines the hyperplane $\widetilde{D}$ of $\widetilde{V}$, and one sees that in this case $|[D,T]|>2$. Thus $s=1$, and $8\le|V_\sigma|\le 16$.

Set $W=C_{V_\tau}(O^2(G_\tau))$, and suppose that $W\nleq D$. Then $V_\tau=WD$, and so $[\widetilde{V},T]=\widetilde{Y}_\rho$. In this case, Lemma 7.1 yields $|V|=8$. On the other hand, suppose that $W\le D$, and suppose further that $W\neq Y_\sigma$. Set $W_1=C_{V_\rho}(O^2(G_\rho))$. We then have $D=WW_1$ and $W\cap W_1\neq 1$. But $O_2(O^2(G_\tau))\nleq Q_\sigma$, as $Q_\tau\cap Q_\sigma$ is not normal in G_ρ. Then $[W\cap W_1,O^2(G_\sigma)]=1$, which is contrary to $Z(P)=1$. This contradiction now shows that $|V|=8$ in any case. Now $P/Q_P\cong SL(2,2)$, by Lemma 7.2. □

Lemma 7.4. *Suppose that b is even. Then $|V|=8$.*

Proof. Fix a critical pair (α,α') and a path $\pi=(\alpha,\beta,\beta+1,\beta+2,\cdots,\alpha'-1,\alpha')$ of length b. Fix $\lambda\in\Delta(\alpha')-\{\alpha'-1\}$. As b is even, we have $Y_\beta=[Y_\alpha,Y_{\alpha'}]=[Y_\alpha,Y_\lambda]=Y_{\alpha'-1}$. As $Y_\beta\neq Y_{\beta+2}$, we then have $b\ge 6$. In particular, it then follows that $[V_{\beta+2},Y_\alpha]=1$.

It will be convenient to establish, and to list, (and to prove) the following facts.

(1) $[V_{\beta+2},V_\lambda]=1$.

(2) $V_\lambda\le G_\beta$ and $V_\lambda\nleq Q_\beta$.

In order to prove (1) and (2), notice that we have $[Y_\alpha,Y_\lambda]\neq 1$. As $V_{\beta+2}\le Q_\lambda$, we have $[V_{\beta+2},V_\lambda]\le Y_\lambda$, while also $[V_{\beta+2},Y_\alpha]=1$. This yields (1). Then $[Y_\beta,V_\lambda]=1$, and so $V_\lambda\le G_\beta$. Supposing $V_\lambda\le Q_\beta$, we obtain $[Y_\alpha,V_\lambda]=[Y_\alpha,Y_\lambda]=Y_{\alpha'-1}\le V_\lambda$, and so $V_\lambda\trianglelefteq\langle Y_\alpha,G_\lambda\rangle=\langle G_{\alpha'},G_\lambda\rangle$, which contradicts our basic hypothesis that $\langle P,H\rangle\notin\mathcal{L}$. Thus (2) holds.

By Lemma 1.2 in [Bernd's N-Groups] there exists $g\in V_\beta-G_\lambda$ with $|V_\lambda/C_{V_\lambda}(g)|\le 4$. Set $D=C_{V_\lambda}(g)Y_\lambda$. Then $D\trianglelefteq\langle g,Q_\lambda\rangle$. Thus D is invariant under $O^2(G_{\alpha'})$, and so $D\le V_{\alpha'-1}\cap V_\lambda$. Thus $|V_\lambda/V_{\alpha'-1}\cap V_\lambda|=2$, and so also $|V_\beta/V_\beta\cap V_{\beta+2}|=2$. Now (1) and (2) and Lemma 7.3 together yield $|V|=8$. □

We now take up the case where b is odd.

Reminder. *For any vertex δ of Γ and for any non-negative integer n, we set $V_\delta^{(n)}=\langle Y_\gamma\ :\ dist(\gamma,\delta)\le n\rangle$.*

Lemma 7.5. *Assume b odd, $b\ge 5$. Let $(\rho,\sigma,\tau,\lambda)$ be a path of length 3 in Γ, with ρ of type P (and σ of type H). Set $B=V_\rho^{(b-1)}$, and assume that there exists $t\in B-G_\sigma(\tau)$ with $|W_\tau/C_{W_\tau}(t)|\le 4$. Then $H/O_2(H)\cong SL(2,2)$, and (P,H) is a weak BN-pair.*

Proof. Set $U=C_{V_\lambda}(t)Y_\lambda$, $U^*=U(Y_\lambda)^t$, $R=N_{Q_\sigma}(U)$, and $R^*=N_{Q_\sigma}(U^*)$. As $t\notin G_\sigma(\tau)$, by assumption, we have $Y_\lambda\nleq U$, and so $|V_\lambda/U|\le 2$.

Suppose first that $Q_\sigma \cap Q_\tau \not\leq R$. Then $\langle U^{Q_\sigma \cap Q_\tau} \rangle = V_\lambda$. Since $(Y_\lambda)^t \leq V_\sigma$ we then have

$$\langle (U^*)^{Q_\sigma} \rangle V_\sigma = \langle U^{Q_\sigma} \rangle V_\sigma = \langle U^{(Q_\sigma \cap Q_\tau)Q_\sigma} \rangle V_\sigma = \langle V_\lambda^{Q_\sigma} \rangle V_\sigma = W_\tau.$$

But this implies that W_τ is t-invariant, which is contrary to $t \notin G_\sigma(\tau)$. We therefore conclude that $Q_\sigma \cap Q_\tau \leq R$.

Suppose next that $R \not\leq Q_\tau$. Then $O^2(G_\tau) \leq \langle R, Q_\lambda \rangle$, and hence $U \leq D_\tau$. Thus $|V_\sigma/D_\tau| \leq 2$, and then 7.3 yields the conclusion of the lemma. Thus, we may assume that $R \leq Q_\tau$, and hence that $R = Q_\sigma \cap Q_\tau$. Then R is not t-invariant, by 1.9, and so $R \neq R^*$. As $|Q_\sigma/(Q_\sigma \cap Q_\tau)| = 2$, we have thus shown:

(1) $R^* = Q_\sigma$, $R = Q_\sigma \cap Q_\tau$, and $U \neq U^*$.

Let $g \in R^* - R$. As $|U^*/U| = 2$ we then have $|U/(U \cap U^g)| = 2$, and since $|V_\lambda/U|/ \leq 2$ we obtain $|V_\lambda/(V_\lambda \cap (V_\lambda)^g)| \leq 4$. As $g \notin G_\lambda$, by (1), 1.17 yields

(2) $|V_\sigma/D_\tau| = |V_\sigma/D_\rho| \leq 4$.

Recall that B denotes $V_\rho^{(b-1)}$. For any vertex γ with $dist(\gamma, \rho) \leq b-1$ we have $[Y_\gamma, V_\delta] = 1$ for some $\delta \in \Delta(\rho)$, and thus $[B, D_\rho] = 1$. Take $P = \rho$ and $H = \sigma$, so that $S = G_\rho \cap G_\sigma$. We shall also write D for D_ρ. We have $t \notin N_G(W_\tau)$, so $W_\tau \neq V_\sigma C_{W_\tau}(t)$. As $|W_\tau/C_{W_\tau}(t)| \leq 4$, by assumption, it then follows that $|V_\sigma/C_{V_\sigma}(t)| = 2$. In particular, t induces a transvection on $\widetilde{V_\sigma}$. As $B \trianglelefteq S$, it now follows from 3.11 and from (2) that $H/Q_H \cong SL(2,2)$ or $O_4^+(2)$, with $\widetilde{V}/C_{\widetilde{V}}(O^2(H))$ a natural module for H/Q_H. In the $SL(2,2)$-case we are done, so we assume henceforth that H/Q_H is isomorphic to $O_4^+(2)$. We next show:

(3) $\widetilde{V}$ is a natural $O_4^+(2)$-module for H/Q_H.

Suppose (3) is false. Then $C_V(O^2(H))$ properly contains Y_H, and so there exists $x \in V$ with $[x, H] = Y_H$. As $[B, D] = 1$, it follows from (2) that $x \in D$, and then $[x, P] \leq Y_P$ by 1.17(b). Here $x \notin Y_P$ as Y_P is not normal in H. As $Y_P = \Omega_1(Z(Q_P))$ and as V is elementary abelian, we then have $[x, Q_P] = Y_P$ and $|S : C_S(x)| > 2$, for a contradiction. This proves (3).

Notice that D is now identified as $[V, S]Y_H$, of order 8. Since $|V/C_V(t)| = 2$, we then have $C_S(D/Y_H) = O_2(H)B$, and $Q_H B/Q_H = Q_H)C_S(D)/Q_H$ is the subgroup of S/Q_H induced by the elements of S/Q_H which induce transvections on $\widetilde{V}$.

Set $J = \langle ((Q_P \cap Q_H))^P \rangle$. Then $J \geq [Q_P, O^2(P)]$. As $D \not\leq Y_P$ we have $Q_P/C_P(D)$ isomorphic to Y_P as modules for P, and so $Q_P = C_{Q_P}(D)J$. As $B \leq J$, this says that J induces the whole of the action of Q_P on V/Y_H. Then, since $S = Q_P Q_H)$, we obtain $S = Q_H J$. Translating this information to the edge (τ, λ), and setting $Q_\tau^* = (Q_\sigma \cap Q_\tau)(Q_\tau \cap Q_\lambda)$ we observe that Q_τ^* is conjugate to J, by 1.17(c). We have thus shown:

(4) $G_\tau \cap G_\lambda = Q_\tau^* Q_\lambda = (Q_\sigma \cap Q_\tau)Q_\lambda$.

As U is normal in $Q_\sigma \cap Q_\tau$, by (1), it now follows from (3) and (4) that

(5) $U = [V_\lambda, G_\tau \cap G_\lambda] = [V_\lambda, Q_\tau Q_\lambda]$.

Note that (5) also holds if λ is replaced by the vertex $\lambda' \in \Delta(\tau) - \{\sigma, \lambda\}$, and U is replaced by $C_{V_{\lambda'}}(t)Y_{\lambda'}$. Set $X = [W_\tau, Q_\tau]V_\sigma$. Then $X = \langle (U)^{G_\tau} \rangle V_\sigma$, by (5). It follows that X is t-invariant, and so $X \trianglelefteq \langle t, G_\sigma \cap G_\tau \rangle = G_\sigma$. Thus $X \leq W_\delta$ for any $\delta \in \Delta(\sigma)$, and thus $[W_\tau, Q_\tau] \leq W_\gamma$ for every vertex γ at distance 2 from τ. For every such γ we have $[V_\gamma^{(b-3)}, W_\gamma] = 1$, and we obtain $[W_\tau, Q_\tau, V_\tau^{(b-1)}] = 1$. Translating back to the edge (ρ, σ), we then have $[V, Q_H), B] = 1$, and so B induces a transvection on V/Z. This is contrary to $B \trianglelefteq S$, and the lemma is thereby proved. $\square$

Lemma 7.6. *Suppose that $\overline{H} \cong SL(2,2)$. Then $b \neq 5$.*

Proof. We'll just copy this from [9.11 in Weak (B,N)-pairs of Rank 2]. $\square$

For the remainder of this section we will assume that b is odd, $b \geq 5$. (The case $b = 3$ will be taken up in the next section.) Under this assumption, it follows from 2.3 that there is a subgroup A of $[Q_P, O^2(P)]O_2(H)$ such that A is a quadratic $F2^*$-offender on $\widetilde{V}$. Then 3.11 implies that $\overline{H}_0$ is a direct product

$$\overline{H}_0 = \overline{L}_1 \times \cdots \overline{L}_r$$

where $|\overline{L}_i| = 3$ and where $|[\widetilde{V}, \overline{L}_i]| = 4$ for all i, $1 \leq i \leq r$. For each such i, set $\widetilde{V}_i = \prod \{[\widetilde{V}, L_j]\}_{j \neq i}$, and set $K_i = C_H(\widetilde{V}_i)$. (Thus, we have $\overline{K}_i = \overline{L}_i$ unless there exist elements of H which induce transvections on $\widetilde{V}$.) We set

$$\mathcal{K} = \{K_1, \cdots, K_r\}$$

and for any H-vertex $\delta = H^g$ of Γ we define $\mathcal{K}_\delta$ to be $\mathcal{K}^g$.

The parameter r will be fixed for the remainder of this section.

A **path** in Γ will always be understood to be non-empty, without stammering. For any path $\pi = (\delta, \cdots, \delta')$ in Γ, we write $|\pi|$ for the **length** of π, and we write $\overline{\pi}$ for the **opposite path** $(\delta', \cdots, \delta)$. If γ and γ' are vertices of π, with γ preceding γ', then $\pi(\gamma, \gamma')$ denotes the **sub-path** $(\gamma, \cdots, \gamma')$ of π. If π' is a path whose initial vertex is incident with the terminal vertex of π, then we write $\pi \circ \pi'$ to denote the path obtained by **concatenation** of π and π', in the given order. Let $H(\pi)$ denote the set of H-vertices of π which are not terminal vertices of π. Then, denote by $\mathcal{S}(\pi)$ the set of vertices δ in $H(\pi)$ which satisfy the following two conditions, for both $\epsilon = 1$ and $\epsilon = -1$.

(1) There exists $K \in \mathcal{K}_\delta$, and an element g of H such that either g is in $K - C_K(\widetilde{V})$ or g induces a transvection on $\widetilde{V}$ which inverts $\overline{K}$, and such that $\delta + \epsilon$ is fused to $\delta - \epsilon$ by g^ϵ.

(2) We have $N_{G_\delta \cap G_{\delta+\epsilon}}(Y_{\delta-\epsilon}) \leq G_{\delta-\epsilon}$.

We will refer to the group K in (1) as the **linkage group** for π at δ.

Lemma 7.7. *Assume that b is odd, $b \geq 5$.*

(a) *Let $K \in \mathcal{K}$ and let $g \in K - C_K(\widetilde{V})$. Set $T = N_S((Y_P)^g)$. Then there exists $g' \in N_K(T)$ such that $(Y_P)^g = (Y_P)^{g'}$.*
(b) *Let (ρ, σ) be an edge of Γ with ρ of type P, σ of type H. Then $V_\sigma = \langle Y_\tau \ : \ \sigma \in \mathcal{S}(\rho, \sigma, \tau)\rangle Y_\rho$.*

Proof. We have $[\widetilde{V}, H_0]$ the direct product of $\{[\widetilde{V}, K_i] \ : \ 1 \leq i \leq r\}$, and the action of H on $\mathcal{K}$ induces an equivalent action on this set of subgroups of $\widetilde{V}$. Let $y \in Y_P - Y_H$. Then T centralizes both $\widetilde{y}$ and $\widetilde{y}^g$, so T centralizes $[\widetilde{y}, g]$, and therefore $\overline{K}$ is T-invariant. Let ϕ be the endomorphism of $[\widetilde{V}, H_0]$ which restricts to the identity on $[\widetilde{V}, K]$ and which is the zero mapping on any $[\widetilde{V}, K_i]$ for $K_i \neq K$. Then $N_S(\overline{K})$ centralizes also $\phi(\widetilde{y})$, and so T centralizes $[\widetilde{V}, K]$. Setting $N = C_H(\widetilde{V})$, we now have $[T, K] \leq N$. As $T \geq O_2(H)$, T is a Sylow 2-subgroup of NT, and the Frattini argument now shows that there exists $g' \in N_{NT\langle g\rangle}(T) - NT$. This proves (a).

In order to prove (b), take $\rho = P$ and $\sigma = H$. Set $V_i = \langle (Y_P)^{K_i}\rangle$. Then $V = V_1 \cdots V_r$, and (b) then follows from (a). $\square$

Lemma 7.8. *Assume that b is odd, $b \geq 5$. Let $\pi = (\rho, \sigma, \tau)$ be a path in Γ, with $\sigma \in \mathcal{S}(\pi)$, and let A be an elementary abelian subgroup of Q_ρ. Assume that there exists $a \in A - G_\tau$ with $[V_\sigma, a, A] = 1$. Then $|A/A \cap G_\tau| = 2$.*

Proof. Let K be the linkage group for π at σ, and let $g \in K$ with $\tau = \rho^g$. Identify H with σ, and set $N = C_H(\widetilde{V})$. Then $[C_A(\overline{K}), g] \leq N$, and so $C_A(\overline{K})$ normalizes Y_τ. As $\sigma \in \mathcal{S}(\pi)$ we then have $C_A(\overline{K}) \leq G_\tau$, and it therefore suffices to show that $|A/C_A(\overline{K})| \leq 2$. This follows from quadratic action, and from the structure of $\overline{H}_0$ and of $\widetilde{V}$ given after 7.6. $\square$

Lemma 7.9. *Assume that b is odd, $b \geq 5$. Let $\pi = (\delta, \cdots, \gamma)$ be a path with δ of type P and with γ of type H. There then exists $\mu \in \Delta(\gamma)$ such that $\gamma \in \mathcal{S}(\pi \circ (\mu))$. Moreover, if (δ, γ) is a critical pair and $|\pi| = b$, then μ may be chosen so that $Y_\delta \nleq G_\mu$.*

Proof. Take $H = \gamma$ and take $P = \gamma - 1$, where $\gamma - 1$ denotes the vertex of π which is adjacent to γ. Thus, we have $S = G_{\gamma-1} \cap G_\gamma$. Choose $K \in \mathcal{K}$, with $[\overline{K}, \overline{Y_\delta}] \neq 1$ if $Y_\delta \nleq Q_\gamma$. Set $N = C_H(\widetilde{V})$, choose $g \in K - N$, and set $m = (\gamma-1)^g$. Set $T = N_S(Y_\mu)$. By Lemma 7(a) we may assume that g normalizes T, and hence that $T \leq G_\mu \cap (G_{\gamma-1})^{g^{-1}}$. Thus $\gamma \in \mathcal{S}(\pi \circ (\mu))$. Now suppose that $Y_\delta \nleq Q_\delta$ so that, by choice, $[Y_\delta, K] \nleq N$. We then have $Y_\delta \nleq T$. Thus, $Y_\delta \nleq G_\mu$, and this completes the proof of the lemma. $\square$

Lemma 7.10. *Assume that b is odd, $b \geq 5$. Let $\pi = (\delta, \cdots, \gamma)$ be a path of length $b - 1$ with both δ and γ of type H. Suppose that $V_\delta \nleq Q_\gamma$. There then exists $\lambda \in \Delta(\delta)$ such that (λ, γ) is a critical pair, and such that $\delta \in \mathcal{S}((\lambda) \circ \pi)$.*

Proof. Immediate from Lemma 7.7(b). $\square$

Lemma 7.11. *Assume that b is odd, $b \geq 5$, and let $\pi = (\rho, \sigma, \tau, \lambda)$ be a path in Γ, with $\sigma \in \mathcal{S}(\pi)$. Then the following hold.*

(a) $Y_\rho \cap V_\lambda = Y_\sigma$.
(b) $W_\rho = \langle x \in W_\rho : |V_\lambda / C_{V_\lambda}(x)| \leq 2 \rangle$.
(c) $[W_\rho, W_\tau] \cap Y_\tau \leq Y_\sigma$.

Proof. Suppose that (a) is false. As $Y_\sigma \leq V_\lambda$ we then have $Y_\rho \leq V_\lambda$, and thus $[Y_\rho, Q_\lambda] \leq Y_\lambda$. As G_τ is 2-transitive on $\Delta(\tau)$, there exists a vertex α at distance $b-2$ from λ such that (α, σ) is a critical pair. As $\sigma \in \mathcal{S}(\pi)$, we have $\rho = \tau^x$ for some $x \in G_\sigma - G_\tau$, such that x lies in the linkage group K for π at σ.

Take $H = \sigma$ and take $P = \tau$, so that $S = G_\sigma \cap G_\tau = Q_\sigma Q_\tau$. Set $R = \langle (Y_\alpha)^{Q_\tau}$. Then $[Y_\rho, R] \leq [V_\lambda, R] = 1$, and since $\sigma \in \mathcal{S}(\pi)$ we then have $R \leq G_\rho$. Notice that $1 \neq \overline{R} \trianglelefteq \overline{S}$. As $\overline{S}$ is transitive on $\mathcal{K}$, we then have $[\overline{R}, \overline{K}] \neq 1$. Then $H = \langle S^x, R \rangle$, and we may conclude that $Q_\rho \cap Q_\sigma = C_{Q_\sigma}(Y_\rho)$ is normal in H. This contradicts —, and so (a) is proved.

Suppose next that (b) is false. Then $W_\rho \not\leq Q_\lambda$, and so $b = 5$. Choose $\gamma \in \Delta(\rho)$ with $V_\gamma \neq \langle x \in G_\gamma : |V_\lambda / C_{V_\lambda}(x)| \leq 2 \rangle$. It then folows from Lemma 5(b) that there exists $\delta \in \Delta(\gamma)$ such that $\gamma \in \mathcal{S}(\delta, \gamma, \rho)$ and such that, for some $x \in G_\delta$, we have $|V_\lambda / C_{V_\lambda}(x)| > 2$. Here $V_\lambda \leq G_\gamma$, V_λ acts quadratically on V_γ, and $V_\lambda \not\leq G_\delta$. In particular, V_λ does not centralize the linkage group K for (δ, γ, ρ) modulo $C_{G_\gamma}(\widetilde{V_\gamma})$. 7.6 then yields $|V_\lambda / V_\lambda \cap G_\delta| = 2$, and then also $[Y_\delta, V_\lambda \cap G_\delta] \neq 1$. Thus, we have $Y_\gamma = [Y_\delta, V_\lambda \cap G_\delta] \leq V_\lambda$, which contradicts (a).

Finally, suppose that (c) is false. Again, we have $b = 5$, as otherwise $b \geq 7$ and $[W_\rho, W_\tau] = 1$. Write $\Delta(\rho) = \{\sigma, \gamma, \gamma'\}$ and $\Delta(\tau) = \{\sigma, \lambda, \lambda'\}$. If $W_\rho \leq Q_\lambda$ and $[W_\rho, V_\lambda] \neq 1$ then $[V_\gamma, V_\lambda][V_{\gamma'}, V_\lambda] = Y_\lambda$. and so Y_λ is contained either in V_γ or in $V_{\gamma'}$, contrary to (a). Thus, if $W_\rho \not\leq Q_\lambda$ then $[W_\rho, V_\lambda] = 1$. As $[W_\rho, W_\tau] \neq 1$, we may then fix notation (replacing λ by λ', or γ by γ', if necessary) so that $V_\gamma \not\leq Q_\lambda$.

By Lemma 7.9 we may choose $\mu \in \Delta(\lambda)$ so that $\lambda \in \mathcal{S}(\tau, \lambda, \mu)$, and so that $V_\gamma \not\leq G_\mu$. Suppose that $Y_\mu \leq Q_\gamma$. Then $[V_\gamma, Y_\mu] = Y_\gamma$, and thus $Y_\gamma \leq V_\lambda$, which is contrary to (a). We conclude that $Y_\mu \not\leq Q_\gamma$, and then 7.9 implies that there exists $\delta \in \Delta(\gamma)$ such that $\gamma \in \mathcal{S}(\delta, \gamma, \rho)$ and with $Y_\mu \not\leq G_\delta$.

Suppose that $[W_\delta, Y_\lambda] = 1$. As $\sigma \in \mathcal{S}(\pi)$ we then have $W_\delta \leq Q_\tau \leq G_\lambda$. As $[W_\delta, V_\gamma] = 1$, we have $[V_\lambda, V_\gamma, W_\delta] = 1$. Then $|W_\delta / W_\delta \cap G_\mu| = 2$, by Lemma 8, and then also $|W_\delta / C_{W_\delta}(Y_\mu)| \leq 4$. Now 7.5 yields $r = 1$ and $|V| = 8$. Since also $b = 5$, we have a contradiction to 7.6. We therefore conclude that $[W_\delta, Y_\lambda] \neq 1$. On the other hand, we have $Y_\lambda \leq Y_\tau \leq Y_\sigma[W_\rho, W_\tau]$, by assumption in (c). Thus $Y_\lambda \leq Y_\sigma[V_\gamma, W_\tau][V_{\gamma'}, W_\tau]$, and so $1 \neq [Y_\lambda, W_\delta] \leq [V_{\gamma'}, W_\tau, W_\delta]$.

By (b), above, both W_τ and W_δ are generated by elements which centralize hyperplanes of $V_{\gamma'}$. Since $W_\tau W_\delta$ is contained in the 2-group $G_\rho \cap G_{\gamma'}$ it follows that $[V_{\gamma'}, W_\tau, W_\delta] \leq Y_{\gamma'}$. Thus, $[Y_\lambda, W_\delta] = Y_{\gamma'}$. Write $\Delta(\delta) = \{\gamma, \beta, \beta'\}$. As $Y_\lambda \leq Q_\delta$ we conclude that, up to a possible permutation of $\{\beta, \beta'\}$, we have $Y_{\gamma'} \leq V_\beta$. But this contradicts (a), with $(\rho, \gamma, \delta, \beta)$ in place of π. The proof of (c) is thereby complete. $\square$

Lemma 7.12. *Assume that b is odd, $b \geq 5$. There then exists a critical pair (α, α'), a path $\pi = (\alpha, \beta, \cdots, \alpha')$ of length b, and vertices $\mu \in \Delta(\alpha') - \{\alpha' - 1\}$ and $\lambda \in \Delta(\mu) - \{\alpha'\}$ such that the following hold.*

(i) *Both β and α' are in $\mathcal{S}(\pi \circ (\mu))$.*
(ii) *Y_α is not contained in G_μ.*
(iii) *If $W_\mu \not\leq G_{\beta+1}$, then $\beta + 2 \in \mathcal{S}(\pi)$ and $Y_{\beta+2} \not\leq [V_{\beta+2}, W_\mu]$.*

Proof. Suppose false, and choose a critical path $\pi = (\alpha, \beta, \cdots, \alpha')$ so that $|\mathcal{S}(\pi)|$ is as large as possible. Then $\beta \in \mathcal{S}(\pi)$ by 7.10 (applied to $\pi(\beta, \alpha')$). By 7.9, we may choose $\mu \in \Delta(\alpha') - \{\alpha' - 1\}$ so that $\alpha' \in \mathcal{S}(\pi \circ (\mu))$ and so that $Y_\alpha \not\leq G_\mu$.

As the lemma is assumed false, we have $W_\mu \not\leq G_{b+1}$. Choose $\lambda \in \Delta(\mu) - \{\alpha'\}$ so that $V_\lambda \not\leq G_{\beta+1}$. Suppose next that $V_{\beta+2} \leq Q_\lambda$. Then $[V_{\beta+2}, V_\lambda] = Y_\lambda$, and since V_λ acts non-trivially on $V_{\beta+2}/Y_{\beta+2}$ we conclude that $Y_{\beta+2} \not\leq [V_{\beta+2}, V_\lambda]$. As the lemma is false, we then have $\beta + 2 \notin \mathcal{S}(\pi)$. Now set $\pi_0 = (\lambda, \mu) \circ \overline{\pi}(\alpha', \beta + 2)$ and apply 7.10 to π_0. Thus, there exists $\gamma_0 \in \Delta(\lambda) - \{\mu\}$ such that $(\gamma_0, \beta + 2)$ is a critical pair and such that, for $\pi' = (\gamma_0) \circ \pi_0$, we have $\lambda \in \mathcal{S}(\pi')$. Notice however that $|\mathcal{S}(\pi')| > |\mathcal{S}(\pi)|$, which is contrary to our choice of π. We conclude that $V_{\beta+2} \not\leq Q_\lambda$.

We now apply 7.10 to the path $\pi_1 = \pi(\beta + 2, \alpha') \circ (\mu, \lambda)$. Thus, there exists a vertex $\gamma_1 \in \Delta(\beta+2)$ such that (γ_1, λ) is a critical pair, and such that, upon setting $\pi^* = (\gamma_1) \circ \pi_1$, we have $\beta + 2 \in \mathcal{S}(\pi^*)$. Observe that $|\mathcal{S}(\pi^*)| \geq |\mathcal{S}(\pi)|$. Thus, we may replace π by π^*, and since $\alpha' \in \mathcal{S}(\pi^*)$ we then obtain $\alpha' - 2 \in \mathcal{S}(\pi)$. But then maximality of $|\mathcal{S}(\pi)|$ shows that, to begin with, and before making the above replacement, we had $\alpha' - 2 \in \mathcal{S}(\pi)$. By iteration it now follows that $\mathcal{S}(\pi) = \{\beta, \beta + 2, \cdots, \alpha' - 2\}$. Moreover, since the lemma fails to hold for π^*, we have (prior to replacement) $Y_{\beta+2} \leq [V_{\beta+2}, W_\mu]$, and thus (after replacement) $Y_\beta \leq [V_\beta, W_{\alpha'-1}] \leq W_{\alpha'-1}$. Lemma 7.11(c) then yields $b > 5$, so that $[W_{\alpha'-1}, W_\mu] = 1$, whence $[Y_\beta, W_\mu] = 1$. But with $\beta + 2 \in \mathcal{S}(\pi)$ we then have $W_\mu \leq Q_{\beta+1}$. Thus $W_\mu \leq G_{\beta+1}$ after all, and the lemma is proved. $\square$

Proposition 7.13. *Assume that b is odd, $b \geq 5$. Then $|V| = 8$.*

Proof. Assume that $|V| > 8$. Choose π and μ as in 7.12, and fix $t \in Y_\alpha - Y_\beta$. Suppose first that $Y_\beta \leq [V_\beta, V_{\alpha'}]$. Then $Y_\beta \leq V_{\alpha'} \leq W_\mu$, and it then follows from 7.12 that $W_\mu \leq G_\beta$. Suppose that W_μ acts quadratically on V_β, or that $V_{\alpha'} \not\leq G_\alpha$. As $\beta \in \mathcal{S}(\pi)$, 7.8 then implies that $|W_\mu / W_\mu \cap G_\alpha| \leq 2$, and so $|W_\mu / C_{W_\mu}(t)| \leq 4$. As 7.5 then yields $r = 1$, we conclude that W_μ is not quadratic on V_β and that $V_{\alpha'} \leq G_\alpha$. In particular, it follows that $[W_\mu, (W_\mu)^g] \neq 1$ for some $g \in V_\beta$, and so $b = 5$. Further, with $V_{\alpha'} \leq G_\alpha$ we have $|V_{\alpha'} / C_{V_{\alpha'}}(t)| = 2$.

Now $V_\beta = (V_\beta \cap G_\mu) Y_\alpha$, and W_m acts quadratically on the hyperplane $V_\beta \cap G_\mu$ of V_β. It follows that the W_μ-orbits on $\mathcal{K}_\beta$ are of length at most 2, and that $|W_\mu / W_\mu \cap G_\alpha| \leq 4$. Thus, we have shown that $[t, W_\mu, W_\mu] \neq 1$, and $|W_\mu / C_{W_\mu}(t)| \leq 8$ in the case that $Y_\beta \leq V_{\alpha'} \leq W_\mu$.

Suppose, on the other hand, that $Y_\beta \not\leq [V_\beta, V_{\alpha'}]$. Then $[t, V_{\alpha'} \cap G_\alpha] = 1$, and so $|V_{\alpha'} / C_{V_{\alpha'}}(t)| = 2$, as in the preceding case. Further, we have $V_{\alpha'} \not\leq Q_\beta$. Notice that $|W_\mu / W_\mu \cap G_\beta| \leq 2$, by 7.12. As $[V_\beta, V_{\alpha'}, W_\mu] = 1$, 7.8 then yields $|W_\mu / W_\mu \cap G_\alpha| \leq 4$, and so once again $|W_\mu/C_{W_\mu}(t)| \leq 8$. Moreover, we either have $|W_\mu/C_{W_\mu}(t)| \leq 4$ or $Y_\beta \leq [t, W_\mu \cap G_\alpha]$. If now $[t, W_\mu, W_\mu] = 1$ we thus obtain $[Y_\beta, W_\mu] = 1$ and $|W_\mu/C_{W_\mu}(t)| \leq |W_\mu/(W_\mu \cap Q_\alpha)| \leq 4$, again contrary to 7.5. We have thus shown that, in any case, we have the following information.

(1) $b = 5$, $[t, W_\mu, W_\mu] \neq 1$, $|W_\mu/C_{W_\mu}(t)| = 8$, and $|V_{\alpha'}/C_{V_{\alpha'}}(t)| = 2$.

Set $\sigma = \mu^t$, and observe that since t induces a transvection on $\widetilde{V_{\alpha'}}$, we have $\alpha' \in \mathcal{S}(\sigma, \alpha', \mu)$. Further, we have the following consequence of 7.2.

(2) $\alpha' \in \mathcal{S}(\sigma, \alpha', \mu)$.

Suppose next that $W_\mu \leq Q_\rho$ for all $\rho \in \Delta(\sigma)$. Then $[W_\mu, W_\sigma] \leq Y_\sigma$, and then 7.11(c) yields $[W_\mu, W_\sigma] \leq Y_{\alpha'}$. For any $\rho \in \Delta(\sigma) - \{\alpha'\}$ we then have $[V_\rho, W_\mu] \leq]yr \cap Y_{\alpha'} = 1$, and so $[W_\mu, W_\sigma] = 1$. This implies that $[t, W_\mu, W_\mu] = 1$, contrary to (1). We conclude that there exists $\rho \in \Delta(\sigma)$ with $W_\mu \not\leq Q_\rho$. Fix such a vertex ρ and set $l = \rho^t$. Then $W_\sigma \not\leq Q_\lambda$.

Notice that $W_\mu \cap W_\sigma \geq C_{W_\mu}(t)V_{\alpha'}$. Then (1) yields:

(3) $W_\mu \cap W_\sigma$ is of index at most 4 in W_μ and in W_σ.

We note also that 7.11(b) yields the following.

(4) $W_\sigma = \langle x \in W_\sigma \ : \ |V_\lambda/C_{V_\lambda}(x)| \leq 2\rangle$.

Suppose that $|V_\lambda/C_{V_\lambda}(W_\sigma)| = 2$. Noting that $Y_\lambda \not\leq [W_\sigma, V_\lambda]$ by 7.11(c), we then have $|[W_\sigma, V_\lambda]| = 2$. Set $U = [W_\sigma, V_\lambda]$. Since $Q_{\alpha'} \cap Q_\mu$ is not normal in $G_{\alpha'}$, by 1.9, where $G_{\alpha'} = \langle G_\sigma \cap G_{\alpha'}, G_{\alpha'} \cap G_\mu \lambda\rangle$, it follows that $Q_{\alpha'} \cap Q_\mu \neq Q_{\alpha'} \cap Q_\sigma$, and hence $(Q_{\alpha'} \cap Q_\mu)(Q_{\alpha'} \cap Q_\sigma) = Q_{\alpha'}$. In particular, $Q_{\alpha'} \cap Q_\mu$ is transitive on $\Delta(\sigma) - \{\alpha'\}$. As $W_\sigma \not\leq Q_\lambda$ we then have $V_\rho \not\leq Q_\lambda$, and so $U = [V_\rho, V_\lambda]$ is t- invariant. As U lies in the center of $Q_{\alpha'} \cap Q_\mu$ we then have $[U, Q_{\alpha'}] = 1$, and then 1.1(a) shows that $U = Y_{\alpha'}$. Lemma 7.1 then implies that $|V| = 8$, which is contrary to the hypothesis. Thus $|V_\lambda/C_{V_\lambda}(W_\sigma)| > 2$.

Now (3) implies that $W_\mu = C_{W_\mu}(W_\sigma)V_\lambda$. Then $[W_\sigma, V_\lambda] = [W_\sigma, W_\mu] \trianglelefteq Q_{\alpha'}$, and so $Y_{\alpha'} \leq [W_\sigma, V_\lambda]$, by 1.8. Then (4) and Lemma 1 imply that $W_\sigma Q_\lambda/Q_\lambda$ is of order 2^r, and (3) then yields $r \leq 2$. As $|V| > 8$, Lemmas 1 and 2 now give:

(5) $|V| = 32$ and $H/Q_H \cong \Omega_4^+(2)$.

For any edge (γ, δ) of Γ, with γ of type H, set $T(\gamma, \delta) = \langle g \in G_\gamma \cap G_\delta \ : \ |[V_\gamma/Y_\gamma, g]| \leq 2\rangle$. For any vertices ϕ and ψ in $\Delta(\alpha')$, set $F(\phi, \psi) = [W_\phi, W_\psi]$. Set $F = F(\sigma, \mu)$. We have seen that $|W_\sigma/C_{W_\sigma}(V_\lambda)| = 4$, that W_σ is generated by $C_{W_\sigma}(V_\lambda)$ together with two elements that induce transvections on V_λ, and that $[W_\sigma, W_\mu] = [W_\sigma, V_\lambda] \geq Y_{\alpha'}$. Thus F is a fours group containing $Y_{\alpha'}$, and we have $FY_\lambda/Y_\lambda = [V_\lambda/Y_\lambda, T(\lambda, \mu)]$. Setting $U = FY_\lambda$, it follows that $U \trianglelefteq G_\mu \cap G_\lambda$. But also F, and hence also U, is normal in $Q_{\alpha'}$, and so $U \trianglelefteq G_\mu$. Then $U \leq V_{\alpha'} \cap V_\lambda$, and U has index 4 in V_λ. Since $|V_\lambda/C_{V_\lambda}(W_\sigma| = 4$, it follows that $U = V_{\alpha'} \cap V_\lambda$, and so $U = D_{\alpha'}$, by 1.17. Similarly, we have $FY_\rho = D_\sigma$, and 7.11(a) yields $F = D_\mu \cap D_\sigma$.

Write $\mathcal{K}_{\alpha'} = \{K_1, K_2\}$ and define $X_i = X_{\alpha'}^{(i)}$ to be the inverse image in $V_{\alpha'}$ of $[V_{\alpha'}/Y_{\alpha'}, K_i]$, $(i = 1, 2)$. As $U \trianglelefteq G_\mu$ we have $U = [V_{\alpha'}, T(\alpha', \mu)]$, and since $Y_\lambda \nleq F$ it follows that $F \leq X_i$ for some i. For definiteness, take $F \leq X_1$. Then $[F, N_S(X_1)] \leq Y_{\alpha'}$. On the other hand we have $U = FY_\lambda = FY_\mu$, and so $[U, Q_\mu] = [F, Q_\mu] = Y_\mu$. As $D_\mu \neq Y_\mu$, $Q_\mu/C_{Q_\mu}(F)$ is a fours group admitting non-trivial action by $O^2(G_\mu)$. Setting $R = O_2(O^2(G_\mu))$, we then have $[F, R] = Y_\mu$, and thus $R \nleq N_S(X_1)$. As $R \trianglelefteq S$, $R/R \cap Q_{\alpha'}$ then contains a fours group.

Let us record these results.

(6) We have $F(\sigma, \mu) := [W_\sigma, W_\mu] = V_\rho \cap V_{\alpha'} \cap V_\lambda = [X_1, T(\alpha', \mu)]$.

(7) $R/R \cap Q_{\alpha'}$ contains a fours group, where $R := O_2(O^2(G_\mu))$.

We have $|\Delta(\alpha')| = 9$, and we may identify $\Delta(\alpha')$ with the set of singular points in the orthogonal space $V_{\alpha'}/Y_{\alpha'}$, for the action of $G_{\alpha'}$. Define a map

$$B : \Delta(\alpha') \times \Delta(\alpha') \longrightarrow \mathbb{F}_2$$

by the formula

$$B(\phi, \psi) = \begin{cases} 0, & \text{if } [W_\phi, W_\psi] = 1 \\ 1, & \text{if } [W_\phi, W_\psi] \neq 1. \end{cases}$$

We require the following elementary lemma (for which we need provide no proof).

Lemma 7.14. *Let $\widetilde{V}$ be an $O_4^+(2)$-space with associated bilinear form $\widetilde{B}$, let t be an orthogonal transvection on $\widetilde{V}$, and let x and v be singular points with $x = x^t$ and $v \neq v^t$. Then the following hold.*

(a) $\widetilde{B}(v, v^t) \neq 0$.

(b) *There exists a singular point y, with $y \neq y^t$, and such that $\widetilde{B}(x, y) = 0$.*

Notice that S has two orbits on $\Delta(\alpha') - \{\mu\}$, each orbit being of length 4. Let $\tau \in \Delta(\alpha')$ with $\tau \notin \sigma^S$. Thus, we have $\tau = \mu^g$ where g is an element of order 3 in $G_{\alpha'}$ which is fixed-point-free on $V_{\alpha'}/Y_{\alpha'}$. For any $\delta \in \Delta(\tau) - \{\alpha'\}$ one may then observe that $V_\delta \cap V_{\alpha'} \cap V_\lambda = Y_{\alpha'}$, and then for any $\lambda' \in \Delta(\mu)$ we have $[V_\delta, V_{\lambda'}] \leq Y_{\alpha'}$. Lemma 7.1 then implies that $[V_\delta, V_{\lambda'}] = 1$, and thus $B(\tau, \mu) = 0$. Since $B(\sigma, \mu) = 1$, where $\sigma = \mu^t$, it follows from 7.13(a) that B is indeed the bilinear form associated with the orthogonal space $V_{\alpha'}/Y_{\alpha'}$. Then 7.13(b) says that we may choose τ as above, with $B(\alpha' - 1, \tau) = B(\mu, \tau) = 0$, and with $t \notin G_\tau$. In particular, we have $[W_\tau, V_{\beta+2}] = 1$, and so W_τ stabilizes the path $\pi(\beta, \alpha')$. As $t \notin G_\tau$, 7.5 shows that $|W_\tau/C_{W_\tau}(t)| \geq 8$. Then $|W_\tau/W_\tau \cap G_\alpha| = 4$ (so that W_τ induces a non-quadratic fours group on V_β/Y_β) and $[Y_\alpha, W_\tau \cap G_\alpha] = Y_\beta$. As $t \in X_\beta^{(i)}$ for some i, $(i = 1$ or $2)$, one may then compute that $[t, W_\tau] = [V_\beta, G_\beta \cap G_{\beta+1}]$, and that $[t, W_\tau, W_\tau] = Y_{\beta+1}$. On the other hand, 7.13(a) shows that $[(W_\tau)^t, W_\tau] \neq 1$, and so $[t, W_\tau, W_\tau] = [(W_\tau)^t, W_\tau] = F(\tau^t, \tau) \leq V_{\alpha'}$, of order

4. Thus $F(\tau^t, \tau) = Y_{\beta+1} \le V_{\alpha'}$. This contradicts 7.11(a), thus completing the proof of 7.12. $\square$

[THERE MUST BE AN EASIER ARGUMENT FOR 7.13]

Section 8: The solvable case, $b = 3$

Lemma 8.1. *Suppose that P and H are solvable, with $p = 2$. Then $b = 3$, and one of the following holds.*

(i) $H/O_2(H) \cong SL(2,2)$ *and* $|V| = 8$.
(ii) $H/O_2(H) \cong Sz(2)$ *and* $|V| = 32$.
(iii) $\overline{H}$ *is isomorphic to a subgroup of* $\cong O_4^+(2)$ *of index at most* 2, *and* $|V| = 32$.
(iv) $O_3(\overline{H})$ *is extraspecial of order* 27 *and of exponent* 3, *and*

(a) *There exists* $P_1 \in \mathcal{P}_H(S)$ *such that* $O_3(\overline{P}_1) = Z(O_3(\overline{H}))$ *and such that* $\langle (Y_P)^{P_1} \rangle$ *is invariant under* P.
(b) $\widetilde{V}$ *is a natural* $SU(3,2)$*-module for* $O^2(\overline{H})O_2(\overline{P}_1)$.
(c) $O^2(\overline{H})O_2(\overline{P}_1)$ *is isomorphic to a subgroup of* $SU(3,2)$ *of index at most* 2.

Proof. Suppose $b > 3$. Then 7.12 yields $|V| = 8$ and $H/O_2(H) \cong SL(2,2)$. Now [AN ARGUMENT EXTRACTED FROM THE WEAK BN-PAIRS PAPER] yields a contradiction. Thus, we have $b = 3$. Set $\gamma = \beta + 1 = \alpha' - 1$, and take $H = \beta$ and $P = \gamma$. Denote by δ the unique element of $\Delta(\gamma) - \{\beta, \alpha'\}$.

As $b > 2$ we have $V \le Q_P$, and so 1.10 implies that $[Q_H, O^2(H)] \not\le V$. Thus, there is at least one non-central chief factor X for H in Q_H/V.

Suppose first that $Q_\beta \cap Q_\gamma \le V_\beta Q_{\alpha'}$. Then $[Q_\beta \cap Q_\gamma, V_{\alpha'}] \le V_\beta Y_{\alpha'} = V_\beta$. As $|Q_\beta/(Q_\beta \cap Q_\gamma)| = 2$, $V_{\alpha'}$ induces a transvection on X. But $V_{\alpha'}Q_\beta \trianglelefteq Q_\beta Q_\gamma = Q_H Q_P = S$, and it follows that $H/C_H(X) \cong SL(2,2)$. Then H/Q_H is dihedral of order $2 \cdot 3^m$ for some m, $m \ge 1$. As $\widetilde{V}$ is a quadratic $F2$-module for H, it follows from 3.11 that $\overline{H} \cong SL(2,2)$. As $\widetilde{V}$ is the closure under H of $\widetilde{Y}_P$, we then have $|\widetilde{V}| \le 8$, and then 7.2 implies that $H/Q_H \cong SL(2,2)$. Suppose that $|\widetilde{V}| = 8$. As P is doubly transitive on $\Delta(\gamma)$ we have $[\widetilde{V}, V_\delta] = [\widetilde{V}, V_{\alpha'}]$, and 7.1 implies that neither Y_β nor yd is contained in $[vb, V_{\alpha'}]$. But $[V_\beta, V_{\alpha'}] \trianglelefteq Q_\gamma$, so $Y_\gamma \cap [V_\beta, V_{\alpha'}] \ne 1$. Then $Y_\gamma \cap [V_\beta, V_{\alpha'}] = Y_\beta$. This contradicts 7.2, applied to the action of V_β on $V_{\alpha'}$, so we conclude that $|V| = 8$ and that outcome (i) of the lemma holds. We may therefore assume that $Q_\beta \cap Q_\gamma \not\le V_\beta Q_{\alpha'}$. Then also:

(1) $Q_\gamma \cap Q_{\alpha'} \not\le V_{\alpha'} Q_\beta$.

Set $A = V_{\alpha'}$. As $[A, Q_{\alpha'}] = Y_{\alpha'} \le Q_\beta$, (1) yields:

(2) $C_{\overline{S}}(\overline{A}) \not\le \overline{A}$.

Suppose that A is an $F1$-offender on $\widetilde{V}$. Then 3.11 shows that $\overline{A}$ is generated by transvections, and since $\overline{A} \trianglelefteq \overline{S}$ it follows that $\overline{A}$ contains all the transvections in $\overline{S}$. This contradicts (2), and so, in fact, $\overline{A}$ is not an $F1$-offender on $\widetilde{V}$, and similarly $VQ_{\alpha'}/Q_{\alpha'}$

is not an $F1$-offender on $V_{\alpha'}/Y_{\alpha'}$. Then by 1.18 and symmetry, we may assume that $[V \cap Q_{\alpha'}, A] = Y_{\alpha'}$. We then have:

(3) There exists $v \in \widetilde{V}$ such that $[v, A] = \widetilde{Y}_P$.

We note also that A is a quadratic $F2$-offender on $\widetilde{V}$, by 2.3, and then 3.10 shows that $\overline{A}$ is generated by 2-transvections.

By 3.11 there is an $\overline{S}$-transitive collection $\mathcal{D} = \{\overline{D}_i\}_{1 \leq i \leq r}$ of subgroups $\overline{D}$ of $\overline{H}$, such that $O^2(\overline{H}) = \overline{D}_1 \times \cdots \times \overline{D}_r$, with $[\widetilde{V}, O^2(\overline{H})] = [\widetilde{V}, \overline{D}_1] \times \cdots \times [\widetilde{V}, \overline{D}_r]$, and such that $\overline{D}_i$ is isomorphic to $\mathbb{Z}_3$, $\mathbb{Z}_5$, or $O_3(SU(3,2))$. Set $U_i = [\widetilde{V}, \overline{D}_i]$. The transitivity of $\overline{S}$ on $\mathcal{D}$ then implies that there are non-identity elements u_i of U_i such that $\widetilde{Y}_P = \langle u_1 \cdots u_r \rangle$. On the other hand, (3) shows that $u_1 \cdots u_r = [v, \overline{a}]$ for some $v \in \widetilde{V}$ and some 2-transvection $\overline{a}$ in $\overline{A}$. It follows that either $r = 1$ or that $|U_i| = 4$ and $r = 2$. In any case we have $|\overline{A}| = 2$ and $\overline{A}$ induces a 2-transvection on $[\widetilde{V}, O^2(H)]$, so (3) implies that $[\widetilde{V}, O^2(H)] = \widetilde{V}$.

If $r = 2$ we have the desired outcome (iii) of the lemma, so we assume now that $r = 1$. Then $|O^2(\overline{H})| > 3$, by (2). Suppose that $|O^2(\overline{H})| = 5$. Then (2) implies that $\overline{H} \cong Sz(2)$, while $|V| = 32$. Then also $H/Q_H \cong Sz(2)$, by 1.11(b), and we have outcome (ii). Finally, suppose that $O^2(\overline{H})$ is an extraspecial group of order 27 and exponent 3, with $|\widetilde{V}| = 64$. Denote by H_1 the inverse image in H of $Z(O^2(\overline{H}))$. Then $\langle (Y_P)^{H_1} \rangle$ is of order 8, and 1.7 then shows that $O^2(P)$ is not invariant under H_1. By 1.4 there then exists $P_1 \in \mathcal{P}_{H_1 S}(S)$ such that $O^2(P)$ is not P_1-invariant, and such that $\langle P, P_1 \rangle \in \mathcal{L}$. Then $O^2(\overline{P}_1) = Z(O^2(\overline{H}))$.

Notice that $O_2(\overline{P}_1)$ is a subgroup of index at most 2 in a quaternion group. Thus $|\Omega_1(\overline{Q_\gamma})| = 2$. For any two distinct vertices δ and δ' in $\Delta(\gamma)$ we then have $[V_\delta, V_{\delta'}] = [V_\delta, \Omega_1(\overline{Q_\gamma})]$, and thus $[V_\delta, V_{\delta'}]$ is G_γ-invariant. Thus $[V_\beta, V_{\alpha'}]$ is invariant under both P and P_1, and (iv) holds. $\square$

Lemma 8.2. *Outcome (iv) of Lemma 8.1 does not hold.*

Proof. Suppose false, and let P_1 be as in 8.1(iv). Set $L = \langle P, P_1 \rangle$. As we have seen, we have $L \in \mathcal{L}$, and Y_P is not normal in L.

Form the amalgam $\Gamma^* = \Gamma(H, L)$ and set $b^* = b(H, L)$. It follows from 1.3(4) and from 8.1(iv) that $|Y_L| = 8$ and that $L/O_2(L) \cong SL(3,2)$. In particular, we have $Y_L \leq \langle (Y_P)^H \rangle$, and so we have $V = \langle (Y_L)^H \rangle$. Then $b^* \geq 3$, and since $b^* \leq b$ we conclude that $b^* = 3$. We may then label a critical path $(\alpha, \beta, \gamma, \alpha')$, and we shall take $H = \beta$ and $L = \gamma$, so that $P_1 = G_\beta \cap G_\gamma$. For δ a vertex with stabilizer H^g we write V_δ for V^g. We note that, since L is doubly transitive on $Y_L^\sharp$, G_γ is doubly transitive on $\Delta(\gamma)$, and hence G_β is transitive on $\Delta^{(2)}(\beta)$. In particular, any path of length 2 from β is a critical path.

Wishing to avoid needless repetition, we ask the reader to check that the argument at the relevant point in the proof of 8.1, above, shows:

(1) $|V/C_V(V_{\alpha'})| = 4$, $C_V(V_{\alpha'}) = [V, O_2(P_1)]$, and $[V, V_{\alpha'}] = Y_L$.

Set $X = X_\beta = \langle C_{V_\delta}(V) \ : \ \delta \in \Delta^{(2)}(\beta) \rangle$. Then $X_{\alpha'} = \Omega_1(X_{\alpha'})$ and $X_{\alpha'} \leq Q_\gamma \leq O_2(G_\gamma \cap G_\beta) = O_2(P_1)$. Here $\widetilde{V}$ is a natural $SU(3,2)$-module for $O^2(H)O_2(P_1)$, and

$\Omega_1(O_2(\overline{P}_1))$ is of order 2, so $[\widetilde{V}, X_{\alpha'}] \leq [\widetilde{V}, V_{\alpha'}]$, and then $[V, X_{\alpha'}]Y_L$. Thus $[V, X_{\alpha'}] \leq V_{\alpha'}$, and so also $[X, V_{\alpha'}] \leq V$, by symmetry. It follows that $[X, O^2(H)] \leq V$. Further, we have $[C_{Q_H}(V), V_{\alpha'}] \leq [Q_\gamma, V_{\alpha'}] \leq C_{V_{\alpha'}}(V)$, by (1), and so $[C_{Q_H}(V), O^2(H)] \leq X$. This now shows:

(2) $[C_{Q_H}(V), O^2(H)] = V$.

We have $[V, Q_H] = Y_H$, so:

(3) $Q_H/C_{Q_H}(V)$ is isomorphic to the dual of $\widetilde{V}$ as a module for H.

Set $D = \langle [V_\delta, Q_\gamma] \ : \ \delta \in \Delta(\gamma) \rangle$. Then (1) shows that $D \leq X \cap X_{\alpha'}$, so $[D, V_{\alpha'}] = 1$, and so $V \not\leq D$. As $\eta(P_1, V) = 3$, it follows from (2) that $\eta(P_1, D/Y_L) = 1$. As $(V \cap D)/Y_L$ is a P_1-invariant subgroup of D/Y_L of order 4, it follows that D/Y_L is an L-module of order 8, dual to Y_L. Thus $V \cap D$ is a hyperplane of D.

Set $W = W_\gamma$. Then $D \leq \Omega_1(Z(W))$. We have $C_S(W) \leq C_{Q_H}(V)$, and since $V \not\leq C_S(W)$ we then have $\eta(P_1, C_S(W)) = 2$. Thus $[C_S(W), O^2(L)] = D$. Suppose that there exists an element x in Q_H such that $[x, H] = Y_H$. From (3) we have $x \in C_H(V)$, so $x \in G_\gamma^{(2)} \leq G_{\alpha'}$. As $|[x, V_{\alpha'}]| \leq 2$ we have $x \in Q_{\alpha'}$, and since $[x, V_{\alpha'}] \neq Y_{\alpha'}$ we conclude that $[x, V_{\alpha'}] = 1$. In this way we have $x \in G_\gamma^{(4)}$ and, in particular, $x \in C_S(W)$. Set $R = \langle x^L \rangle$. Then $Y_L \leq R \leq \langle x \rangle D$. As $[x, P_1] \leq Y_H \leq Y_L$, and as $C_{D/Y_L}(P_1) = 1$, lemma — [FROM J-MODULES PRELIM LEMMAS] implies that R/Y_L is of order 2. But then $[x, Q_L] = Y_L$, contrary to $|S/C_S(x)| = 2$. We conclude that no such element x exists, and this yields the following result.

(4) $C_H(O^2(H)) = Y_H$.

Let $d \in D - V$. In particular, we have $d \in C_{Q_H}(V)$. If $[d, Q_H] = Y_H$ then (2) implies that there exists $d' \in Vd$ such that $[d', O^2(H)] = 1$, and we contradict (4). Thus $[d, Q_H] = V$, and it follows from (3) that $|[Q_L \cap Q_H, d]| \geq 16$. As $[d, Q_L] \leq Y_L$, of order 8, we have a contradiction, and the lemma is proved. $\square$

[THE REMAINDER OF THIS SECTION (AND THE NEXT SECTION) WILL BE A CLOSE COPY OF BERND'S PAPER ON N-GROUPS, SECTION 10, TO PRODUCE THE AMALGAM FOR M_{12}, $Aut(M_{12})$, the Tits group, OR ${}^2F_4(2)$. AFTER THAT, IMITATE MICHAEL IN ORDER TO PRODUCE THE GROUPS.]