On the generic groups of p-type

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Contents

Definition 0.1 *Let G be a finite group, p a prime dividing. the order of G and $S \in \mathrm{Syl}_p(G)$. Then G is of generic p-type provided that* *dgpty*

(a) *If L is a p-local subgroup of G with $S \leq L$, then $F^*(L) = O_p(L)$.*

(b) *G is generated by the p-locals containing S.*

(c) *all p-locals of G are $\mathcal{K}$-groups.*

dqt

Definition 0.2 *1. A quasisimple group K is called a C_2 - group if and only if*
K is a quasisimple group of Lie type in characteristic 2 or
$K = PSL(2, q)$ for q a Fermat or Mersenne prime or $q = 9$
or $K = PSL(3,3), PSL(4,3), PSU(4,3), 2U(4,3)$ or $G_2(3)$
or $K/Z(K) = M_{11}, M_{12}, M_{22}, M_{23}, M_{24},\ J_2,\ J_3, J_4,\ HS, Suz, Ru,\ Co_1, Co_2,\ Fi_{22}, Fi_{23}, Fi'_{24}, F_3,\ F_2,$ or F_1
except $2A_8, Sp(4,3)$ and $[X]L_3(4)$ for exp $X = 4$ are not C_2 groups.

2. $L_2(G) = \{K :$ for some involution x of G, K is a component of $C_G(x)/O_{2'}(C_G(x))\}$

3. G is of even type if and only if the following conditions hold:
(a) Every element of $L_2(G)$ is a C_2 - group
(b) $O_{2'}(C_G(x)) = 1$ for every involution x of G; and
(c) $m_2(G) \geq 3$.

4. Let G be of even type and let S be a Sylow $2-$subgroup of G. Then
$\sigma(G) = \{p : p$ is an odd prime and $m_p(M) \geq 4$ for some maximal 2-local M of G with $|S : S \cap M| \leq 2\}$.

5. G is of quasithin type if G is a simple group of even type with $\sigma(G)$ empty.

dhead

Definition 0.3 $\mathrm{Head}(P) \overset{def}{=} O^p(P)O_p(P)/O_p(P)$.

1 Random Observations

Let G be a finite group, S the Sylow 2-subgroup of G and B the intersection of the maximal 2-locals containing M.

Borel

Lemma 1.1 *Let G be a finite group such that $F^*(G)$ is the direct product of simple groups of simple groups of Lie type in characteristic 2. Suppose that all the 2-locals of G containing S are of characteristic 2-type. Then S acts transitive on the set of components of G, $B = N_G(S \cap F^*(G))$ and $BF^*(G) = G$.*

Remark: False for $D_4(q).3$ and $D_4(q).Sym(3)$

Proof: Let $E_1, \dots, E_n$ be the components of G, $E = F^*(G) = E_1 E_2 \dots E_n$ and $T = E \cap S$. Suppose that S does not act transitively on the set of components of G. Then $\langle E_1, S\rangle$ is contained in a 2-local which is not of 2-type, a contradiction.

Let M be any maximal 2-local of G containing S. As M is of 2-type and $C_E(O_2(M)) \neq 1$ we conclude $O_2(M) \cap E \neq 1$.

Let Q_i be the projection of $O_2(M) \cap E$ onto E_i and $Q = Q_1 \cdot Q_2 \cdot \dots \cdot Q_n$. Then Q is a 2 group normalized by M and so $O_2(M) \leq Q \leq O_2(M)$, $Q = O_2(M)$ and $M = N_G(Q)$.

Suppose now that $n = 1$.

Let $M_i = N_M(E_i)$ and M_i^* a maximal 2-local subgroup of E_i containing $M \cap E_i$. Then $\langle M_i^{*M}\rangle \cap E_i = M_i$ and so $\langle M_i^*, M\rangle$ is contained in a 2-local of G. Thus $M_i^* = M \cap E_i$.

TO BE CONTINUED

Remark 1.2 *It seems that in groups of charateristic 2-type, B-irreducible subgroups actually have B as a maximal subgroup. For example if G has a parabolic P with $P/O_2(P) \cong Sym(5)$ then the the inverse image of the $Sym(4)$ seems always to be in the Borel group.*

2closed

Lemma 1.3 *For $L \in \mathcal{L}(= \mathcal{L}(S)$ put $Z_L = \langle \Omega_1(Z(S))^L$, $C_L = C_L(Z_L)$ and $L^* = N_L(S \cap C_L)$. Let $\mathcal{R} \subseteq \mathcal{L}$ put $R = \langle L^* \mid L \in \mathcal{R}\rangle$.*

(a) *$L = L^* C_L$ for all $L \in \mathcal{L}$.*

(b) *Let $L \in \mathcal{L}$ and $P \in calN(L, S)$. Then $P \leq L^*$ or $O^2(P) \leq C_L$.*

(c) *Let $L \in calL$. Then $O_2(L^*) = S \cap C_L$*

(d) *If $R \in \mathcal{L}$, then C_R is 2-closed and $R = R^*$.*

(e) *Let $\mathcal{R} = \mathcal{L}$*

e.1. Suppose $R \in \mathcal{L}$. Then for all $L \in \mathcal{L}$, $L = (R \cap L)(C \cap L)$.

e.2. Suppose that $R \notin \mathcal{L}$. Then there exists $\mathcal{R}_i \subseteq \mathcal{L}$, $i = 1, 2$ so that $R_i \in \mathcal{L}$ but $O_2(\langle R_1, R_2\rangle) = 1$.

Proof: (a) follows by the Frattini argument.

To prove (b) let $L \in \mathcal{R}$. Then $L^* \leq R$, $Z_L \leq Z_R$, and $S \cap C_R \leq C_L$. Thus $S \cap C_R = (S \cap C_L) \cap C_R$ and $S \cap C_R$ is normalized by L^*. As R is generated by the L^*'s, $L \in \mathcal{R}$, $S \cap C_R$ is normal in R and so also in C_R. Thus C_R is 2-closed and $R = R^*$.

(c) and (d) are obviuos.

(e.1) follows since from (a) as $L^* \leq R \cap L$ and $C_L \leq L \cap C$.

For (e.2) let for $\mathcal{R}_1$ be maximal in $\mathcal{L}$ with $R_1 \in \mathcal{L}$ and let $\mathcal{R}_2 = \{L\}$ for some $L \in \mathcal{L} \setminus \mathcal{R}_1$.

**gomi*

Lemma 1.4 *Let $R = R_{\mathcal{L}}$ and suppose that $R \in \mathcal{L}$.*

(a) *$N_G(Z_L)$ is the unique maximal 2-local of G containing R.*

(b) *Let* $L \in \mathcal{N}(R,S)$ *with* $O^2(L) \trianglelefteq \trianglelefteq R$ *and* $P \in \mathcal{N}(S)$ *with* $P \not\leq R$. *If* $\langle P, L\rangle \in \mathcal{L}$, *then* $O^2(P) \cap S \leq O_2(L)$.

(c) *Let* $P \in calN(S)$ *so that* P *does not normalize* Z_R. *Then there exists* $L \in \mathcal{N}(R,S)$ *with* $O^2(L) \trianglelefteq \trianglelefteq R$ *and* $\langle L, P\rangle \notin calL$.

Proof: (a) Let $R \leq M \in \mathcal{L}$. Then $M^* \leq R$ and so $R = M^*$ and $Z_M = Z_{M^*} = Z_R$. Thus $M \leq N_G(Z_R)$.

(b) Let $M = \langle P, L\rangle$. As $P \not\leq R$, $O^2(P) \leq C_M$. By 3.6 $[Z, L] \neq 1$ and so $S \cap O^2(P) \leq S \cap C_M \leq O_2(L)$.

(c) As $O_2(R)$ is the intersection of the $O_2(L)$'s, L as in the statement of (c) we conclude that $O^2(P) \cap S \leq O_2(R)$. Hence $O_2(R)$ is a Sylow 2-subgroup of $O^2(P)O_2(R)$. By (a) $\langle P, R\rangle$ is not a 2-local and we conclude that $\langle \Omega_1(O_2(R)^{O}_{2}(P)\rangle$ is an FF-module for $O^2(P)O_2(R)$. But this contradicts $[Z, P] = 1$.

$L+$

Lemma 1.5 *Let* $\mathcal{N}^+(S) = \{L \in \mathcal{N}(S) \mid [Z, L] \neq 1\}$ *and for* $L \in \mathcal{L}$ *put* $L^+ = \langle \mathcal{N}^+(L, S)\rangle$. *Then*

(a) $O_2(L^+) = S \cap C_L = O_2(L^*)$

(b) $L = L^+(L \cap C)$.

(c) $Z_L = Z_{L^+}$.

Proof: Put $T = S \cap C_L$ and $R = N_L(T)$. Then by 3.6 $F_2^*(R) \leq R^+$ an so $O_2(L^+) = O_2(F_2^*(R)$. As $O_2(L/C_L) = 1$, $O_2(F_2^*) = T$. So (a) holds.

For (b) suppose first that $C_L \neq O_2(L)$. By the Frattini argument, $L = RC_L$ and by induction $R = R^+(R \cap C)$. Hence $L = R^+C_L(R \cap C) = L^+(L \cap C)$.

So suppose that $C_L = O_2(L)$.Then $R = L$. Let $E = S \cap F_2^*(L)$ and $H = N_L(T)$. By the Frattini argument, $L = F_2^*(L)H$ and by induction, $H = H^+(H \cap C)$. Hence $L = F_2^*(L)H^+(H \cap C) = L^+(L \cap C)$..

(c) follows directly from (b)

Rgomi

Lemma 1.6 *Let* $\mathcal{N}^+(S) = \{L \in \mathcal{N}(S) \mid [Z, L] \neq 1\}$ *and* $D = \bigcap\{O_2(L^*) \mid L \in \mathcal{L}\}$.

(a) *Let* $P \in \mathcal{N}^+(S)$ *with* $P \not\leq N_G(D)$. *Then there exists* $L \in \mathcal{N}^+(S)$ *so that* $\langle P, L\rangle \notin \mathcal{L}$.

(b) *Let* $R^+ = \mathcal{N}^+(S)\rangle$ *and suppose that* $R^+ \in \mathcal{L}$.

(b.a) *For all* $L \in \mathcal{L}$, $L = (L \cap R^+)(L \cap C)$.

(b.b) *Suppose that* $R^+ \leq L \in \mathcal{L}$. *Then* $R^+ = L^+$, $Z_L = Z + R^+$ *and* $O_2(R^+) = C_L \cap S$.

(b.c) $O_2(R) = D$.

(b.d) $N_G(Z_{R^+})$ *is the unique maximal 2-local of* G *containing* R^+.

Proof: Suppose (a) is false. Let $L \in calN^+(S)$ and put $M = \langle L, S\rangle$. By assumption $M \in calL$ and so by 1.3b, $M = M^*$. Let $Y \in \mathcal{N}(M)$ with $O^2(Y) \trianglelefteq \trianglelefteq M$. Then by 3.6, $[Z, Y] \neq 1$ and so $Y \in \mathcal{N}^+(S)$. Hence the Gomi argument implies that P normalizes D.

(b.a) follows directly from 1.5b

Since

$$\mathcal{N}^+(S) \subseteq \mathcal{N}^+(R^+, S) \subseteq \mathcal{N}^+(L, S)\mathcal{N}^+(S),$$

$R^+ = L^+$. Thus by 1.5a, $O_2(R) = C_L \cap S$. Furthermore, by 1.5c, $Z_L = Z_{L^+} = Z_{R^+}$

(b.c) follows from 1.5a.

(b.d) follows directly from (b.b) □

dpred

Definition 1.7 *Let $L \in \mathcal{L}(S)$. Then a p-reduced normal subgroup of L is a elementary abelian normal p-subgroup Y of L so that $O_p(L/C_L(Y)) = 1$,(i.e all normal subgroups of L which act unipotently on Y already centralize Y.*

bpred

Lemma 1.8 *Let $L \leq \mathcal{L}(S)$.*

(a) *There exists a unique maximal p-reduced normal subgroup Y_L of L.*

(b) *Let $R \in (L, S)$ and X a p-reduced normal subgroup of R. Then $\langle X^L\rangle$ is a p-reduced normal subgroup of L. In particular, $Y_R \leq Y_L$.*

(c) *Let $S_L = C_S(Y_L)$ and $L^f = N_G(S_L)$. Then $S_L = O_p(L^f)$ and $Y_L = \Omega_1 Z(S_L)$.*

Proof: (a) Let Y_L be the subgroup generated by the p-reduced normal subgroups of L. Let N be a normal subgroup acting unipotently on Y_L. Then N also acts unipotently on all the generators of Y_L. Hence N centralizes all the generators of Y_L and so Y_L. Thus Y_L is p-reduced.

(c) Let $Y = \langle X^L\rangle$ and $C = C_L(Y)$. Let $N/C = O_p(L/C)$. Then $N = (N \cap S)C$ and in particular, $N = (N \cap L)C$. As X is p reduced, $N \cap L$ centralizes X. The same is true for C and so also for N. Since N is normal in L and $Y = \langle X^L\rangle$, N centralizes Y. Thus $N = C$ and Y is p-reduced.

(b) Put $C = C_L(Y_L)$. By Frattini, $L = L^f C$. Since $O_p(L/C) = 1$ we conclude $O_p(L_f) \leq C$ Hence $O_p(L_f) \leq C \cap S = S_L$ and so $O_p(L_f) = S_L$). Let $X = \Omega_1(Z(S_L))$. Then clearly $Y_L \leq X$ and L_f normalizes Y. Put $Y = \langle Y^L\rangle = \langle Y^C\rangle$. Clearly X is p-reduced for S_L and so by (b) applied to C, Y is p-reduced for C. Let N be a normal subgroup of L acting unipotently on Y. Since $Y_L \leq Y$ and Y_L is p-reduced for L, $N \leq C$. As Y is p-reduced for C, N centralizes C and so Y is p-reduced for L. By maximality of Y_L we get $Y \leq Y_L$. But $Y_L \leq X \leq Y$ and so $Y_L = X = Y$. □

2 Preliminaries

e

Lemma 2.1 *Let r and s be positive real numbers and put $e = \frac{rs^2-r-s}{s^2}$.*

(a) *Suppose that* $s > 1$. *Then* $e > O$ *if and only if* $r > \frac{s}{s^2-1}$. *In particular* $e > 0$ *if* $r \geq 2$ *and* $s \geq 1.3$.

(b) $e \leq 1$ *if and only if* $(r-1)(s-1) \leq 1$.

Proof: (a) is easily computed and for (b) note that the following are equivalent: $e \leq 1$, $rs^2 - r - s - s^2 \leq 0$, $(rs - r - s)(s + 1) \leq 0$, $rs - r - s \leq 0$, $(rs - r - s) + 1 \leq 1$ and $(r-1)(s-1) \leq 1$.

$G = Qt$

Lemma 2.2 *Let* $P \in \mathcal{P}(S)$ *be of weak* $L_2(2)^k$ *type. Put* $\Delta = \{L_i \mid 1 \leq i \leq k\}$ *and let* $Q \trianglelefteq S$ *such that*

(i) $|Z_P/C_{Z_P}(A)| < |A/C_A(Z_P)|^2$ *for some* $A \leq Q$ *with* $[Z_P, A] \neq 1$.

(ii) Q *contains an involution* t *acting fixed point freely on Delta.*

Then $O^2(P) \leq \langle C_Q(\Delta)^e, t\rangle$ *for some* $e \in P$.

Proof: Let $\Delta\{L_i \mid 1 \leq i \leq k\}$. Choose A as in (i) with $|A|$ minimal. Then it easy to see that A acts trivially on Δ. Next let T be maximal in $C_Q(\Delta)$ so that T fullfills $|Z_P/C_{Z_{P(T)}}| < |T/C_T(Z_P)|^2$. By [CD] T is unique and so $T \trianglelefteq S$. Let $E = O^2(P)C_P/C_P$. Then S acts irreducible on E and $E = E_1 \times \ldots \times E_k$ with $|[Z_p, E_i]| = 4$. We claim that each of the E_i is a Wedderburn component for T on E. Indeed, let E^* be a Wedderburn component for T on E and suppose that $E^* = E_1 \ldots E_t$. Then $k = lt$ for some integer l, $C_T(E^*) = C_T(E_1)$, $|T/C_T(E^*)| = 2$ and $|T/C_T(Z_P)| = |T/C_T(E)| \leq 2^l$. On the otherhand $Z_P/C_{Z_P}(T) = 2^k$. Thus $k < 2l$ and as l divides k, $l = k$.

We conclude that:

(1) Each T invariant subspace in E is a sum of some of the E_i's.

As t acts fixed point freely on Δ, t inverts an element $e \in O^2(P)$ with projects nontivially on each of the E_i's. Thus (1) implies

(2) $E = \langle \overline{e}^T \rangle$.

Let $L = \langle T^e, t\rangle$. Then $T^{e^{-1}} = (T^e)^t \leq L$ and so also $[T, e] \in L$. Since $C_E(T) = 1$, $\overline{e} \in [T, \overline{e}]$ and (2) implies that $E \leq \overline{L}$. Hence $P = LS$ and $O^2(P) \leq \langle T^P\rangle = \langle T^L\rangle \leq L$. As $T \leq C_Q(\Delta)$ the lemma is proved.

s

Lemma 2.3 *Let* H *be a group,* V, B *and* $Z_i \in I$ *subgroups of* H *and* s *a positve real number. Supppose that*

(i) $V = \langle Z_i \mid i \in I\rangle$ *and for all* $i \in I$, $Z_i \trianglelefteq V$.

(ii) *For all* i *in* I *and* $D \leq B$, B *normalizes* Z_i *and* $|D/C_D(Z_i)|^s \leq |Z_i/C_{Z_i}|$.

Then $|B/C_B(V)|^s \leq |V/C_V(B)|$.

Proof: Without loss $I - \{1, \ldots, n\}$. Let $B_1 = B$ and $B_{i+1} = C_{B_i}(Z_i)$. Then $B_{n+1} = C_B(V)$. Moreover, by (ii) applied to $D = B_i$,

$$|B_i/B_{i+1}|^s \leq |Z_i/C_{Z_i}(B_i)| \qquad (1).$$

Thus

$$|B/C_B(V)|^s \leq \prod_{i=1}^{n} |Z_i/C_{Z_i}(B_i)| \qquad (2).$$

As by defintion B_{i+1} centralizes Z_i we get

$$|Z_i/C_{Z_i}(B_i)| = |Z_i C_V(B_i)/C_V(B_i) \leq |C_V(B_{i+1}/C_V(B_i)| \qquad (3).$$

Thus

$$\prod_{i=1}^{n} |Z_i/C_{Z_i}(B_i)| \leq |C_V(B_{i+1})/C_V(B_i)| = |V/C_V(B)|. \qquad (4).$$

The lemma now follows from (2) and (4).

qrc

Lemma 2.4 *Let* $V = \langle W_i \mid i \in I \rangle$, *where* W_i *is a normal subgroup of* V *for all* $i \in I$. *Let* B *be a subgroup of* A *normalizing all the* W_i*'s. If* $A \neq B$ *define* r *by* $|A/B|^r = |V/C_V(A)|$ *and* t *by* $|V/C_V(A)|^t = |A/C_A(V)|$. *Let* $I = \{1, 2, \ldots n\}$ *and define* $A_0 = B$ *and inductively* $A_i = C_{A_{i-1}}(V_i)$. *Choose notation so that* $B = A_0 > A_1 > \ldots > A_k = C_A(V)$. *Define* s_i *by* $|A_{i-1}/A_i|^{s_i} = |W_i/C_{W_i}(A_{i-1})|$ *and* $s = \min_{i=1}^{k} s_i$. *Then*

(a) $|B/C_B(V)|^s \leq |V/C_V(B)|$.

(b) *If* $A \neq B$, *then* $trs \leq r + s$.

(c) *Suppose that* $A \neq B$ *and equality holds in* (b). Then

(c.a) $s_i = s$ for all $1 \leq i \leq k$.

(c.b) $C_V(B) = C_V(A)$.

(c.c) $|B/V_B(V)|^s = |V/C_V(B)|$.

Proof: (a) follows from 2.3.

Note that $|A/B|^{rt} = |V/C_V(A)|^t = |A/C_A(V)| = |A/B||B/C_B(V)|$ and therefore $|B/C_B(V)| = |A/B|^{rt-1}$. Suppose that $A \neq B$. By (a) we conclude

$$|A/B|^r = |V/C_V(A)| \leq |V/C_V(B)| \leq |B/C_B(V)|^s = |A/B|^{(rt-1)s}$$

and so $(rt-1)s \leq r$ and $rts \leq r + s$.

(c) follows by investigating the places where " $<$ " was used.

msn

Lemma 2.5 *Let H be a finite group, P a p-subgroup of H and suppose that P is subnormal in all proper subgroups of H containing P, but is not subnormal in H. Then A is contained in a unique maximal subgoup of H.*

Proof: Suppose that A is contained in two distinct maximal subgroups M_1 and M_2. Choose the M_i's so that M_1 contains a Sylow p-subgroup of H and so that $|M_1 \cap M_2|_p$ is maximal. Let D be a Sylow p-subgroup of $M_1 \cap M_2$ and put $B_i = \langle A^h \mid h \in H, A^h \leq M_i \rangle$. Then by asumption $B_i \leq O_p(M_i) \leq M_j$.

Suppose that D is not a Sylow p-subgroup of M_2. Then $M_{M_2}(D) \not\leq M_1$ and $|N_{M_2}(D) \cap M_1|_2 > |D|$, a contradiction. Thus D is a Sylow p-subgroup of M_2 and so $B_2 \leq D$ and $N_G(D)$ normalizes B_2. Thus $N_G(D) \leq M_2$ and so D is also a Sylow p-subgroup of M_1. Hence $B_1 \leq D$ and $B_1 = B_2$, a contradiction.

almp

Lemma 2.6 *Let H be a finite group, p a prime, S a Sylow p-subgroup of H and suppose that S lies in a unique maximal subgroup M of H. Let $P \leq S$ and suppose that $P \not\leq O_p(H)$. Then there exist a subgroup L of H and $h \in H$ so that*

(a) *$P \leq L$ and $P \not\leq O_p(L)$*

(b) *$M^h \cap L$ is the unique maximal subgroup of L containing P.*

(c) *$S^h \cap L$ is a Sylow p-subgroup of L.*

Proof: If M is the unique maximal subgroup of H containg P, then the lemma holds with $L = H$ and $h = 1$. Hence there exists a proper subgroup K of H such that $P \leq K$ and $K \not\leq M$. Choose K so that $|M \cap S|_p$ is maximal and then with K minimal. Let $T = M \cap K$ and $R = \langle P^G \cap T \rangle$. Let $S^* \in \mathrm{Syl}_p(M)$ with $T \leq S^*$. Then M is the unique maximal subgroup of H containing S^* and so $T \neq S^*$. Thus $T < N_{S^*}(T) \leq N_H(R)$ and $|M \cap K|_p < |M \cap N_H(R)|_p$. Thus by the choice of K, $N_H(R) \leq M$. In particular, $N_K(R) \leq K \cap M$ and so T is a Sylow p-subgroup of K. Hence $O_p(L) \leq T \leq M$. If $R \leq O_p(K)$, then $R \trianglelefteq K$, contradiction. $P^* \in P^H \cap T$ with $P^* \not\leq O_p(K)$. By the minimal choice of $|K|$, $M \cap K$ is the unique maximal subgroup of K containing T and so we can apply induction. Thus there exists $L^* \leq K$ with $P^* \leq L^*$, $P^* \not\leq O_p(L^*)$ and $h^* \in K$ so that $(M \cap K)^{h^*} \cap L^*$ is the unique maximal subgroup of L^* containing P^*. Let $x \in H$ with $P^{*x} = P$ and put $h = h^*x$ and $L = L^*x$. The clearly (a) and (b) hold. □

anc

Lemma 2.7 Remark: Quadratic groous normalize components

mqm

Lemma 2.8 *Let $A \leq H$ and V a faithful $GF(p)H$-module. Suppose that*

(i) *A is contained in a unique maximal subgroup of H.*

(ii) *$[V, A, A] = 1$.*

(iii) *$A \not\leq O_p(H)$*

(iv) *One of the following holds:*

1. $V = \langle Z^H \rangle$ *for some* $Z \leq V$ *with* $[Z, A] = 1$.
2. $V = C_V(A)[V, H]$.

Let $t \in A \setminus O_p(H)$. *Then each of the following holds:*

(a) *Then* $C_V(t) = C_V(A)$.

(b) $|V/C_V(A)| \geq |A/A \cap O_p(H)|^c$, *where* c *is the number of non-trivial chief-factors for* H *on* V.

(c) $[V, t] \cap C_V(H) = 1$ *and* $|[V, t]^2| = |V/C_V(H)$.

(d) *Suppose that (iv)1 holds and* $O_p(L)$ *normalizes* Z. *Then one of the follwing holds:*

1. $[V, A \cap O_p(H)] \leq C_V(H)$.
NI2 $p = 2$, $H/O_p(H) \cong Dih(2r^k)$, r *an odd prime* $C_H([V, A \cap O_p(H)]) \not\leq O_p(H)$.

(e) $[V, H] \cap C_V(H) \leq [V, A]$

(f) $W = C_W(H)[W, H]$ *for each* H*-section on* V. *In particular,* H *has no central chief-factor on* $V/C_V(H)$.

Proof: Note first that (iv)1. implies (iv)2. So we assume from now on that (iv)2. holds. Let M be the unique maximal subgroup of H containg A and $N = \mathrm{Core}_M(G)$. By a Frattini argument, N is p-closed with $O_p(H)$ as the Sylow p-subgroup. Hence $t \notin N$ and so there exists $h \in H$ with $t \notin M^h$. Put $B = A^h$. Then $H = \langle t, B \rangle$ and so $[V, H] = [V, t][V, B]$. By (iv)2. we conclude and (ii) we conclude

$$V = C_V(A)[V, B] = C_V(t)[V, B].$$

Thus

$$C_V(B) = [V, B](C_V(A) \cap C_V(A) = [V, B]C_V(H).$$

Hence also

$$C_V(A) = [V, A]C_V(H)$$

and so by (iv)2.,

$$V = C_V(H)[V, H].$$

That is (f) holds for $W = V$. Moreover, $C_V(t) = C_V(A)(C_V([V, B]) \cap C_V(t)) = C_V(A)$ and so (a) holds. Let $Y = [V, A] \cap C_V(H) = [V, B] \cap C_V(H) = [V, A] \cap [V, B]$. Then $[V, A] = [V, H] \cap [V, A] = [V, t]([V, A] \cap [V, B]$ and so $[V, A] = [V, t]Y$. On the otherhand,

$$|[V,t]| = |[V,B,t]| = |[V,B]/([V,B] \cap C_V(t)) = |[V,B]/Y| = |[V,A]/Y|$$

and so $[V,A] = [V,t] \oplus Y$. In particular $[V,t] \cap C_V(H) = 1$. Moreover $|[V,H]| = |[V,t]^2|Y|$. $C_{[V,H]}(A) = [V,A]$ and so $C_{[V,H]}(H) = Y$. Thus (c) and (e) hold. Let W be an non-trivial chief-factor for H on V. Since $H = A\langle t^H \rangle$, $A/O_p(H)/O_p(H)$ acts faithfully on W. Also $W = [W,A] \oplus [W,B]$ and so $|W/C_W(A)| = |[W,A]|$. Let $x \in W \setminus C_W(A)$. By (a) $|AO_p(H)/O_p(H)| = |[x,A]| \leq |[W,A]| = |W/C_W(A)|$. Thus (b) holds. Clearly (iv)2 is inherited by quotients of V so it is enough to verify (f) for H-submodules W of V. By (d) applied to $V/[W,H]$, $W \leq [V,A][W,H]$ and so $W = ([V,A] \cap W)[W,H]$ fulfills (iv)2. Thus (f) holds.

It remains to prove (d). Let $h \in H \setminus M$. As A is quadratic, A centralizes $[Z^h, A \cap O_p(H)]$. As $O_p(H)$ normalizes Z^h, also A^h centralizes $[Z^h, A \cap O_p(H)$. Since $M \neq M^h$, $H = \langle A, A^h \rangle$ and $[Z^h, A \cap O_p(H) \leq C_V(H)$.

$$Y = \langle Z^h \mid h \in H \setminus M \rangle.$$

Then $[Y, A \cap O_p(H)] \leq C_V(H)$.

Suppose first that $|AO_p(H)/O_p(H) \geq 3$. We claim then that B normalizes Y. For this let $h \in H \setminus M$ and $b \in B$. We need to show that $Z^{hb} \leq Y$. If $hb \notin M$, this is true by definition of Y. So suppose that $hb \in M$. Since $|AO_p(H)/O_p(H) \geq 3$ there exists $c \in B$ with $c \notin O_p(H) \cup O_p(H)b$. If $hc \in M$, then $b^{-1}c \in B \cap M$. But by 2.9 (10), $b^{-1}c \in O_p(H)$, a contradcition. Thus $hc \notin M$. Similarly $hbc \notin M$. Thus $Z^h Z^{hbc} Z^{hc} \leq Y$. Since $Z^h Z^{hbc} Z^{hc} = Z^h [Z^h, bc][Z^h, c]$, the quadratic action of B implies that $\langle bc, c \rangle$ normalizes $Z^h Z^{hbc} Z^{hc}$. Hence $Z^{hb} \leq Y$ as claimed.

Suppose next that $|AO_p(H)/O_p(H)| = 2$. Then $p = 2$ and $H/O_2(H) \cong Dih(2r^k)$. If $k = 1$, then $M = AO_p(H)$ normalizes Z and so $V = ZY$ and again d1 and as a matter of fact also d2 holds. So suppose $k > 1$ and define L as in d2. Then $L \leq M$. Also let H^* be minimal with $A \leq H^*$ and $H^* O_p(H) = M$. Let $V^* = \langle Z^{H^*} = Z^M$. Then $V = V^* Y$. Also $A \cap O_p(H) \leq O_p(H^*)$ and so by induction $R \overset{def}{=} C_{H^*}([V^*, A \cap O_p(H)] \not\leq O_p(H^*)$. Since $[V, A \cap O_p(H)] = [V^*, A \cap O_p(H)][Y, A \cap O_p(H]$ we have $[V, A \cap O_p(H), R] = 0$. Since $R \not\leq O_p(H)$, d2 holds in this case.

mq

Lemma 2.9 *Let H be a finite group, p a prime, A a p-subgroup of H and V a faithful $GF(p)H$-module. Suppose that $A \not\leq O_p(H)$, that A acts quadratically on V and that A lies in a unique maximal subgroup of H. Then one of the following holds for $\overline{H} = H/O_p(H)$:*

1. $\overline{H} \cong SL_2(p^k)$.
2. $p = 2$ *and* $\overline{H} \cong Sz(2^k)$.
3. $p = 2$ *and* $\overline{H} \cong Dih(r^k)$, *$r$ an odd prime.*

Proof: Let M the the unique maximal subgroupp of H containing A and $D = \bigcap_{h\in H} M^h$. Note that M contains a Sylow p-subgroup S of H and so $O_p(H) \leq D$. Replacing V by the direct sum of the H-composition factors on V and H by $\overline{H}$ we may assume that $O_p(H) = 1$. Moreover, if $|A| = 2$, 3. holds so we may assume $|A| > 2$.

Let T be an A invariant Sylow p-subgroup of D. Then $H = DN_H(T)$. If $H = N_H(T)$ we get $N_H(T) \leq M$ and so $H = DM \leq M$, a contradiction. Hence $T \trianglelefteq H$ and so $T \leq O_p(H) = 1$. Thus D is a p'-group. Let R be a maximal subgroup of H and suppose that $D \not\leq R$. Then $H = DR$ and so R contains a Sylow p-subgroup of H. Hence $A \leq R^h$ for some $h \in H$ and thus $R^h \leq M$. But then $H = DR = DR^h \leq M$, a contradiction. Thus $D \leq R$. It follows that

(1) $D \leq \Phi(H)$ and D is a nilpotent p' group. $mq-1$

Let N be a normal subgroup of H. If $H \neq NA$ then $NA \leq M$ and so $N \leq D$. Put $L = O^p(H)$ and suppose that $L \leq D$. Then $H = DS \leq M$, a contradiction. Thus $L \not\leq D$, $H = LA$. Hence:

(2) Each normal subgroup of H is either contained in D or contains L. In particular, L/D is characteristicly simple. $mq-2$

Since H acts faithfully on $[V, O^p(H)]$ and on $V/C_V(O^p(H))$ we may assume that

(3) $V = [V, H]$ and $C_V(H) = 0$. $mq-3$

Let $1 \neq a \in A$ and pick $g \in H$ with $a \not\leq M^g$. Then $H = \langle a, A^g\rangle$ and so by (3) $V = [V, a] + [V, A^g]$ and $C_V(a) \cap C_V(A^g)$. Since A is quadratically on V we also have

$$[V, a] \leq [V, A] \leq C_V(A) \leq C_V(a).$$

We conclude that

(4) $[V, a] = [V, A] = C_V(A) = C_V(a)$ and $|V| = |[V, A]|^2$ $mq-4$

With a similar argument:

(5) $C_V(b) = [V, b]$ for each non-trivial quadratic element b in H. $mq-4'$

We may assume without loss that A is a maximal quadratic subgroup of H and so

(6) $A = C_H([V, A]) \cap C_H(V/[V, A])$ $mq-5$

From (4) and (6) we conclude that

(7) $C_H(a) \leq N_H(A)$ and $A \cap A^h = 1$ for all $h \in H \setminus N_H(A)$. $mq-6$

Let $h \in H$ with $A \cap M^h \neq 1$ and let $b \in A \cap M^h$. Choose $k \in M^h$ so that $\langle b, A^{hk} \rangle$ is a p-group. Then $C_V(b) \cap C_V(A^{hk}) \neq 0$ and so also $V_V(A) \cap C_V(A^{hk}) \neq)$. Thus $H \neq A, A^{hk} \rangle$ and so $M = M^{hk} = M^h$. We proved

$mq - 7$

(8) Let $h \in H$. Then $h \in M$ or $A \cap M^h = 1$.

If p is odd, then by (5)

$$\dim[V, A] = \min\{\dim[V, b] \mid 1 \neq b \in H, [V, b, b] = 0\}$$

Hence by the work of Thompson and Ho, $H \cong SL_2(p^k)$ or $p = 3$ and $H \cong 2{}^{\cdot}Alt(5)$. But in latter case, A lies in more than one maximal subgroup of H, a contradiction.

Thus we may assume from now on that

$mq - 8$

(9) $p = 2$ and $|A| \geq 4$.

In particular, by (7)

$$O_{p'}(H) = \langle C_{O_{p'}(H)}(a) \mid 1 \neq 1 \in A \rangle \leq C_H(A).$$

and we conclude:

$mq - 7$

(10) $D = Z(H)$ and $L = E(L) = E(H)$.

Note that the exceptionell case in 2.7 is not possible and so A normalizes the components of L and thus

$mq - 8$

(11) L is quasisimple.

None of the groups in **??** is a minimal parabolic and so L is an alternating group or a Lie type in characteristic 2. Since S lies in a unique maximal subgroup of H we get $L \cong Alt(2^k + 1), L_2(2^k), SU_3(2^k), Sz(2^k), SL_3(2^k)$ or $Sp_4(2^k)$. In the last two cases A has to induce a graph automorphism on L, which contradicts the quadratic action of A on V. If $L \cong Alt(2^k + 1)$, A either is contained just has one non-trivial orbit and that one has lenght four or all orbits of A have length at most 2. Since A lies in a unique maximal subgroup of H we conclude that $L = H \cong Alt(5) \cong SL_2(4)$. If $L \cong SU_3(2^k)$, A lies in the normalizer of a Sylow 2-subgroup and in a $SL_2(2^k)$, a contradiction, which completes the proof of the lemma.

$factorize$

Lemma 2.10 *Let G be a finite group, $M \leq G$, p a prime with $F^*(M) = O_p(M)$ and $T \in \mathrm{Syl}_p(M)$. Let $Z_M = \langle \Omega_1(Z(T))^M \rangle$, $C_M = C_M(Z_M)$ and $J(M) = \langle J(T)^M \rangle$.*

(a) $C_M \leq N_G(Z_T)$

(b) *Z_M is a faithful $J(M)C_M/C_M$-module and $J(M)C_M/C_M = \mathrm{P}^*(J(M)C_M/C_M), Z_M)$.*

(c) $M/J(M) \cong N_M(J(T)/N_{J(M)}(J(T)$

(d) *Suppose that T is normal in a Sylow p-subgroup S of G. Then $N_G(Z(T)) \in \mathcal{L}(S)$ and $N_G(J(T)) \in \mathcal{L}(S)$.*

Proof: Obvious.

vqnhg

Lemma 2.11 *Let G be a finite group, $N \trianglelefteq H \leq G$, p a prime, $S \in \mathrm{Syl}_p(H)$, V an elementary abelian normal p-subgroup of H, and $C_S(V) \leq Q \leq S \cap N$ Suppose that $\mathcal{A}(Q)^G \cap \not\subseteq N$, then there exists an elementary abelian subgroup A of S with $H \not\leq N$, $[V, A] \neq 1$ and $|V/C_V(A)| \leq |A/C_A(V)|$.*

Proof: Let $D \in \mathcal{A}(Q)$ and $g \in G$ with $D^g \leq H$ and $D^g \not\leq N$. As S is a Sylow p-subgroup of H there exists $h \in H$ with $D^{gh} \leq S$. Put $A = D^{gh}$. As N is normal in H, $A \not\leq N$. Since $C_N(V) \leq Q \leq N$, $[V, A] \neq 1$. Moreover, $VC_A(V) \leq Q$ and so $|VC_A(V)| \leq |A|$.

i2lt

Lemma 2.12 *Let L be an alternating group or simple group of Lie-type in characteristic 2. Let $H \leq L$ with $|L|_2/|H|_2 \leq 2$. Then all non abelian composition factors of H are alternating or a simple groups of Lie type.*

Proof: Let $T \leq \mathrm{Syl}_2(H)$, and $S \leq \mathrm{Syl}_2(L)$ with $T \leq S$. Then $S' \leq T$.

Suppose first that $L = \mathrm{Alt}(\Omega_)$. If H is intransitive or imprimitive we are done by induction. So suppose that H is primitive. If H has a non-trivial abelian normal subgroup A, then $H = H_iA$ for any $i \in \Omega$. Thus T_i has index two in a Sylow 2-subgroup of L_i and again we are done by induction.

Hence we may assume that H has no non-trivial solvable normal subgroup. Since $|S/T| \leq 2$, T contains an element x of cycle type $(2, 2)$. Since $x \notin O_2(H)$, $1 \neq x \cdot x^h$ has odd order for some $h \in H$. Its is now straight forward to verify the lemma.

So suppose L is a group of Lie type. and not an alternating group. If $O_2(H) \neq 1$, then H is contained in a parabolic subgroup of L and the lemma follows by induction. Hence we may assume that $O_2(H) = 1$.

If S is abelian, $L \cong L_2(q)$ and the result is readily verified in this case.

So we may assume that S is not abelian. In particular, S' and so also H contains a long root group R with $R \leq Z(S)$. As $R \not\leq O_2(H)$, there exists $h \in H$ with $X \overset{def}{=} \langle R, R^h \rangle \cong SL_2(q)$, where $q = |R|$. Let r be the highest ling root in the root system associate to L. Without loss $\omega_r \in X \leq H$. It is now easy to verify that $L = \langle S' \omega_r \rangle$ and so $L \leq H$, a contradiction.

Remark: this is rather scetchy

3 CS generated modules

In this section G is a finite group, p a prime and V a (finite dimensional) $GF(p)$G-module.

dcvg

Definition 3.1 (a) ${}_GV = \langle C_V(S) \mid S \in \mathrm{Syl}_p(G)\rangle$.

(b) V *is called* CS*-generated provided that* $V = {}_GV$.

ec

Lemma 3.2 *Let* $L \lhd \lhd G$. *Then* ${}_GV \leq {}_L(V)$.

Proof: Let $S \in \mathrm{Syl}_p(G)$. Then $S \cap L \leq \mathrm{Syl}_p(L)$ and $C_V(S) \leq C_V(S \cap L)$.

dualtrans

Lemma 3.3 *Let* p *be a prime,* G *a finite group,* L *a normal subgroup of* G, $S \in Syl_2(G)$. *Then* S *normalizes a complement to* $C_V(L)$ *in* $C_V(S \cap L)$.

Proof: Remark: This is a standard result in cohomology, the map π below is called the corestriction map, a reference should be included

Let $T = S \cap L$, $\mathcal{X}$ a set of right coset representatives for T in L and define

$$\begin{array}{rccc} \pi : & C_V(T) & \to & V \\ & v & \to & \sum_{x \in \mathcal{X}} v^{Tx} \end{array}$$

Then clearly $\pi(v) = \pi(v^l)$ for al $l \in L$ and so $\pi(C_V(T) \leq C_V(L)$. On the otherhand π restricted to $C_V(L)$ is just multiplication by L/T. Thus $\pi\mid_{C_V(L)}$ is an isomorphism and $C_V(T) = C_V(L) \oplus \ker \pi$. Moreover, it follows immediately from the definition of π that for all $v \in C_V(T)$ and $s \in S$, $\pi(v^s) = \pi(v)^s$. Thus S normalizes $\ker \pi$.

cc

Lemma 3.4 *Let* $L \lhd \lhd G$ *with* $[C_V(S), L] = 1$, *then* $[C_V(L \cap S), L] = 1$.

Proof: Clearly we may assume that $L \trianglelefteq G$. By 3.3 there exists an S invariant complement D to $C_V(L)$ in $C_V(S \cap L)$. Moreover, $C_D(S) \leq C_V(S) \leq C_V(G) \leq C_V(L)$ and so $C_D(S) = 0$. This implies $D = 0$ and $C_V(S \cap L) = C_V(L)$

c2

Lemma 3.5 *Let* L *be subnormal subgroup of* G. *If* $[C_V(S), L] = 1$ *then* $[{}_GV, L] = 1$.

Proof: By 3.4 $C_V(S \cap L) \leq C_V(L)$. So L centralizes ${}_LV$ and hence the lemma follows from 3.2.

c

Lemma 3.6 *Let* $L \lhd \lhd G$. *Then* $L \cap C_G({}_GV) = C_L({}_LV)$.

Proof: Let $L^* = C_L({}_LV$ and $L_* = C_G({}_GV)$. By 3.2 $L^* \leq L_*$. Moreover, L_* is subnormal in G and centralizes $C_V(S)$. Thus by 3.4 L_* centralizes $C_V(L_* \cap S)$. By 3.2 ${}_LV \leq {}_{L^*}V = C_V(L_* \cap S)$ and so $L_* \leq L^*$.

[V, L]

Lemma 3.7 *Let* $L \trianglelefteq G$ *with* $G = LC_G(L)$. *If* V *is* CS*-generated then* $[V, L]$ *is a* CS*-generated* G*-module and* $V = [V, L]_G C_V(L)$

Proof: Let $S \in \mathrm{Syl}_p(G)$, $T = S \cap L$, $R = S \cap C_G(L)$ and put $W = {}_LC_V(R)$. Then by Gaschütz theorem $W = [W, L]C_W(L)$. Moreover, $C_W(T) = C_{[W,L}(T)C_W(L)$. It follows that $[V, L] = \langle C_{[W,T}(T)^G\rangle$ and $[V, L]$ is a CS geneated G-module. Moreover, $V = \langle W^G\rangle = [V, L]\langle C_W(L)^G\rangle$ and so $V = [V, L]_GC_V(L)$.

gaschuetz

Lemma 3.8 *Suppose that* $G = \Pi_{i\in I}L_i$ *for some subgroups* $L_i \leq G$ *such that* $[L_i, L_j] = 1$ *whenever* $i, j \in I, i \neq j$. *For* $\Delta \subseteq I$ *let* $L_\delta = \langle L_i \mid i \in \Delta$ *and*

$$V_\delta = [{}_GC_V(L_{I\setminus\Delta}, L_{i_1}, L_{i_2}, \dots L_{i_r}]$$

where $r = |\Delta|$ *and* $\Delta = \{i_1, \dots l_r\}$. *(Note that by the Three Subgroup Lemma this defintion is independent form the order in which the* i_j*'s are chosen). Also put* $V_\emptyset = C_V(G)$.

Suppose that V *is a* CS*-generated* $GF(p)G$*-modules. Then*

$$* \qquad V = \sum_{\Delta\subseteq I} V_\Delta.$$

Moreover, each of the V_Δ*'s is* CS*-generated as* G*-module.*

Proof: By 3.7 The V_δ^s are CS-generated as G-module and it remains tp prove (*). For this we may assume without loss that V is not the direct sum of two proper CS-generated G-submodules. Let $\Delta = \{i \in I \mid [V, L_i] \neq O$ and let $i \in \Delta$. 3.7 implies $V = [V, L_i]_GC_V(L_i)$ with both summands CS generated. Hence $V = [V, L_i]$ and $V = V_\delta$.

4 Groups with $m_{2'}(G) \leq 3$

gwm3
epc2r3

Lemma$_{QT}$ 4.1 *Let* p *be an odd prime,* P *a* p *group of exponent* p*, class at most two and rank at most three. Then* $P \cong E_{p^i}, i \leq 3, Ex(p^{1+2i}), i \leq 2$ *or* $C_p \times Ex(p^{1+2})$.

Proof: [As, 3.1,3.2]

l3p

Lemma$_{QT}$ 4.2 *Let* p *be an odd prime,* G *a irreducible subgroup of* $GL_3(p)$ *and* $\Lambda = Z(GL_3(p))$ *Then there exists an irreducible normal subgroup* H *of* G *so that one of following holds.*

1. $H = SL(V) \cong SL_3(p)$.
2. $H = \Omega(V, q)$ *for some non degenerate quadratic form* q *on* V.
3. $H \cong Alt(5)$, $p^2 \equiv 1 \mathrm{mod} 10$ *and* $G \leq \Lambda \times H$.
4. $H \cong L_3(2)$, $p^3 \equiv 1 \mathrm{mod} 7$ *and* $G \leq \Lambda \times H$.
5. $H \cong 3^{\cdot}Alt(6)$, $p \equiv 1, 19 \mathrm{mod} 30$ *and* $G \leq \Lambda H$.

6. *H is cyclic of order dividing* p^3-1 *but not* $p-1$ *and* $H=G$ *or* $|G/H|\cong C_3$.

7. $H\cong Ex(3^{1+2})$ *and* $G\Lambda/H\Lambda\leq SL_2(3)$.

8. *G is monomial*

Proof: [As, 3.12]

msp4p

Lemma$_{QT}$ **4.3** *Let p be an odd prime, V a four dimensional non-degenerate symplectic space over* $GF(p)$ *and G a maximal subgroup of* $Sp(V)$. *Then one of the following holds.*

(a) *G is the normalizer of a singular* 1*-space in V and* $G\sim Ext(p^{1+2}):(C_{p-1}\times SL_2(p))$.

(b) *G is the normalizer of a singular* 2*-space in V and* $G\sim E_{p^3}:GL_2(p)$

(c) $G\sim SL_2(p^2).2$ *and* G' *fixes a non-degenerated 2-dimensional sympectic form over* $GF(p^2)$ *on V.*

(d) $G\cong SL_2(p)\wr C_2$ *and G fixes a decompostion of V into the orthorgonal sum of two non-degenerated 2-dimensional subspaces.*

(e) $G\sim GL_2(p).2$ *and G fixes a decomposition of V into the direct sum of two singular 2-spaces.*

(f) $G\sim GU_2(p).2\sim(C_{p+1}\cdot SL_2(p)).2$ *and the subgroup of index 2 fixes a non-degenerate 2-dimensional unitary form over* $GF(p^2)$ *on V.*

(g) $G\cong SL_2(p)$ *and V is the third symmetric power of the natural module for G.*

(h) $G\sim Ext_-(2^{1+4}).Alt(5)(.2)$.

(i) $G\sim 2{}^{\cdot}Alt(6)(.2)$ *and V is the half-spin module for G*

(j) $p=7$, $G\sim 2{}^{\cdot}Alt(7)$ *and V is the half-spinmodule for G*

Proof: See [Mi, Theorem 10]. We remark that this list can be easily checked if one is only interested in K-groups. Namely let W be the natural $\Omega_5(p)$ module for $PSp_4(p)$, $H=Sp_4(p)$ and $\overline{H}=H/Z(H)$. We may assume that G acts irreducible on W.

If Sol() $\neq 1$ let A be a minimal solvable normal subgroup of $\overline{G}$. If A is cyclic, $|A|$ divides p^5-1 and $|H|$. Hence $|A|$ divides $p-1$ and A acts as a scalar on W, a contradiction. So A is not cyclic and it is now easy to see that (h) holds.

If $\mathrm{Sol}(\overline{G})=1$, let E be a component of G. Since $O_2^{\pm}(p)$ is solvable, $[W,E]C_W(E)/C_W(E)$ is at least three dimensional. It follows that $C_H(G)$ is solvable and so $EZ(H)=F^*(G)$ and E acts irreducibly on W. If $Z(H)\not\leq E$, $m_2(Z(H)E)\geq 3$, a contradiction to $m_2(Z(H))=2$. Thus $Z(E)=Z(H)$. Let V be the natural $Sp_4(p)$ module for H. If E does not act irreducible on V then since $V\bigwedge V=W\oplus GF(p)$, E is not irreducible on W. So E acts irreducible on W. Using the list of finite simple groups its now easy to verify that one of (g),(i) or (j) holds or that $E\cong 2{}^{\cdot}Alt(5)$. But in the latter case, G is contained in a subgroup of type (i) or (j).

sp4p

Lemma$_{QT}$ **4.4** *Let p be an odd prime, V a four dimensional non-degenerate symplectic space over $GF(p)$ and $G \leq Sp(V)$ with $O_p(G) = 1$.*

(a) *If $G = O^{p'}(G) \neq 1$, then one of the following holds:*

1. $G \cong Sp_4(p), SL_2(p^2), SL_2(p) \times SL_2(p)$ *or* $SL_2(p)$
2. $p = 7$ *and* $G \cong 2\cdot Alt(7)$.
3. $p = 5$ *and* $G \sim 2\cdot Alt(5), Ext_-(2^{1+4}).Alt(5),\ Ext_(2^{1+4}).C_5$.
4. $p = 3$ *and* $G \sim 2\cdot Alt(5), Ext_-(2^{1+4}).Alt(5),\ Ext_(2^{1+4}).C_3$.

(b) *If G is quasisimple then one of the following holds:*

1. $G \cong Sp_4(p), SL_2(p^2)$ *or* $SL_2(p)$.
2. $G \cong 2\cdot Alt(5)$ *or* $2\cdot Alt(6)$.
3. $G \cong 2\cdot Alt(7)$ *and* $p = 7$.

Proof: [As, 3.13]

mp3Q

Lemma$_{QT}$ **4.5** *Let p be an odd prime, G a group with $F^*(G) = O_p(G) \overset{def}{=} Q$, $m(Q) \leq 3$ and $G^* = G/Q$.*

(a) *If $G = O^{p'}(G) \neq Q$, then one of the following holds:*

1. $G^* \cong SL_2(p)$ *or* $SL_3(p)$.
2. $G^* \cong SL_2(p) \times SL_2(p), SL_2(p^2),$ *or* $Sp_4(q)$ *and* $m_p(G) > 3$.
3. $p = 7$ *and* $G^* \cong 2\cdot Alt(7)$.
4. $p = 5$ *and* $G \sim SL_2(5), Ext_-(2^{1+4}).Alt(5)$ *or* $Ext_-(2^{1+4}).C_5$.
5. $p = 3$, $G \sim 2\cdot Alt(5)$ *or* $Ext_-(2^{1+4}.Alt(5)$ *and* $m_3(G) > 3$.
6. $p = 3$ *and* $G \sim Ext_(2^{1+4}).C_3$

(b) *If G^* is quasisimple then one of the following holds:*

1. $G^* \cong Sp_4(p)$, *or* $SL_2(p^2)$ *and* $m_p(G) > 3$.
2. $G^* \cong L_2(p), SL_2(p) or SL_3(p)$
   Remark: $SL_3(p)$ **also should have** $m_p(G) > 3$
3. $G^* \cong Alt(5), 2\cdot Alt(5)$ *or* $2\cdot Alt(6)$. *Moreover, if $p = 3$ then $m_3(G) > 3$.*
4. $G^* \cong L_3(2)$ *and* $p^3 \equiv 1 \bmod 7$
5. $G^* \cong 3\cdot Alt(6)$ *and* $p \equiv 1, 19 \bmod 30$
6. $G \cong 2\cdot Alt(7)$ *and* $p = 7$.

Proof: By [As, 3.13] we only need to show that $m_p(G) > 3$ in a.5, b.1 and for $p = 3$ in $b.3$. As in Aschbacher's proof let G be a minimal counterexample and D a critical subgroup of Q. As $G^* = O^p(G^*)$, $G = O^3(G)$.

Let t be an involution in G with $t^* \in Z(G^*)$. By minimality $G = DC_G(t)$ and without loss $D = [D, t]$. It follows that $D \cong Ext(p^{1+4})$. In particular, as $m(Q) \leq 3$, $\Omega_1(C_Q(D)) = Z(D)$. As G acts irreducible on $D/Z(D)$, $Q = DC_Q(D)$. Since G centralizes $\Omega_1(C_Q(D))$, $G = O^3(G)$ centralizes $C_Q(D)$.

Considering the p-part of the Schur multiplier of G^* we see that $C_G(t)' \cong G^*$ or $p = 3$ and $C_G(t)' \cong 3{\cdot}SL_2(3^2)$. In any case there exists $X \leq C_G(t)'$ so that X is an elementary abelian p-group and $XD'/D' \cong C_p$. Moreover $[D, X, X, X] \leq D'$ and so $[Y, X] \leq D'$ for some $Y \leq D$ with $Y \cong E_{p^3}$. Since $Y = [Y, t] \times D'$ we have $[Y, X] = 1$ and so $YX \cong E_{p^4}$.

Definition 4.6 *Let p be an odd prime, Q a p-group and H a group acting on Q.* *dcr*

(a) $\mathcal{CR}_Q(H)$ *is the set of maximal, H-invariant, class 2 and exponent p, normal subgroups of Q.*

(b) *We say that Q is H-homogeneous of rank n provide that there exists $A \in \mathcal{CR}_H(Q)$ so that $A \cong E_{p^n}$ and H acts irreducible on A.*

Lemma 4.7 *Let p be an odd prime, Q a p-group, H a group acting on Q. Let $D \in \mathcal{CR}_Q(H)$ and $T = C_Q(D)$. Then $\mathcal{CR}_T(H) = \{Z(D)\}$. For $i \geq 0$ put $T_i = \Omega_i(T)$. Then $T_{i+1}/T_i = \Omega_1((T/T_i) = \Omega_1(Z(T/T_i)) \in \mathcal{CR}T/T_i(H)$ and if $i \geq 1$, T_{i+1}/T_i is isomorphic to HQ-submodule $T^p_{i+1}T_{i-1}/T_{i-1}$ of T_i/T_{i-1}.* *homo*

Proof: Let $A = Z(D)$. Clearly $A \leq \Omega_1(Z(T))$. Let $A^* \in \mathcal{CR}_T(H)$. Then DA^* has class two und exponent p and so by maximality of D, $A^* \leq D \cap T \leq A$. By maximality of A^*, $A \leq A^*$ and so $A = A^*$ and $\mathcal{CR}_T(H) = \{A\}$. Let $C/A \in \mathcal{CR}_{Q/A}(H)$ and $B/A = Z(C/A)$. Then B is of class two and $\Omega_1(B) = A$ by maximality of A. As p is odd the map

$$\begin{aligned} \phi : B/A &\to A \\ bA &\to b^p \end{aligned}$$

is a HQ-homomorphism. As $\Omega_1(B) = A$, ϕ is one to one thus $B/A \cong B^p$ as HQ-module. Let $c, e \in C$ The $c^p \in A \leq Z(T)$ and so $c^p = (c^p)^e = (c^e)^p$ Put $d = cc^{-e}$. As $c^e \in cB$,$\langle c \rangle B$ has class two and p is odd, $d^p = c^p(c^e)^{-p} = 1$. It follows that $d \in \Omega_1(B) = A$. Hence $cA = c^eA$ for all $e \in C$ and so $cA \in Z(C/A) = B$. Thus $C = B$ and $B/A \in \mathcal{CR}_{Q/A}(H)$. Since T centralizes $B^p \leq A$, T/Acentralizes B/A. The lemma now follows by induction on $|T|$.

Corollary 4.8 *Let p be an odd prime, Q a p-group, H a group acting on Q and $D \in \mathcal{CR}_Q(H)$. Then $C_H(D)/C_H(Q)$ is p-group.* *ccr*

Proof: Note first that $C_H(D)$ centralizes $Q/C_Q(D)$ and $Z(D)$. Let T and T_i be as 4.7. Then by 4.7, $C_H(D)$ centralises all factors of the normal series

$$1 = T_0 \leq T_1 \leq T_2 \cdot T_k = T \leq Q.$$

Thus $C_H(D)/C_H(Q)$ is a p-group.

Lemma 4.9 *Let p be a prime with $p \geq 5$, $A \cong C_{p^2} \times C_{p^2}$ and $t \in Aut(A)$ with $t^p = 1$. Then t centralizes $\Omega_1(A)$. In particular, $Aut(A)$ has no subgroup isomorphic to $SL_2(p)$.*

Proof: Identify t which its image in the ring $End(A)$. Since $|A| = p^4$ we have $(t-1)^4 = 0$ and since $p \leq 4$ we get

(1) $(t-1)^p = 0$

Since $|A^p| = p^2$ we have

(2) $p(t-1)^2 = 0$

Since $t^p = 1$ we have

(3) $t^p - 1 = 0$

Consider the polyomial $f(x) = x^{p-1} + x^{p-2} + \ldots + x + 1 \in Z[x]$. Since $f(x) \equiv (x-1)^{p-1} \mathrm{mod} p$, $f(x) = (x-1)^{p-1} + p \cdot g(x)$ for some $g(x) \in Z[x]$. Write $g(x) = h(x)(x-1) + d$ for some $h(x) \in Z[x]$, $d \in Z$. Then $p = f(1) = p \cdot d$ and so $d = 1$ and $f(x) = (x-1)^{p-1} + p \cdot h(x)(x-1) + p$. Since $f(x)(x-1) = x^p - 1$ we obtain

(4) $x^p - 1 = (x-1)^p + h(x)p(x-1)^2 + p(x-1)$

Substituting t for x in (4) and using (1) to (3) we obtain

(5) $0 = p(t-1)$

Hence t centralizes $A^p = \Omega_1(A)$.

$pe = 3$

Lemma$_{QT}$ 4.10 *Let G be a finite, perfect $\mathcal{K}$-group with $O_2(G) = 1$ and $m_{2'}(G) \leq 3$.*

(a) *G is the central product of its* Sol*-components.*

(b) *If G is a* Sol*-component of G then one the following holds:*

(b1) *G is quasisimple and if $G/Z(G)$ is a group of Lie type in characteristic 2 or an alternating group then $G/Z(G)$ is one of the following:*
$Alt(n), 5 \le n \le 11$;
$L_n(q), n \le 4$;
$L_n(2), n \le 7$;
$Sp_{2n}(q), n \le 3$;
$G_2(q)$;
$U_n(q), n \le 4$;
$Sz(q)$;
$\Omega_8^-(q)$;
${}^3D_4(q)$;
${}^2F_4(q)$.

(b2) $F^*(G) = F(G)$. *Let p be a prime dividing $|[F(G), G]|$ and put $Q = [O_p(G), G]$. Then one of the following holds:*

1. $G/F(G) \cong 2{\cdot}Alt(5)$ *or* $SL_2(p)$, *and* $Q \cong Ext(p^{1+2})$ *or Q is of G homogenous of rank 2.*
2. $G/F(G) \cong SL_3(p)$; $L_3(2)$ *(* $p^3 \equiv 1 \mathrm{mod} 7$*);* $L_2(p)$; $(2{\cdot})Alt(5)$; *or* $(2{\cdot})3{\cdot}Alt(6)$ *(* $p \equiv 1, 19 \mathrm{mod} 30$ *and Q is G-homogenous of rank 3.*
3. $G/F(G) \cong SL_2(p), 2{\cdot}Alt(5), (3{\cdot})2{\cdot}Alt(6)$ *or* $2{\cdot}Alt(7)$ *(and $p = 7$) and* $Q \cong Ext(p^{1+4})$.

(c) *Let E be quasisimple so that $E/Z(E)$ is alternating or a group of Lie type in characteristic 2. Suppose that G is a central product of r copies of E with $r \ge 2$. Then $r \le 3$ and one of the following holds:*

(b1) $E/Z(E) \cong L_2(q)$,$L_3(2)$ *or* $Sz(q)$.

(b2) $E \cong 3{\cdot}Alt(6)$ *or* $SL_3(4)$, $r = 2$ *and* $|Z(G)| = 3$.

Proof: (a) Let L be a Sol-component of G.

Suppose first that L does centralize all its distinct conjugates under G. Then $|L^G| \le 3$ and as Sym(3) is solvable, G normalizes L. As L is a $\mathcal{K}$-group, $Out(L/Sol(L))$ is solvable and so $G = LC_G(L/Sol(L))$. Bu induction $C_G(L/Sol(L)^\infty$ is the central product of its Sol-components.

Hence we may in any case assume that there exist distinct Sol-components L_1 and L_2 of G with $[L_1, L_2] \ne 1$. Note that $[L_1, L_2] \le Sol(G)$ and by induction $G = L_1L_2$. Moreover, L_i is normal in G. If $[F(G), L_1, L_2] = 1$ and $[F(G), L_2, L_1] = 1$ we get $[L_1, L_2] \le C_G(F^*(G) \le F(G)$ and so $[L_1, L_2] = [L_1, L_2, L_2] = [L_1, L_2, L_1, L_2] \le [F(G), L_1, L_2] = 1$, a contradiction. Hence we may assume that $[O_p(G), L_1, L_2] \ne 1$ for some odd prime p. Put $Q = O_p(G)$ and $D \in \mathcal{CR}_Q(G)$. Then $[D, L_1] \ne 1 \ne [D, L_2]$. We conclude that $D \cong Ext(p^{1+4})$ and $[D, L_1, L_2] = 1$. Moreover, $[D, Q] \le D'$, $Q = C_Q(D)D$, $C_Q(D)$ is cyclic and so $[C_Q(D), G] = 1$. Thus $[Q, L_1, L_2] = [D, L_1, L_2] = 1$, a contradiction.

(b) If $E(G) \neq 1$, then G is clearly a component of G and it is now easy to verify that (b1) holds.

So suppose that $E(G) = 1$. Then by definition $F^*(G) = F(G)$. Let p and Q be as in (b2). Let $D \in \mathcal{CR}_Q(G)$, $D^* = D/D'$ and $\overline{G} = G/C_G(D^*)$. Let R be minimal in G with respect to $D \leq R$ and $G = RC_G(D)$. Then $C_R(D)D/D$ is nilpotent and so $C_R(D)$ is nilpotent. In particular, $F^*(R/O_{p'}(R))$ is a p-group.

Assume that $\mathrm{Sol}(\overline{G}) \neq O_p(\overline{G})Z(\overline{G})$. Then its easy to see that $D \cong Ext(p^{1+4})$ and $\overline{G} \sim Ext_-(2^{1+4}).Alt(5)$. Moreover, by 4.55, applied to $R/O_{p'}(R)$, $p > 3$.

Assume that $O_p(\overline{G} \neq 1$. Then $D \cong E_{p^3}$, $C_p \times Ext(p^1 + 2)$ or $Ext(p^{1+4}$. Mostly without loss,(**TO BE CONTINUED**) $G = R$ and $O_{p'}(G) = 1$.

Suppose that $D \cong Ext(p^{1+4})$ and let A/D' be a minimal G invariant subgroup of D/D'. If $|A/D'| = p$ we get conclude that $[A, G'] = 1$ and so $[A, G] = 1$ and $[A, D] = 1$, a contradiction. Hence $|A/D^prime| = p^2$ and $\overline{G} \sim p^3 SL_2(p)$ or $p^3 2{\cdot}Alt(5)$. Let t be an involution in which inverts A/D'. Then $C_G(t) \sim p^{1+3} SL_2(p)$ or $p^{1+3} 2{\cdot}Alt(5)$ and so contains a normal E_{p^4}, a contradiction.

Suppose that $D \cong C_p \times Ext(p^{1+2})$. Then $G = G'$ centralizes $Z(D)$ and $Z(D)/D'$ and so $Z(D) \leq Z(G)$. By 4.7 we conclude that G also centralizes $C_Q(D)$ and so $C_Q(D) = C_Q(G)$. Let t be an involution in G inverting $D/Z(D)$. Then $Q/C_Q(D)$ has order p^4 and is inverted by t. Thus $Q/C_Q(D)$is abelian and $Q' \leq Z(G)$. In particular Q has class two and so $\Omega_1(Q) = D$. Let $x, y \in Q$ so that t inverts x and y and $Q = C_Q(D)D\langle x, y\rangle$. Since t inverts x^p, $x^p \in D$ and since $x^p \neq 1$, we conlude that $D = \langle x^p, y^p\rangle Z(D)$ and so $Q = C_Q(D)\langle x, y\rangle$. Hence $Q' = \langle [x, y]\rangle$ is cyclic and so $Q\prime \cap D = D'$. Thus $[Q, D] \leq D'$ and $[D^*, Q] = 1$, a contradiction.

Thus $D \cong E_{p^3}$ and so $Q/C_Q(D) \cong E_{p^2}$. We will use 4.7 without further reference. In particular we are done if G normalizes a hyperplane in Q. So suppose $|C_D(G)| = p$. Let T and T_i be as in 4.7. Let t be an involution in G inverting $D/C_D(G)$. Assume first that $T = D$. The t inverts $Q/C_D(G)$ and thus $Z(Q) = Q' = C_D(G)$. It follows that Q is extra special, a contradiction to $D \in \mathcal{CR}_Q(G)$. Thus $T \neq D$. Let $A/D = C_{T/D}(G)$. Note that $C_Q(t) = C_T(t)$ is cyclic and $A = C_A(t)D$. Thus t inverts $Q/C_Q(A)$. It is now easy to see in $Aut(A)$ that $C_Q(A) = T$ and $A = C_A(G)D$. If $T_2 \neq A$ put $B = T_2$ otherwise let $B = Q$. Note that since G is perfect, $Q = [Q, t]$ and $T = [T, t]Q'$. But $|Q'[T, t]/[T, t] \leq p$ and so if $A = T_2$, $A = T$. Hence in any case $|B/A| = p^2$, $[B, Q]D = A$ and t inverts $B/C_A(G)$. In particular, $B' \leq C_A(G)$. Since t centralizes $Hom(B/A, A/C_A(G))$, $[B, Q] \leq C_A(G)$. If $Q/C_Q(B)$ has exponent p we conclude that $[B, Q]$ has exponent p and $[B, Q] \leq D$, a contradiction. Thus $Q/C_Q(B) \cong C_{p^2} \times C_{p^2}$ and hence $Q^p = T$. Hence $[B, T] = C_D(G)$,

Assume that $\mathrm{Sol}(\overline{G} = Z(\overline{G}$. Then as G is a Sol-component, $\overline{G}$ is quasisimple.

Remark: Lots of case with $L/F(G) \cong 2{\cdot}Alt(5)$ or $Ext_-(2^{1+4}$ need to be worked into the statement of the theorem,4.9 has to be used to exclude smilar cases for $SL_2(p)$ TO BE CONTINUED

ParAlt

Lemma 4.11 *Let $G \cong Sym(\Omega)$ or $Alt(\Omega)$, $|\Omega| = n$ finite, and H a maximal subgroup of G such that $|G/H|$ is odd.*

(a) *For an integer k let $b_2(k) = \{2^i \mid a_i \neq 0\}$ where $k = \sum_{i=1}^n a_i 2^i$ with $a_i \in \{0,1\}$. Then one of the following holds.*

1. $H = N_G(\Lambda)$ *where* $\Lambda \subset \Omega$ *and* $b_2(|\Lambda|) \subseteq b_2(\Omega)$
2. $H = N_G(\Pi)$, *where* Π *is a partition of* Ω *into* m *parts of size* l *and* l *is a power of 2 dividing* n.
3. $G = Alt(7)$ *and* $H \cong L_3(2)$.
4. $G = Alt(8)$ *and* $H \sim 2^3 : L_3(2)$.

(b) *If* $G = Alt(7)$, *then* $H = L_3(2), Alt(6), Sym(5)$ *or* $Sym(3) \wr Sym(4)$.

(c) *If* $G = Sym(7)$, *then* $H = Sym(6), Sym(5) \times C_2$ *or* $Sym(3) \times Sym(4)$.

(d) *If* $G = Sym(9)$ *then* $H = Sym(8)$.

(e) *If* $G = Sym(10)$, *then* $H = Sym(8) \times C_2$ *or* $C_2 \wr Sym(5)$.

(f) *If* $G = Sym(11)$, *then* $H = Sym(8) \times Sym(3)$, $Sym(9) \times C_2$ *or* $Sym(10)$.

(g) *If* $G = Alt(n)$, $n \geq 9$, *then* $H = H^* \cap Alt(n)$ *for some maximal subgroup* H^* *of* $Sym(n)$ *which contains a Sylow 2-subgroup of* $Sym(n)$.

Proof: Remark: Maybe we should find a reference, below is a the sketch of aproof

If $G = Sym(\Omega)$, this easliy follows since the subgroup of H generated by the 2-cycles in H is a direct product of natural embedded symmetric groups. So we may assume that $G = Alt(\Omega)$ and $N_{Sym(\Omega)}(H) \leq Alt(\Omega)$. Moreover, we may assume that H acts primitively on Ω. Let $X \subset \Omega$ with $|X| = 4$ and $A_X \overset{def}{=} O_2(\mathrm{Alt}(X)) \leq H$. Let $h \in H$.

If $|X \cap X^h| = 3$, then $\langle A_X, A_X^h \rangle = Alt(X \cup X^h)$ and so $H = G$, a contradiction. If $|X \cap X^h| = 1$, then $|X \cap X^a| = 3$ for all $a \in A_X^h$, a contradiction to by previuos case.

Thus $|X \cap X^h| \in \{0, 2, 4\}$ for all $h \in H$.

Let V be the power set of Ω viewed as a vector space over $GF(2)$ and endowed with the natural symmetric form. It follows that $U \overset{def}{=} \langle X^H \rangle$ is a singular subspace of V and all sets in U have size divisible by 4. Moreover if $|X \cap X^h| = 2$, then $X + X^h$ is in $\langle A_X, A_X^h \rangle$ conjugate to X and X^h. Since $X \cap X^h$ is not a set of imprimitivety, there exists $l \in H$ with $|X \cap X^h \cap X^l \cap X^{hl}| = 1$ It follows that $|X \cap X^h \cap Y = 1$ or some $Y \in \{X^l, X^{hl}$. Let $Z = X \cup X^h \cup Y$. Since $|X \cap Y| = |X^h \cup Y| = 2$ we get $|Z| = 7$. Put $L = \langle A_X, A_X^h, A_Y \rangle$ then $L \cong L_3(2)$. If $n \leq 7$ we are done. If $n \geq 8$, there must exists $k \in H$ with $Z \cap X^k \neq \emptyset$ and $X^k \not\subset Z$. Since X^k is perpendicular to $\langle X^L \rangle$ we get that $|Z \cap X^k| = 3$ (and indeed $Z \cap X^k = Z \setminus X^r$ for some $r \in L$. Let $W = Z \cup X^k$ and $K = \langle L, A_X^k$. Then $K \cong 2^3 : L_3(2)$. We $n = 8$ then $K = H$ an we are done. If $n \geq 9$ then there exists $s \in H$ with $W \cap X^s \neq \emptyset$ and $X^s \not\subset W$. Since K acts transitively on W, we conclude that X^s intersects each subset of sixe seven in W in O or 3 elements, a contradiction, which completes the proof of the lemma.

5 Subnormal Subgroups

OqRcQ

Lemma 5.1 *Let G be a finite group, L a subnormal subgroup of G, Q a normal q-subgroup of G and R a subgroup of G which centralizes L and $N_Q(L)$. Then $O^q(R)$ centralizes Q.*

Proof: Without loss $R = O^q(R)$. Suppose the lemma is false and let X be minimal in Q such that L and R normalize X, and R does not centralize X. Then $[X, R, R] \neq 1$ and so $X = [X, R]$. As $O^q(L)$ is subnormal in $Q^q(L)X$ and X is a q-group we conclude that $[X, O^q(L)] \leq L$. Thus R centralizes $[X, O^q(L)]$ and hence $[X, Q^q(L)] \neq X$. But this implies $[X, L] \neq X$ and so by minimal choice of X, $[X, L, R] = 1$. The three subgroup lemma implies $[X, R, L] = 1$ and thus $[X, L] = 1$ and $X \leq N_Q(L)$. We conclude that $[X, R] = 1$ and the lemma is established.

pi

Lemma 5.2 *Let G be a finite group, π a set of primes and L a subnormal subgroup of G such that $L = O^\pi(L)$. Then $E_\pi(N_G(L)) = E_\pi(G)$.*

Proof: Note first that $N_G(L) = N_G(LO_\pi((G)))$, $E_\pi(G/O_p(G)) = E_\pi(G)/O_\pi(G)$ and $E_\pi(N_G(L)/O_\pi(G)) = E_\pi(N_G(L)/O_p(G)$. Thus we may assume that $O_\pi(G) = 1$.

Put $H = N_G(L)$. Since $E(G)$ normalizes L we have $E(G) \leq E(H)$. Let R be the group generated by $O_\pi(H)$ and the π-components of H which are not contained in $E(G)$. Then R centralizes $E(G)$ and $F(G) \cap H$. By the previous lemma applied with Q a Sylow subgroup of $F(G)$ we conclude that R centralizes $F(G)$ and $F^*(G)$. Thus $R \leq F^*(G)$ and since $E_\pi(H) = E(G)R$, $E_\pi(H) = E_\pi(G) = E(G)$.

OqF

Corollary 5.3 *Let G be a finite group, p, q distinct primes and L a subnormal subgroup of G such that $L = O^p(L)$ and $L/O_p(L)$ is a q-group. Then $O^q(F^*_p(N_G(L)) = O^q(F^*_p(G))$.*

Proof: Apply the previous lemma with $\pi = q'$.

USN

Lemma 5.4 *Let G be a finite group and L a subgroup of G such that $L = O^p(L)$, $O_p(L) \neq 1$ and $L_/O_p(L)$ is either quasi-simple or a q-group. Then L is subnormal in at most one maximal p-local subgroup of G containing $N_G(L)$.*

Proof: Let M_1 and M_2 be maximal p-locals of G containing $N_G(L)$. By the previous lemma $E_p(M_1) = E_p(N_G(L) = E_p(M_2)$. As $O_p(L) \neq 1$, $O_p(E_p(N_G(L)) \neq 1$ and so $N_G(E_p(N_G(L)))$ is a p-local containing M_1 and M_2. Thus $M_1 = M_2$.

6 Nice Modules

NM
davh

Definition 6.1 *Let H be group and V a faithful $GF(p)H$-module. Then*

1. *$a_V(H)$ is defined by $|V/C_V(H)|^{a_V(H)} = |H|$.*

2. *$qa_V(H) = \min\{a_V(A) \mid 1 \neq A \leq H, [V, A, A] = 1]\}$, where $qa_V(H) = \infty$ if H has no nontrivial quadratic subgroups.*

3. *$ra_V(H)$ is the minimum of the $qa_W(H)$, where W runs through the non-trivial composition factor for H on V*

4. *Let a be a positive real number. Then V is called an Fa module if $qa_V(H) \leq a$ and an F^*a module if $qa_V(H) < a$.*

5. *An FF-module is an $F1$-module.*

oqu

Lemma 6.2 *Let G be a finite group, p an odd prime, $S \in Syl_2(G)$ and V a faithful $GF(2)$-module. Suppose that*

(i) $G = O_p(G)S$.

(ii) $[V, S, S] = 0$.

The there exists a set of hyperplanes $\mathcal{H}$ of S and G-submodules V_H, $H \in \mathcal{H}$ so that

(a) $V = C_V([O(G), S]) \oplus oplus_{H\in\mathcal{H}} V_H$

(b) *For all H in $\mathcal{H}$, H centralizes V_H.*

Proof: We may assume without loss that V is not the direct sum of two proper G-submodules. Put $P = O_p(G)$ and $Q = [P, S]$. If $Q = 1$ we are done. So suppose $Q \neq 1$ and let E be a normal subgroup of G in Q minimal with respect to $[E, Q] \neq 1$. Let $F = C_E(QS)$. Then by minimality of E, G acts irreducibly on E/F. In particular, $[E, P] \leq F$, S inverts E/F and $|E/F| = p$. Since $F \leq Z(Q) \cap E \leq Z(E)$, E is abelian. Then also $[\Omega_1(E), S] \neq 1$ and hence E is elementary abelian. Let $T = C_S(E)$. Then $|S/T| = 2$.

Suppose first that $F = 1$. Then $E = [E, S] \leq \langle S^E \rangle \leq C_G([V, T])$. Since $C_V(E) = 0$, $T = 1$ and the lemma holds.

Suppose next that $F \neq 1$ and ley $\mathcal{D}$ be the set of all hyperplanes D in E with $C_V(D) \neq)$. Then

$$V = \oplus_{D\in\mathcal{D}} C_V(D).$$

As V is indecomposable, G acts transitively on $\mathcal{D}$. Moreover, T is a Sylow 2 subgroup of $C_G(E)$ and so $G = N_G(T)C_G(E)$. In particular, $N_G(T)$ acts transitively on $\mathcal{D}$. We may assume that $[C_V(D), T] \neq 0$ for some $D \in \mathcal{D}$ and so $[C_V(D), T] \neq 1$ for all $D \in \mathcal{D}$. As $[C_V(D), T, S] = 0$, S normalizes $C_V(D)$ and D. Since $F \neq 1$ and $F \trianglelefteq G$, $F \notin \mathcal{D}$. Hence $E = FD$ and $[E, S] = [D, S] \leq D$. It follows that $[E, S] \leq \bigcap_{D\in calD} D$, contradicting the minimal choice of E.

FFMP

Lemma 6.3 *Let H be finite group such that the Sylow subgroup is contained in a unique maximal subgroup of H. Let V be a faithful $GF(2)$ FF-module for H. Then H has a normal subgroup $L = L_1 \times L_2 \times \ldots \times L_k$ such that*

(a) $L_i \cong SL_2(q)$ *or* $\mathrm{Sym}(q + 1)$, *q power of* 2.

(b) *Put* $\overline{V} = V/C_V(L)$ *and* $V_i = [V, L_i]$. *Then* $\overline{V} = \overline{V_1} \oplus \overline{V_2} \oplus \ldots \oplus \overline{V_k}$ *and* $\overline{V_i}$ *is a natural* $SL_2(q)$*-module for* $\overline{L_i}$.

(c) $H = LS$ *and* S *transitively permutes the* L_i*'s.*

$F*2$

Lemma 6.4 *Let* H *be finite simple group such that the Sylow subgroup is contained in a unique maximal subgroup of* H. *Let* V *be a faithful faithful* $GF(2)$ F^*2*-module for* H. *Then either* V *is an* FF*-module or* H *has a normal subgroup* $L = L_1 \times L_2 \times \ldots \times L_k$ *such that*
Remark: maybe we should do all F2 modules, even the non-quadratic ones

(a) $L_i \cong \mathrm{Alt}(q+1)$, $SL_3(q)$ *or* $O_4^{\pm}(q)$, q *a power of two.*

(b) *Put* $\overline{V} = V/C_V(L)$ *and* $V_i = [V, L_i]$. *Then* $\overline{V} = \overline{V_1} \oplus \overline{V_2} \oplus \ldots \oplus \overline{V_k}$ *and either* $L_i \cong \mathrm{Alt}(q+1)$ *and* $|\overline{V_i}|$ *is natural module or* $L_i \cong SL_3(q)$ *and* $\overline{V_i}$ *is the direct sum of a natural module and its dual.*

(c) $H = LS$ *and* S *transitively permutes the* L_i*'s.*

(d) *If* $L_i \cong SL_3(q)$, *then some element of* $N_H(L_i)$ *induces a graph automorphism on* L_i.

dtendec

Definition 6.5 *Let* K *be a field,* H *a group and* V *a* KH*-module. Then a tensor decomposition of* V *for* H *is a tuple* $(F, V_i, i \in I)$ *such that*

(a) $F \leq \mathrm{End}_K(V)$ *is a field with* $K \leq F$.

(b) H *acts* F*-semilinear on* V.

(c) *Put* $E = C_H(F)$ *(the largest subgroup of* H *acting* F*-linear on* V*). Then* V_i *is an* FE*-promodule.*

(d) *As* FE*-modules,* V *and* $\bigotimes_F \{V_i \in I\}$ *are isomorphic.*

qtp

Lemma 6.6 *Let* Q *be a group with* $|Q| \geq 3$. $1 \neq Z \leq Z(Q)$, K *a field with* $\mathrm{char} K = p$, p *a prime,* V *a faithful* KQ*-module with* $[V, Z, Q] = 0$ *and* $(F, V_i, i \in I)$ *a tensor decomposition of* V *for* Q. *Then* Q *acts* F*-linear and one of the follwing holds:*

1. *There exists* $i \in I$ *so that* $[V_i, Z, Q] = 0$ *and* Q *acts trivially on all other* V_j*'s.*

2. $p = 2$, Q *is* F*-linear and there exist* $i, j \in I$, $a_k \in \mathrm{End}_F(V_k)$ *with* $a_k^2 = 0$ *(k=i,j)and a monomorphism* $\lambda : Q \to (F, +)$ *so for* $q \in Q$,

   (a) *For* $k = i, j$, q *acts on* V_k *as* $1 + \lambda(q)a_i$.

   (b) Q *centralizes all* V_s*'s with* $s \neq i, j$.

Proof: Note first that as Z acts quadratically on V, Z is an elementary abelian p-group. Also $[V, Z, Q] = 0$ and $[Q, Z] = 1$. So the three subgroup lemma implies that $[V, Q, Z] = 1$.

Suppose that Q does not act F-linear. Note thet z induces some field automorphism σ on F. Let F_σ be the fixed field of σ in F. As z is quadratic on V, $f - f^\sigma \in F_\sigma$ for all $f \in F$. It easy to see that this implies $F = F_\sigma$ or $p = 2$ and F_σ has inded two in F. Moreover, $[V, z]$ is an F_σ-subspace centralized by Q. So Q is F_σ and $F_\sigma \neq F$. Since $[V, C_Q(F)]$ is an F-spave centralizes by z, $C_Q(F) = 1$. Thus $|Q| = 2$ in contradcition to the assumptions.

Suppose from now on the Q is F-linear. Since Z is a p-group, we mau assume that the V_i's are actually FZ-modules and not only promodules. If Q acts trivially on some V_k, V is a direct sum of copies of the FQ-module $\bigotimes_F \{V_i \mid i \in I - k\}$. So the latter has the same properties as V. Thus we may assume fom now on that Q acts non-trivially on each V_i. If $|I| = 1$, then 1. holds

Suppose next that $|I| = 2$ and say $I = \{1, 2\}$. Note that

$$[C_{V_1}(Z) \otimes V_2, Z] = C_{V_1} \otimes [V_2, Q].$$

Q acts as scalars on $[V_2, Z]$ and $[V_1, Z]$. Hence we may choose the promodules V_1 and V_2 so that $[V_i, Z, Q] = 0$ for $i = 1, 2$. For $q \in Q$ let q_i be the endomorposim $q - 1$ of V_i. Then $z_i q_i = 0$. Moreover, in $\mathrm{End}_F(V_1 \otimes V)$,

$$z - 1 = (1 + z_1) \otimes (1 + z_2) - 1\otimes = z_1 \otimes 1 + 1 \otimes z_2 + z_1 \otimes z_2.$$

Thus $[V, z, q] = 0$ implies

$$z_1 \otimes q_2 = -q_1 \otimes z_2$$

If $z_1 = 0$ then as V is faithful, $z_2 \neq 0$. Thus the previuos equation implies $q_2 = 0$ for q, a contradcition to the assumption that Q does not centalize V_2. Hence both z_1 and z_2 are not zero. Choosing $q = z$ we see that $p = 2$. Hence for arbitray q, $q_1 = \lambda(q) z_1$ and $q_2 = \lambda(q) z_2$ for some $\lambda(q) \in F$. Thus 2. holds in this case.

Suppose now that $|I| \geq 3$. Say $1, 2 \in I$ and but $W = \bigotimes_F \{V_i \mid i \in I \setminus \{1, 2\}$. Then $V \cong (V_1 \otimes V_2) \times W$. Then by the prviuos case Q acts faithfully on $V_1 \otimes V_2$ $z - 1$ and $q - 1$ are linear dependent on $V_1 \otimes V_2$. Let $\lambda = \lambda(q)$ be as above. Then on $v_1 \otimes v_2$

$q - 1 = (1 + \lambda z_1) \otimes (1 + \lambda z_2) - 1 \otimes 1 = \lambda(z_1 \otimes 1 + 1 \otimes z_2 + \lambda z_1 \otimes z_2)$.

On the otherhand $z - 1 = z_1 \otimes 1 + 1 \times z_2 + z_1 \otimes z_2$ and we conclude that $\lambda = 0, 1$ and so $|Q| = 2$, a contradiction. □

Definition 6.7 *Let H be a finite group, F a finite field, V a finite dimensional FH-module and s a postive real number.*

(a)
$$\mathrm{P}_s(H, V) = \{A \leq H \mid |A|^s |C_V(A)| \geq |B|^s |C_V(B)| \textit{ for all } B \leq A\}$$

.

(b)

$$\mathrm{P}^*_s(H,V) = \{A \in \mathrm{P}_s(H,V) \mid |A|^s|C_V(A)| > |B|^s|C_V(B)| \textit{ for all } C_A(V) < B < A\}$$

.

(c) $\mathrm{PQ}_s(H,V) = \{A \in \mathrm{P}_s(H,V) \mid [V,A,A] = 0$

(d) $\mathrm{PQ}^*_s(H,V) = \{A \in \mathrm{P}^*_s(H,V) \mid [V,A,A] = 0$

bpgv

Lemma 6.8 *Let H be a finite group, F a finite field, V a finite dimensional FH-module, s a postive real number and $A \leq H$.*

(a) *$A \leq \mathrm{P}_s(H,V)$ if and only if $|W/C_W(A)| \leq |A/C_A(W)|^s$ for all $W \leq V$.*

(b) *$A \in \mathrm{P}^*_s(H,V)$ if and only if $|V/C_V(A)| \leq |A|^s$ and for each $W \leq A$ one of the following holds:*

1. *[W,A]=0.*
2. *$C_A(W) = C_A(V)$.*
3. *$|W/C_W(A)| < |A/C_A(W)|^s$.*

(c) *Let $A \in P_s(H,V)$ and W an FA-submodule in V. Then $A \in \mathrm{P}_s(N_H(W),W)$.*

(d) *Let $A \in P^*_s(H,V)$ and W an FA-submodule in V. Then $A \in \mathrm{P}^*_s(N_H(W),W)$.*

Proof: (a) Suppose first that $A \in \mathrm{P}_s(H,V)$ and let W be a F-subspace of V. Let $B = C_A(W)$. Then $W \leq C_V(B)$. Since $A \in P_s(H,V)$ we have $|C_V(B)/C_V(A)| \leq |A/B|^s$ and thus

bpgv -1

(1)

$$|W/C_W(A)| \leq |C_V(B)/C_V(A)| \leq |A/B|^s = |A/C_A(W)|^s.$$

.

Suppose next that $|W/C_W(A)| \leq |A/C_A(W)|^s$ for all $W \leq V$ and let $B \leq A$. Put $W = C_V(B)$. Then $B \leq C_A(W)$ and

bpgv -1

(2)

$$|C_V(B)/C_V(A)| \leq |W|/|C_W(A)| \leq |A/C_A(W)|^s \leq |A/B|^s.$$

(b) Suppose first that $A \in \mathrm{P}^*_s(H,V)$ and let W be a F-subspace of V. Let $B = C_A(W)$ Then $W \leq C_V(B)$. If $A = B$, then 1. holds. If $B = C_A(V)$, then 2. holds. So assume $C_A(V) < B < A$. Then by minimalty of $|A|$ the middle "$\leq$" in (2) becomes a "$<$" and so 3.holds.

Suppose next that $|V/C_V(A) \leq |A/C_A(V)|^s$ and that 1.,2. or 3. holds for each $W \leq V$. Let $B < A$. Put $W = C_V(B)$. If 1. holds then, $C_V(A) = C_V(B)$ and so clearly $|A|^s|C_V(A)| > |B|^s|C_V(B)|$. If 2. holds then $B \leq C_A(V)$ and so $|A|^s|C_V(A)| \geq |V||C_A(V)|^s \geq |C_V(B)||B|^s$. If 3. holds then the middle "$\leq$" in **??** becomes a "$<$" and (b) is proved.

Finally (c) follows from (a), and (d) from (c) and (b).

bqpgv

Lemma 6.9 *Let H be a finite group, F a finite field V a finite dimensional FH-module and s a postive real number with $s \leq 2$. Let $A \in \mathrm{PQ}_s(G, V)$*

(a) *Suppose that Δ is a System of imprimitivity for A on V and $U \in \Delta$.*

  (a.a) *One of the following holds:*

   1. *A normalizes U.*
   2. *$|F| = 2 = |U|$ and $s \geq 1$.*
   3. *$|F| \in \{2, 4\}$, $|U| = 4$ and $s = 2$.*

  (a.b) *If in addition $A \in \mathrm{P}*(H, V)$ and either (a.a.2) with $s = 1$ or (a.a.3) holds, then $|A| = 2$ and A centralizes $\angle\Delta \setminus U^A\rangle$.*

(b) *Suppose that $V = \otimes_{i=1}^n V_i$ for some FH-module $V_i, 1 \leq i \leq n$ and that $[V_1, A] \neq 0 \neq [V_2, A]$ and $\dim_F V_i > 1$. Then $n = 2$, $s = 2$, $\dim_F V_1 = 2 = \dim_F V_2$, $C_A(V_1) = C_A(V_2) = C_A(V)$ and $|A/C_A(V)| = q$.*

Proof: (a) Let $W = \langle U^A \rangle$ and suppose that A does not normalize U. Since A acts on W, we get char F=2, $[U, N_A(U)] = 0$ and $|U^A| = 2$. Thus $|A/C_A(W)| = 2$. Hence by 6.8c, $W/C_W(A) \leq 2^s$. Since $U \cap C_W(A) = 0$ we get $|U| \leq 2^s$ and so 2. or 3. holds. Suppose that $A \in \mathrm{P}*(G, V)$ and either 2. with $s = 1$ or 3. holds. Then $|W/C_W(A)| = |A/C_A(W)|^s$. Thus by 6.8b, $C_A(V) = C_A(W)$. Since $|V/C_V(A)| \leq |A/C_A(W)|^s$ we conclude $V = WC_V(A)$ and so (a) is proved.

(b) If $|A| \geq 3$, this follows this is an easy consequence of 6.6. If $|A| = 2$ we get $|V/C_V(A)| \leq 2^s \leq 4$ and again (b) is easily verified.

lbfs

Lemma 6.10 *F a finite field , A a finite group, V a n-dimensional FA-module with $[V, A] \neq 0 = [V, A]$ and s defined by $|V/C_V(A)| = |A/C_V(A)|^s$. Then $s \geq \frac{1}{\dim_F[V,A]} \leq \frac{1}{n-1}$.*

Proof: We may assume that A acts faithfully on V. Let $m = \dim_F V/C_V(A)$ and $k = \dim[V, A]$. Then $A \leq |F|^{km}$ and so

$$|V/C_V(A)| = |F|^m \leq |A|^s \leq |F^{kms}.$$

Thus $m \leq kms$ and $s \leq \frac{1}{k} \leq \frac{1}{n-1}$.

Lemma 6.11 *Let H be a finite group, p a prime and V an irreducible, faithful $GF(p)H$-module. Let s be a positive integer with $s \leq 2$ and $L = \langle \mathrm{PQ}^*_s(H,V)\rangle$. Suppose that $L \neq 1$ and that L acts irreducible on V. Let $A \in \mathrm{PQ}^*_s(H,V)$ and $F = \mathrm{End}_L(V)$, then one of the following holds:* $s2$

1. $p = 2,3$, $L \cong SL_2(p)$, $|A| = p$, $|F| = p$, $\dim_F V = 2$ *and* $s \geq 1$.
2. $p = 2$, $L \cong Dih(D_{10}$, $|A| = 2$, $|F| = 4$, $\dim_F V = 2$ *and* $s = 2$.
3. $p = 2$, $L \cong SU_3(2)'$, $|A| = 2$, $|F| = 4$,$\dim_F V = 3$ *and* $s = 2$.
4. $p = 2,3$, $L \cong SL_2(p) * SL_2(p)$, $|A| = p$, $|F| = p$, $\dim_F V = 4$ *and* $s = 2$.
5. $p = 2$, $L \cong SL_2(F) \times SL_2(F)$, $|A| = |F|$, $|F| \geq 4$, $\dim_F V = 4$ *and* $s = 2$.
6. $p = 2$, $L \cong O^4_+(F)$, $|A| \leq 2|F|$, $|V/C_V(A)| = |F|^2$, $|F| \geq 4$, $\dim_F V = 4$ *and* $s \geq \frac{4}{3}$.
7. $p = 3$, $L \sim \mathrm{Ext}_-(2^{1+4}).Alt(5)$, $|A| = 3$, $|F| = 3$, $\dim_F V = 4$ *and* $s = 2$.
8. $p = 2$, $L \cong Sym(5)$ *or* $Sym(3) \wr Sym(5)$, $|A| = 2$ *or* $A \leq L'$, $F| = 2$, $\dim_F V = 4$, $s = 2$ *and* $|\mathrm{End}_{L'}(V)| = 4$.
9. $p = 2$, $s = 2$, $F \leq 4$. *There exists a system of imprimitivity Δ for L on V with $L/C_L(\Delta) = Sym(\Delta)$. Let $U \in \Delta$, then $|U| = 4$. If $A \leq C_L(\Delta)$ then $|A| = 2$. $C_L(\Delta)$ is a $Sym(\Delta)$ invariant subgroup of $Sym(3)^\Delta$. If $|F| = 2$ then $C_L(\Delta)$ induces $Sym(3)$ on U and if $|F| = 4$ then $C_L(\Delta)$ induces C_3 on U.*
10. *Let $K = E(L)$. Then K is quasi simple, K acts irreducible on V, $F = \mathrm{End}_K(V)$. Moreover, L acts primitively and tensor indecomposable on V.*
11. $s > 1$. *There exists a central extension L^* so that $V \cong V_1 \otimes V_2$ for some faithful FL^* modules V_1 and V_2. Let $\{i,j\} = \{1,2\}$, $\mathrm{P}_i = \{A \in \mathrm{PQ}^*_s(H,V) \mid [V_j, A] = 0\}$ and $L_i = \langle \mathrm{P}_i\rangle$. Then $\mathrm{PQ}^*_s(H,V) = P_1 \cup P_2$, $L = L_1L_2$ and $[L_1, L_2] = 1$. Let $K_i = E(L_i)$ Then V_i is an irreducible FK_i module module and $F = \mathrm{End}_{K_i}(V_i)$. $P_i \in \mathrm{PQ}^*_{\frac{s}{n_j}}(L_i, V_i)$. Let $A_i \in P_i$, $n_i = \dim FV_i$ and and let s_i be defined by $|V_i/C_{V_i}(A_i)| = |A_i|^{s_i}$. Then $s_i \leq \frac{s^2}{n_i+s} \leq \frac{4}{n_i+2}$ and $\frac{n_j}{s} + 1 \leq n_i \leq s(n_j - 1)$.*

Proof:
We will first prove:

$s2-2$

(1) Suppose V can be regarded as a vector space over a field F so that L acts F-semilinear but not F-linear on V. Then $|A| = p = 2$, $|F| = 4$ or 16, $|V| = 4$ or 16 and L is one of $Dih(6), Dih(10), Sym(3) \times Sym(3)$, $Sym(5)$ or $Sym(3) \wr Sym(5)$. Moreover if $s \neq 2$, then $s \geq 1$, $|F| = |V| = 4$ and $L \cong Sym(3)$.

Choose $A \in \mathrm{PQ}^*(L,V)$ which does not act F-linear on A. Since A acts quadratically on V we conclude that $|A| = 2$. Moreover, $|V| = |V/C_V(A)|^2 \leq |A|^{2s} = 2^{2s} \leq 16$. Thus $L \leq \Gamma GL_2(4)$. (1) now follows by inspecting the irreducible subgroups of $\Gamma GL_2(4) \cong Sym(3) \wr Sym(5)$ generated by involutions.

$s2-3$

(2) Suppose there exist a central extension L^* of L, a field F and FL^*-moduln V_1 and V_2 so that $V \cong V_1 \otimes_F V_2$ as $GF(p)L^*$ modules. Then one of the following holds:

1. $s = 2$, $p = 2$, $\dim_F V_i = 2$, $|A| = |F|$ for all $A \in \mathrm{PQ}^*_s(L,V)$ and $L \cong SL_2(F) \times SL_2(F)$

2. $s > 1$. Let $\{i,j\} = \{1,2\}$, $\mathrm{P}_i = \{A \in \mathrm{PQ}^*_s(H,V) \mid [V_j, A] = 0\}$ and $L_i = \langle \mathrm{P}_i \rangle$. Then $\mathrm{PQ}^*_s(H,V) = P_1 \cup P_2$, $L = L_1L_2$ and $[L_1, L_2] = 1$. $P_i \in \mathrm{PQ}^*_{\frac{s}{n_j}}(L_i, V_i)$. Let $A_i \in P_i$, $n_i = \dim FV_i$ and and let s_i be defined by $|V_i/C_{V_i}(A_i)| = |A_i|^{s_i}$. Then $s_i \leq \frac{s^2}{n_i+s} \leq \frac{4}{n_i+2}$ and $\frac{n_j}{s} + 1 \leq n_i \leq s(n_j - 1)$.

Suppose first that there exists $A \in \mathrm{PQ}^*_s(H,V)$ with $[V_1, A] \neq 0 \neq [V_2, A]$. Using 6.9b it is then easy to see that refs2-31. holds. So suppose that no such A exists. Then clearly $\mathrm{PQ}^*_s(H,V) = P_1 \cup P_2$, $L = L_1L_2$ and $[L_1, L_2] = 1$.

Note that V is as an L_i module the direct sum of n_j copies of V_i. Hence for all $B \leq L_i$, $|C_V(B)| = |C_{V_1}(B)|^{n_j}$ and so $(|B|^{\frac{s}{n_j}}|C_{V_1}(B)|)^{n_j} = |B|^s|C_V(B)|$. Thus $P_i \in \mathrm{PQ}^*_{\frac{s}{n_j}}(L_i, V_i)$. Moreover, we see that $s_in_j \leq s$. Thus $s_i \leq \frac{s}{n_j}$. By 6.10 we have $s_i > \frac{1}{n_i-1}$ and so $\frac{s}{n_j} \geq s_i \geq \frac{1}{n_i-1}$ and thus $n_i \geq \frac{n_j}{s} + 1$. Hence also $n_j \geq \frac{n_i}{s} + 1 = \frac{n_i+s}{s}$. Therfore $s_i(\frac{n_i+s}{s}) \leq s_in_j \leq s$ and $s_i \leq \frac{s^2}{n_i+s}$. Hence refs2-32 holds.

$s2-4$

(3) If V is tensor-decomposable as L-module, then 4.,5. or 11. holds.

In case (2)1, 4. or 5. holds. So suppose (2)2. holds. Since $P_i \leq \mathrm{PQ}_{\frac{s}{n_i}}(L_i, V_i)$ can imply induction to (L_i, V_i). Moreover, either $\frac{s}{n_i} < 1$ or $\frac{s}{n_i} = 1$ and $n_i = 2$. If $n_i = 2$, then $s_i = 1$ and $s_in_j \leq s$ implies $n_j = 2$. It follows that 4. or 11 holds in this case.

We may and do assume form now on that V is tensor indecomopsable.

Suppose that L acts irreducible but does not primitively on V and let Δ be a system of imprimitivity for L on V. Since L acts irreducble on V, L acts transitively on Δ. Thus there exists $U \in \Delta$ and $1 \neq A \in \mathrm{PQ}^*_s(H,V)$ so that A does not normalizes U. If $|U| = 2$, L centralizes the sum of the non-zero elements in $\bigcup \Delta$, a contradiction to the irreducible action of L. Hence by 6.9a we conclude that $|U| = 4$, $s = 2$, $|A| = 2$ and A centralizes $\langle \Delta \setminus U^A$. In particular, A acts a 2-cycle on Δ and we conclude that $L/C_L(\Delta) = Sym(\Delta)$.

Thus

$s2-1$

(4) If L acts irreducible but not primitively on V, then $p = 2$, $s = 2$ and L is a subgroup of $SL_2(2) \wr Sym(n)$, where $n = \dim V/2$.

Suppose next that L acts irreducible and primitively on V.

Let K be a normal subgroup of L minimal with respect to $[K, L] \neq 1$. As L acts primitively, V is a as K-module isomorphic to the direct sum of isomorphix irreducible $GF(p)K$-modules. In particular $KC_{GL(V)}(K)$ acts irreducible on V and so $F \stackrel{def}{=} \mathrm{End}_{KC_{GL(V)}(K)}(V)$ is a field. By (1) we may assume that L acts F-linear on V. As V is tensor indecoposable we conclude that K acts irreducible on V. If K is cyclic, we conclude that V is 1-dimensional over F and so L is cyclic, a contradicion, since $O_p(H) = 1$. Thus K is not cyclic and we may assume that all cylic normal subgroup of L are contained in $Z(L)$. In particular $C_L(K) \leq Z(L)$.

Assume that K is a q-group for be a prime q. Then $q \neq p$. Pick $A \in \mathrm{PQ}^*(L, V)$ with $[K, A] \neq 1$. Then $p = 2$ or 3. Moreover, $[K, A] \not\leq Z(K)$ and so $1 \neq [A, K, K] \leq Z(L)$.

Suppose that $p = 2$, then by 6.2 and the irreducible action of K, A is cyclic. But then $|A| = 2$ and so $|[V, A]| = |[V/C_V(A)| = 2^r \leq 2^s \leq 4$ for some integer $r \leq s \leq 2$. Hence there exist $1 \neq k \in [A, K, K]$ with $|V| = |[V, k]| \leq 2^{4r}$. Also note that since $Z(K) \neq 1$, $|F| \geq 4$ and so $\dim_F V \leq 2r$. Since K is non-abelian and acts irreducible on V, we conclude that $r = 2$ and

$s2-5$

(5) $|A| = 2 = p$, $s = 2$, $K \cong \mathrm{Ext}(3^{1+2})$, $|V| = 2^6$, and $L = KA \cong SU_3(2)'$

Suppose next that $p = 3$. Then $q = 2$ and $[K, A]$ is extraspecial. If A is not cyclic we obtain a contradiction to 6.9b applied to an irreducible submodule for $[K, A]A$ in V. Hence A is cyclic and similarly $[K, A] \cong Q_8$. Moreover $|C_V(A)|^2 = |V|$ and so $|V| \leq 3^{2s} \leq 3^4$. As L is irreducible and tensor indecomospable on V one of the following holds:

$s2-6$

(6) 1. $|A| = p = 3$, $s \geq 1$, $|V| = 3^2$ and $L \cong SL_2(3)$.

2. $|A| = p = 3$, $s = 2$, $|V| = 3^4$ and $L \sim \mathrm{Ext}_-(2^{1+4}){\cdot}Alt(5)$.

Suppose next that K is not nilpotent. Then $K = E(K)$ and L acts transitively on the components of L.

Assume that K is not quasisimple. Then there exist a component R of K and $A \in \mathrm{PQ}_2^*(L, V)$ so that A does not normalize R. Since A acts quadratically this implies $p = 2$, $R \cong SL_2(F)$ and $|R^A| = 2$. Moreover, using 6.9b we get:

$s2-7$

(7) Put $q = |F|$. Then $p = 2$, $s \geq \frac{4}{3}$, $q > 2$, $|A| \leq 2q$, $\dim_F V = 4$, $|V/C_V(A)| = q^2$, and $L \cong \Omega_4^+(F) \sim SL_2(F) \times SL_2(F) : 2$.

Assume finally that K is quasi simple. Then

$s2-8$

(8) $K = E(L)$ is quasi simple, $C_L(K) = Z(L)$, L acts irreducibly, primitively, tensor indecompsable and F-linear on V.

Lieq

Lemma 6.12 *$F2$-modules for groups of Lie type and maybe also the non-quadratic $F2$-modules*

Lemma 6.13 *Let Ω be a finite set, $G = Sym(\Omega)$, and $V(\Omega) = GF(2)[\Omega]$ the natural permutation module $GF(2)G$-permutaion module. Define $V_O(Omega) = [V(\Omega), G]$, $\overline{V(\Omega)} = V(\Omega)/C_{V(\Omega)}(G)$ and $\overline{V_0(\Omega)} = V_0(\Omega)/C_{V_0(\Omega)}(G)$. Let V be one of the modules, $V(\Omega), V_0(\Omega), \overline{V(\Omega)}$ and $\overline{V_O(\Omega)}$.*

(a) *Let A be a non-trivial elementary abelian subgroup of G with $|V/C_V(A)| \geq |A|$. Then there exists commuting transpositions $t_1, t_2, \dots t_k$ so that one of the following holds*

1. $A = \langle t_1, t_2, \dots, t_k \rangle$.
2. $|\Omega| = 2k$, $V = V_0(\Omega)$ *or* $\overline{V_O(\Omega)}$ *and* $A = \langle t_1t_2, t_2t_3, \dots, t_{i-1}t_i, t_{i+1}, t_{i+2}, \dots, t_k \rangle$, *where* $1 \leq i \leq k$.
3. $|\Omega| = 2k + 4$, $V = V_0(\Omega)$ *or* $\overline{V_O(\Omega)}$ *and* $A = \langle t_1, t_2, \dots, t_k, (ab)(cd), (ac)(bd) \rangle$, *where a, b, c, d are the four common fixed points of $t_1, \dots, t_k$.*
4. $|\Omega| = 4|$, $V = \overline{V(\Omega)}$ *and* $A \leq Alt(\Omega)$.
5. $|\Omega| = 8$, $V = \overline{V_O(\Omega)}$, $|A| = 8$ *and A acts regularly on* Ω.

(b) *Suppose $|\Omega| \neq 8$ and let $H \leq G$ with $H = \langle \mathrm{P}(H, V)$. Let Ψ an orbit for H on Ω. Then one of the following holds:*

1. $H/C_H(\Psi) = Sym(\Psi)$.
2. $H/C_H(\Psi) = Alt(\Psi)$.
3. $|\Psi|$ *is even and* $H/C_H(\Psi) = N_{Sym(\Psi)}(\Delta) \cong C_2 \wr Sym(|Psi|/2)$, *where Δ is a partion of Ψ into sets of size 2.*
4. $|\Psi| = 4$ *and* $H/C_H(\Psi) \cong E_4$.
5. $|\Psi| = 6$ *and* $H/C_H(\Psi) \cong Alt(5)$.
6. $|\Psi| = 8$ *and* $H/C_H(\Psi) \sim 2^3 : L_3(2)$.

Proof: (a) By induction on $|A|$, V and $|\Omega|$. Suppose that $A \notin \mathrm{P}(G, V)$ and let $1 \neq B \leq A$ with $B \in \mathrm{P}(A, V)$ with $|B||C_V(B)| > |A||C_A(V)| \leq |V|$. Then by induction $\Omega = 2k$ and $B = \langle t_1, t_2, \dots, t_k \rangle$. But then $A \leq C_G(B) = B$ and so $A = B$, a contradiction.

Hence $A \in \mathrm{P}(G, V)$. Let $B = C_V([V, A])$. Then $1 \neq B \in \mathrm{P}(G, V)$. Suppose $B \neq A$ and apply (a) to B. In case (a3) $A \leq C_G(B) \leq A$, a contradiction. In case (a1) and (a2), $C_G(B) = \langle t_1, t_2, \dots t_k \rangle \times Sym(\Omega')$. If $|\Omega| = 2k$, then $C_G(B)$ acts quadratically on V, a contradiction to $A \neq B$. Thus $|\Omega| \neq 2k$ and $A = B \times D$, where $D = B \cap Sym(\Omega')$. We may view $V_O(\Omega')$ as a subspace of V. Then $A \leq \mathrm{P}(A,\ V_O(\Omega')$ and so $D \in \mathrm{P}(Sym(\Omega', V_O(\Omega_{\prime})$. In particular we can apply (a) to D. Since $C_D([V, A]) = 1$ we get that $C_D(V(\Omega') = 1$. But this implies that (a3) with $k = 0$ holds for D on $V_O(\Omega')$. Thus also (a3) holds for A on V.

So we may assume that $[V, A, A] = 0$. Suppose that A has an orbit of length larger then four on Ω. If $|\Omega| = 4$, (a3) or (a4) holds. So assume $|\Omega| > 4$. If A has an orbit of lenght less then four on Ω then $[V_\Omega, A, A]$ has an element of lenght four, a contradiction to $[V, A, A] = 0$. Thus all orbits of A have length at least four. Moreover, $[V(\Omega), A, A]$ has an element of lenght four and $[V_\Omega, A, A]$ has an element of length eight. We conclude that $|\Omega| = 8$ and $V = \overline{V_0(\Omega)}$. If A has an orbit of lenght eigth on Ω, (a5) holds. So suppose that A has two orbits of length four. If $1 \neq a \in A$ acts trivially on on of the orbits of A on Ω, then $[V, a, A \neq 0$. Thus $|A| = 4$, but $|V/C_V(A)| = 8$, a contradiction.

Hence we may assume that all the orbits of A on V have length at most 2. If A has a fixed point on Ω we are done by induction. Hence we may assume that A acts fixed point freely on Ω. Suppose that there exists $v \in V(\Omega)$ with $0 \neq [v, A] \leq C_{V(\Omega)}(G)$. Then it os easy to see that $C_A(v) = 1$ and so $|A| = 2$ and $|\Omega| = 2$. So we may assume that no such v exists. Hence $|V/C_V(A)| \geq 2^{k-1}$, where $k = \Omega|/2$ and thus $|A| \geq 2^{k-1}$ and (a2) holds.

(b) Let $A \in \mathrm{P}(H, V)$ so that A does not act trivially on Ψ.

Suppose first that some element of H induces a transposition on Ψ. If H acts primitively on Ψ, (b1) holds. So suppose that Δ is a system of imprimitivity for H on Ψ. Since A is generated by elements of support less or equal to four, we conclude that elements of Δ have size two and A on its action on Δ is generated by transopsition. As H acts transitively on Δ, $H/C_H(\Delta) = Sym(\Delta)$. Moreover, all the transposition in H act trivially on Δ and so $C_{Sym(\Psi)}(\Delta) \leq H/C_H(\Psi)$ and (b3) holds.

So suppose that no element of H induces a transposition on Ψ.If A fulfils (a3) or (a4) then $|\Psi| = 4$ and (b4) holds.

So we may assume that A fulfils (a2). Then $\Psi = \ \mathrm{Supp}(\langle t_1, t_2, \dots t_k\rangle$ and we may assume without loss that $\Psi = \Omega = \{1, \dots, 2k\}$ and $t_i = (2i-1, 2i)$. It is easy to see that $k \geq 3$. Suppose that Δ is a system of imprimitivity for H on Ψ and without loss that A acts non trivially on Δ. Let $D \in \Delta$. Then $|D| = 2$ and say $D = \{1, 3\}$. Then $|D^{t_1 t_3} \cap D| = 1$, a contradiction.

Thus A acts primitively in Ψ. Hence if H contains a 3-cycle, (b2) holds. So we may assume that H contains no three cycle. Let $A^* \in \mathrm{P}(H, V)$ with $A \neq A^*$ and so that A^* does not normalize A. Let $a \in A$ and $a \in A^*$ with $|\mathrm{Supp}(a)| = |Supp(a^*)| = 4$ and $A \neq A^{a^*}$. If $|\mathrm{Supp}(a) \cap Supp(a^*)| = 1$, then $(aa^*)^2$ is a three cycle, a contradiction. Hence $|\mathrm{Supp}(a) \cap Supp(a^*)| \neq 3$, for all such a and a^*.

Suppose $a^* = (1, 2)(3, 5)$. Then $(12)(34)a^*$ is a three cycles, a contradiction.

Suppose that $a^* = (1, 3)(2, 5)$. If $k \geq 4$ we obtain a contradiction by choosing $a = (34)(78)$. Thus $k = 3$, $A^* = \langle (1, 3)(2, 5), (1, 3)(4, 6)\rangle$ and $\langle A, A^*\rangle \cong Alt(5)$. It follows that $H = \langle A, A^*\rangle$ and (b5) holds.

Up to conjugation under $N_{Sym(\Psi)}(A)$ we now may assume that $a^* = (1, 3)(5, 7)$. If $n \leq 5$ we obtain a contradiction by choosing $a = (1, 2)(9, 10)$. Thus $k = 4$. By the previous case neither $(13)(26)$ nor $(13)(28)$ can be in A^* and we conclude that the orbits of A^* on Ψ are $13, 24, 57$ and 68. In particular, A and A^* normalize $\{1, 2, 3, 4\}$ and $\langle A, A^*\rangle \sim 2^4 Sym(3)$. It is now readily verified that (b6) holds.

LPGV

Lemma 6.14 *Let G be a finite group with $F^*(G)$ quasisimple. Let V be a faithful $GF(p)G$-module and $\mathcal{A}$ a G invariant subset of $\mathrm{P}(G, V)$. Let $S \in Syl_p(G)$ and put $J = J_{\mathcal{A}}(S) = \rangle \mathcal{A} \cap S\langle$. $L \leq G$ with $L = N_G(O_p(L)$ and $J \leq L$ and suppose that K is p-component* *of L so that J does not normalize K. Then $p = 2$, $\langle \mathcal{A} \rangle \cong \mathrm{O}^+_{2n}(2^k), n \geq 3, k \geq 2$ and $K/O_2(K) \cong SL_2(2^k)$ all non-trivial composition factors for $\langle \mathcal{A} \rangle$ on V are natural $\mathrm{O}^+_{2n}(2^k)$-modules. In particular, if $n = 3$, then $\mathrm{P}(O_p(L), V) = 1$.*

Remark: If $n > 3$, then it can be shown that K is not subnormal in $C_G(C_V(S)$, where $S \in Syl_p(L)$.

Proof: Let $H = F^*(G)$. We may assume without loss that H centralizes all proper G-submodules in V. That is $V = [V, H]$ and G actss irreducible on $V/C_V(H)$. In particular by the Three Subgroup Lemma, $O_p(G) = 1$.

If $p = 2$ and $H/Z(H)$ is an alternating group we obtain a contradiction from 6.13. So we may assume that:

$LPGV-1$

(1) H is a group of Lie type in characteristic p.

We may assume without loss that H centralizes all proper G-submodules in V. That is $V = [V, H]$ and G acts irreducibly on $V/C_V(H)$. In particular by the Three Subgroup Lemma, $O_p(G) = 1$.

If $O_2(L) \cap H = 1$, then $[O_2(L), K] = 1$ and so by the $P \times Q$-lemma, $[C_V(O_2(L), K] \neq 1$. But $L \cap \mathcal{A} \subseteq \mathrm{P}(L, C_V(O_2(L))$ and K maps onto a component of $L/C_L(C_V(O_2(L))$, a contradiction.

Hence $O_2(L) \cap H \neq 1$. Let $M = N_G(O_2(L) \cap H)$. Then $L \leq M$ and $N_{O_2(M)}(O_2(L)) \leq O_2(L)$ and so $O_2(M) \leq O_2(L)$. Hence $O_2(M) \cap H = O_2(L) \cap H$ and $M \cap H$ is a parabolic subgroup of H. We have proved:

$LPGV-3$

(2) There exists a parabolic subgroup M of G with $L \leq M$ and $O_2(M) \cap H = O_2(L) \cap H$.

It follows immediately from (2) that

$LPGV-4$

(3) H has rank at least three.

Note that $C_V(H) = 0$ unless $H \cong Sp_{2n}(q)$ and $V/C_V(H)$ is a natural $Sp_{2n}(q)$-module. In which case we have $C_V(X)C_V(H)/C_V(H) = C_{V/C_V(H)}(X)$ and so $\mathrm{P}(G, V) \subset \mathrm{P}(G, V/C_V(H))$. Hence we may assume without loss that $C_V(H) = 0$ and so V is irreducible as G-module.

$LPGV-2$

(4) One of the following holds

1. $\langle \mathcal{A} \rangle = H$
2. $p = 2$, $\langle \mathcal{A} \rangle =\cong \mathrm{O}^{\pm}_{2n}(2^k), n \geq 3$ and V is a natural $\Omega^{\pm}_{2n}(2^k)$ module for H.

Let $P \in \cap P(G, S)$ so that $[C_V(O_2(P)), O^2(P)] \neq 1$. Then J induces inner automorphisms on Head(P) and (4) follows from the structure of P and V.

Suppose that $O_2(M) = O_2(L)$. Then $L = M$ is a parabolic of G and so the $p-$componets of L are normal in $H \cap L$. Using (4),we conclude that the lemma holds. So we may assume that

(5) $O_2(M) \neq O_2(L)$ and $O_2(L) \not\leq H$. $LPGV-5$

Note that $[O_2(L), L\cap H] \leq O_2(L)\cap H \leq O_2(M)$ and so $L/O_2(M) = C_{M/O_2(M)}(O_2(L))$. In particular, $[J\cap H, O_2(L) \leq O_2(M)$. Without loss $S \leq M$ and $S\cap L \leq Syl_p(L)$. Since $J \not\leq O_2(M)$ there exists $P \in \mathcal{P}(M,S)$ with $J \not\trianglelefteq P$. Then $J \not\leq O_2(P)$ and $[J\cap H, O_2(L) \leq O_2(P)$. Let $\overline{P} = P/O_2(P)$

Suppose that $J \leq H$. Then $N_P(S\cap H)$ normalizes J and we conclude that $Z(\overline{S\cap P}) \leq \overline{J}$, or $p=2$ and $\overline{P} \cong Sym(3)\wr C_2$. As $O_2(L)$ centralizes $\overline{J}$ and $O_2(L) \not\leq H$ one of the following has to hold

(6) $LPGV-6$

1. $p=2$, $H \cong SL_n(q)$, $O_2(L)$ induces a graph automorphism on H and $\overline{P\cap H} \cong L_2(q)$ or $SL_3(q)$
2. $p = 2$ $H \cong SU_n(q)$, $O_2(L)$ induces a field automorphism of order two on H and $\overline{P\cap H} \cong L_2(q)$ or $SU_3(q)$
3. $p=2$ and $O_2(L)H \cong \mathrm{O}2n^{\pm}(q)$.
4. $p=2$ and $G = O_2(L)H = Aut(L_n(2))$.

In case (6)1 or (6)2, P is uniquely determined. Let R be the maximal parabolic of M with $P \not\leq M$. Then we conclude that $J \trianglelefteq R$ and so $[J,[R,O_2(L)] \leq O_2(M)$. By the structure of M this implies $J \leq O_2(M)$, a contradiction. In case (6)3 it is easy to see that L is the normalizer of a non-singular isotropic space and so all p-components of L are normal in L. In case (4), since J does not normalize K and $J \leq H$, M most have parbolic E with $E/O_2(E) \cong L_3(2)\wr C_2$ and $J \not\leq O_2(E)$. Let T be a 2-componet of E. As $[J,O_2(L)] \leq O_2(E)$ and $O_2(E)$ does not normalizes T, $T\cap J \leq O_2(E)$. Hence $[T\cap S, J] \leq O_2(E)$ and J i normal in both minimal parabilocs of E, a contradiction.

We have proved:

(7) $J \not\leq H$, $p=2$ and $JH \cong O^{\pm}_{2n}(q)$. $LPGV-7$

If $O_2(L) \leq JH$ we are done by the argument in (6)3 we are done. So suppose $O_2(L) \not\leq JH$. Then $O_2(L)$ induces field automorphisms on H and on Head(P). In particular $q \geq 2$. If $J \leq HO_2(P)$, we get that $\overline{S\cap P)} = \overline{J\cap H}$, a contradiction. Thus $J \not\leq HO_2(P)$ and so P is uniquely determined. But now the argument in (6)1&2 yields a contradiction.

$qusp$

Lemma 6.15 *Let H be a finite group such that $L = F^*(H)$ is quasi simple but neither a group of Lie type in charcateristic 2 nor an alternating group. Let V be a faithful irreducible $GF(2)H$-module and $1 \neq A \leq G$ with $[V,A,A] = 1$ and let B be a maximal quadratic subgroup of H containing A. Moreover assume that there exists at least one fours group in H acting quadratically on V.*

(a) *One of the following holds.*

Remark: Information should be written down more clearly

1. $L \cong \mathrm{Mat}_{12}$ *and* V *is* 10*-dimensional.*
   1.1. $|B| = 4$, $A \leq L$, $N_L(A) \sim 2^5.\mathrm{Sym}(3) \sim N_L(B)$, $[V, B] = C_V(B)$ *is* 5*-dimensional and either*
      1.1.1. $A = B$
      1.1.2. $|A| = 2$ *and* $[V, A]$ *is* 4*-dimensional.*
   1.2. $|B| = 4$, $B \not\leq L$, $N_L(B) \sim C_2 \times \mathrm{Sym}(5)$, $C_V(B) = [V, B]$ *is* 5*-dimensional and either*
      1.2.1. $A \not\leq L$ *and* $C_V(A) = C_V(B) = [V, B] = [V, A]$
      1.2.2. $A = B \cap L$ *and* $[V, A]$ *is* 4*-dimensional.*
2. $L \cong 3{\cdot}\mathrm{Mat}_{22}$ *and* V *is* 12*-dimensional.*
   2.1. $|A| = 2$, $A \leq L$ *and* $[V, A]$ *is* 4*-dimensional.*
   2.2. $|A| = |B| = 2$, $|A| \not\leq L$ *and* $[V, A] = C_V(A)$ *is* 6*-dimensional.*
   2.3. $|A| \geq 4$, $|B| = 8$, $B \leq L$, $N_L(B) \sim C_3 \times 2^3.L_3(2)$ *and* $C_V(A) = C_V(B) = [V, B] = [V, A]$ *is* 6*-dimensional.*
   2.4. $|A| \geq 4$, $|B| = 16$, $B \leq L$, $N_L(B) \sim 2^4 : 3{\cdot}Alt(6)$ *and* $C_V(A) = C_V(B) = [V, B] = [V, A]$ *is* 6*-dimensional.*
3. $L \cong Mat_{22}$ *and* V *is* 10 *dimensional.*
   3.1. $|A| = |B \cap L| = 2$ *and* $[V, A]$ *is* 4*-dimensional.*
   3.2. $|A| = 2$, $|B| = 4$, $A \not\leq L$, $C_L(A) \sim 2^3.L_3(2)$ *and* $[V, A]$ *is 3-dimensional.*
   3.3. $|A| = |B| = 4$, $A \not\leq L$, $N_L(A) = N_L(A \cap L)$ *and* $C_V(A) = C_V(B) = [V, B] = [V, A]$ *is 5-dimensional.*
4. $H \cong Mat_{24}$ *and* V *is* 11*-dimensional.*
   4.1. $|A| = 2$, $|B| = 4$, $N_G(A) \sim 2^{1+3+\bar{3}}.L_3(2)$ *and* $[V, A]$ *is* 4*-dimensional.*
   4.2. $|A| = |B| = 4$, $N_G(A) \sim 2^8.(Sym(3) \times Sym(3)) \leq 2^6 : (Sym(3) \times L_3(2))$ *and either*
   V *is the Golay code module and* $C_V(A) = [V, A]$ *is* 6*-dimensional or*
   V *is the Todd module and* $C_V(A) = [V, A]$ *is* 5*-dimensional*
   4.3. $|A| \leq 4$, $|B| = 4$, $N_L(A) \leq N_L(B) \sim 2^{2+4} : 3 : Sym(5) \leq 2^6 : 3{\cdot}Sym(6)$ *and either*
   V *is the Golay code module and* $C_V(A) = C_V(B) = [V, B]$ *is* 6*-dimensional or*
   V *is the Todd module and* $[V, A] = C_V(B) = [V, B]$ *is* 5*-dimensional*
5. $L \cong 3{\cdot}U_4(3)$, V *is* 12*-dimensional.*
   5.1. $|A| = 2$, $A \leq L$ *and* $[V, A]$ *is* 4*-dimensional.*

5.2. $|A| = |B| = 2$, *A inverts $Z(L)$ and $[V, A] = C_V(A)$ is 6-dimensional.*

5.3. $|A| = 2$, *$A \not\leq L$, $C_L(A) \cong C_3 \times U_4(2)$ and $|[V, A]| = 4$.*

5.4. $|A| = 2$, *$A \not\leq L$, $|B| = 2^5$ and $C_V(A) = [V, A] = C_V(B) = [V, B]$ is 6-dimensional and $C_L(A) \sim 2^4(\mathrm{Sym}(3) \times \mathrm{Sym}(3)$.*

5.5. $|B \cap L| = 16$, *$N_L(B) \sim 2^4 : 3{}^{\cdot}\mathrm{Alt}(6)$ and either*
$C_V(A) = [V, A] = C_V(B) = [V, B]$ is 6-dimensional or
$|A| = 4$, $|A \cap L| = 2$ and $[V, A] = [V, A \cap L]$ is 4 dimensional.

6. *$L \cong J_2$ and V is 12-dimensional.*

6.1. $|A| = 2$, *$|B| = 4$, $N_L(A) \sim 2^{1+4}Alt(5)$ and $[V, A]$ is 4-dimensional.*

6.2. $|A| = |B| = 4$, *$N_L(A) \sim 2^6.Sym(3)$ and $[V, A] = C_V(A)$ is 6-dimensional.*

6.3. $|B| = 4$, *$N_L(A) \leq N_L(B) \cong Alt(4) \times Alt(5)$ and $C_V(A) = [V, A] = C_V(B) = [V, B]$ is 6-dimensional.*

6.4. $|A| = |B| = 2$, *$A \not\leq L$ and $[V, A]$ is 6-dimensional.*

7. *$G \cong Co_1$ and V is 24-dimensional.*

7.1. $|A| = 2$, *$|B| = 4$, $N_L(A) \sim 2^{1+8}\Omega_8(2)$ and $[V, A]$ is 8-dimensional.*

7.2. $|A| = |B| = 4$, *$N_L(A) \sim 2^{14}.Sym(3) \times Alt(8)$ and $[V, A] = C_V(A)$ is 12-dimensional.*

7.3. $|B| = 4$, *$N_L(A) \leq N_L(B) \sim (Alt(4) \times G_2(4)).2$ and $C_V(A) = [V, A] = C_V(B) = [V, B]$ is 12-dimensional.*

7.4. $|A| = |B| = 2$, *$N_L(A) \sim 2^{11}Aut(M_{12})$, and $[V, A]$ is 12-dimensional.*

8. *$G \cong Co_2$ and V is 22-dimensional.*

8.1. $|A| = 2$, *$|B| = 4$, $N_L(A) \sim 2^{1+8}Sp_6(2)$ and $[V, A]$ is 6-dimensional.*

8.2. $|A| = 2$, *$|B| = 4$, $N_L(A) \sim 2^{1+4+6}Alt(8)$ and $[V, A]$ is 8-dimensional.*

8.3. $|A| = |B| = 4$, *$N_L(A) \sim 2^{15}.L_3(2)$ and $[V, A] = C_V(A)$ is 11-dimensional.*

8.4. $|A| = |B| = 2$, *$N_L(A) \sim 2^{10}Aut(Alt(6))$, and $[V, A]$ is 11-dimensional.*

9. *$L \cong 3{}^{\cdot}Sz$ and V is 24-dimensional.*

9.1. $|A| = 2$, *$|B| = 4$, $N_L(A) \sim 2^{1+6}\Omega_6(2)$ and $[V, A]$ is 8-dimensional.*

9.2. $|A| = |B| = 4$, *$N_L(A) \sim 2^{14}.Sym(3) \times Alt(5)$ and $[V, A] = C_V(A)$ is 12-dimensional.*

9.3. $|B| = 4$, *$N_L(A) \leq N_L(B) \sim (Alt(4) \times L_3(4)).2$ and $C_V(A) = [V, A] = C_V(B) = [V, B]$ is 12-dimensional.*

9.4. $|A| = |B| = 2$, *$A \not\leq L$ and $[V, A]$ is 12-dimensional.*

(b) *Suppose in addition that $q \leq 2$, where $|A|^q = |V/C_V(A)|$. Let c be the case in (a) fulfilled by A and $a = |A|$. Then (c, a, q) is one of the following* **Remark: this doesn't look very nice**

1. $(2.3, 8, 2)$.

2. $(2.4, 8, 2)$ *or* $(2.4, 16, \frac{3}{2})$.
3. $(5.3, 2, 2)$.
4. $(5.5.1, 8, 2)$, $(5.5.1, 16, \frac{3}{2})$ *or* $(5.5.1, 32, \frac{6}{5})$.
5. $(5.5.2, 4, 2)$

Inparticular, $L \cong Mat_{22}, 3{\cdot}Mat_{22}$ *or* $3{\cdot}U_4(3)$*; and* $q \geq \frac{3}{2}$ *unless* $L \cong 3{\cdot}U_4(3)$ *and* $|A| = 32$.

Proof: This can be verified using [MS] and [At].

dpromo

Definition 6.16 *Let* H *be a group and* F *a field. Then an* FH *promodule for* H *is a pair* (ϕ, V) *there* V *is a vector space over* F *and* $\phi : H \to GL_K(V)$ *is a map so that the induced map* $\phi^* : H \to PGL_K(V)$ *is a homomorphism.*

VZQ

Lemma 6.17 *Let* p *a prime and* H *be a finite group* p*-connected group with* $O_p(H) = 1$*. Let* $S \in \mathrm{Syl}_p(H)$ *and* Z *and* Q *non-trivial normal subgroups subgroups of* S *with* $Z \leq Z(Q)$ *and* $|Q| \geq 3$*. Let* $L = O^p(H)$*.*

(a) *Suppose* $p = 2$ *and* H *is a transitive subgroup of* $Sym(\Omega)$ *such that* Z *acts trivially all* Q *orbits of size larger than two. Then one of the following holds:*

1. *The exists a system of blocks* $\mathcal{D}$ *for* H *on* Ω *such that*
   (a) *If* $\Delta \in \mathcal{D}$*, then* Q *normalizes* Δ*,* $Q = ZC_Q(\Delta)$ *and* $|Q/C_Q(\Delta)| = 2$*.*
   (b) *For* $\Delta \in \mathcal{D}$ *let* $L_\Delta = C_L(\bigcup \mathcal{D} - \Delta)$*. Then* $L = \times_{\Delta \in \mathcal{D}} L_\Delta$*.*
2. $L \neq O(L)$*. Let* $\mathcal{D}$ *be the set of orbits of* $O(H)$ *on* $|\Omega|$*. Then* $H/O(H)$ *acts faithfully on* H*. Let* Δ *be an orbit for* L *on* $\mathcal{D}$ *and for* $X \leq H$ *let* $X^\Delta = N_X(\Delta)/C_X(\Delta)$*.Then*
   (a) Q *normalizes* Δ*.*
   (b) $L^\Delta = F^*(H^\Delta)$ *is simple.*
   (c) $1 \neq Z^\Delta \leq Z(Q^\Delta)$*,* Z^Δ *and* Q^Δ *are normal in* S^Δ*,* S^Δ *is a Sylow* 2*-subgroup of* H^Δ*,* $|Q^\Delta| \geq 4$*, and each orbit for* Q^Δ *on* Δ *is either centralized by* Z^Δ *or has size at most* 2*.*
   (d) *One of the following holds:*
      1. $H^\Delta = \mathrm{Alt}(\Delta)$ *or* $\mathrm{Sym}(\Delta)$*.*
      2. Δ *can be viewed as projective space over the field with two elements so that* $H^\Delta = PGL(\Delta)$*. Moreover if* K *is a component of* $L/O(L)$*, then* $N_S(K)$ *induces only inner autmorphism on* K*.*
      3. $|\Delta| = 6$ *and* $H^\Delta \cong \mathrm{Alt}(5)$ *or* $\mathrm{Sym}(5)$*.*
      4. $|\Delta| = 10$ *and* $H^\Delta \cong \mathrm{Sym}(6)$ *or* $\mathrm{Aut}(\mathrm{Alt}(6))$*.*
      5. $|\Delta| = 12$ *and* $H^\Delta = \mathrm{Mat}_{12}$ *or* $\Delta = 24$ *and* $H^\Delta \cong Aut(\mathrm{Mat}_{12}$*.*
      6. $|\Delta| = 22$ *and* $H^\Delta = \mathrm{Mat}_{22}$ *or* $\mathrm{Aut}(\mathrm{Mat}_{22}$*.*

7. $|\Delta| = 24$ *and* $H^\Delta = \mathrm{Mat}24$. **Remark: This needs careful checking**

(b) *Let* K *be a field with* $\mathrm{char}K = p$ *and suppose that* H *is an irreducible subgroup of* $GL_K(V)$ *with* $[V, Z, Q] = 0$. *Let* W *a Wedderburn componet for* L *on* V. *For* $X \leq H$ *let* $X^W = N_X(W/C_X(W)$. *Then one of the following holds.*

1. $p = 2$ *and there exists a system of blocks* $\mathcal{D}$ *for* H *on* V *such that*
   (a) *If* $U \in \mathcal{D}$, *then* Q *normalizes* U, $Q = ZC_Q(U)$ *and* $|Q/C_Q(U| = 2$.
   (b) *For* $U \in \mathcal{D}$ *let* $L_U = C_L(\bigcup \mathcal{D} - U)$. *Then* $L = \times_{U \in \mathcal{D}} L_U$.
2. $p = 2$ *and there exists a system* $\mathcal{D}$ *of* H*-blocks on* V *with* $C_H(\mathcal{D}) = O(H)$ *and so that the action of* $H/O(H)$ *on* $\mathcal{D}$ *is described as in (a)2.*
3. $L = E(L)$ *and*
   (a) Q *normalizes* W.
   (b) L *acts irreducible on* W.
   (c) $1 \neq Z^W \leq Z(Q^W)$, Z^W *and* Q^W *are normal in* S^W, S^W *is a Sylow* 2-*subgroup of* H^Δ, $|Q^W| \geq 3$, $[W, Z, Q] = 0$ *and* $F^*(H^W) = L^W$.
   (d) *One of the follwing holds.*
      1. L^W *is quasi-simple.*
      2. $p = 2$, $L^W = L_1 L_2$, *where* $L_1 L_2$ *are the components of* L^W. Q *normalizes* L_1 *and* L_2 *and as* $L^W Q^W$ *module* $W = W_1 \otimes_F W_2$ *for some faithful* $FL_i Q^W$ *modules* W_i. *Moreover* Q^W *acts linear dependently on* W_i.
      3. $p = 2$, $L^W Q^W \cong L_2(q) \wr C_2$ *and* W *is the natural* $\Omega_4^+(q)$*-module for* $L^W Q^W$.
4. *One of the following holds:*
   1. $p = 2$, $L = O_3(L)$, $L^W \cong \mathrm{Ext}(3^{1+2})$,$Z^W \cong C_2$, $Q^W \cong C_4$ *or* Q_8 *and* $|W| = 2^6$.
   2. $p = 3$, $L = O_2(L)$, $L^W \cong Q_8$, $Z^W = Q^W \cong C_3$ *and* $|W| = 3^2$.
5. $p \in \{2, 3\}$. *Let* $\{2, 3\} = \{p, q\}$ *and* $M = O_q(H)^W / Z(O_q(H)^W$. *Then*
   (a) $O_q(L)^W \cong \mathrm{Ext}(q^{1+2n})$ *or* $C_4 \circ \mathrm{Ext}(2^{1+2n})$, $n \geq 2$
   (b) $Z^W \cong C_p$ *and* $Q^W \cong C_3, C_4$ *or* Q_8.
   (c) L *acts irreducible on* M.
   (d) $|[M, Q]| = q^2$.
   (e) $O_q(H)$ *acts irreducible on* W.
   (f) **Conjecture** *If* $p = 2$, *then* $L/C_L(M) = Sp_{2n}(3)$ *and if* $p = 3$, *then* $L/C_L(M) \cong \Omega^\pm_{2n}(2), \mathrm{Alt}(2n+1), \mathrm{Alt}(2n+2), Sp_{2n}(2)$ *or* $SU_n(2^2)$. *Also there are restricions on* n *from the fact that* Q *is normal in* S.

Proof: (a) The proof is divided into a series of steps

$VZQ - 1$

(1) Let Δ be a block for Q on Ω.

(a) One of the following holds:

1. Q normalizes Δ.
2. Z centralizes Δ and so also $\bigcup \Delta^Q$.
3. $|\Delta^Z| = |\Delta^Q| = 2$ and $N_Q(\Delta)$ centralizes Δ and so also $\bigcup \Delta^Q$.

(b) One of the following holds:

1. Q normalizes Δ.
2. $N_Z(\Delta)$ centralizes Δ and so also $\bigcup \Delta^Q$.

Clearly (a) implies (b). For (a) suppose that Z does not centralize Δ. If Z normalizes Δ then Z has a non-trivial orbit on Δ and Q has to normalize that orbit. Since Δ is a block, Q normalizes Δ in this case. If Z does not normalize Δ, pick $z \in Z$ with $\Delta \neq \Delta^z$. Then $\Delta \cup \Delta^z$ is a union of non-trivial z orbits and so Q normalizes $\Delta \cup Delta^z$. Let $\omega \in \Delta$. Then $N_Q(\Delta)$ normalizes $\Delta \cap \{\omega, \omega^z\} = \{\omega\}$. Hence 3. holds in this case.

(2) Let Δ be an L-invariant H-block. Then $VZQ-2$

(a) $\Omega = \bigcup \Delta^S$.

(b) Z does not centralize Δ.

(c) If Z normalizes Δ and $|Q/C_Q(\Delta)| = 2$, then (a)1. in the lemma holds.

(d) If Q does not normalize Δ, then (a)1. in the lemma holds.

Since $H = LS$,(a) holds. Since $Z \trianglelefteq S$, (a) implies (b). If the assumptions of (d) hold, then by (b) and (1)(a), also the assumptions of (c) are with Δ replaced by Δ^Z. So it remains to prove (c). By (b) and (1)(a), Q normalizes Δ. Let $\mathcal{D} = \Delta^H$, $Q_D = C_Q(D)$ and note that $Q/Q_D = 2$. Let Γ be the union of the blocks in Δ^H centralized by Q_D. We claim thhat Γ is a H-block. Otherwise there exists $s \in S$ with $Q_D^s \neq Q_D$ and a block in Δ^H centralized by $Q = Q_D Q_D^s$, a contradiction to (b). Hence Γ is a block and replacing Δ by Γ we may assume $\Gamma = \Delta$. Define L_Δ as in (a)1. of the lemma. Let $R = \langle L_\Delta \mid \Delta \in \mathcal{D}$. Then R is a normal subgroup of H and $R = \times_{\Delta \in \mathcal{D}} L_\Delta$. It remains to show that $R = L$. Let $\mathcal{D} = \{\Delta, \Delta_1, \Delta_2, \ldots, Delta_n\}$. Put $L_0 = L$ and inductively for $1 \leq i \leq n$, $L_i = [L_i, Q_{\Delta_i}]$. We claim that $L = L_i C_L(\Delta)$. This is obvious for $i = 0$ Since H is 2 connected, $L = [L, Q]$ and so by induction, $L = [L_{i-1}, Q] C_L(\Delta)$. Since $Q = Q_\Delta Q_{\Delta_i}$ and $Q_\Delta \leq C_L(\Delta)$ we conclude, $L = [L_{i-1}, Q_{\Delta_i}] C_L(\Delta) = L_i C_L(\Delta)$. Thus $L = L_n C_L(\Delta)$. But $L_n \leq L_i$ for all i and L_i centalizes Δ_i. Thus $L_n \leq L_\Delta$ and so $L = L_\Delta C_L(\Delta)$ But this clearly implies $L = R$ completing the proof of (2).

(3) Let $F \leq Q$ with $|F/F \cap Z| \leq 2$. Then an orbit for F on Ω has length at most for 2. In particular, F is elementary abelian. $VZQ-2a$

Either $Z \cap F$ acts trivially on a given F-orbit or not. In both cases the orbit has size at most two.

(4) Let P be a subgroup of odd order in H normalizes by Q. Let Δ be an orbit for PQ on Δ such that P acts transitively and Z non-trivially on Δ. Then $|Q/C_Q(\Delta)| = 2$. $VZQ-3$

By the Sylow theorem and the Frattini argument, Q fixes a point $\omega \in \Delta$. Also $P = [P,Q]C_P(Q)$ and replacing P by $[P,Q]$ and Δ by $\omega^{[P,Q]}$ we may assume that $P = [P,Q]$. Let R be a maximal Q invariant normal subgroup of P. If R is transitive on Ω, then by induction on $|P|$, Z centralizes P. Hence $Z/C_Z(\Delta)$ acts semiregulary on Δ and all orbits of Z on Ω have size two. Also Q and hence $[R,Q]$ normalizes all orbits of Z. Thus $[R,Q]$ centralizes Δ. Since $P = [P,Q]$, $[R,P$ centalizes Δ and so $R/C_R(\Delta)$ acts regularly. But then R centralizes Δ, a contradition. So R is not transitive. Let $\mathcal{D}$ be the set of orbits for R on Δ. Then the abelian group $M \stackrel{def}{=} P/R$ acts regularly on $\mathcal{D}$ and $\mathcal{D}$ and and M are ismorphic as Q-sets. Suppose that Z centralizes M, then $P = C_P(Z)R$ and M acts non-trivially on each member of $\mathcal{D}$. But then Q normalizes each member of $\mathcal{D}$. Thus Z acts non-trivially on M and $\mathcal{D}$. Similarly, if $C_Q(M)$, acts non-trivialy on Δ, Z is forced to act trivially on $\mathcal{D}$. Thus $Q/C_Q(Delta)$ acts faithfully on M and $\mathcal{D}$. Let $z \in Z \setminus C_Z(M)$. Since $z \in Z(Q)$ and Q acts irrducibly on M, z inverts M. Let $m \in M^{\#}$. Then Q normalize $\{m, m^{-1}$ and as Q is irreducible, $M = \langle m \rangle$ and $|Q/C_Q(M) = Q/C_Q(\Delta)| = 2$.

(5) Suppose (a)1. does not hold and let $\mathcal{D}$ be the set of orbits for $O(H)$ on Ω. Then $H/O(H)$ acts faithfully on $\mathcal{D}$. $VZQ-5$

Suppose not. Then since H is 2-connected, L centralizes $\mathcal{D}$ .Let $\Delta \in \mathcal{D}$. By (2), Q normalizes Δ. Also Z acts non-trivially on Δ and $O(G)$ acts transitively. Thus by (4), $|Q/C_Q(\Delta)| = 2$ and by (2) (a)1. holds.

We assume form now on that (a)1. does not hold. Replacing Ω by the set of orbits of $O(H)$ on Ω and H by $H/O(H)$ we also may assume that $O(H) = 1$. Thus $L = \times_{i=1}^{m} L_i$ for some non-abelian simple groups L_i. Let Δ be an orbit for L on Ω. We wish to whow that a2 holds. a2a and a2c follow from (2). Let $M = L^{\Delta}$. Then $M = \times_{i=1}^{n} E_i$, where $\{E_1, \ldots, E_n\}$ consists of whose $L_i^{\Delta} (\cong L_i)$ which act non-trivially on Δ. Suppose for a contradiction that $n \geq 2$. Let $1 \neq z \in Z \cap Z(S)$. Then z centralizes the Sylow 2-subgroup $M \cap S$ of M and so z normalizes all L_i and E_i. If Q does not normalize the componets of M, then $|[S \cap M, Q]| \geq |S \cap M_i| \geq 4$ and so $|M \cap Q| \geq 4$. So replacing Q by $(M \cap Q)Z$ in this case, we may assume that Q does normalize the components of M.

Let $E = E_1$ and $F = C_M(E_1)$. Since $z \in Z(S)$, $E = [E, z]$. Suppose that $C_Q(E)^{\Delta} \neq 1$ and pick $t \in C_Q(E)^{\Delta}$ with $|t| = 2$. Then z normalises all the non-trivial orbits for t on Ω. Since E centralizes t, the same is true for $E = [E, t]$. But the $E = E'$ centralizes each non-trivial orbit of t, a contradiction. Thus $C_Q(E)^{\Delta} = 1$.

Suppose that E does not act transitively on Δ. Since M acts transitively, M does not normalize any orbit of E. As $M = [M, z]$ there exists an orbit Γ for E on Δ with $\Gamma \neq \Gamma^z$.

Thus by (1), $P = C_Q(\Gamma)$ has index two in Q. But then $[E,P]$ centralizes Δ and so $[E,P] = 1$ and $P^\Delta \leq C_Q(E)^\Delta = 1$, a contradiction to $|Q/P| = 2$.

Thus E acts transitively on Δ. By symmetry also F is transitively on Δ and so E is regular. Let F be a group of order four in Q^Δ with $z^\Delta \in F$. Let $\omega \in \Delta$. Let $F = \{1, f_1, f_2, f_3\}$ and $\omega^{f_i} = \omega^{e_i}$ for some $e_i \in E$. Let $E_i = \{e \in E \mid e^{f_i} = e_i^{-1}\}$. Note that E_i is a coset of the proper subgroup $C_E(f_i)$ in E. Let $e \in E$. By (3), there exists $f_i \in F$ with $\omega^e = \omega^{ef_i} = \omega^{f_i e^{f_i}} = \omega^{e_i e^{f_i}}$. As E is regular we get $e_i e^{f_i} = 1$ and so $e \in E_i$. Thus $E = E_1 \cup E_2 \cup E_3$ is covered three proper cosets. But this implies that E has a subgroup of index two or three, a contradiction as E is non-abelian simple. Thus a2c holds.

To prove a2d we assume without loss that $\Delta = \Omega$ so $L = F^*(H)$ is simple. Let $V = GF(2)\Omega$ be the permutation module associate to Ω. Then $[V, Z, Q] = 0$ and so V is a faithful $GF(2)H$-module with a quadratic fours group. Hence by 6.15, L is a group of Lie type in characteristic 2, or $L = \mathrm{Mat12}, \mathrm{Mat22}, \mathrm{Mat24}, J_2, CO_1$ or Co_2. Let $1 \neq z \in Z$ and $R = \langle Q^{C_H(z)}$. Then R normalizes all non trivial orbits of z on Ω and $[V, z, Q] = 0$.

Suppose that L is one of the sporadic groups. Then H has a unique class of 2-central involution. If L is $J_2, C0_1$ or CO_2 we get that $O_2(C_L(z)) \leq R$ and so $V, z, O_2(C_L(z))] = 1$, a contradcition. Hence $L = \mathrm{Mat12}, \mathrm{Mat22}$ or Mat_{24}. **TO BE CONTINUED**

(b) Again we divide the proof into a series of steps and use a similar strategy as in the proof of (a)

$VZQ - 11$

(6) Let U be a block for Q on V.

(a) One of the following holds:

1. Q normalizes U.
2. Z centralizes U and so also $\sum U^Q$.
3. $p = 2$, $|U^Z| = |U^Q| = 2$ and $N_Q(U)$ centralizes U and so also $\sum U^Q$.

(b) One of the following holds:

1. Q normalizes U.
2. $p = 2$ and $N_Z(U)$ centralizes U and so also $\sum U^Q$.

Clearly (a) implies (b). For (a) suppose that Z does not centralize U. If Z normalizes U, then $0 \neq [U, Z] \leq U$ and Q centralizes $[U, Q]$. Since U is a block, Q normalizes U in this case. If Z does not normalize U, pick $z \in Z$ with $U \neq U^z$. Since $z \in Z(Q)$, $U + U^z$ is a block for Q.Also Q centralizes $[U, z]$ and so normalizes $U + U^z$. As a $N_Q(U)$ module, $U \cong U + U^z/U^z = [U, z] + U^z/U^z \cong [U, z]/[U, z] \cap U^z$ and so $N_Q(U)$ centralizes U. Hence 3. holds in this case.

$VZQ - 12$

(7) Let U be an L-invariant H-block. Then

(a) $V = \sum U^S$.

(b) Z does not centralize U.

(c) If Z normalizes U and $|Q/C_Q(U)| = 2$, then (b)1. in the lemma holds.

(d) If Q does not normalize U, then (b)1. in the lemma holds.

The proof is essentially the same as the one for (2).

(8) Suppose exists an H-block which is not L-invariant, then $(b1)$ or $(b2)$ in the lemma holds. $VZQ-13$

Let $calD$ be a block system for H on V with L acting non-trivially on $\mathcal{D}$ and let $\mathcal{D}$ be maximal with this property. Then $p = 2$, $C_H(\mathcal{D} \leq O(H)$ and we can apply (a) to $H/C_H(\mathcal{D})$ and $\mathcal{D}$. In case (a)1., (b)1. holds. In case of (a).2 the maximality of $\mathcal{D}$ implies that $O(H)$ acts trivially on $\mathcal{D}$. Thus (b)2. holds.

We assume from now on without loss that neither (b)1. nor (b)2. hold.

(9) Let W be a Wedderburn component for L on V. Then Q normalizes Q and W is irreducible as L-module. $VZQ-15$

By (7)d, Q normalizes W. As V is irredicble for H, W is irreucible for $N_H(L)$. As W is L-homogenous and $N_H(L)/L$ is a p-group, L is irreducible on W.

(10) Suppose that $L = E(L)$. Then (b3) holds. $VZQ-16$

If $Q/C_Q(W)| = 2$, then (b1) holds. Hence ((b3a),(b3b) and (b3c) holds It remains to verify (b3d). Let $L_1, L_2, \ldots L_n$ be the components of $L/C_L(W)$. If $n = 1$, (b3d1) holds. Put $F = \mathrm{End}_K L(W)$ and let P the largest subgroup of Q normalizing the components of L^W. As in part (a), P^W has order at least three and $(Z \cap P)^W \neq 1$. Then W has a tensor compostion $(F, W_i, 1 \leq i \leq n)$, where W_i is an $C_{LP}(F)$ module centralized by all L_j, $j \neq i$. Then by 6.6,$p = 2$, $n = 2$ and P^W acts linearly dependently on W_1 and W_2. If $Q = P$, (b3d2) holds. So suppose that $|Q/P| = 2$ and let $q \in Q \setminus P$. Note that Q is F-linear. Let $1 \neq z \in P\mathcal{Z}$. Let U be an irreducible FU subspace in W with $U \neq U^z$. Then $U = W_1 \otimes a_2$ for some $a_2 \in W_2$. Also U^q is an irreducible FL_2P subspace and so $U^q = a_1 \otimes W_1$ for some $w_1 \in W_2$. Similarly $U^z = b_1 \otimes W_2$ and $U^{zq} = W_2 \otimes W_1$. Thus $(U + U^z) \cap (U + U^z)^q = (Fa_1 + Fb_1) \otimes (Fa_2 \otimes Fb_2)$. On the otherhand , q centralizes $[U, z] \leq U + U^z$ and we conclude that $\dim_F U = 2$. We conclude that W_1 and W_2 are 2-dimensional and by say Dicksson's theorem, (b3d3) holds.

(11) Suppose that W is tensor decomposable for LQ. Then (b3) holds. $VZQ-14$

By 6.6, $p = 2$ and Q is elementary abelian and $C_{L^W}(q) = C_{L^W}(Q$ for all $1 \neq q \in Q$. Thus $O(H)^W \leq Z(L^W)$ and so $L = E(L)$. So the claim follows from (10).

Suppose from now on that W is tensor indecomposable. Let M be a normal subgroup of H minimal with respect to $[M, L] \neq 1$. Note that $M/C_M(L)$ is characteristicly simple. Hence either $M = E(M)$ or M is a q-group for some prime q. If $M = E(M)$, it is easy to see that M is not a p' group and so $M = L$ since H is p-connected. So in view of (10) we may assume that M is a q- group.

$VZQ - 17$

(12) M acts irreducible on W and $M^W \cong \mathrm{Ext}(q^{1+2n})$ or $C_4 \circ Ext(2^{1+2n})$, $n \geq 2$.

If M is not homogenous on W. Then L acts non-trivially on the Wedderburn components of M on V, a contradiction to (8). Hence M is homogenous. As W is tensor indecoposable, this implies that M is irreducible on W. Let $F = \mathrm{End}_{KM}(W)$. Then by 6.6, Q and so also $L = [L, Q]$ is F-linear on W. Thus $[Z(M^W), L = 1$, $C_L(M) = Z(L)$ and $C_M(L) = Z(M)$. By a standard argument the structure of M^W is as described.

$VZQ - 18$

(13) One of the following holds:

1. $p = 2$, $q = 3$ and $[M^W, Q]Q^W \cong SU_3(2)$ or $\mathrm{Ext}(3^{1+2})C_4$
2. $p = 3$, $q = 2$ and $[M^W, Q]Q^W \cong SL_2(3)$.

Let $P = [M^W, Q]$, $R = PQ^W$ and Y and irreducible R-submodule in W. Then P and so also R acts faithfuly on Y. Then P is extra-special. Let $1 \neq z \in Z^W$. Then as z acts quadratically on W, Hall-Higmann implies $p = 2$, or $p = 3$ and $q = 2$. Suppose that $P \neq [P, z]$. Then $[P, z]$ and $C_P(z)$ are normal in R and $P = [P, z] \circ C_P(z)$. But then Y is tensor decomposable for R. Then the argument in (11) gives a contradiction. Thus $P = [P, z]$. A be a maximal abelian z-invariant normal subgroup of P. Let $\mathcal{A} = \{D \leq A | A = Z(P)D, D \cap Z(P) = 1\}$. Then P acts transitively on $\mathcal{A}$ and z fixes a unique member of $\mathcal{D}$, namely $[A, z]$. Also $Y \bigoplus_{D \in \mathcal{A}} C_Y(D)$. If $p = 3$ we conclude that $|\mathcal{A}| = 1$ and so $|P| = 8$ and 2. holds. So suppose $p = 2$. Let $|P| = q^{1+2n}$. Then $|A| = q^{1+n}$, $|\mathcal{A}| = q^n$ we conclude that $\dim_F[Y, z] = \frac{q^n-1}{2}$, $\dim_F C_Y(z) = \frac{q^n+1}{2}$ and $\dim_F C_Y(z)/[Y, z] = 1$. Let $q \in Q^W \setminus \langle z \rangle$. If $|q| = 2$, we may assume that q normalizes A. But then $[Y, z, t] = 0$ implies that t normalizes all the orbits of z on $\mathcal{A}$, a contradicition. Thus $|q| = 4$ and we may assume $q^2 = z$. Since $[Y, q, t] = 0$, $|[Y, q] + [Y, z]/[Y, z]$ has dimension at most 1 over F. Hence there exists an q invariant F-hyperplane U in Y with $[U, q] \leq [Y, t] \leq C_U(q)$. Thus $[U, q, q] = 0$ and $[U, q^2] = 1$. Thus $Y/C_Y(z) = 1$ is 1-dimensional. So $\frac{q^n-1}{2} = 1$. $q^n = 3$ and $|P| = 3^3$. Hence 1. holds in this case.

$VZQ - 19$

(14) Either L acts irreducible on $M^W/Z(M^W)$ or (b4) holds.

Let $Z(M^W) < P \leq M^W$ be minimal with respect to being L-invariant. Put $\overline{M} = M^W/Z(M^W)$. If Q does not normalize P, then by (13), $|\overline{U}| \leq q^2$. Thus $L/C_L(U)$ is a solvable $\{p, q\}$ group. Since H is p-connected we conclude that $L/C_L(U)$ is a p' group and so a q-group. Since L is irreducible on U we conclude $[U, L] = 1$. Since H is irreducible on $M/Z(M)$ we conclude $[M, L] \leq Z(M)$. Thus $O^q(L) \leq C_L(W) \leq Z(L)$ and $L = O_q(L)Z(L)$. Since $[Z(L), Q] = 1$, p-connectivity of H implies, $L = O_q(L)$. Thus (b4) holds in this case.

So we may assume that Q normalizes U. If U is abelian, then by (13), Q centralizes U and so also L centalizes U, a contradiction. Hence U is not abelian and $M^W = PC_{M^W}(P)$. Thus 6.17-14 implies $P = M^W$.

(15) If L acts irreducible on $M^W/Z(M^W)$ then (b5) holds. $VZQ-20$

This follows form (13). □

Lemma 6.18 *Let p be a group, H a finite p-minimal group with $O_p(H) = 1$. Let $S \in \mathrm{Syl}_p(H)$ and Z and Q non-trivial normal subgrous of S with $Z \le Z(Q)$. Let R be maximal in Q with $[V, R] \le [V, Z]$. Let V be a faithful $GF(p)H$-module so that* $VZQm$

(i) $[V, Z, Q] = 0$.

(ii) $V = [V, O^p(H)]$.

(iii) $V/C_V(O^p(H))$ *is irreducible as H-module.*

Then $|Q/R| \le V/C_V(Z)$. Moreover if $T \trianglelefteq S$ with $Z \le T$. Then either $T \le R$ or $[V, T] = [V, Q]$

Proof: Remark: Some parts of the proof are still very sketchy, also the proof is a lot longer than it should be and to much of a case by case analysis Let $Y = C_V(L)$ and $\bar{V} = V/Y$. Then $\bar{V}$ is irreducible as H-module.

Let $C = C_H(\bar{V})$.Then $C \cap L]$ centralizes U and V/U and so $C \cap L$ is a p-group. Since $O_p(H) = 1$ we conclude $C \cap L = 1$. Thus $O^p(C) = 1$, C is p-group and $C = 1$.

Hence H acts faithfully on $\bar{V}$ and we can apply 6.17(b) to $\bar{V}$.

Let W be a LQ submodule in V minimal with respect to $[W, L] \ne 0$. Then $W = [W, L]$. For $X \in LQ$ let $X/C_X(W)$. Let $1 \ne z \in Z(S) \cap Z$.

(1) Suppose that $|Q^W/Z^W| \le \bar{W}/C_{\bar{W}}(Z)$ and $[W, T] \in \{[W, Z], [W, Q]$. Then the lemma holds. $VQZm-1$

Since $\bar{V}$ is irreducible and $H = LS$, $\bar{V} = \langle \bar{W}^S \rangle$ Thus there exists $s_i \in S, 1 \le i \le k$ with $\bar{V} = \oplus_{i=1}^k \bar{W}^{s_i}$. Then $V = [V, L] = [\sum_{i=1}^k W^{s_i}, L] = \sum_{i=1}^k W^{s_i}$. Let $P = \bigcap_{i=1}^k ZC_S(W^k)$. Then $P \le R$ and

$$|Q/R| \le |Q/P| \le |Q^W/Z^W|^k \le \bar{W}/C_{\bar{W}}(Z)^k = |\bar{V}/C_{\bar{V}}(Z) \le V/C_V(Z)$$

Also $[W, T] = [W, Z]$ implies $[V, T] = [V, Z]$, while $[W, T] = [W, Q]$ implies $[V, T] = [V, Q]$

(2) $C_{LQ}(\bar{W}) = C_{LQ}(W)$. $VQZm-2$

Let $B = C_{LQ}(\bar{W})$. Then $B \cap L$ centralizes Y and $W + Y/Y$ and so acts as a p-group on W. Since no composition factor of L on L is a p-group, $B \cap L$ centralizes W. Thus $[B, L, W] = 0$ and $[W, B, L] = 0$. Thus by the three subgroup lemma $[W, L, B] = 0$. As $W = [W, L]$ we conclude $[W, L] = 0$ and so (2) holds.

$VZQm-3$

(3) If $|Q^{\bar{W}}| \leq p^2$, the lemma holds.

By **??**, $|Q^W| \leq p^2$.Also $Z^W \neq 1$ and Z does not centralize $\bar{W}$. Thus (3) follows from **??**.

$VZQm-4$

(4) If $O_{p'}(L) \neq 1$, then $Y = 0$.

By Mascke, $V = C_V(O_{p'}(L)) \oplus [V, O_{p'}(L)]$. Also $Y \leq C_V(O_{p'}(L))$ and as $\bar{V}$ is irreducible, $V = Y + [V, O_{p'}(L))]$. Thus $V = [V, L] = [V, O_{p'}(L)), L]] = [V, O_{p'}(L)]$ and (4) holds.

Suppose first that 1. in 6.17(b) holds for $\bar{V}$. Then $|Q^{\bar{W}}| = 2$ and we are done by (3).

Suppose next that 2. in 6.17)(b) holds. Let $D/Y \in \mathcal{D}$ and $\Delta = D^L$. Without loss $W \leq \sum \Delta$. Since H is p-minimal we conclude from 6.17(a2) that $L^\Delta \cong \mathrm{Alt}(n)$ with $n = 2^k + 1, k \geq 2$ or $n = 6$. If $n \leq 6$ it is easy to see that $Q^\Delta \leq 4$ and so also $|Q^W| \leq 4$. So we may assume that $m = 2^k + 1, k \geq 2$. Let $E \in \Delta$ with $E \neq E^z$. Then $N_Q(E)$ centralizes E. Let $M = N_{LQ}(E)$. Then $M^\Delta \cong \mathrm{Alt}(2^n)$ or $\mathrm{Sym}(2^n)$ and so $M^E = \langle N_Q(E)^M \rangle O(L)$. Hence $M = C_M(E)O(L)$. If $O(L)$ centralizes E. Then $\bar{V}$ is a permutation module for L, a contradiction to $C_{\bar{V}}(L) = 0$. Thus $O(L) \neq 1$ and by (4), $Y = 0$. It follows that $[D, Z] = [D, Q]$. Let F be the unique fixed point for z on Δ. Since F and E are conjugate under L, all p-elements in $N_{LQ}(F)$ act trivially on F. So $[F, Q] = 0$ and $[V, Z] = [V, Q]$.

Suppose that 3. in 6.17(b) holds. By (3) we may assume that $|Q^W| > p^2$. Then p-minimality and quadratic action implies that the components for L are one of $SL_2(q), SU_3(q), Sz(q)$,Alt(q+1)$, Sp$ or $L_3(q)$ Here q is a power of p, $p = 2$ in the last four cases, and a graph automorphism is induced on the components in the last two cases.

If 3d2 or 3d1in 6.17(b) holds then $Y = 0$. Let $F = \mathrm{End}_L(W)$. Then $|Q^W| \leq 2 \cdot |F|$, $|W/C_W(Z)| \geq |F|^2$ and $[W, T] = [W, Q]$ if $|T^W| \geq 4$. Thus we are done by **??**.

So suppose that L^W is quasi simple. If Q^W is not elementary abelain then W is a strongly quadratic module in the sense of Stroth and so $\bar{W}$ is the natural module. Because of the graph automorphism, $L = Sp_4(q)\prime$ is impossible in this case. Thus $Y = 0$ and the lemma is readily verifed in this case.

So suppose that Q^W is elementary abelian. Then its is easy to check that $|C_{\bar{W}}(Z)|^2 = \bar{W}$ and $|Q^W| \leq |\bar{W}/C_{\bar{W}}(Z)|$. In particular, Q acts quadratically on W. Let $J \leq H^W$ minimal with $Q^W \leq J$ and $Q^W \not\leq O_p(J)$. Suppose first that $O_p(J) = 1$. Then (for example by 2.9), $J \cong SL_2(\tilde{q})$ or $Sz)\tilde{q})$. Thus there exists $j \in J$ with $J = \langle T^{Wj}, T^W \rangle$. Thus $[W, J] = [W, T]^j + [W, T]$ and $[W, Q] = ([W, T]^j \cap [W, Q]) + [W, T]$. But $[W, T]^j \cap [W, Q] \leq C_W(J) \cap [W, T]^j \leq [W, T]$ and so $[W, Q] \leq W$. So we may assume that $O_p(J) \neq 1$ and J is not generated by two conjugate of T^W in J. In particular, $L^W \cong Sp_4(q)$. We conclude that either $[W, T] \leq [W, Z]$ or $Y \cap W \leq [W, T]$. In the latter case, $[W, Q] \leq [W, T]$ and the lemma holds in this case.

Suppose finally that 4. or 5. in 6.17(b). In view of (3) we may assume that $Q^W \cong Q_8$. So $p = 2$ Also by (4), $Y = 0$. Let $X = \langle Q^{O_3(L)} \rangle$. Then $X^W \cong SU_3(2)$ and W is a direct sum of natural modules for X^W, Again it is easy to verify the assumptions of **??** and the lemma is proved. □

7 An interesting choice of an amalgam for generic p-type groups

gpt

Hypothesis 7.1 *p is a prime, G is a finite groupe of generic p-type and $S \in Syl_p(G)$.* hgpt

Definition 7.2 (a) *$\mathcal{W}$ is the set of sets $\{M_1, M_2\}$ such that* dcalw

(a) $M_i \in \mathcal{L}(J(S)$

(b) $O_p(\langle M_1, M_2 \rangle) = 1$.

(b) *Define an partial ordering "$\leq$"l on $\mathcal{W}$ by defining $(H_1, H_2) < (M_1, M_2)$ if and only if one of the the follwing holds.*

1. *Some Sylow p subgroup of $H_1 \cap H_2$ is properly contained in a Sylow p -subgroup of $M_1 \cap M_2$.*
2. *$H_1 \cap H_2$ and $M_1 \cap M_2$ have a common Sylow subgroup T and $C_{H_1 \cap H_2}(\Omega_1(Z(T)) < C_{M_1 \cap M_2}(\Omega_1(Z(T))$*
3. *$H_1 \cap H_2 < M_1 \cap M_2$.*
4. *$H_1 \cap H_2 = M_1 \cap M_2$ and (possible after interchanging M_1 and M_2 and H_1 and H_2, $M_1 < H_1$ and $M_2 \leq H_2$.*

"$\leq$" is defined as "$<$" or "$=$"

(c) *$\mathcal{W}^*$ is the set of maximal elements of $\mathcal{A}$ under the order defined in (b).*

We leave it as an easy exercise to the reader to verify that $(\mathcal{W}, \leq)$ is a partially ordered set.

Lemma 7.3 *Let $(M_1, M_2) \in \mathcal{W}^*$, $M_{12} = M_1 \cap M_2$, $T \in \mathrm{Syl}_p(M_{12})$ and put $Z_0 = \Omega_1 Z(T))$. Then*

(a) *For $i = 1, 2$, $|\mathcal{M}(M_i)| = 1$.*

(b) *Suppose R is a p-subgroup of M_1 with $T < R$. Then $\mathcal{M}(R) = \mathcal{M}(M_1)$ and $T \in Syl_p(M_2)$.*

(c) *Suppose that $T \notin \mathrm{Syl}_p(G)$. Then $C(G,T) \in \mathcal{L}$, $C(G,T)$ lies in a unique maximal p-local M of G, $|\mathcal{M}(S)| = 1$ and either T is a Sylow p-subgroup in M_1 and M_2, or $M = M_i^*$ for some i.*

(d) *M_{12} is a maximal subgroup of M_1 and of M_2.*

(e) *One of the following holds:*

1. $C_{M_1}(Z_0) = C_{M_{12}}(Z_0) = C_{M_2}(Z_0)$.

2. *There exists* $\{i,j\} = \{1,2\}$ *so that*

   (a) $C_{M_i}(Z_0) \not\leq M_j$, $\mathcal{M}(M_i) = \mathcal{M}(C_{M_i}(Z_0) = \mathcal{M}(C_G(Z_0))$.

   (b) $C_{M_j}(Z_0) \leq M_i$.

Proof: (a) Suppose M_1 is contained in two distinct maximal p-locals L_1,L_2. Then $M_1 \cap M_2 < M_1 \leq H_1 \cap H_2$. But this contradicts the maximal choice of (M_1, M_2).

(b) Let $M \in \mathcal{M}(R)$. Then T is properly contained in a Sylow $M_1 \cap M$ and so by that maximality of (M_1, M_2), $M_1 \leq M$. If T is not a Sylow p-subgroup of M_2, then we conclude $\mathcal{M}(M_1) = \mathcal{M}(N_L(T)) = \mathcal{M}(M_2)$, a contradcition. Thus (b) holds.

(c) Assume without loss that $T < S$. Then by maximality $N_S(T)$ lies in a unique p-local subgroup M of G. Clearly $C(G,T) \leq M$ and it is easy to see that (c) holds.

(d) Let $M_{12} < L_1 \leq M_2$ and put $M = \langle L_1, M_2 \rangle$. If $M \in \mathcal{L}$, then $(M, M_2) \in \mathcal{W}$ and $M_{12} < L_1 \leq M \cap M_1$, a contradiction to the maximality of (M_1, M_2). Thus $O_p(M) = 1$ and $(L_1, M_2) \in \mathcal{W}$. Also $L_1 \cap M_2 = M_{12}$, $L_1 \leq M_1$ and $M_2 \leq M_2$. So by maximality $L_1 = M_1$.

(e) Suppose that $C_{M_1}(Z_0) \not\leq M_2$ and let $M \in \mathcal{M}(C_{M_1}(Z_0)$. Suppose that $M_1 \not\leq M$. Since $T \leq M_1 \cap M$, maximality implies that T is a Sylow p-subgroup of $M_1 \cap M$. But then part 2. of the definition of "¡" gives a contradiction. Thus (ea) holds. Clearly (ea) implies (eb). □

cjt

Lemma 7.4 *Let* $M \in \mathcal{L}(S)$ *and* $1 \neq x \in Z_M \cap ZJ(S)$ *Suppose that* $Z_M \not\leq O_p(C_G(x)0$. *Then* **TO BE CONTINUED**

Proof: Assume without loss that M is a maximal p-local. Put $Q = C_S(Z_M)$. Note that $C_G(x) \in \mathcal{L}(B(S)$. Pick $L \in \mathcal{L}(Q)$ so that $Z_M \not\leq O_p(L)$, $|L|_p$ is maximal and $|L|$ is minimal. Let T be a Sylow p-subgroup of $|L|$ with $Q \leq T$. Let R be an T invariant subgroup of L with $[R, Z_M \not\leq O_p(R)$. Then by minimality of L, $L = RS$. In particular, $L \in \mathcal{N}(T)$. Also $Z_M \leq D \overset{def}{==} \bigcap\{O_p(P) \mid P \in \mathcal{M}(L,T)\}$.

Case 1 T is not a Sylow p-subgroup of G.

Let C be a non-trivial characteristic subgroup of T. Then $N_G(C)$ has a larger p-part then L and so by choice of L, $Z_M \leq O_p(N_G(C))$. In particular, C is not normal in L. In particular, $[Z_L, Z_M] \neq 1$.

Suppose that $F^*(L)$ is not a p-group. Then no element of $O_p(L)$ is of p-type. Pick $E \in \mathcal{L}$ with $Q \leq L$, $F^*(E)$ is not a p-group, $|E|_p$ maximal and $|E|$ minimal. Then $Z_M \not\leq O_p(E)$. Let R be a Sylow p-subgroup of E containing Q and $R \lhd R^*$ for some p-group R^*. Let $1 \neq r \in R \cap Z(R^*)$. Then $Q \leq C_G(r)$ and $C_G(r)$ has larger p-part then E. Thus r is of p-type and so $r \not\leq O_p(E)$. Thus $[O_p(E), O^p(E)] = 1$. **TO BE CONTINUED**

8 Some general amalgam results

geamre
amalgam

Hypothesis 8.1 1. *G is a group.*

2. *p is a prime.*

3. *G_1 and G_2b are finite subgroups of G.*

4. *$G = \langle G_1, G_2 \rangle$*

5. *$S \leq G_1 \cap G_2$ so that S is a Sylow p-subgroup of G_1 and G_2*

6. *Both $F^*(G_1)$ and $F^*(G_2)$ are p-groups.*

Let $O_S(G)$ be the largest subgroup of S which is normal in G. Let $Z = \Omega_1 Z(S)$. Let $\Gamma = \Gamma(G; G_1, G_2)$ be the coset graph for G with respect two G_1, G_2. In equal the vertices are the right cosets of G_1 and G_2 in G and two cosets are adjacent if they are distinct and have non-empty intersection. For $\gamma \in \Gamma$, let G_γ be the stabilizer of $\gamma \in G$, $Q_\gamma = O_p(G_\gamma)$, $Z_\gamma = \Omega_1(Z(T)) \mid T \in \mathrm{Syl}_p(G_\gamma)$, $\triangle(\gamma)$ is the set of neighbors of γ, $G_{\gamma\delta} = G_\gamma \cap G_\delta$. $G_\gamma^{(1)} = \bigcup_{\delta \in \triangle(\gamma)} G_{\gamma\delta}$, $V_\gamma = \langle Z_\delta \mid \delta \in \triangle(\gamma)$, $C_\gamma = C_{G_\gamma}(Z_\delta)$, $E_\gamma = O^p(G_\gamma)$, $Q_\gamma^* = [Q_\gamma, E_\gamma)]$, $X_\gamma = \Omega_1 Z(Q_\gamma)$, $X_\gamma^* = C_{Q_\gamma}(Q_\gamma^*)$, Y_γ is the largest p-reduced normal subgroup of G_γ

For $\gamma \in \Gamma$ let $b_\gamma = \min\{d(\gamma, \delta) \mid Z_\gamma \not\leq G_\delta^{(1)}$. Let $b = \min_{\gamma \in \Gamma} b_\gamma = \min\{b_{G_1}, b_{G_2}$. Let $\alpha, \alpha' \in \Gamma$ with $d(\alpha, \alpha') = b$ and $Z_\alpha \not\leq G_{\alpha'}^{(1)}$. Let

$$(\alpha, \alpha+1, \alpha+2, \ldots, \alpha+b) = (\alpha'-b, \ldots, \alpha'-1, \alpha')$$

be a shortest path form α to α'. Put $\beta = \alpha + 1$. Without loss $\{G_\alpha, G_\beta\} = \{G_1, G_2\}$.

Let $q_\delta = qa_{Z_\delta}(G_\delta)$, $r_\delta = \min\{r \mid |AQ_\beta/Q_\beta|^r = |V_\beta/C_{V_\beta}(A)\}$ for some $A \leq S$ with $A \not\leq Q_\beta$ and $[V_\beta, A, A] = 1$. Let c_β the number of non-trivial chief factors for G_β on V_β.

connected

Definition 8.2 *Let H be a group and T a subgroup of H.*

1. *H is connected with respect to T if T is not normal in H and for each normal subgroup N of H, either $N \cap T$ is normal in H or $H = NT$.*

2. *H is p-connected if H is connected with respect to some Sylow p-subgroup of H.*

3. *H is p-minimal with H is not p-closed and a Sylow p-subgroup of H lies in a unique maximal subgroup of H.*

rrc

Lemma 8.3 *If G_β is connected then, $r_\beta \geq ra_{V_\beta} c_b$.*

Proof: $A \leq G_\beta$ with $[V_\beta, A, A] = 1$ and put $r = ra_{V_\beta}$. Let U be a non-trivial chief factor for G_β on S Then as $G_\beta \in \mathcal{N}^*(S)$, $C_A(U) = A \cap Q_\beta$. So by definition of $ra_{V_\beta}(S)$, $|AQ_\beta/Q_b|^r \leq |U/C_U(A)|$. Multiplying together these inequalities over all such U in a chief series we obtain $|AQ_\beta/Q_b|^{rc_\beta} \leq |V/C_V(A)|$ and so $r_b \geq rc_\beta$. □

QRC

Lemma 8.4 *Suppose that $b \geq 2$ and allow for the case that $O_S(G) \neq 1$.*

(a) *Suppose that $q_\alpha > 1$ and $[V_\beta, J(S) \neq 1$. Then b is odd or ∞ and $(q_\alpha - 1)(r_\beta - 1) \leq 1$.*

(b) *Suppose that $C_\alpha \cap Q_\beta$ is not normal in G_α and put $Q = \langle C_\alpha \cap Q_b^{G_\beta}\rangle$. Then Q acts quadratically on Z_α, $|[Z_\alpha, Q]| \leq |Q/C_Q(Z_\alpha)|$, Z_α is an FF module and $[C_{Z_\alpha}(Q), E_\beta] = 1$.*

Proof: (a) If b is even,8.17 shows that Z_α or $Z_{\alpha'}$ is FF, a contradiction to $q_\alpha > 1$. Thus b is odd or ∞. In particular, $b \geq 3$ and V_β is abelian.

Since $[V_\beta, J(S) \neq 1$, there exists $A \in \mathcal{A}(S)$ with $[V_\beta, A] \neq 1$. By the Thompson replacement lemma we may assume that $[V_\beta, A, A] = 1$. Suppose $A \leq Q_\beta$ and let $\delta \in \triangle(\beta)$. Then $q_\delta > 1$ implies $[Z_\delta, A] = 1$ and $[V_\beta, A] = 1$, a contradiction. Thus $A \not\leq Q_\beta$. Put $B = A \cap Q_\beta$. We will apply 2.4 with $I = \triangle(\beta)$ and $W_i = Z_i$ for $i \in I$. Define r, t and s as in the 2.4. Since $A \in \mathcal{A}(S)$, $|V_\beta/C_{V_\beta}(A)| \leq |A/C_A(V_\beta)$ and so $t \geq 1$. Also $s \geq q_a > 1$ and $r \geq r_\beta$. By 2.4b to obtain $trs \leq r + s$, $rs \leq r + s$, $(s-1)(r-1) \leq 1$ and $(q_\alpha - 1)(r_\beta - 1) \leq 1$.

(b) Let $D = C_{Z_\alpha}(E_\alpha)$. If $D = Z_\alpha$, then Z_α and $Q = C_\alpha \cap Q_\beta$ are normal in G_b in contrast to our assumptions. Thus $Z_\alpha \neq D$ and we can choose $D \leq E \leq Z_\alpha$ with $E \trianglelefteq S$ and $|E/D| = p$. Let $W = \langle E_\beta^G \rangle$. Note that $[E, Q] \leq D$ and so is centralized by E_b and normalized by S. Thus $[E, Q] \trianglelefteq G_\beta$, $[E, Q] = [W, Q]$ Since $[W, E_\beta] \neq 1$ and $c_\beta = 1$, $[V_\beta, E_b] \leq W$ and so $V_\beta = Z_\alpha W$. Hence $[V_\beta, C_{Q_\beta}(Z_a)] \leq [W, Q$ and so $[Z_a, Q] \leq [V_\beta, Q] = [W, Q] = [E, Q] \leq Z_\alpha$. $[C_\alpha \cap Q_\beta$ centralizes D, Q centralizes D and $[E, Q]$. Hence $[E, Q] = \{[e, q] \mid q \in Q\}$, where $e \in E \setminus D$. Thus $|[E, Q]| = |Q/C_Q(e) \leq |Q/C_Q(Z_\alpha)|$. If $C_{Z_\alpha}(Q) \neq D$, we can choose $[E, Q] = 1$ and we get $[Z_\alpha, Q] = 1$ and so $Q = C_\alpha \cap Q_\beta$ is normal in G_β,a contradiction. □

ocf

Lemma 8.5 *Suppose that b is odd, $b \geq 3$ and $L \leq G_{\alpha'}$ with*

(i) *$L = (G_{\alpha'-1} \cap L)O^p(L)$.*

(ii) *$G_{\alpha'} = \langle G_{\alpha'-1}, L\rangle$.*

(iii) *L has at most one non-central composition factor on $\langle Z_{ap-1}^L \rangle$.*

Then one of the following holds

1. *$[Z_{ap-1}, [Q_{\alpha'}, O^p(L)] \neq 1$ and Z_α is an FF-module for G_α/C_α.*
2. *$[Z_{ap-1}, [Q_{\alpha'}, O^p(L)] = 1$ and*
   (a) *$V_\beta = Z_\alpha C_{V_b}(Q_b)$.*
   (b) *$C_a \cap Q_\beta \trianglelefteq G_\beta$.*
   (c) *$C_{V_b}(Q_b)$ is an FF module for $\langle \mathrm{Q}_a^{G_\beta}\rangle$.*

Proof: Let $V = \langle Z_{ap-1}^L \rangle$ and $Q = [Q_{\alpha'}, O^p(L)$. Then by (i), $V = \langle Z_{ap-1}^{O^p(L)}\rangle$ and we may assume without loss that $L = O^p(L)$. Note also that $Q_{\alpha'}$ normalizes $Z_{\alpha'}$ and V.

Suppose first that $[Z_{\alpha'-1}, Q] \neq 1$. If $[V, Q, L] \neq 1$, then by (iii), $V = Z_{\alpha'-1}[V, Q]$ and so $V = Z_{\alpha'-1}$, a contradiction to (ii). Thus $[V, Q, L] = 1$ and by [St1] (1) holds.

So we may assume that Q centralizes $Z_{\alpha'-1}$ and V. Hence (iii) implies that $[V, Q_{ap}, L] = 1$ and $[V, L, Q_{ap}] = 1$. Thus $V = Z_{\alpha'-1}C_V(Q_{\alpha'})$ and so L normalizes $Z_{ap-1}C_{V_{\alpha'}}(Q_{\alpha'})$.

Therefore (ii) implies that $G_{\alpha'}$ normalizes $Z_{ap-1}C_{V_{\alpha'}}(Q_{\alpha'})$ and so $V_{\alpha'} = Z_{ap-1}C_{V_{\alpha'}}(Q_{\alpha'})$. Thus $C_{Q_{\alpha'}}(V_{ap}) = C_{\alpha'-1} \cap Q_{\alpha'}$ and (a) and (b) are proved. Moreover we get $[V_\beta \cap Q_{\alpha'}, V_{\alpha'}] = 1$ and $[V_{\alpha'} \cap Q_\beta, V_\beta = 1$. Hence (c) follows from 8.17.

$q < 2$

Lemma 8.6 *Suppose that G_β is a mimimal parabolic and allow for the case that $O_S(G) \neq 1$. Then one of the following holds:*

1. *S centralises Z_α.*
2. *$Z_\alpha \not\leq Q_\beta$.*
3. $q_a \leq 2$
4. *Z_α is the dual of an FF-module*
5. *There exists a non-tivial characteristic subgroup C of $B(S)$ with $C \trianglelefteq G_\beta$ and $G_\alpha = N_{G_\alpha}(C)C_\alpha$. Moreover, either $C = J(S)$ or $Q_b^* \leq B(S) \leq C_\alpha$.*
6. *Put $G_\beta^* = B(S)O^2(G_\beta)$. T $O_2(G_\beta^*) \leq B(S) \leq C_\alpha$ and non-trivial characteristic subgroup of $B(S)$ is normal in G_β^*. Moreover, $Z \trianglelefteq G_\beta$.*
7. *Z and Z_α are normal in G_β and centralized by E_β. Futhermore, $S \cap C_\alpha$ is a Sylow p-subgroup of $C_{G_\beta}(Z_a)$.*

Proof: Without loss $Z_a \leq Q_b$. If $[J(S), Z_\alpha] \neq 1$, $r(S, Z_1) \leq 1$. So we may assume that $J(S) \leq C_\alpha$. Thus $Z_\alpha \leq C_S(J(S))$ and $B(S) \leq C_\alpha$. Hence

$q < 2 - 1$

(1) $G_\alpha = N_{G_\alpha}(B(S))C_a = N_{G_\alpha}(C)C_a$ for any characteristic subgroup C of $B(S)$.

If E_β centralizes V_β, then 7. holds. So suppose $[V_\beta, E_\beta] \neq 1$. If $J(S) \trianglelefteq G_\beta$, 5. holds. Hence we may assume that $J(S) \ntrianglelefteq G_\beta$. in particular, $[V_\beta, J(S)] \neq 1$. By 6.3, $r_{V_b}(G_\beta) \geq 1$. If $c_\beta \geq 2$, then 8.3 implies $r_\beta \geq 2$. By refQRCa, $(q_\alpha - 1)(r_b - 1) \leq 1$ and so 3. holds. If $c_\beta = 1$, then 8.4b implies that 4. holds or $C_\alpha capQ_\beta$ is normal in G_β. So suppose the latter.

Since $J(S) \leq C_\alpha$, $J(S)$ centralizes $Q_\beta/Q_\beta \cap C_\alpha$. Since $J(S) \not\leq Q_\beta$, $E_\beta \leq \langle J(S)_\beta^G \rangle$ and so E_β centralizes $Q_\beta/Q_\beta \cap C_\alpha$. Thus $Q_\beta^* \leq C_\alpha \cap Q_\beta$ and $[V_\beta, Q_b^*] = 1$. Thus $[C_{Q_\beta}(Q_\beta^*), E_\beta] \neq 1$ and by Thompson's $P \times Q$-lemma, $[X_\beta, E_\beta] \neq 1$. Thus by 8.10 (and the remark following 8.10), $O_p(E_b) \leq B(S)$. Now either there exists a non-trivial charcteristic subgroup of $B(S)$ which is normal in G_β^* or there does not. In the first case (1) implies that 5. holds and in the second 6. holds.

□

Lemma 8.7 *Suppose $b > 1$, $s_{Z_\alpha}(S) \geq 1$, $C_{G_\beta}(V_b)$ is p-closed and $[V_\beta \cap Q_{\alpha'}, V_{\alpha'} \cap Q_\beta] = 1$. Then V_β is $F2$ for G_β.*

Proof: We may assume without loss that $V_\beta Q_{\alpha'}/Q_{\alpha'} \geq V_{\alpha'}Q_\beta/Q_a p$. Since $s_{Z_\alpha}(S) \geq 1$ we can apply 2.3 with $s = 1$, $V = V_\beta$ and $B = V_{\alpha'} \cap Q_\beta$ and conclude

$$|B/C_B(V_b)| \leq |V_b/C_{V_b}(B)|$$

By assumption $V_\beta \cap Q_{\alpha'} \leq C_{V_b}(B)$ and so

$$|V_{\alpha'}/C_{V_{\alpha'}}(V_b)| \leq |V_{\alpha'}/B||B/C_B(V_\beta) \leq |V_{\alpha'}Q_\beta/Q_b| \cdot |V_\beta/V_b \cap Q_{\alpha'}| \leq |V_\beta Q_{\alpha'}/Q_{\alpha'}|^2.$$

Hence V_β is F^2. □

mqrc

Lemma 8.8 *Let (P_0, P_1, P_2) be an amalgam over S. Let $Z_0 = \langle Z^{P_0} \rangle$. For $i = 1, 2$ put $L_i = \langle P_0, P_i \rangle$ and $Z_i = \langle Z^{L_i} \rangle$. Suppose that*

(i) *P_1 and P_2 are in $\mathcal{P}(S)$.*

(ii) *For $\{i, j\} = \{1, 2\}$, $O^2(P_i) \not\leq O_2(P_j)$.*

(iii) *For $i = 1, 2$, $Z \leq O_S(L_i)$*

The one of the following holds for some $i \in \{1, 2\}$

1. *$J(S) \trianglelefteq P_0$.*
2. *$J(S) \trianglelefteq P_i$, $[Z_0, O^2(P_i)] \neq 1$ and $r(S, Z_i) \leq 1$.*
3. *$Z_i \not\leq Q_j$*
4. *$r(S, Z_j) \leq 2$ or $r^*(S, Z_j) \leq 2$*

Proof: Without loss $J(S) \triangleleft P_O$ and since $J(S)$ is not normal in all the P_i's we may assume that $J(S) \trianglelefteq P_1$. If $[Z_0, O^2(P_1)] \neq 1$ we conclude that $[Z_1, J(S)] \neq 1$ and 2. holds. So we also may assume that $[Z_0, O^2(P_1)] = 1$. Then Z_0 is not normal in P_2 and hence $[Z_0, O^2(P_2)] \neq 1$.We apply 8.6 to $G_\alpha = L_2$ and $G_\beta = P_1$. As $J(S) \trianglelefteq P_1 = G_\beta$ we conclude that either 3. holds or 4. holds or $[Z_2, Q_1^*] = 1$. In the latter case $Q_1^* \not\leq O_2(P_2)$ implies $[Z_2, O^2(P_2) = 1$, a contradiction to $[Z_0, O^2(P_2)] \neq 1$.

cb

Lemma 8.9 *Let L be a subgroup of G_β which acts transitively on $\triangle(\beta)$. Put $D_\beta = \bigcap_{\delta \in \triangle(\beta)} Z_\delta$ and l minimal with $[Z_\alpha, Q_\beta, l] \leq D_\beta$. Suppose that $V_\beta \leq Q_\beta$. Then for all $0 \leq i < l$, L acts non-trivially on $[V_\beta, Q_\beta, i]/[V_\beta, Q_\beta, i+1]$.*

Proof: Put $Z_i = [Z_\alpha, Q_\beta, i]$ and $V_i = [V_\beta, Q_\beta, i]$. As L acts transitively on $\triangle(\beta)$, $V_i = \langle Z_i^L \rangle$. Let i be so that L acts trivially on V_i/V_{i+1}. Then $V_i = Z_i V_{i+1}$ and so $V_i/Z_i = [V_i/Z_i, Q_\beta]$. Hence $V_i = Z_i$ and $Z_i \leq D_\beta$. Thus $i \geq l$.

Baumann

Lemma 8.10 *Let G be a finite group, p a prime, p-subgroup of G, $V = \langle\Omega_1(Z(O_p(G))$, $B(S) = C_S(\Omega_1(Z(J(S))$, $J(G) = \langle J(S)^G\rangle$, $B(G) = \langle B(S)^G\rangle$, $\overline{G} = G/C_G(V)$, and $\widetilde{V} = V/C_V(O^p(B(G))$ and suppose that each of the following holds:*

(i) $C_G(V)$ *is p-closed.*

(ii) *If $A \in \mathrm{P}(\overline{G}, V)$ then $|\tilde{V}/C_{\tilde{V}}(A)| \geq |A|$.*

(iii) *If U is an FF-module for $G/O_p(G)$ module with $\tilde{V} \leq U$ and $U = C_U(B(S))\tilde{V}$, then $U = C_U(O^p(J(G)))\tilde{V}$.*

Then $O_p(B(G)) \leq B(S)$.

Proof: and $Y = \Omega_1 ZJ(S)$. Let $A \in \mathcal{A}(S)$. Then $\overline{A} \in \mathrm{P}(\overline{G}, V)$ and so by (ii), $|\tilde{V}/C_{\tilde{V}}(A)| \geq |\overline{A}|$. By (i), $|\overline{A}| = |A/A \cap Q|$ and so $V(A \cap Q) \in \mathcal{A}(S)$. Thus $Y \leq V(A \cap Q) \leq Q$. Put $W = \langle Y^G\rangle V$. We conclude that $W \leq \Omega_1 ZJ(Q)$ and so W is elementary abelian and $(A \cap Q)V$ centralizes W. Hence $W \leq (A \cap Q)V$ and $W = V(A \cap W) = VC_W(A)$. It follows that A centralizes W/V. Since A was arbitray in $\mathcal{A}(S)$, $J(G)$ centralizes W/V. As $Y = \Omega_1 ZJ(S \cap J(G))$, Sylow's theorem implies that $J(G)$ acts transitively on Y^G. Thus $W = YV$ and so $[W, Q] = [Y, Q] \leq Y$. Hence $[W, Q] \leq C_W(B(G))$. Let $D = C_W(O^p(B(G))$ and $U = W/D$. Then $O_p(G)$ centralizes U. Since $\tilde{V} \cong VD/D$ and $U = YV/D$, we can apply (iii) to conclude that $W = DV$ and $U \cong \widetilde{V}$. Since $A \in \mathcal{A}(S)$, $|W/W \cap A| \leq |A/C_A(W)| = |A/A \cap Q|$. One the otherhand by (i), $|A/A \cap Q| \leq |\widetilde{V}/C_{\widetilde{V}}(A)| = |U/C_U(A)| \leq |W/C_W(A)D|$. Thus $|W/C_W(A)| \leq |W/C_W(A)D|$ and $D \leq C_W(A)$. Hence $[D, A] = 1$, $D \leq Y$ and $[D, B(G)] = 1$. Therefore $[W, O_p(B(G)] \leq [D, B(G)][V, Q] = 1$ and so $O_p(B(G)) \leq C_S(Y) = B(S)$. □

Remark 8.11 *Assume (i) in 8.10. Then (ii) and (iii) hold as well unless $\overline{J(G)}$ has a component K with $K \cong Alt(2n), n \geq 3$; $SL_n(q)$, $n \geq 3$; $SU_n(q), n \geq 6$; $Sp_{2n}(q), n \geq 2$; $\Omega^+_{2n}(q), n \geq 3$; or $\Omega^-_{2n}(q), n \geq 4$; and some compostion factor for K on V is a natural module.*

pump

Lemma 8.12 *pushing up minimal parabolics, odd elements*

pusym

Lemma 8.13 *pushing up sym(10) over $\langle(12), (34), (56), (78), (9, 10)\rangle$*

trpu

Lemma 8.14 *some trivial pushing up result, at least including $L_5(2)$ over the O_2 of a point stabilizer, saying that $b = 4$ and non trivial center; or $b = 2$ and O_2 basicly a natural module*

qaniqb

Lemma 8.15 *Suppose that G_α is a p-minimal. Then $Q_\alpha \not\leq Q_\beta$.*

Proof: This follows from 8.12**Remark: This needs some thought** □

znnab1

Lemma 8.16 *Suppose that each of the follwing holds:*

(i) $\alpha, \beta = \{\gamma, \delta\}$.

(ii) *G_γ is p-minimal and $[X_\gamma, E_\gamma] \neq 1$.*

(iii) *G_δ is p-connected or $C_S(X_\delta) = Q_\delta$.*

Then one of the following holds.

(a) *$[X^*_\delta, E_\delta] = 1$ and $Z \trianglelefteq G_\delta$.*

(b) *$J(S) \not\leq Q_\delta$ and X_δ is an FF-module for G_δ.*

(c) (a) *$J(S) \trianglelefteq G_\delta$.*
  (b) *$O_p(B(G_\gamma)) \leq B(S) = B(Q_\delta)$.*
  (c) *E_γ is a $SL_2(p^r)^k$-block, $\mathrm{Alt}(2^r+1)^k$-block or $SL_2(3^r)^k$- double block.*
  (d) *If G is finite and $S \in Syl_p(G)$, then G contains a p-local R with $B(S) \leq R$ and $C_R(O_p(R)) \not\leq O_p(R)$.*

Proof: We may assume that $[X^*_\delta, E_\delta] \neq 1$. Then by Thompsons's $A \times B$-lemma, $[X_\delta, E_\delta] \neq 1$. Hence if G_δ is p-connected, $C_S(X_\delta) = Q_\delta$. Thus by (ii) $C_S(X_\delta) = Q_\delta$.

If $J(S) \not\leq Q_\delta$, then (b) holds.

So suppose $J(S) \leq Q_\delta$. Then $X_\delta \leq ZJ(S)$ and so $B(S) \leq C_S(X_\delta) \leq Q_\delta$ and $B(S) = B(Q_\delta)$. By 8.10, $O_p(B(G_\gamma)) \leq B(S)$. Thus (ca) and (cb) hold in this case.

Since G_γ is p-minimal, $G_\gamma = B(G_\gamma)S$. Let R be normal subgroup of $B(G_\gamma)$. Let U be unique maximal subgroup of G_γ containing S. Let C be a non-trivial characteristic subgroup of $B(S)$. Then C is normal in G_δ and so C is not normal in G_γ. Since $S \leq N_{G_\gamma}$, this implies $N_{G_\gamma} \leq U$. Let $W = W_\gamma = \rangle \Omega_1 Z(J(S))^{G_\gamma}$. Then W is an FF-modules for $B(G_\gamma)$ and $O_p(B(G_\gamma))$ centralizes V. Hence $W/C_W(E_\gamma)$ is a natural $SL_2(p^r)^k$ or $Sym(2^r+1)^k$ module for $B(G_\gamma)$. Let E be minimal with $B(S) \leq E$, and $O^p(E)$ maps onto on normal $SL_2(q)'$'s or $Alt(q+1)$'s. Then $E \not\leq U$ and so $C \ntrianglelefteq E$. Hence by 8.12 $O^p(E)$ is an $L_2(p^r)$-block, $Alt(2^r+1)$ block or $SL_2(q)$-double block. It is now easy to see that $O^p(E)$ is normal in E_γ and that (cc) holds.

Suppose now that G is finite and $S \in Syl_p(G)$. Assume first that E_γ is a $SL_2(p^r)^k$- or $Alt(2^r+1)^k$-block. Then there exists $\lambda \in \triangle(\delta)$ with $[W_\gamma, W_\lambda] \neq 1$. Then $W_\lambda \leq B(Q_\delta) = B(S) \leq B(G_\gamma))$. Suppose that $[X_\delta, Q_\gamma] \neq 1$**TO BE CONTINUED**

PFF

Lemma 8.17 *Let $\lambda, \mu \in \Gamma$ and F_λ, F_μ normal p-subgroups of G_λ and G_μ, respectively. Suppose that*

(i) *$F_\lambda \leq G_\mu$ and $F_\mu \leq G_\lambda$.*

(ii) *$[F_\lambda, F_\mu] \neq 1$.*

(iii) *For $\rho \in \{\lambda, \mu\}$, $C_{G_\rho}(F_\rho)$ is p-closed*

(iv) *$[F_\lambda, F^\mu \cap Q_\lambda] = 1$ and $F_\mu, F_\lambda \cap Q_\mu] = 1$.*

Then one of the follwing holds

1. F_λ *is an* F^*1 *module for* G_λ.

2. F_μ *is an* F^*1 *module for* G_μ.

3. *Both* F_λ *and* F_μ *are* FF*-modules.*

Proof: By (iii) and (iv) $F_\lambda \cap Q_\mu = C_{F_\lambda}(F_\mu)$ and $F_\mu \cap Q_\lambda = C_{F_\mu}(F_\lambda)$. $|F_\lambda / F_\lambda \cap Q_\mu|$ is either less, larger or equal to $F_\mu / F_\mu \cap Q_\lambda$. In the first case $|F_\lambda / C_{F_\lambda}(F_\mu)| < F_\mu Q_\lambda / Q_\lambda|$ and 1. holds. Similarly the second case implies 2. and the third 3. □

vbvap

Lemma 8.18 *Supposse that* $b \geq 3$, b *is odd and* $r_\alpha \geq 1$.

(a) $(r_a - 1)(r_b 1) \leq 1$.

(b) *Suppose that equality holds in* (b). Then

- (b.a) $|V_{\alpha'} Q_\beta / Q_a| = V_\beta Q_{\alpha'} / Q_{\alpha'}|$
- (b.b) $C_{V_{\alpha'}}(V_\beta \cap Q_{\alpha'}) = C_{V_{\alpha'}}(V_\beta)$.
- (c.b) Let $\delta \in \triangle(\beta)$ with $[Z_\delta, V_{\alpha'}] \neq 1$. Then $V_{\alpha'} \cap Q_\beta \not\leq Q_\delta$ and $|(V_{\alpha'} \cap Q_b) Q_\alpha / Q_a|^s = |Z_\delta / C_{Z_\delta}(V_{\alpha'})|$.
- (c.d) $|V_\beta Q_{\alpha'} / Q_{\alpha'}|^r = |V_{\alpha'} / C_{V_{\alpha'}}(V_\beta)|$.

Proof: By 2.4 we have

vbvap1

(1) $|V_{\alpha'} \cap Q_\beta / C_{V_{\alpha'}}(V_\beta)|^{r_\alpha} \leq |V_\beta / C_{V_\beta}(V_{\alpha'} \cap Q_\beta)|$.

and

vbvap2

(2) $|V_\beta \cap Q_{\alpha'} / C_{V_\beta}(V_{\alpha'})|^{r_\alpha} \leq |V_{\alpha'} / C_{V_{\alpha'}}(V_\beta \cap Q_{\alpha'})|$.

Suppose first that $V_{\alpha'} \leq Q_\beta$. Since $r_\alpha \geq$, (1) implies $|V_{\alpha'} / C_{V_{\alpha'}}(V_\beta)| \leq |V_\beta / C_{V_\beta}(V_{\alpha'})|$. If $V_{\alpha'} \not Q_\beta$ the situation is symmetric in α' and β and we may assume in any case that

vbvap3

(3) $|V_{\alpha'} / C_{V_{\alpha'}}(V_\beta)| \leq |V_\beta / C_{V_\beta}(V_{\alpha'})|$

TO BE CONTINUED

rsc

Lemma 8.19 *Suppose that* $r_\beta \geq 1$, $s_\alpha \geq \frac{3}{2}$ *and* $s^*_\alpha > 1$. *Then*

(a) $\frac{3}{2} \leq s_\alpha \leq 2$.

(b) $1 \leq r_\beta \leq \frac{3}{2}$.

(c) $c = 2$ *or* 3.

(d) *If* $c = 3$*, then* $s_\alpha = \frac{3}{2}$ *and* $r_\beta = 1$.

(e) *If* $r_\beta = \frac{3}{2}$*, then* $c = 2$*,* $s_\alpha = \frac{3}{2}$ *and* $(s_\alpha - 1)(r_\beta c_\beta - 1) = 1$.

(f) *If* $s_\alpha = 2$*, then* $c = 2$*,* $r_\beta = 1$ *and* $(s_\alpha - 1)(r_\beta c_\beta - 1) = 1$.

(g) $[Z_\alpha, Z_{\alpha'}] = 1$.

Proof: As $s^*_\alpha > 1$, 2.4 implies $c_\beta \geq 2$. All but the last statement are now an immediate consequece of 8.4. The last statement follows from 8.17.

p3/2

Lemma 8.20 *Suppose that* b *is odd and* $\beta^+, \beta^- \in \Gamma_2$ *with* $d(\beta^+, \beta^-) = b-1$ *For* $\epsilon \in \{+, -\}$ *let* $\Lambda^\epsilon \subseteq \triangle(\beta^\epsilon)$*. Define* $V^\epsilon = \langle Z_\lambda | \lambda \in \Lambda^\epsilon \rangle$ *and* $B = V^\epsilon \cap \bigcap_{\lambda \in \Lambda^{-\epsilon}} G_\lambda$*. Finally, let* s *be a postive real number so that for all* $\epsilon \in \{+, -\}$*, all* $\lambda \in \Lambda^{-\epsilon}$*, and all* $A \leq B^\epsilon$*,* $|Z_\lambda / C_{Z_\lambda}(A)|^s \leq |A/C_A(Z_\lambda)|$*. Then*

(a) (aa) $|B^+/C_{B^+}(V^-)| \leq |V^-/V_{V^-}(B^+)|^{\frac{1}{s}} \leq |V^-/C_{V^-}(B^+)|^{\frac{1}{s}}$

(ab) $|V^+/C_{V^+}(V^-)| \leq |V^+/B^+||B^+/C_{B^+}(V^-)|$

(ac) $|V^+/C_{V^+}(V^-) \leq |V^+/B^+||V^-/C_{V^-}(V^+)|^{\frac{1}{s}}$.

(b)(b.a) $|V^+/C_{V^+}(V^-)|^{\frac{s^2-1}{s^2}} \leq |V^+/B^+||V^-/B^-|^{\frac{1}{s}}$.

(b.b) $|B^+/C_{B^+}(V^-)|^{\frac{s^2-1}{s}} \leq |V^+/B^+|^{\frac{1}{s}}|V^-/B^-|$.

(c) *Suppose* $s > 1$ *and* $V^+ = B^+$*, then* $|V^+/C_{V^+}(V^-)| \leq |V^-/B^-|^{\frac{s}{s^2-1}}$.

(d) *Suppose* $s > 1$ *and that* r *is a positive real number with* $|V^-/B^-|^r \leq |V^+/C_{V^+}(V^-)|$*. Put* $e = \frac{rs^2-r-s}{s^2}$.

(d.a) $|V^-/B^-|^e \leq |V^+/B^+|$.

(d.b) $|B^-/C_{B^-}(V^+)| \geq \frac{|V^-/B^-|^r}{|V^+/B^+|}$

(d.c) *If* $e > 0$*, then* $|B^+/C_{B^+}(V^-)|| \leq |V^+/B^+|^{\frac{rs}{rs^2-r-s}}$

(e) *Suppose* $s > 1$ *and* r *is a positive integer so that for* $\epsilon \in \{+, -\}$*,* $|V^\epsilon/B^\epsilon|^r \leq |V^{-\epsilon}/C_{V^{-\epsilon}}(V^\epsilon)|$*. Put* $e = \frac{rs^2-r-s}{s^2}$ *and suppose that* $e > 0$.

(e.a) $|V^-/B^-|^e \leq |V^+/B^+||V^-/B^-|^{\frac{1}{e}}$

(e.b) *If* $V^- \neq B^-$*, then* $V^+ \neq B^+$ *and* $e \leq 1$.

Proof: The first inequa lityin (aa) follows from 2.3 while the second is obvious. (ab) is obviuos and (ac) follows from (aa) and (ab).

Interchanging " + " and " − " in (ac) and substituting the result into (ac) we obtain

$$|V^+/C_{V^+}(V^-)| \leq |V^+/B^+||V^-/B^-|^{\frac{1}{s}}|V^+/C_{V^+}(V^-)|^{\frac{1}{s^2}}.$$

Thus (b.a) holds. Simimalrly interchanging " + " and " − " in (ac) and substituting the result into (ab) one obtains (bb).

(c) follows easily from (b.a). (ea) follows from (da) and using symmetry in " + " and " − ". (eb) follows from (eb). So it remains to prove (d). By assumption $|V^-/B^-|^r \leq |V^+/C_{V^+}(V^-)|$. As $s > 1$ we can raise this inequality to the $\frac{s^2-1}{s^2}$ power and obtain

$$|V^-/B^-|^{\frac{r(s^2-1)}{s^2}} \leq |V^+/C_{V^+}(V^-)|^{\frac{s^2-1}{s^2}}.$$

Thus (da) follows from (ba). For (db) note that

$$|V^-/B^-|^r \leq |V^+/C_{V^+}(V^-)| \leq |V^+/B^+||B^+/V_{V^+}(V^-).$$

Finally (d.c) follows from (d.a), (b.b) and a simple computation. □

LLp

Lemma 8.21 *Suppose $b > 1$ and G_β is p-minimal. Let $M_{\alpha\beta}$ be the unique maximal subgroup of G_β containing $G_{\alpha\beta}$. Put $\beta^+ = \beta, \beta^- = \alpha'$. Then one of the follwing holds*

1. *For each $\epsilon \in \{+,-\}$ there exists $L^\epsilon \leq G_{\beta^\epsilon}$ and $\mu^\epsilon \in \triangle(\beta^\epsilon)$ so that for $V^\epsilon = \langle Z_{\mu^\epsilon}^{L^\epsilon}$ each of the following holds.*

   (a) $V^{-\epsilon} \not\leq 0_p(L_\epsilon)$.

   (b) $V^{-\epsilon} \leq G_{\mu^\epsilon}$ *and $G_{\beta^\epsilon\mu^\epsilon}$ contains a Sylow p-subgroup of L^ϵ*

   (c) $L^\epsilon \cap M_{\beta^\epsilon\mu^\epsilon}$ *is the unique maximal subgroup of L^ϵ containg $V^{-\epsilon}$.*

   (d) $[V^{-\epsilon}, Z_{\mu^\epsilon}] = 1$.

2. *There exists $\epsilon \in \{+,-\}$, $L^\epsilon \leq G_{\beta^\epsilon}$, $\mu^\epsilon \in \triangle(\beta^\epsilon$ and $\mu \in \triangle(\beta^{-epsilon})$ so that with $V^\epsilon = \langle Z_{\mu^\epsilon}^{L^\epsilon}$ each of the follwing holds.*

   (a) $V_\epsilon \leq G_\mu$, $Z_\mu \leq L^\epsilon$ *and* $Z_\mu \not\leq 0_p(L_\epsilon)$.

   (b) $Z_\mu \leq G_{\mu^\epsilon}$ *and $G_{\beta^\epsilon\mu^\epsilon}$ contains a Sylow p-subgroup of L^ϵ*

   (c) $L^\epsilon \cap M_{\beta^\epsilon\mu^\epsilon}$ *is the unique maximal subgroup of L^ϵ containg Z_μ.*

   (d) $[Z_\mu, Z_{\mu^\epsilon}] = 1$.

3. *There exist $\mu^+ \in \triangle(\beta^+)$ and $\mu^- \in \triangle(\beta^-)$ so that $Z_{\mu^+} \leq G_{\mu^-}$, $Z_{\mu^-} \leq G_{\mu^+}$ and $[Z_{\mu^+}, Z_{mu^-}] \neq 1$.*

Proof: Suppose that 3. does not hold. For $\epsilon \in \{+,-\}$ choose $L^\epsilon \leq G_{\beta^\epsilon}$ and $\mu^\epsilon \in \triangle(\beta^\epsilon)$ so that $|L^+||L^-|$ is minimal with respect to

(i) For all ϵ, $V^{-\epsilon} \leq L^\epsilon \cap G_{\beta^\epsilon\mu^\epsilon}$.

(ii) For all ϵ, $G_{\beta^\epsilon\mu^\epsilon} \cap L^\epsilon$ contains a Sylow p-subgroup of L^ϵ and $M_{\beta^\epsilon\mu^\epsilon} \cap L^\epsilon$ the unique maximal subgroups of L^ϵ containg that Sylow p-subgroup.

(iii) For at least one ϵ, $V^{-\epsilon} \not\leq O_p(L^\epsilon)$.

Note that (i),(ii) and (iii) are fulfilled with $L^\epsilon = G_{\beta^\epsilon}$, $\mu^+ = \alpha + 2$ and $\mu^- = \alpha' - 1$ and so we can make such a minimal choice.

Case 1 For some $\epsilon \in \{+, -\}$ and some $\mu \in \mu^{\epsilon L^\epsilon}$, $[V^{-\epsilon}, Z_\mu] \neq 1$ and $V^{-\epsilon} \leq G_\mu$.

For ease of notation we assume without loss that $\epsilon = -$.

$LLp - 11$

(1) In case 1, $Z_\mu \not\leq O_p(L^+)$ and $[Z_{\mu^+}, Z_\mu] = 1$.

Suppose $Z_\mu \leq O_p(L^+)$ and pick $\rho \in \mu^{+L^+}$ with $[Z_\rho, Z_\mu] \neq 1$. Then $Z_\mu \leq G_\rho$, $Z_\rho \leq G_\mu$ and so 3. holds, contrary to our assumption. As $Z_\mu \leq G_{\mu^+}$, the same argument shows $[Z_{\mu^+}, Z_\mu] = 1$.

$LLp - 12$

(2) In case 1, 2. holds.

By 2.6 there exists $L \leq L^+$ and $h \in L^+$ such that $Z_\mu \leq L$, $Z_\mu \not\leq O_p(L)$, $(G_{\beta^+\mu^+} \cap L^+)^h \cap L$ contains a Sylow p-subgroup of L, and $(M_{\beta^+\mu^+} \cap L^+)^h \cap L$ is the unique maximal subgroup of L containing Z_μ. Thus 2. holds with $\epsilon = +$, L in place of L^ϵ.

Case 2 Case 1 does not hold.

$LLp - 13$

(3) In case 2, for all ϵ, $V^{-\epsilon} \not\leq O_p(L^\epsilon)$ and $[V^{-\epsilon}, Z_{\mu^\epsilon}] = 1$.

If the first statement is false pick $\mu \in \mu^{\epsilon L^\epsilon}$ with $[Z_\mu, V^{-\epsilon}] \neq 1$, if the second statement is false put $\mu = \mu^\epsilon$. Then in any case $V^{-\epsilon} \leq G_\mu$ and the assumption of Case 1 are fulfilled.

$LLp - 2$

(4) In case 2. 1. holds.

We prove is basicly the same as for (2). By 2.6 there exists $L \leq L^\epsilon$ and $h \in L^\epsilon$ such that $V^{-\epsilon} \leq L$, $V^\epsilon \not\leq O_p(L)$, $(G_{\beta^\epsilon\mu^\epsilon} \cap L_\epsilon)^h$ contains a Sylow p-subgroup of L, and $(M_{\beta^\epsilon\mu^\epsilon} \cap L_\epsilon)^h \cap L$ is the unique maximal subgroup of L containing V^ϵ. Hence (i), (ii) and (iii) are still fulfilled if we replace L^ϵ be L, μ^ϵ by $\mu^{\epsilon h}$ and leave $L^{-\epsilon}$ and $\mu^{-\epsilon}$ as they are. Thus the minimal choice of $|L^+||L^-|$ implies $L = L^\epsilon$ and so 1. holds holds. □

znnab

Lemma 8.22 *Assume that each of the following holds for each $\{\gamma, \delta\} = \{\alpha, \beta\}$ and each critical pair (α, α')*

(I) $Z_{\alpha\beta} \not\trianglelefteq G_\gamma$.

(ii) *If $N \lhd G_\gamma$ with $N \cap O_p(G_\alpha\beta) \not\leq Q_\gamma$ then $G_\gamma = NG_{\alpha\beta}$.*

(iii) *Let $\mathcal{O} = \mathcal{O}_{\gamma\delta} = \{A \leq Q_\delta \mid |Z_\gamma/C_{Z_\gamma}(A) \leq |AQ_\gamma/Q_\gamma| \neq 1, [Z_\gamma, A, A] = 1\}$. Then $Z_\gamma/C_{Z_\gamma}(A) = |AQ_\gamma/Q_\gamma|$ for all $A \in \mathcal{O}$.*

(iv) *Either $\mathcal{O} = \emptyset$ or $A_{\gamma\delta} \overset{def}{=} \bigcap_{A \in \mathcal{O}} [Z_\gamma, A] \neq 1$.*

(v) $Z_\beta Z_\alpha \ntrianglelefteq G_\alpha$

(vi) *One of the following holds*

(vi.1) *If* $\alpha-1 \in \triangle(\alpha)$ *such* $Z_{\alpha'}$ *does normalize* $Z_{\alpha-1}Z_\alpha$, *then* $Z_{\alpha-1} \not\leq Q_{\alpha'-1}$.

(vi.2) *There exists* $\alpha-1 \in \triangle(\alpha)$ *with* $G_\alpha = \langle G_{\alpha\alpha-1}$ *and* $Z_{\alpha-1} \not\leq Q_{\alpha'-1}$.

Then

(a) $\mathcal{O}_{\alpha\beta} \neq \emptyset \neq \mathcal{O}_{\alpha\beta}$.

(b) *If* $b \geq 2$, *then* $A_{\beta\alpha} \trianglelefteq G_\alpha$.

(c) $b \leq 2$.

Proof: By (iii), $Z_{\alpha'} \in \mathcal{O}_{\alpha\beta}$. By (ii) and (vi), there exists $\alpha-1 \in \triangle(\alpha)$ so that $Z_{\alpha'}$ does not normalize $Z_{\alpha-1}Z_\alpha$. Hence by (vi) we may choose $\alpha-1$ so that $Z_{\alpha-1} \not\leq Q_{\alpha'-1}$. In particular,

$znnab-1$

(1) $Z_{\alpha'-1} \in \mathcal{O}_{\alpha-1\alpha}$

Thus (a) holds.

Let $H = N_(G_\alpha)(Z_\alpha Z_\alpha - 1))$, $\mathcal{G} = \{g \in G_\alpha \mid Z_{\alpha'}^g \not\leq H\}$ and $T = \langle Z_{\alpha'^g} \mid g \in \mathcal{G}\rangle$. Let $g \in G$. Then $g \in \mathcal{G}$ or $Z_{\alpha'}^g \leq H$. Hence $\langle H, T\rangle \geq G_{\alpha-1\alpha} Z_{\alpha'}^{G_\alpha}\rangle = G_\alpha$, where the last euality follows from (ii). Since both H and T normalize T, we conclude that $T = \langle Z_{\alpha'}^{G_\alpha}\rangle$ and inparticular

$znnab-2$

(2) $G_\alpha = G_{\alpha-1\alpha}\langle Z_{\alpha'^g} \mid g \in \mathcal{G}\}$.

Suppose now that $b > 1$ and $A_{\beta\alpha} \ntrianglelefteq G_\alpha$. Then by (2) we may assume that $Z_{\alpha'}$ does not normalize $A_{\alpha-1\alpha}$. But (1) and the definition of $A_{\alpha-1\alpha}$ imply $A_{\alpha-1\alpha} \leq [Z_{\alpha-1}, Z_{\alpha'-1}]$. Hence $A_{\alpha-1\alpha} \leq Z_{\alpha'-1}$ and $b > 1$ provides the contradiction, $[A_{\alpha-1\alpha}, Z_{\alpha'}] = 1$. Thus (b) holds.

Suppose now that $b > 2$. Then by (b) applied to $(\alpha-1, \alpha'-1)$ in place of (α, α'), $A_{\alpha\alpha-1} \trianglelefteq G_{\alpha-1}$. Hence by (2) we may now assume that $Z_{\alpha'}$ does not normalize $A_{\alpha\alpha-1}$. On the otherhand by (1) there exist $\alpha-2 \in \triangle(\alpha-1)$ so that $Z_{\alpha'-2} \in \mathcal{O}_{\alpha-2\alpha-1}$ Hence

$$A_{\alpha\alpha-1} = A_{\alpha-2\alpha-1} \leq [Z_\alpha - 2, Z_{\alpha'} - 2] \leq Z_{\alpha'-2}.$$

Since $b > 2$ we conclude $[A_{\alpha\alpha-1}, Z_{\alpha'}] = 1$, a contradiction and so also (c) is established.

$vvi1$

Lemma 8.23 *Suppose that (i) to (v) in 8.22 holds. Suppose in addition that*

(a) *If* $A \in \mathcal{Q}$ *and* B *is an elementary abelian subgroup of* Q_δ *with* $[Z_\gamma, A, B] = 1$ *and* $A \leq B$. *Then* $[Z_\gamma, B] \leq [Z_\gamma, A][C_{Z_\gamma}(A), B]$

(b) *If* $A \in \mathcal{Q}$ *then there exists* $\lambda \in \triangle(\gamma)$ *with* $G_\gamma = \langle G_{\lambda\gamma}, A\rangle$.

Then (vi.2) in 8.22 and so also the conclusions of 8.22 hold.

Proof: By (b) there exists $\alpha - 1 \in \triangle(a)$ with $G_\alpha = \langle G_{\alpha-1\alpha}, Z_\alpha\rangle$. Suppose that $Z_{\alpha-1} \leq Q_{\alpha'-1}$. Then by (a) applied with $\gamma = \alpha'$, $A = Z_\alpha$ and $B = Z_{\alpha-1}Z_\alpha$, we conclude that

$$[Z_{\alpha'}, Z_{\alpha-1}Z_\alpha] \leq [Z_{\alpha'}, Z_\alpha][Z_{\alpha'} \cap Q_\alpha, Z_{\alpha-1}] \leq Z_{\alpha-1}Z_\alpha.$$

Thus $Z_{\alpha-1}Z_\alpha$ is normalized by $\langle G_{\alpha-1\alpha}, Z_\alpha\rangle = G\alpha$, a contradiction to (v).

cznnab

Lemma 8.24 *Suppose that (i) to (v) in 8.22 holds. In addition assume that for each $A \in \mathcal{Q}$ and each elementary abelian subgroup B each Q_δ with $[Z_\gamma, A, B] = 1$ and $A \leq B$ the following statements hold*

(a) $|B/C_B(C_{Z_\gamma}(A) \leq |C_{Z_\gamma}(A)/C_{Z_\gamma}(B)$.

(b) *If* $[C_{Z_\gamma}(A), B] = 1$ *then* $[Z_\gamma, B] \leq [Z_\gamma, A]$.

(c) *Suppose that* $[C_{Z_\gamma}(A), B] \neq 1$. *Then for elementary abelian subgroup C of Q_δ with $B \leq C$ and $[Z_\gamma, B, C] = 1$, $[C_{Z_\gamma}(A), C] \leq [C_{Z_\gamma}(A), B]$*

(d) *There exists $\lambda \in \triangle(\gamma)$ with $L_\gamma = \langle O_p(G_{\lambda\gamma}), A\rangle$.*

Then the conclusions of 8.22 hold.

Proof: We may assume that (vi.2) in 8.22 does not hold. Thus by (d) we can choose a critical pair (α, α') and $\alpha - 1 \in \triangle(a)$ with $G_\alpha = \langle G_{\alpha-1\alpha})Z_\alpha\rangle$ and $Z_{\alpha-1} \leq Q_{\alpha'-1}$. If $[Z_{\alpha'}, Z_{\alpha-1}Z_\alpha] \leq [Z_{\alpha'}, Z_\alpha]$ we get that $Z_{\alpha-1}Z_\alpha$ is normalized by $\langle G_{\alpha-1}, Z_{\alpha'}\rangle = G_\alpha$, a contradiction to (v). Then by (b) we may assume that $[Z_{\alpha'} \cap Q_\alpha, Z_{a-1}] \neq 1$. Put $X = Z_{\alpha'} \cap Q_\alpha$. Then by (a) $[X \in \mathcal{Q}_{\alpha-1\alpha}$ and so 8.22(a) holds.

Moreover, $A_{\alpha-1\alpha} \leq [Z_\alpha - 1, X] \leq Z_{\alpha'}$ and so $A_{\alpha-1\alpha}$ is normalized by $G_{\alpha-1}\alpha$ and $Z_{\alpha'}$ and so 8.22b holds.

Suppose that $b > 2$. By (d) there exists $\alpha - 2 \in \triangle(\alpha - 1)$ with $G_{\alpha-1} = \langle G_{\alpha-2\alpha-1}, X\rangle$. If $Z_{\alpha-2} \not\leq Q_{\alpha'-2}$, then $A_{\alpha-2\alpha-1} \leq [Z_\alpha - 2, Z_{\alpha'} - 2] \leq Z_{\alpha'} - 2$. As $b > 2$ we get that $G_{\alpha\alpha-2}$, X and $Z_{\alpha'}$ normalize $A_{\alpha-2\alpha-1}$. But then $A_{\alpha-2\alpha-1}$ is normal in $G_{\alpha-1}$ and G_α.

Hence $Z_{a-2} \leq Q_{\alpha'-2}$. If $Z_{\alpha-2} \not\leq Q_{\alpha'-1}$, then since also $Z_{\alpha'-1} \leq Q_{\alpha-1}$ we conclude from 8.22(iii) that $Z_{\alpha'} - 1 \leq calQ_{\alpha'-2\alpha'-1}$. But then $A_{\alpha-2\alpha-1} \leq [Z_\alpha - 2, Z_{\alpha'} - 1]$ a we get the same contradiction to the previous paragaph.

Thus $Z_{\alpha-2} \leq Q_{\alpha'-1}$ and so by (c) applied with $C = Z_{\alpha-2}$ and $\gamma = \alpha'$ we conclude that $[Z_{\alpha-2}, X] \leq [Z_{\alpha-1}, X] \leq Z_{\alpha-1}$. Hence $Z_{\alpha-2}Z_{\alpha-1}$ is normalized by $G_{\alpha-2\alpha-1}$ and X, a contradiction to 8.22(v).

znnabmp

Lemma 8.25 *Suppose that G_α and G_β are minimal parabolics and $Z \ntrianglelefteq G_\alpha$ and $Z \ntrianglelefteq G_\beta$. Then $b \leq 2$ or $Z_\alpha Z_\beta \trianglelefteq G_\alpha$*

Proof: We assume without loss that $G_{\alpha\beta}$ is Sylow 2-subgroup of G_α and G_β. Put $T_{\alpha\beta} = \langle Z_{\alpha'}^{G_{\alpha\beta}}\rangle Q_a$ and $Z_{\alpha\beta} = C_{Z_\alpha}(T_{\alpha\beta}$. Note that $T_{\alpha\beta}$ only depends on α and β but not on $Z_{\alpha'}$. Let $\alpha - 1 \in \triangle(a)$ with $Z_{\alpha\beta} \cap Z_{\alpha\alpha-1} \leq D_\alpha$. For $O \leq i \leq b$, put $W_i = \langle Z_{\alpha'-i}^{G_{\alpha\beta}}\rangle$. Then $W_b = Z_\alpha$ and $W_0 Q_\alpha = T_{\alpha\beta}$. Put $T = T_{\alpha-1\alpha}$ and supose that $W_1 Q_{\alpha-1} \neq T$. Then there exists a $U \trianglelefteq T$ so that $Z_\alpha = \langle U^{G_{\alpha\alpha-1}}\rangle$ and $[W_1, U] = 1$. Hence $U \leq Q_{\alpha'-1}$. It is now easy to see that $Z_{\alpha'} \cap Q_\alpha \leq T_{\alpha-1\alpha}$ and so $[U, Z_{\alpha'}] \leq [Z_\alpha, Z_{\alpha'}][U, Z_{\alpha'} \cap Q_\alpha] \leq Z_\alpha[U,T]$. Hence $[U, W_0] \leq Z_\alpha[U,T]$ and W_O. Let $L = \langle T, W_O$. Then $O^2(L)$ centralizes UZ_α/Z_α. As $Z_\beta = \langle U^{G_{\alpha-1\alpha}}\rangle$ we conclude that G_α normalizes $Z_\beta Z_\alpha$. **Remark: It is easy to see that V_α/Z_α is an FF-module. This will kill any problem $O_{2\Phi}$ might cause, also this shows that basicly $T_{\alpha\beta} = T_{\beta\alpha}$**

Hence $W_1 Q_{\alpha-1} = T$. Choose $\alpha - i - 1 \in \triangle(\alpha - i)$ with $Z_{\alpha-i-1\alpha-i} \cap Z_{\alpha-i+1\alpha-i} \leq Z(G_{\alpha-i}$. Then a similar argument shows inductively that $W_i Q_{\alpha-i} = T_{\alpha-i\alpha-i+1}$. Hence $Z_\alpha Q_{\alpha-b} = T_{\alpha-b\alpha-b+1}$. Therfore we may assume that $Z_{\alpha'} Q_a = T_{\alpha\beta}$. The above argument now shows that $Z_{\alpha'-1} Q_{\alpha-1} = T$ and we conclude that if $b > 1$, then $[Z_{\alpha-1}, T] = [Z_{\alpha-1}, Z_{\alpha-1}] \leq D_\alpha$. Moreover, if $b > 2$, $[Z_\alpha - 2, Z_{\alpha'} - 2] \leq D_\alpha - 1 \cap D_\alpha$, a contradiction and the lemma is proved.

Lemma 8.26 *Let $M_i \in calL(S)$, $1 \leq i \leq 3$ and suppose that that*

(i) *For $i = 2, 3$, $O^2(M1i \cap S \leq Q_{23}$*

(ii) *$O^2(M_1) \cap S = (O^2(M12 \cap S)(O^2(M13 \cap S)$.*

Then Q_{23} is a Sylow 2-subgroup of $O^2(M_1)Q_{23}$ and $Q_1 \cap Q_{23} = O_2(O^2(M_1)Q_{23})$ is normal in M_1

Proof: Let $L = O^2(M_1)Q_{23}$. Then by (ii) and (i)

$$Q_{23} \leq L \cap S = (O^2(M_1) \cap S)Q_{23} = (O^2(M12 \cap S)(O^2(M13 \cap S)Q_{23} = Q_{23}$$

. Since $L \trianglelefteq LS = M_1$, $O_2(L) \leq Q_1$.

Hence $O_2(L) = Q_1 \cap L = Q_1 \cap Q_{23}$.

9 Amalgams involving uniqueness groups

minparun

Hypothesis 9.1 (i) *Hypothesis 8.1 holds with G finite.*

(ii) *G_α is a minimal parabolic.*

(iii) *$E_\beta B(S)$ lies in a unique maximal p-local M_β of G.*

(iv) *$Q_\beta^* \leq O_p(M_b)$.*

(v) *$G_\beta = E_\beta G_{\alpha\beta}$*

(vi) *$M_{\alpha\beta} \stackrel{def}{=} M_\beta \cap G_\alpha$ is the unique maximal subgroup of G_α containing S.*

(vii) $G_\beta \in \mathcal{CL}(S)$.

Put $Q_{\alpha\beta} = O_2(M_{\alpha\beta})$, $X_b = \Omega_1(Z(Q_b))$ and $X^*_\beta = \Omega_1(C_{Q_\beta}(Q^*_b))$
Put $D_\beta = \bigcap_{\delta\in\triangle(\beta)} Z_\delta$ and $R = [Z_\alpha, Z_{\alpha'}]$.
The next two lemmas reveal how the assumptions on E_β can be used

Qab

Lemma 9.2 (a) $Q^*_\beta \leq O_2(M_\beta) \leq Q_{\alpha\beta}$.

(b) *Let $\gamma \in \triangle(\beta)$ and R_α be a normal subgroup of G_α. Then*

$$R_\gamma \cap Q_\beta \leq (R_\alpha \cap Q_\beta)Q^*_\beta \leq (R_\alpha \cap Q_\beta)Q_{\alpha\beta} \leq (R_\alpha \cap Q_\beta)O_2(M_\beta) \leq R_\alpha Q_{\alpha\beta}.$$

(c) *Let $\gamma \in \triangle(\beta)$. Then $Q_\gamma \cap Q_\beta \leq Q_\alpha O_2(M_\beta) \leq Q_{\alpha\beta}$.*

(d) *Let $R_{\alpha\beta}$ be a normal subgroup of $G_{\alpha\beta}$ contained in Q_β. Then for all $\gamma \in \triangle\beta$,*

$$R_{\alpha\beta} \leq \langle R^{G_\beta}_{\alpha\beta} \rangle \leq O_2(M_\beta)R_{\gamma\alpha}.$$

Proof: By hypothesis, $Q^*_\beta \leq O_2(M_\beta)$. As $G_{\alpha\beta}$ contains a Sylow 2-subgroup of M_β, $O_2(M_\beta) \leq G_{\alpha\beta}$ and (a) holds.

Since E_β acts transitively on $\triangle(\beta)$ we have $R_\gamma \cap Q_\beta \leq (R_\alpha \cap Q_\beta)[Q_\beta, E_\beta]$ and so (b) follows from (a).

Since $Q_\alpha \leq Q_{\alpha\beta}$, (c) follows from (b) applied to $R_\alpha = Q_\alpha$.

As $R_{\alpha\beta} \leq \langle R^{E_\beta}_{\gamma\beta} \rangle \leq [Q_\beta, E_\beta]R_{\gamma\alpha} \leq O_p(M)R_{\gamma\alpha}$, (d) holds.

$Mtrick$

Lemma 9.3 *Suppose $1 \neq D \leq ZJ(S)$ and $E_\beta] \leq N_G(D)$. Then*

(a) $N_{G_\alpha}(D) \leq M_{\alpha\beta}$

(b) *Let $\delta \in \Gamma$ such that $d(\beta, \delta) = b - i$ with $1 \leq i < b$. Suppose that $N_{G_\delta}(D)$ normalizes no non-trivial 2-subgroup of G_δ/Q_δ. Then*

  (ba) $V^{(i+1)}_\beta \cap G_\delta \leq Q_\delta$

  (bb) $V^{(i)}_\beta \leq Q_\delta$.

  (bc) *If $N_{G_\delta}(D)$ contains a Sylow $p-$subgroup of G_δ, then $V^{(i+1)}_\beta \leq Q_\delta$.*

(c) *If b is odd and $b \geq 3$, then $E_{\alpha'}$ does not normalize D.*

(d) *Suppose that b is even, $b \geq 3$ and $E_{\alpha'-1}$ normalizes D, then*

  (da) $V^{(3)}_\beta \cap G_{\alpha'-1} \leq Q_{\alpha'-1} \leq G_{\alpha'}$.

  (db) *If $G_{\alpha'-1}$ normalizes D, then $V^{(3)}_\beta \leq Q_{\alpha'-1} \leq G_{\alpha'}$.*

Proof: As $B(S)$ and E_β normalize D, $N_G(D) \leq M_\beta$. Thus (a) holds.

For (b) let $\gamma \in \triangle(\beta)$ with $d(\gamma, \delta) = b - i - 1$. Then by 9.2(d) $V_\beta^{(i+1)} \leq V_\gamma^{(i-1)} O_2(M_\beta)$. By minimality of b, $V_\gamma^{(i)} \leq Q_\delta$. Since $N_{G_\delta}(D) \leq M_\beta$, $N_{G_\delta}(D)$ normalizes the 2-group $G_\delta \cap O_2(M_\beta)$. Thus by assumption, $G_\delta \cap O_2(M_\beta) \leq Q_\delta$. Hence $V_\beta^{(i+1)} \cap G_\delta \leq V_\gamma^{(i-1)}(O_2(M_\beta) \cap G_\delta) \leq Q_\delta$. So (ba) holds. Clearly (ba) implies (bb). In case (bc) $O_2(M_\beta) \leq G_\delta$ and so $V_b^{(i+1)} \leq G_\delta$.

Suppose b is odd and $E_{\alpha'}$ centralizes D. Then by (bb) applied with $\delta = \alpha'$ and $i = 1$, $V_\beta \leq Q_{\alpha'}$, a contradiction.

(d) follows from (ba) and (bc) applied with $\delta = \alpha' - 1$ and $i = 2$. □

uznn

Lemma 9.4 *Suppose that $[Z, E_\beta] \neq 1$. Then Z_β is an FF-module.*

Proof: 8.16 □

1za

Lemma 9.5 *Suppose that $[Z_\alpha, Z_{\alpha'}] \neq 1$ and $[Z, E_\beta] = 1$.*

(a) *Let $L_\alpha = \langle Z_{\alpha'}^{G_\alpha} \rangle Q_\alpha$. Then $L_\alpha / C_\alpha \cong SL_2(q)^k$, where k is a postive integer and q a power of 2.*

(b) *Z_α is a natural module for L_α / C_a.*

(c) *$Z_{\alpha'} Q_\alpha$ is a Sylow p-subgroup of $\langle Z_{\alpha'}^{L_\alpha} \rangle Q_\alpha$.*

Proof: As $[Z_\alpha, Z_{\alpha'}] \neq 1$ we may assume that $Z_{\alpha'}$ acts as an offending subgroup on Z_α. Since $[Z, E_\beta] = 1$, $C_{Z_\alpha}(L_\alpha) = 1$.Moreover, by 9.2c $Z_{\alpha'} \leq Q_{\alpha\beta}$, which excludes the possibilty that Z_α is a a natural $\mathrm{Sym}(q+1)^k$-modules for $q \geq 4$. Thus the lemma follows from 6.3. □

Define $Z_{\alpha\beta} = C_{Z_a}(S \cap L_\alpha)$ and $Z_\beta^* = \langle Z_{\alpha\beta}^{G_\beta} \rangle$. In the next two lemmas we will assume $[Z_\alpha, Z_{\alpha'}] \neq 1$. Let V be an irreducible L_α submodule in Z_α not centralized by $Z_{\alpha'}$ and similarly choose $V' \leq Z_{\alpha'}$. Put $R = [V, V']$.

zb∗

Lemma 9.6 *Suppose that $[Z_\alpha, Z_{\alpha'}] \neq 1$ and $[Z, E_\beta] = 1$. Then one of the following holds:*

1. *$Z_{\alpha\beta}$ is normal in G_β.*
2. *$Z_{\alpha\beta} \leq X_\beta^*$ and $[X_\beta^*, E_\beta] \neq 1$.*
3. *$q = 2$ and $k \geq 2$. Moreover, if $U_{\alpha\beta}$ be maximal in $Z_{\alpha\beta}$ with $[U_{\alpha\beta}, G_{\alpha\beta}] \leq Z_\beta$ and $U_\beta = \langle U_{\alpha\beta}^{G_\beta} \rangle$, Then U_β / Z_β is an FF-module for G_β / Q_β*

Proof: We may assume that $Z_{\alpha\beta}$ is not normal G_β and so is not centralized by E_β.

Suppose first that $q > 2$ or $k = 1$. Then $Q_\beta^* \leq Q_{\alpha\beta} \leq L_\alpha$ and so $Z_{\alpha\beta} \leq X_\beta^*$. Thus $[X_\beta^*, E_\beta] \neq 1$ and the $P \times Q$ lemma implies $[X_\beta, E_\beta] \neq 1$.

So suppose now that $q = 2$ and $k > 1$. Let $\alpha - 1 \in \triangle(\alpha)$ with $\langle G_{\alpha\alpha-1}, V' \rangle = G_\alpha$. By 9.5c, $[Z_{\alpha\beta}, Z_{\alpha'}] = 1$ and so

(1) (a) $Z^*_\beta \le Q_\delta$ for all $\delta \in \Gamma$ with $d(\beta,\delta) < b$. $zb*-1$

(b) $[Z^*_{\alpha'-1}, V'] = 1$, even if $b = 2$.

In particular, $[Z^*_\beta, Z_{\alpha'}] = 1$ and as S acts transitively on the $L^{(i)}_\alpha$ and normalizes $C_{Q_\beta}(Z^*_b)$ we conclude

(2) (a) $S \cap L_\alpha = C_{Q_\beta}(Z^*_\beta)Q_\alpha$. $zb*-1a$

(b) $Z^*_\beta \cap Z_\alpha = Z_{\alpha\beta}$.

By definition of $U_{\alpha\beta}$ we have $[U_{\alpha\beta}, Q_\beta] \le Z_\beta$ and thus

(3) $[U_\beta, Q_\beta] \le Z_\beta$. $zb*-2$

In particular, $D \overset{def}{=} [U_{\alpha-1}, U_{\alpha'-1} \cap Q_{\alpha-1}] \le Z_{\alpha-1}$. On the otherhand, by (1)a,$U_{\alpha-1} \le Z^*_{\alpha-1} \le Q_{\alpha'-2} \le G_{ap-1}$ and so $D \le U_{\alpha'-1} \le Z^*_{ap-1}$ and so by (1)c, $[D, V'] = 1$. Hence by choice of $\alpha - 1$, D is centralized by G_α and $G_{\alpha-1}$. Thus

(4) $[U_{\alpha-1}, Z^*_{\alpha'-1} \cap Q_{\alpha-1}] = 1$. $zb*-3$

Suppose that $U_{\alpha-1} \le Q_{\alpha'-1}$. As $[R, U_{\alpha-1}] = 1$ we conclude that $[U_{\alpha-1}, V'] \le R \le Z_\alpha$. Thus

$$U_{\alpha-1}Z_\alpha \trianglelefteq \langle G_{\alpha-1\alpha}, V' \rangle = G_\alpha.$$

Hence also $[U_{\alpha-1}, Q_\alpha] \trianglelefteq G_\alpha$. By (4), $Z_\alpha \not\le U_{\alpha-1}$ and since Z_α is the unique minimal normal subgroup of G_α in Q_α we conclude that $[U_{\alpha-1}, Q_\alpha] = 1$. Thus $[U_\beta, Q_\alpha] = 1$. Since $E_\beta \le \langle Q_\alpha^{G_\beta} \rangle T$ we get $[U_\beta, E_\beta] = 1$. Note also that $[U_{\alpha\beta} \le ZJ(S)$ and that there exists $1 \ne D \le U_{\alpha\beta}$ with $C_{G_\alpha}(D) \not\le M_{\alpha\beta}$. Hence we obtain a contradiction to 9a. We proved

(5) (a) $[U_\beta, E_\beta] \ne 1$. $zb*-4$

(b) $U_{\alpha-1} \not\le Q_{\alpha'-1}$.

If $[U_{\alpha-1} \cap Q_{\alpha'-1}, U_{\alpha'-1}] = 1$, then 8.17 and (4) imply that 3. holds. Thus we may assume:

(6) $Z_{\alpha'-1} = [U_{\alpha-1} \cap Q_{\alpha'-1}, U_{\alpha'-1}] \le U_{\alpha-1}$ $zb*-5$

Suppose that $b = 2$. Then by (6) and (2)b, $Z_\beta = Z_{\alpha'-1} \le U_{\alpha-1} \cap Z_\alpha \le Z^*_{\alpha-1} \cap Z_\alpha = Z_{\alpha\alpha-1}$. But this contradicts the choice of $\alpha - 1$. Hence

(7) $b \ge 4$. $zb*-6$

By (6), there exists $\lambda \in \triangle(\alpha'-1)$ and $t \in U_{\alpha-1} \cap Q_{\alpha'-1}$ with $[t, U_{\alpha'-1\lambda}] = Z_{\alpha'-1}$.

Suppose t normalizes one of the $Z_\lambda^{(i)}$ and let X be the sum of the $Z_\lambda^{(j)}, j \neq i$. Then $U_{\alpha'-1\lambda} = U_{\alpha'-1\lambda} \cap Z_\lambda^{(i)} \oplus U_{\alpha'-1\lambda} \cap X$, t centralise $U_{\alpha'-1\lambda} \cap Z_\lambda^i$ and so $Z_{\alpha'-1} = [U_{\alpha'-1\lambda}, t] \leq [X, t] \leq X$, a contradiction.

$zb*-7$

(8) t acts fixed-point freely on $\{L_\lambda^{(i)} \mid 1 \leq i \leq k\}$.

Thus by 2.2 and (2)a there exists $\mu \in \triangle(\lambda)$ with $O^2(G_\lambda) \leq \langle C_{Q_\mu)}(Z_\mu^*), t\rangle$. As t centralizes Z_α, (8) implies that $Z_\alpha \leq Q_\lambda$. Moreover, $U_\mu \leq Q_{\alpha+2} \leq G_\beta$ and so $[V_\beta, U_\mu] \leq U_\mu \cap V_\beta$. Since $b \geq 4$, we conclude from (1)a that $U_{\alpha-1}$ and so also t centralizes $[V_\beta, U_\mu]$. Since $C_{Q_\lambda}(O^2(G_\lambda)) = 1$ the choice of μ implies $[V_\beta, U_\mu] = 1$ and so

$zb*-8$

(9) $U_\mu \leq Q_\beta \cap Q_\alpha \leq G_{\alpha-1}$.

Since $d(\mu, \alpha') = 3 < b$, (2) implies $[\langle U_\mu^{G_\lambda}\rangle, V' = 1$. Thus $[t, U_\mu \cap Q_{\alpha-1}] \leq Z_{\alpha-1}(V') = 1$. From $C_{U_\mu}(t) \leq C_{Q_\lambda}(O^2(G_\lambda)) = 1$ we get

$zb*-9$

(10) $U_\mu \cap Q_{\alpha-1} = 1$

Thus

$$|U_{\alpha-1}/C_{U_{\alpha-1}}(U_\mu)| \leq |U_{\alpha-1}| = |U_\mu| = |U_\mu Q_{\alpha-1}/Q_{\alpha-1}|$$

and 3. holds. □

pred

Lemma 9.7 *Suppose that $[Z_\alpha, Z_{\alpha'}] \neq 1$ and $Z_{\alpha\beta}$ is normal in G_β. Then $b = 2$, E_β centralizes $Z_{\alpha\beta}$ and G_a is of L_2-type.*

Proof: By8.15 $Q_\alpha \not\leq Q_\beta$. As Q_α centralizes $Z_{\alpha\beta}$ and $E_\beta \leq \langle Q_\alpha^{G_\beta}\rangle$ we conclude that E_β centralizes $Z_{\alpha\beta}$. Note that $V \cap Z_{\alpha\beta} \neq 1$ and so by 9, $C_{G_\alpha}(V \cap Z_{\alpha\beta} \leq M_{\alpha\beta}$. Thus $k = 1$ and G_α is of L_2-type. It remains to show that $b = 2$.

Suppose that $b > 2$. Let $\alpha - 1 \in \triangle(\alpha)$ with $\langle G_{\alpha-1\alpha}, V\rangle = G_\alpha$ and note that $R = Z_\beta^* = Z_{\alpha'-1}^*$ is normalized by G_β and $G_{\alpha'-1}$. Hence 9(d) implies that $V_{\alpha-1} \leq G_{\alpha'}$. As $V_{\alpha-1}$ centralizes R we conclude that $[V_{\alpha-1}, Z_{\alpha'}] \leq R$ and G_α normalizes $V_{\alpha-1}$, again a contradiction. □

$2za$

Lemma 9.8 *Suppose that $[Z_\alpha, Z_{\alpha'}] = 1$, $b > 1$ and $r_\beta > 1$. Then there exists a normal subgroups L_α of G_α and normal subgroups $L_\alpha^{(i}, 1 \leq i \leq k$ of L_α such that*

(a) $C_\alpha \leq L_\alpha$ *and* $C_\alpha \leq L_\alpha^{(i)}$

(b) $\overline{O^2(L_\alpha)} = \overline{L_\alpha^{(1)}} \times \ldots \times \overline{L_\alpha^{(k)}}$

(c) $G_\alpha = L_\alpha S$, *S transitively permutes the $L_\alpha^{(i)}$'s and L_α is the largest subgroup of G_α normalizing all the $L_\alpha^{(i)}$'s.*

(d) *Put* $Z_\alpha^{(i)} = [Z_\alpha, L_\alpha^{(i)}]$. *Then* $Z_\alpha = Z_\alpha^{(1)} \oplus \ldots \oplus Z_\alpha^{(k)}$.

(e) *One of the following holds*

1. $\overline{L_\alpha^{(i)}} \cong SL_2(q)$, *$q$ a power of 2 and* $Z_\alpha^{(i)}$ *is a natural* $SL_2(q)$*-module for* $L_\alpha^{(i)}$.
2. $\overline{L_\alpha^{(i)}} \cong C_3$, $|Z_\alpha^{(i)}| = 4$ *and* $s_{Z_\alpha}(O_2(M_\beta)) < 2$.
3. $\overline{L_\alpha^{(i)}} \cong SL_3(q)$, *$q$ a power of 2;* $Z_\alpha^{(i)}$ *is direct sum of a natural* $SL_3(q)$*-module for* $L_\alpha^{(i)}$ *with its dual; some element of S induces a graph automorphism on* $\overline{L_a^{(i)}}$ *and* $c_\beta = 2$

Proof: Suppose first that $c_\beta = 1$. Then the lemma holds by 8.4 and 6.3, where the Sym$(q+1)$ case is excluded as in 9.5.

So suppose that $c_\beta \geq 2$. Then $r_\beta c_\beta - 1 > 1$ and so by 2.4a, $r_\alpha < 2$. Thus we can apply 6.4 with the Sym$(q+1)$-case excluded as usual. Note that in case (e3) we actually have $r_a = \frac{3}{2}$. As $r_\beta > 1$, 2.4 implies $c_\beta = 2$ and all parts of the lemma are proved. □

Put $Z_{\alpha\beta} = C_{Z_\alpha}(L_\alpha \cap S)$ and $Z_\beta^* = \langle Z_{\alpha\beta}^{G_\beta} \rangle$.

Lemma 9.9 *Suppose that* $[Z_\alpha, Z_{\alpha'-1}] = 1$, $b > 1$ *and the conclusions of 9.8 hold for case e3 hold. Then* $Q_\beta Q_\alpha / Q_\alpha \leq Z(S \cap L_\alpha / Q_\alpha)$, $[X_\beta, E_\beta] \neq 1$ *and X_β is an FF-module.* *l3q.2*

Proof: Suppose that E_β centralizes $Z_{\alpha\beta}$ and let D be the intersection of $Z_{\alpha\beta}$ with one of the irreducible L_α submodule in Z_α. Then $D \neq 1$, $N_{G_\alpha}(D) \not\leq M_{\alpha\beta}$ and $E_\beta B(S)$ centralizes D, a contradiction to 9a.

Thus E_β does not centralize $Z_{\alpha\beta}$.

Recall that $c_\beta = 2$ in case 9.8e3. Thus 8.9 applied to $L = E_b$ shows that $[Z_\alpha, Q_\beta, 2] \leq D_\beta$. By 8.15 $Q_\alpha \not\leq Q_\beta$. Hence $E_\beta \leq \langle Q_a^{G_\beta} \rangle$ and so $[D_\beta, E_\beta] = 1$. In particular $Z_{\alpha\beta} \not\leq D_\beta$ and so $Z_{\alpha\beta} \not\leq [Z_\alpha, Q_\beta, 2]$. As S normalizes $[Z_\alpha, Q_\beta, 2]$ we conclude from the action of S on Z_α that $[Z_\alpha, Q_\beta, 2] < Z_{\alpha\beta}$. Since Q_β is normal in S this implies that $Q_\beta \leq L_\alpha$ and then that Q_β acts quadratically on each of the irreducible L_α submodules in Z_α. As S normalizes Q_β and induces a graph automorphism on the $L_\alpha^{(1)}$ we get $Q_\beta Q_\alpha / Q_a \leq Z(S \cap L_\alpha / Q_\alpha)$ and $Z_{\alpha\beta} \leq X_\beta$. Hence $[X_\beta, E_\beta] \neq 1$ and so by **??** X_β is an FF-module.

Lemma 9.10 *Suppose that* $[Z_\alpha, Z_{\alpha'-1}] = 1$, $b > 1$ *and the conclusions of 9.8 hold for case e1 or e2 hold. Then one of the following is true:* *l2k*

1. $k = 1$, $[Z_{\alpha\beta}, E_\beta] = 1$ *and V_β is an FF-module for G_β*
2. $k = 1$, $b = 3$ *and V_β is an $F2$-module.*
3. $[Z_{\alpha\beta}, E_\beta] \neq 1$ *and X_β is an FF-module.*

4. *$q = 2$, $k \geq 2$ and $[Z_{\alpha\beta}, E_\beta] \neq 1$. Let $U_{\alpha\beta}$ be maximal in $Z_{\alpha\beta}$ with $[U_{\alpha\beta}, Q_\beta] \leq Z_\beta$ and put $U_\beta = \langle U_{\alpha\beta}^{G_\beta} \rangle$. Then U_β is an FF-module for G_β.*

Proof: By 9a, $[Z_{\alpha\beta}, E_\beta] = 1$ implies, $k = 1$.

Suppose that $q > 2$ or $k = 1$. Then $Q_\beta^* \leq O_2(M_\beta) \leq Q_{\alpha\beta} \leq L_\alpha$ and so $Z_{\alpha\beta} \leq X_\beta^*$. So if in addition $[Z_{\alpha\beta}, E_\beta] \neq 1$, then **??** implies that 3. holds. Hence we may assume from now on that

(1) One of the following holds: $l2k-1$

(Case 1) $k = 1$ and $[Z_{a\beta}, E_\beta] = 1$.

(Case 2) $q = 2$, $k \geq 2$ and $[Z_{\alpha\beta}, E_\beta] \neq 1$.

Put $D_\beta^* = Z_{\alpha\beta} \cap D_\beta$ and note that in case Case 1, $D_\beta^* = Z_{\alpha\beta}$ while in case Case 2 9a implies $D_\beta^* = Z_\beta$. In Case 1 let $U_{\alpha\beta} = Z_\alpha$ and in Case 2 let $U_{\alpha\beta}$ be maximal in $Z_{\alpha\beta}$ with $[U_{\alpha\beta}, Q_\beta] \leq D_\beta^*$. Put $U_\beta = \langle U_{\alpha\beta}^{G_\beta} \rangle$. It follows easily from the definitions and 9.2c that:

(2) (a) $[U_{\alpha\beta}, E_\beta] \neq 1$ $2z*b$

(b) $[U_\beta, Q_\beta^*] \leq [V_\beta, O_2(M_\beta)] \leq D_\beta^* \leq Z_\alpha$

(c) $[U_\beta, Q_\beta \cap Q_{\alpha+2}] \leq D_\beta^*$.

By 9d applied with $D = D_\beta^* \cap D_{\alpha'-1}^*$ we get

(3) $D_\beta^* \cap D_{\alpha'-1}^* = 1$ $zbza$

By (2)c, $[U_\beta \cap Q_{\alpha'}, U_{\alpha'} \cap Q_\beta] \leq D_\beta^* \cap D_{\alpha'-1}^* = 1$ and so

(4) $[U_\beta \cap Q_{\alpha'}, U_{\alpha'} \cap Q_\beta] = 1$ $vqvq$

We may and do assume from now on that U_β is not an FF-module and and will show that 2. holds.

Suppose that $U_{\alpha'} \leq Q_\beta$. As $b \geq 3$, $U_{\alpha'}$ acts quadratically on Z_α. Let V be an irreducible L_α submodule in Z_α with $V \not\leq Q_{\alpha'}$. Assume first that $U_{\alpha'}$ normalizes V. Then

$$|V/C_V(U_{\alpha'}) = q \geq |U_{\alpha'}/C_{U_{\alpha'}}(V)|.$$

If q =2, this clearly implies that $U_{\alpha'}$ is an FF-module. If $q > 2$ we are in Case 2 and so $V \leq U_\beta$ and by (4), $U_\beta \cap Q_{\alpha'} \leq C_V(U_{\alpha'})$. Hence $|VQ_{\alpha'}/Q_{\alpha'}| \geq q$. Again $U_{\alpha'}$ is an FF-module, a contradiction.

Thus $U_{\alpha'}$ does not normalizes V and quadratic action implies $|U_{\alpha'}/C_{U_{\alpha'}}(V)| \leq 2$, again a contradiction. Thus

(5) $U_{\alpha'} \not\leq Q_\beta$ and the situation is symmetric in β and α'. sym

Suppose that $[U_\beta, U_{\alpha'} \cap Q_\beta] = 1 = [U_{\alpha'}, U_\beta \cap Q_{\alpha'}]$. Then by 8.17 we get that U_β is an FF-module. Thus

(6) $D_\beta = [U_\beta, U_{\alpha'} \cap Q_\beta] \leq U_{\alpha'}$ or $D_{\alpha'} \leq [U_{\alpha'}, U_\beta \cap Q_{\alpha'}] \leq U_\beta$ *vvqa*

Hence we may assume $[U_\beta, U_{\alpha'} \cap Q_\beta] \neq 1$ and so

(7) $D^*_\beta = [U_\beta, U_{\alpha'} \cap Q_\beta] \leq U_{\alpha'}$. *vvq*

Pick $\mu \in \triangle(\beta)$ and $t \in U_{\alpha'} \cap Q_\beta$ with $[U_{\mu\beta}, t] \neq 1$. Then by (4), $Z_\mu \not\leq Q_{\alpha'}$ and we may assume that $\mu = \alpha$. Hence

(8) There exists $t \in U_{\alpha'} \cap Q_\beta$ with $[U_{\alpha\beta}, t] \neq 1$. In particular, $t \notin Q_\alpha$ *vbq*

In particular, by 9.2c, $O_2(M_\beta) \not\leq Q_\alpha$, as $O_2(M_\beta)$ is normal in $M_{\alpha\beta}$ we conclude (compare also (8) in 9.6).

(9) (a) In case 1, $O_2(M_\beta)Q_a = S \cap L_\alpha$. *QMbQS*

(b) In Case 2, t acts fixed point freely on $\{L^{(i)}_\alpha \mid 1 \leq i \leq k\}$.

In particular, (also use 2.2 in Case 2) there exists $\alpha - 1 \in \triangle(\alpha)$ with

(10) $E_\alpha \leq \langle O_2(M_{\alpha-1}) \cap L\alpha, t\rangle$. *O2G*

By (4) and (8) we have $|U_\beta Q_{\alpha'}/Q_{\alpha'}| \geq |U_{\alpha\beta} Q_{\alpha'}/Q_{\alpha'}| = |U_{\alpha\beta}/C_{U_{\alpha\beta}}(t)| \geq q$. We record

(11) $|U_\beta Q_{\alpha'}/Q_{\alpha'}| \geq q$. *vbqa*

Define $Y_\alpha = \bigcap_{\delta \in \triangle(\alpha)} U_\delta Z_\alpha$.

Suppose now that $[U_{\alpha-1}, V_{\alpha'-2}]$=1. Then $U_{\alpha-1} \leq Q_{\alpha'-2} \cap Q_{\alpha'-1}$. Put $A = U_{\alpha-1} \cap (U_\beta Q_{\alpha'})$. Then $A \leq U_\beta(U_\beta U_{\alpha-1} \cap Q_{\alpha'}) \leq U_\beta(Q_{\alpha'-1} \cap Q_{\alpha'})$. Thus by (2)

$$[A, t] \leq [U_\beta, t][Q_{\alpha'-1} \cap Q_{\alpha'}, t] \leq D^*_\beta D^*_{\alpha'}.$$

Let X be maximal in A with $[X, t] \leq D^*_\beta$. As $|D^*_{\alpha'}| = q$ we have $|A/X| \leq q$. Since $D^*_\beta \leq X$, t normalizes X. By (2), $O_2(M_{\alpha-1})$ also normalizes XZ_α. As E_α is transitive on $\triangle(\alpha)$ we conclude from (10) that $XZ_\alpha \leq Y_\alpha$. Put $a = |U_{\alpha-1}/A|$. Then $|U_{\alpha-1}Y_a/Y_a| \leq |U_{\alpha-1}/A||A/X| \leq aq$. Hence

$$|U_\beta Y_a/Y_\alpha| \leq aq.$$

Note that $U_{\alpha-1} \leq Q_{\alpha'-2} \cap Q_{\alpha'-1} \leq G_{\alpha'}$. Since $Y_{\alpha'-1} \leq V_{\alpha'-2}$ we conclude from $|U_\beta Y_a/Y_\alpha| \leq qa$ and edge-transitivity that

$$|U_{\alpha'}/C_{U_{\alpha'}}(U_{\alpha-1}U_\beta)| \leq |U_{\alpha'}Y_{\alpha'-1}/Y_{\alpha'-1}| = |u_\beta Y_a/Y_\alpha| \leq aq.$$

On the otherhand by definition of a, an isomorphism theorem and (11)

$$|U_{\alpha-1}U_\beta Q_{\alpha'}/Q_{\alpha'}| = |U_{\alpha-1}U_\beta Q_{\alpha'}/U_\beta Q_{\alpha'}||U_\beta Q_{\alpha'}/Q_{\alpha'}| \geq aq.$$

By the last two equations, $U_{\alpha'}$ is an FF-module, a contradiction. Hence

$va-1va-2$

(12) $[U_{\alpha-1}, V_{\alpha'-2}] \neq 1$

Suppose that $V_{\alpha'-2} \leq Q_{a-1}$. Then by (5), $V_{\alpha-1} \leq Q_{\alpha'-2}$. Note that by (10), $C_{D^*_{\alpha-1}}(t) = 1$. Thus

$$1 \neq [U_{\alpha-1}, V_{\alpha'-2}] \leq D^*_{\alpha-1} \cap D^*_{\alpha'-2} \leq C_{D^*_{\alpha-1}}(t) = 1$$

a contradiction to (12). Thus

$va-1qa-1$

(13) $V_{\alpha'-2} \not\leq Q_{\alpha-1}$

In particular, $(\alpha'-2, \alpha-1)$ has the same properties as (β, α') and we conclude from (5) that

$va-1qa-2$

(14) $U_{\alpha-1} \not\leq Q_{\alpha'-2}$

Suppose that $1 \neq x \leq D^*_{\alpha'-2} \cap U_{\alpha-1}$. As t centralizes x, $x \in X \leq Y_\alpha$ and so E_α normalizes xZ_α.

Suppose first that $[x, Q_\alpha] \neq 1$. Since E_α normalizes $[x, Q_a]$, $Z_\alpha^{(i)} \leq [x, Q_a]$ for some i. Put $L = O^p(L_\alpha^{(i)})$ and $Q = [Q_\alpha, L]$. Then $[x, Q_\alpha, L] = Z_\alpha^{(i)}$ and $[x, L, Q_\alpha] = 1$. Thus be the three subgroup lemma, $[x, Q] = Z_\alpha^{(i)} = [x, L]$. Since $[x, Q, Q] = 1$ we colcude that $xQ = x^Q = x^L$ and so by the Frattini argument, $L = C_L(x)Q$. Since $x \leq D_{\alpha'-2}$, x is centralised by $E_{\alpha'-2}$ and the Thompson subgroup of $G_{\alpha'-1\alpha'-2}$. By the proof of (ba), $Mtrick$ $t \in V_{\alpha'} \cap G_\alpha \leq V^{(3)}_{\alpha'-2} \cap G_\alpha \leq Q_p(M_{ap-2} \cap G_\alpha$. As $C_L(x)$ normalizes $Q_p(M_{ap-2} \cap G_\alpha$ we get $[t, L] \leq Q_\alpha$. In case 1 this is impossible since $t \notin Q_a$ and in Case 2 this contradicts **??**b.

Suppose next that $[x, Q_\alpha] = 1$, but $x \notin Z_\alpha$. Then its is easy to see that $q > 2$ and $C_{E_a}(x)Q_\alpha/Q_\alpha$ is ismorphic to $D_{2\cdot q\pm 1}$ and again $C_{E_a}(x)$ normalizes no non-trivial 2-subgroup in G_a/Q_a and we get the same contradiction as above.

Hence $x \in Z_\alpha$ and so $D^*_{\alpha'-2} \leq Z_\alpha$. Note that t centralizes $D^*_{\alpha'-2}$. In Case 2 we have n $x \in Z_\alpha$, $[x, O_2(M_{\alpha-1} \cap L_\alpha)] \leq Z_{\alpha-1}$ and $s_{Z_\alpha}(O_2(M_{\alpha-1} \cap L_\alpha) < 2$ implies, $[x, O_2(M_{\alpha-1} \cap L_\alpha)] = 1$. Hence by (10), $[x, E_\alpha] = 1$ a contradiction to $C_{Q_\alpha}(E_\alpha) = 1$.

In case Case 1 we conclude that $D^*_{\alpha'-2} = D^*_\beta$. If $b > 3$, 9bb implies that $V_{\alpha-1} \leq V_\beta^{(3)} \leq Q_{\alpha'-2}$, a contradiction. We have proved

dcu

(15) If $D^*_{\alpha'-2} \cap U_{\alpha-1} \neq 1$, then $b = 3$ and Case 1 holds.

Assume that $b > 3$. Then t centralizes $[U_{\alpha'-2} \cap Q_{\alpha-1}, U_{\alpha-1}]$ and as by (10) $C_{D^*_{\alpha-1}}(t) = 1$ we get $[U_{\alpha'} - 2 \cap Q_{\alpha-1} U_{\alpha-1}] = 1$. Thus by (6) and **??** that $D^*_{\alpha'-2} = [U_{\alpha-1} \cap Q_{\alpha'-2}, U_{\alpha'-1}] \leq U_{\alpha-1}$ a contradiction to (15). Thus

(16) $b = 3$. *l2kb3*

Suppose that $k > 1$. By (6) applied to $(\alpha - 1, \beta)$ in place of (β, α') we get $Z_{\alpha-1} = D^*_{\alpha-1} \leq U_\beta$ or $Z_\beta = D^*_b \leq U_{\alpha'-1}$. In the first case $[Z_{\alpha-1}, O_2(M_\beta) \leq Z_\beta$ and as above so $[Z_{\alpha-1}, O_2(M_\beta \cap L_\alpha) = 1$. But this implies $Z_{\alpha-1} \leq Z_{\alpha\beta}$ and $Z_{\alpha\alpha-1} = Z_{\alpha\beta}$ a contradiction to (10). The second case yields the same contradicion.

Thus $k = 1$ and so $V_\beta = U_\beta$. By (4) and **??**, V_β is $F2$ and so 2. holds. □

We remark that an example for case 2 of the previous theorem occurs in ${}^2F_4(q)$. In that example V_β is exactly $F2$ (that is not F^*2)

10 Connected parabolics not normalizing Z

UII
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Hypothesis 10.1 (a) *Hypothesis 8.1 holds.*

(b) *$C_{G_\alpha}(Y_\alpha)$ is p-closed.*

(c) *G_β is p-minimal.*

(d) *Y_α is neither an FF nor an dual FF-modul.*

Remark: "b" in this section is defined with respect to Y_γ not Z_γ

dmab

Definition 10.2 *$M_{\alpha\beta}$ is the unique maximal subgroup of G_β containing S.*

zc

Lemma 10.3 *b is odd, $Z \trianglelefteq G_\beta$ and $[X_\beta, E_\beta] = 1$.*

Proof: By 8.17 b is odd and as p-minimal groups have no $F1^*$-module, $Z \trianglelefteq G_\beta$. Since Y_α is not FF, $J(S) \not\leq Q_\beta$. If $[X_\beta, E_\beta] \neq 1$, we conclude that X_β is FF. As G_β is p-minimal this gives the contradiction, $Z \ntrianglelefteq G_\beta$. □

qbniqa

Lemma 10.4 *$Q^*_\beta \not\leq Q_\alpha$ and $Q_\alpha \not\leq Q_\beta$.*

Proof: Suppose that $Q^*_\beta \leq Q_\alpha$. Then $[V_\beta, Q^*_\beta] = 1$ and so by Thompson's $P \times Q$-Lemma, $[X_\beta \cap V_\beta, L_\beta] \neq 1$, a contradiction to 10.3. The second statement holds since

$$Z_\alpha \leq Q_{\alpha'-2} \cap Q_{\alpha'-1} \leq Q^*_{\alpha'-1} Q_{\alpha'}. \square$$

sb

Lemma 10.5 (a) *$r_{V_\beta}(G_\beta) \leq 1$.*

(b) *$c_\beta \geq 2$.*

(c) *$q_\alpha \leq 2$.*

Proof: (a) holds since G_β is p-minimal. Since $Q_\alpha \not\leq Q_\beta$ and $Q^*_\beta \not\leq Q_\alpha$, $Q_\alpha \cap Q_b$ is not normal in G_β. Thus by 8.4b, (b) holds. Hence by 8.4a also c. is true. □

Lemma 10.6 *Suppose that $b > 1$.* $nLLp$

(a) *8.21.1 or 8.21.2 holds.*

(b) *For each ϵ in 8.211. or 2., L^ϵ has at least two non trivial chief-factors on V^ϵ.*

(c) *In case* **??** $q_\alpha < \frac{1+\sqrt{17}}{4}$.

(a) Suppose that 8.21.3 holds. Then by 8.17 one of Z_{μ^+} and Z_{μ^-} is FF. But then Z_α is FF, a contradiction.

(b) Suppose L^ϵ has at most one non-central chief factor on V^ϵ. Since L^ϵ and G_{β^ϵ} are p-minimal, 2. implies $L^\epsilon = O^p(L^\epsilon)(G_{\beta^\epsilon\mu^\epsilon} \cap L^\epsilon)$ and $G_{\beta^\epsilon} = \langle G_{\beta^\epsilon\mu^\epsilon}, L^\epsilon\rangle$. Thus we can apply 8.5 to $(\mu^\epsilon, \beta^\epsilon$ in place of (α, β). Since by assumption α is not a dual FF- module we conclude that $V_\beta \leq Z_\alpha X_\beta$. But then $[V_\beta, Q_\alpha] \leq X_\beta$ and so $[V_\beta, E_\beta] \leq X_\beta$ and $[V_\beta, E_\beta] = 1$, a contradiction.

(c) Suppose that $q_a \geq \frac{1+\sqrt{17}}{4}$. Put $\Lambda^+ = \mu^{+L^+}$ and $\Lambda^- = \{\mu\}$. Abusing notation define V^+, V^-,B^+ and B^- as in that lemma. Note that V^+ is the same V^+ as defined before, but V^- now is Z_μ. Also $B^+ = V^+$ and $B^- = Z_\mu \cap O_p(L^+)$. In particular, $V^- \neq B^-$ and $V^+ = B^+$. We wish to apply 8.20e with $r = 2$ and $s = q_\alpha$. By **??** and since L^+ is p-minimial, $|Z_\mu/B^-|^2 \leq |V^+/C_{V^+}(Z_\mu)|$. Also $|V^+/B^+|^2 = 1 \leq |Z_\mu/C_{Z_\mu}(V^+)|$ and so the asumptions of 8.20e are indeed fullfiled for this choice of r and s. Also $e > 0$ by 2.1a. Thus 8.20e gives the contradiction $V^+ \neq B^+$.

$zf1$

Proposition 10.7 *There exists $1 \neq x \in Z_\alpha$ and $\lambda \in \Gamma$ with $d(\alpha, \lambda) = b$ and $Z_\alpha \not\leq O_p(C_{G_\lambda})(x)$.*

Proof: Suppose the lemma is false. Then by 10.3 $b > 1$ and we can apply 8.21. In case 8.21.1 we assume without loss that $\alpha \in \mu^{+L^+}$ with $Z_\alpha \notin O_p(L^-)$. Put $Q = O_p(L^+)$.

In case 8.21.2 we assume $\epsilon = -$ and $\alpha = \mu$. Put $Q = G_\alpha$ and $V^+ = Z_\alpha$.

In each case note that by 8.21 the assumptions of 2.8 with $H = L^-$, $V = V^-$, $A = Z_\alpha$ and $Z = Z_{\mu^-}$ are fulfilled.

$znn - .5$

(1) $V^- \cap Q \leq G_\alpha$ and $C_{V^-}(Z_\alpha) = C_{V^-}(V^+) \leq V^- \cap Q$

In case 8.21.2 there is nothing to prove. So suppose 8.21.1 holds. Then $O_p(L^+) \leq G_\alpha$ and so the first statement holds. The second follows from 2.8a.

$zn1 - 1$

(2) $[Z_\alpha \cap O_p(L^-), V^- \cap Q] = 1$

Suppose $1 \neq x \in [Z_\alpha \cap O_p(L^-), V^- \cap Q]$. Then $x \in Z_\alpha$. Thus by 2.8d, $Z_\alpha \not\leq O_p(C_L(x))$ and so also $Z_\alpha \not\leq O_p(C_{G_{\alpha'}}(x)$, a contradiction.

Since L^- has at least two non-central chief-factors on V^- and as Z_α is not FF we now compute

$$|V^-/V^-\cap Q||V^-\cap Q/C_{V^-}(V^+)| = |V^-/C_{V^-}(V^+)| = |V^-/C_{V^-}(Z_\alpha)| \geq |Z_\alpha/Z_\alpha\cap O_p(L^-)|^2 \geq |Z_\alpha/C_{Z_\alpha}(V^-\cap Q)|^2 \quad (1)$$

Hence

$$|V^-/V^- \cap Q| \geq |V^- \cap Q/C_{V^-}(V^+)|. \quad (2)$$

In case of 8.21.2 we conclude $V^- = V_{V^-}(Z_\alpha)$, a contradiction. Thus

$zn1-3$

(3) 8.21.1 holds.

In particular, the situtation is symmetric in $+$ and $-$ and $Q = O_p(L^+)$. Since by 8.21.1, L^+ has two non-central chief factor on V^+,

$$|V^+/C_{V^+}(V^-)| \geq |V^-Q/Q|^2 = |V^-/V^- \cap Q||V^-/V^- \cap Q|$$

and so by (2)

$$|V^+/C_{V^+}(V^-)| \geq |V^-/V^- \cap Q||V^- \cap Q/C_{V^-}(V^+)| = |V^-/C_{V^-}(V^+)|$$

But the same inequality holds with the roles of $+$ and $-$ are interchanged. Hence equality holds here and also in (1). But has Z_α is not FF this is only posibble if $V^- \cap Q$ centralizes Z_α. But then all the numbers compared in (1) are equal to 1 and so $V^- = C_{V^-}(V^+)$, a contradiction which completes the proof of **??**.

$ffb1$

Theorem 10.8 *Suppose G is of generic p-type, $S \in \mathrm{Syl}_p(G)$ and V is a maximal member of $\{Y_L \mid L \in \mathcal{L}(S)$. Then either V is an FF-or dual FF-module for S or $V \not\leq O_p(C_G(Z))$.*

Proof: Let $M = N_G(V)$ and $L = N_G(C_S(V))$. Then M is the unique maximal p-local of G containing L. Let $G_\alpha = L$ and H a p-minimal member of $\mathcal{L}(S)$ not contained in M. Suppose that V is neither FF nor dual FF for S. Then the assumptions of this section are fulfilled. Hence by **??** there exists a p-local subgroup H with $O_p(L) \leq H$ and $V \not\leq O_p(H)$. Choose such an H with $|H \cap M|_p$ maximal and then $|H|$-minimal. Let R be a Sylow p-subgroup of $H \cap M$ with $O_p(L) \leq R$. Since $O_p(L)$ is a Sylow $p$subgroup of $C_G(V)$, $O_p(L) = C_R(V) \trianglelefteq R$ and so $R \leq L$. Without loss $R \leq S$.

Since $O_p(L) \leq R$ and V is not FF, $J(R) \leq O_p(L)$. Hence $L \leq N_G(J(R))$ and so $N_G(J(R)) \leq M$. Thus $N_H(J(R)) \leq M$ and in particular, $N_H(R) \leq M$. Thus R is a Sylow p-subgroup of H.

Let $W = Z_H$ and suppose that $[W, V] \neq 1$. Since $W \leq O_p(H) \leq R \leq S$, $|V/C_V(W) > |W/C_W(V)$. Thus V is $F{*}1$ on W. By the minimality of H, $V \leq O_p(P)$ for all $P \in \mathcal{L}(H, S)$ with $P \neq H$ and contradiction to **??**

Hence V centralizes W By minimiality of H, $H = \langle V^H \rangle R$ and so $\Omega_1(Z(R)) = W \leq Z(H)$. Thus $V \not\leq O_p(N_G(W)$. By maximaliy if $|H \cap M|$, R is a Sylow p-subgroup of $M \cap N_G(W)$. Thus $N_S(R) \leq N_S(W) \leq R$, $R = S$ and $W = Z$. Thus the theorem is proved $\square$

$CVanV$

Lemma 10.9 *There exists* 1
$neqA \leq SC_\alpha/C_\alpha$ with

(a) $[Z_\alpha, A, A] = 1$

(b) $|Z_\alpha/C_{Z_\alpha}(A)| \leq |A^2|$.

(c) $\langle C_{Z_\alpha}(a) \mid a \in A^\# \rangle \neq Z_\alpha$.

(d) *If 8.212 holds,then* $|Z_\alpha/C_{Z_\alpha}(A)| < |A^{\frac{3}{2}}|$.

Remark: We proof contains more information than stated in the lemma Proof: Let L^ϵ, μ^ϵ and μ as in 8.21.

In case of 8.211. may assume without loss that $|V^+/C_{V^+}(V^-)| \leq |V^-/C_{V^-}(V^+)|$.Pick $\mu \in \mu^{+L^+}$ with $Z_\mu \not\leq O_p(L^-)$ and put $B^- = V^- \cap O_p(L^+)$

In case of 8.212 we assume without loss $\epsilon = -$. Put $V^+ = Z_\mu$ and $B^+ = Z_\mu \cap O_p(L^-)$.

In general pick $t \in Z_\mu \setminus O_p(L^-)$. By 8.21 the assumptions for 2.8 are fulfilled with $H = L^-$, $A = V^+$,$V = V^-$ and $Z = Z_{\mu^-}$. We conclude that $C_{V^-}(t) = C_{V^-}(V^+)$. Thus

$$\langle C_{Z_\mu}(a) \mid a \in B^- \setminus C_\mu \langle \leq Z_\mu \cap O_p(L^-).$$

Suppose now that 8.211. holds and define s by $|B^-/C_{B^-}(Z_\mu)|^s = |Z_\mu/C_{Z_\mu}(B^-)|$. Note that that $C_{B^-}(Z_\mu) \leq C_{B^-}(t) \leq C_{B^-}(V^+)$. Let c be the number of non-central chief-factors for L^+ on V^+. By 2.8 $|V^-/B^-|^c \leq |V_+/C_{V^+}(V^-)$. Then by 2.4b, (with $A = V^-, V = V^+, "s = s", t \geq 1, r \geq c \geq 2)$ we get that $s \geq 2$. Thus the lemma holds in this case with $A = B^- C_\mu/C_\mu($ and μ in place of $\alpha)$.

Suppose next that 8.212 holds. As L^- has at least two non-trivial chief-factors on V^-, we conclude from **??** that

$$|Z_\mu/B^+|^2 \leq |V^-/C_V^-(Z_\mu).$$

On the otherhand has Z_α is not FF, 2.4a implies

$$|B^+/C_{B^+}(V^-)| < |V^-/V_{V^-}(B^+)| \leq (V^-/C_{V^-}(Z_\mu).$$

Combining the last two inequalites we get $|Z_\mu/C_{Z_\mu}(V^-)| \leq |V^-/C_{V^-}(Z_\mu)|^{\frac{3}{2}}$. Hence the lamma holds also in this case with $A = V^- C_\mu/C_\mu$. □

$zair$

Lemma 10.10 *Either Z_α is irreducible as G_α module or some non-trivial chief-factor for G_α on Z_α is FF.*

Proof: Since $[Z, E_\beta] = 1$, $C_{Z_\alpha}(E_\alpha)$. Since Z_α is CS-generated, we conclude $Z_\alpha = [Z_\alpha, E_\alpha]$. So if G_α a unique non-central chief-factor, Z_α is irreducible. If Z_α has more than one non-central chief-factor, then as Z_α is $F2$ and G_α is p-connected, at least one chief-factor is FF. □

Proposition 10.11 *Let U be a non-trivial chief-factor for E_α on Z_α. Let $E = E_\alpha/C_{E_a}(U)$. Then one of the following holds:*

1. *E is solvable and one of the following holds:*

   1.1. *$p = 2$, $E \cong C_3$ and $|U| = 2^2$.*

   1.2. *$p = 3$, $E \cong Q_8$ and $|U| = 3^2$.*

   1.3. *$p = 2$, $E \cong C_5$ and $|U| = 2^4$.*

   1.4. *$p = 2$, $E \cong \mathrm{Ext}(3^{1+2})$ and $|U| = 2^6$.*

   1.5. *$p = 3$, $E \cong \mathrm{Ext}(2_+^{1+4})$ and $|U| = 3^4$.*

2. *E is perfect but $\mathrm{Sol}(E) \not\leq Z(E)$ and one of the folloing holds.*

   2.1. *$p = 2$, $E = (C_3 \wr Alt(n))'$, $n \geq 5$ and $|U| = 2^{2n}$.*

   2.2. *$p = 3$, $E = \mathrm{Ext}(2_-^{1+4}).Alt(5)$ and $|U| = 3^4$.*

3. *E is quasisimple and one of the following holds.*

   3.1. *E is group of Lie type in charcateristic p.*

   3.2. *$p = 2$ and $E/Z(E)$ is an alternating group.*

   3.3. *$p = 2$, $E \cong 3{\cdot}U_4(3)$ and $|U| = 2^{12}$.*

4. *$E = E_1E_2$ for some components E_1, E_2 of E, E_1 and E_2 are isomorphic groups of Lie type in charactaristic p, $U = U_1 \otimes U_2$ for some U_i module E_i such that (E_1, U_1) and (E_2, U_2) isomorphic. Moreover, if n is the dimension of U_i over $\mathrm{End}_{E_i}(U_i)$ then U_i is a quadratic $F\frac{2}{n}$-module for E_i.*

Proof: Let W be a non-trivial chief-factor for G_α on Z_α. By 10.9 Z_α is quadratic $F2$ and since G_α is p-connected, W is quadratic F_2. Let Let $H = G_\alpha/C_{G_\alpha}(W)$ and $L = \langle \mathrm{PQ}_2^*(G_\alpha/C_{G_\alpha}(V), V)$. As p-connected $O^p(H) \leq L$. Let V be a Wedderburn component for H on W. Since $N_H(V)$ is irreducible on V and $N_H(V)/L$ is a p-group, V is irreducible for L. Hence we can apply 6.11 to $\overline{L} = L/C_L(V)$. In particular we see that (except in case 6.114 with $p = 2$) $O^p(L)$ is irreducible on V and clearly any chiefactor for E_α on Z_α arises in such a way. Moreover, since G_α is p-connected, Case 8 of 6.11 does not arise and in case 9, $C_L(\Delta)$ is a 3-group. Thus it remains to show that in cases 10, 11 the componets of L are groups of Lie type or $E(L)/Z(E(L))$ is an alternating group. But this is clear in case 11 and so we may assume that $E(L)$ is quasi simple and neither an alternating group, a group of Lietype in characteristic p nor $3{\cdot}U_4(3)$

Then G_α has no FF-module and so W is the unique non-trivial composition factor for G_α on Z_α and as $Z \trianglelefteq G_\beta$ we get that Z_α is irreducible. We conclude that $E_\alpha C_\alpha/C_a$ the central product of its components $L^(i), 1 \leq 1 \leq n$ and Z_α the the direct sum of the $Z_\alpha^i = [Z_\alpha, L^(i)]$. By 6.15b $L^(i)$ is isomorphic to $3{\cdot}.Mat_{22}$

Let A be as in 10.9 and put $X = \langle C_{Z_\alpha}(a) \mid a \in A^\# \rangle \neq Z_\alpha$. Pick $V = Z^{(}_\alpha i)$ so that $V \not\leq X$ and pick $t \in V \setminus X$. Then $C_A(t) = 1$ and so A acts faithfully, quadratic and $F2$ on V. Thus by 6.15b, $A \geq 2^3$ and 6.152.3 or 2.4 hold. Let $a \in A^\#$. Then $C_V(a) \neq C_V(A)$ and so $C_V(A) < X \cap V < V$. Since $X \cap V$ is invariant under $N_{G_\alpha}(A)$ we conclude that case 2.4 with $|A| = 2^3$ holds. Note that V is actually a 6-dimensional space $GF(4)$. Each $a \in A^\#$ $C_V(a)/C_V(a)$ is 1-dimensional over $GF(4)$ and differnt a's give different 1-spaces. Hence $X/C_V(A)$ contains 7 different $GF(4)$-1-spaces and so $X = V$, a contradiction. □

11 The case $b = 1$ with G_α connected and G_β minimal

b1c

Hypothesis 11.1 (a) *Hypothesis 8.1 holds, except for the $S \leq G_\alpha \cap G_\beta$ we only assume $Q_\alpha \leq S$ and $S \in \mathrm{Syl}_p(G_\beta)$.* *hyb1c*

(b) *G_α is p-connected.*

(c) *$b = 1$, that is $Z_\alpha \not\leq Q_\beta$.*

dVMab

Definition 11.2 (a) *V is a normal subgroup of G_β minimal with respect to $[V, E_\beta] \neq 1$.*

(b) *$M_{\alpha\beta}$ is the unique maximal subgroup of G_β containing S.*

qmzff

Lemma 11.3 *Suppose that G_β is p-minimal. Then either $[Q_\alpha, E_\alpha] \leq Z_\alpha$ or Q_a/Z_a has a unique non-central chief-factor and that chief-factor is FF.*

Proof: Let $D = [V, Q_\beta]$. Then $[D, E_\beta] = 1$. Also note that $V = [V, E_\beta]$ and since $E_\beta \leq \langle Z_\alpha^{G_\beta} \rangle$ we conclude that $V = \langle [V, Z_\alpha]^{G_\beta} \rangle$. Thus $D = \langle [V, Z_\alpha, Q_\beta]^{G_\beta} \rangle$. Since $[V, Z_\alpha, Q_\beta]$ is normalized by $SE_\beta = G_\beta$ we conclude that $D = [V, Z_\alpha, Q_\beta] \leq Z_\alpha$. Let $\overline{V} = V/D$. Then $[V, Z_\alpha, Q_\alpha] \leq [Z_\alpha, Q_\alpha] = 1$. So let R be maximal in Q_α with $[\overline{V}, R] \leq [\overline{V}, Z_\alpha]$. Then by 6.18,

$$|Q_\alpha/R| \leq |\overline{V}/C_{\overline{V}}(Z_\alpha)| \leq |V/C_V(Z_\alpha) = |VQ_\alpha/Q_\alpha|$$

Also $[R, V] \leq [V, Z_\alpha]D \leq Z_\alpha$. Let $\tilde{Q}_\alpha = Q_\alpha/Z_\alpha$, we conclude

$$|\tilde{Q}_\alpha/C_{\tilde{Q}_\alpha}(V)| \leq |VQ_\alpha/Q_\alpha|.$$

.

Futhermore, $[V, Z_\alpha] \neq 1$ and so $V \not\leq Q_\alpha$. It remains to show that G_α has at most one non-central chief-factor on $\tilde{Q}_\alpha$. So suppose $[\tilde{Q}\alpha, E_\alpha] \neq 1$ and let P be a normal subgroup of G_α minimal with respect to $[P, E_\alpha] \not\leq Z_\alpha$. Then $[P, V] \not\leq Z_\alpha]$ and so $P \not\leq R$. By 6.18, we conclude $[\overline{V}, P] = [\overline{V}, Q_\alpha]$ and so $[Q_\alpha, V] \leq [P, V_\alpha] \leq P$. Hence $[Q_\alpha, E_\alpha] \leq P$ and the lemma is proved. □

zaf2

Lemma 11.4 *Z_α is a cubic $F2$-module for G_α.*

Proof: Remark: 1. There should be a much nicer proof which does not go through the list of finite simple groups
2. The structure of L has determined in proof should be recorded as an independent lemma

Assume that Z_α is not FF and let L be minimal such that

(i) $Z_\alpha \leq L$.

(ii) $Z_\alpha \not\leq O_p(L)$.

(iii) $G_\alpha \cap L$ contains a Sylow p-subgroup T of L.

(iv) $C_L(O_p(L)) \leq O_p(L)$.

By minimality of L, $L = \langle Z_\alpha^L \rangle$. Let R be a normal subgroup of L with $L \neq RZ_\alpha$. Then again by minimality $Z_\alpha \leq O_p(RZ_0)$. Thus $[R, Z_\alpha] \leq O_p(R) \leq O_p(L)$ and $[R, L] \leq O_p(L)$. In particular L is p-connected. Let V be a non-central chief-factor for L on $O_p(L)$. Since $O_p(L) \leq T \leq G_\alpha$, Z_α acts quadratically on $O_p(L)$ and so also on W. Let $\tilde{L} = L/C_L(W)$. If $|\tilde{Z}_\alpha| = 2$, then $L/O_2(L)$ is a dihedral group. If $|\tilde{Z}_\alpha| \geq 3$, we can apply 6.17 to $\tilde{L}$ and W. So in any case we conclude that one of the following holds (where we used the minimalty of L to rule out some of the cases)

1. $p = 2$ and $\tilde{L} \cong Dih(2r)$, r and odd prime.

2. $F^*(\tilde{L})$ is quasisimple.

3. $p = 3$ and $\tilde{L} \cong SL_2(3)$.

Suppose first that Z_α lies in a unique maximal subgroup M of L. Put

Put $A = Z_\alpha$, $B = A \cap O_p(L)$ and $Q = \langle B^L$. Let $l \in L \setminus M$. Then $L = \langle A, A^l \rangle$ and so as $[Q, A] \leq B$, $Q = BB^l$. Moreover, $B \cap B^l = C_{B^l}(L) = C_{B^l}(A)$. And so

$$B^l/C_B^l(A) = B^l/B \cap B^l = |Q/B| = |Q/C_Q(A)| \geq |AQ/Q| = |A/B|$$

where the last inequality holds has L is $F*1$-modules.

Now $|B/C_B(B^l)| \leq |B/C_B(A^l)| = |B/B \cap B^l = |B^l/C_{B^l}(A)$.

Hence B^l is $F2$ on A. Since $[A, B^l] \leq Q$ and B^l is quadratic on Q, B^l is cubic on A. Thus the lemma holds in this case.

So we may assume form now on that A lies in more one one maximal subgroup of L. In particular, $K = F^*(\tilde{L})$ is quasi simple. Let $T \leq M < L$. Then by minimality of L, $A \leq O_p(M) \leq T$. Put $Q_M = \langle A^M \rangle$. If Q_M is not abelian, then $[A, A^m] \neq 1$ for some $m \in M$. But then A is FF on A^l or A^l is FF on A, a contradiction. Hence Q_M is abelain for all such M and so acts quadratically on Q. Let $1 \neq \tilde{a} \in \tilde{A} \cap Z(\tilde{T})$. We conclude

(1) $\tilde{A}$ lies in an abelian normal subgroup of $C_{\tilde{L}}(\tilde{a})$ which acts quadratically on Q. $zaf2-1$

Suppose next K is not a group of Lie type in characteristic p. Then $p = 2$ or 3. If $p = 3$, then $|\tilde{A}| = 3$ and $\tilde{A}$ lies subgroup of L is morphic to $SL_2(3)$, a contradiction to the minimality of L. So $p = 2$. Since $|\tilde{A}| \geq 2$, 6.15 and (1) apply $\tilde{L} \cong 3{}^{\cdot}\mathrm{Mat}_{22}, \mathrm{Aut}(\mathrm{Mat}(22))$ or Mat_{24}. But in each of these cases there exists a overgroup of $\tilde{M}$ which does not have a non trivial quadratic normal subgroup.

We conclude

$zaf2-2$

(2) L is a group of Lie type in characteristic p of rank at least two.

Suppose that $\tilde{A}$ is contained in a root group X if $\tilde{A}$. Then $X \leq T$ and X the Sylow subgroup of some $(S)L_2(q)$ in $\tilde{L}$. But this contradcits the minimal choice of L. Hence $\tilde{A}$ is not contained in a root group. By (1) and as A is contained in $O_p(M)$ for all $T \leq M \leq L$ we conclude that $p = 2$, $L \cong Sp_{2n}(q)$ or $F_4(q)$ and $A \leq Z(T)$. The minimality of L implies $L \cong Sp_4(q)$. But $Sp_4(q)$ has no module on which the O_p's of both parabolic acts quadratically. □

12 Elementary results on p-connected groups

$p-con$
dcn

Definition 12.1 *$\mathcal{N}(S)$ is the set of all p-connected $L \in \mathcal{L}(S)$ wh*

Remark: change this to $\mathcal{N}^*$ and use $\mathcal{N}$ for $\mathcal{P} \cup \mathcal{E}$

NS

Lemma 12.2 *Let $L \in \mathcal{L}(S)$. Put $E = O^2(L)$. Then L is in $\mathcal{N}(S)$ if and only if one of the following holds:*

1. *L is solvable, $E/O_2(E)$ has odd order and for all maximal S invariant normal subgroups N of E, $C_S(E/N) = O_2(L)$.*

1. *E is perfect, and $E/O_{2,2'}(E)$ is the direct product of simple groups which are transitively permuted by S.*

Proof: It is trivial to verify that (1) and also (2) imply $L \in \mathcal{N}(S)$. So assume now that $L \in \mathcal{N}(S)$ and let K be the unique maximal normal subgroup of E with $K/O_2(K)$ of odd order. Note that $O_2(E) \leq K$ and by the odd order theorem, K is solvable .

Suppose first that $K = E$. Let and let N be a maximal S invariant normal subgroup of E. Then $NC_S(E/N)$ is normalized by $ES = L$. Since $E \not\leq NC_S(E/N)$ we conclude that $C_S(N) \leq O_2(L)$. Thus (1) holds in this case.

Suppose next that $E \neq K$ and let E^*/K be a minimal L invariant subgroup of E/K. Then E^*/K does not have odd order, $S \cap E^* \not\leq K$, $S \cap E^* \not\leq O_2(L)$ and so $E \leq E^*$ and $E = E^*$. As $E = O^2(E)$, E/K is not a 2-group and so E/K is not solvable. Thus E/K is the direct product of simple groups transitively permuted by S. Since $E' \cap S \not\leq O_2(L)$, $E = E'$.

The following is an extended version of a lemma from [St2] which describes the structure of rank 2 groups.

gsr2

Lemma 12.3 *Let $P_1, P_2 \in \mathcal{N}(S)$. Put $L = \langle P_1, P_2\rangle$. Let L_0 be a normal subgroup of L maximal with respect to $O^2(P_i) \not\leq L_0$ for $i = 1$ and $i = 2$. Let L_1/L_0 be a minimal normal subgroup of L/L_0. Then* **Remark: change L_1, L_0 notation**

(a) *$S \cap N = O_2(L)$ and $L_0/O_2(L)$ has odd order.*

(b) *Let $O^2(P_i) \leq L_1$ for at least one $i \in \{1, 2\}$.*

(c) *If $O^2(P_j) \not\leq L_1$, then $P_j \leq N_L(L_1 \cap S)$ and $O_2(O^2(P_i)) \leq O_2(P_j)$.*

(d) *Suppose that $L_1/L_0 = E_1 \times E_2 \times \ldots \times E_r$ is the direct product of alternating groups or simple groups of Lie type in characteristic 2. Then P_j acts transitively on the E_l's and one of the following holds:*

(d.1) *$O^2(P_j) \not\leq L_1$ and $O^2(P_i)L_0/L_0$ is the product of some of the E_l's.*

(d.2) *$O^2(P_j) \not\leq L_1$, $E_1 \cong D_4(q)$ and some element on P_j induces a graph automorphism of order 3 on E_1*

(d.3) *$O^2(P_j) \leq L_1$, $j = 1, 2$, $L = L_1S = \langle E_1^S\rangle S$ and $E_1 = \langle E_1 \cap P_1, E_1 \cap P_2\rangle$.(modulo L_0)*

Proof: As $O_2(L)L_0 \cap P_i = O_2(L)(L_0 \cap P_i) \leq O_2(P_i)$ the maximality ofL_0 implies $O_2(L) \leq L_0$. Let N be a normal subgroup of L and $k \in \{1, 2\}$.

We next prove that

(1) Suppose that $S \cap N \leq O_2(P_k)$. Then P_k normalizes $S \cap N$. $gsr2-1$

Indeed this is clear as $S \cap N = O_2(P_k) \cap N$ in this case.

(2) If $O^2(P_k) \not\leq N$, then P_k normalizes $S \cap N$ $gsr2-2$

As $O^2(P_k) \not\leq N$ we have $S \cap N \leq O_2(P_k)$ and so (2) follows from (1).

By definition of $\mathcal{N}(S)$ and $O^2(P_i) \not\leq L_0$ we have $S \cap L_0 \leq O_2(P_i)$. By (1) applied to $N = L_0$ and $k = 1, 2$ we conclude that $L_0 \cap S$ is normal in $L = \langle P_1, P_2\rangle$ and so (a) holds. (b) follws from the maximal choice of L_0. The first part of (c) follows from (2) while the second follows from the first.

To prove (d) we assume without loss that $L_0 = 1$. Note that $P_i \cap L_1$ is a parabolic subgroup of L_1 and $P_i = (P_i \cap L_1)S$. Thus either P_i normalizes $S \cap L_1$ or we may choose notation so that $P_i = ((P_i \cap E_1) \times \ldots (P_i \cap E_l))S$, where $P_i \cap E_1$ is a parabolic of E_1 with $O^{2'}(P_i \cap E_1) = P_i \cap E_1$.

Suppose now that $O^2(P_j) \not\leq L_1$. Pick E_1 so that $S \cap N_L(E_1)$ is a Sylow 2-subgroup of $N_L(E_1)$. Then as $L_1 \cap S$ is not normal in L, (c) implies that P_i does not normalise $L \cap S$. If $E_1 \leq P_i$, (d.1) holds. So we may assume that $P_i \cap E_1$ is a proper parabolic subgroup of E_1. Suppose that (d.2) does not hold and that E_1 is a group of Lie type in characteristic two.

Then no element of odd order in $N_{P_j}(E_1)$ induces a non-trivial graph automorphism on E_1 and so $O^2(N_G(P_j))$ normalizes $P_i \cap S$. Hence $N_G(P_j) = O^2(N_G(P_j))(N_S(E_1))$ normalizes $P_i \cap E_1$ and so $L \neq \langle (P_i \cap E_1)^{P_j} P_j = \langle P_1, P_2 \rangle$, a contradiction. If E_1 is an alternating of degree at least six, then $N_{Aut(E_1)}(S \cap E_1)$ is a 2-group and we obtain a similar contradiction.

So assume now that $O^2(P_j) \leq L_1$ for $j = 1, 2$. Then it is easy to verify that (d.3) holds.

13 Establishing Geometries

EG

Throughout this section we assume

(i) $U_0, U_1 \in \mathcal{N}(S)$

Remark: redefine $\mathcal{N}$ as $\mathcal{P} \cup \mathcal{E}$?

(ii) all non-abelian composition factors of elements of $\mathcal{L}(S)$ are alternating groups, rank one group of Lie type over $GF(q)$, $G_2(q)$'s or classical groups over $GF(q)$, where q is a power of two.

(ii) $U_0 \not\leq U_1$ and $U_1 \not\leq U_0$.

compfact

Lemma 13.1 *Let $H \leq G$ with $F^*(H) = O_2(H)$ and $|S/S \cap H| \leq 2$. Then all non-abelian composition factors of elements of $\mathcal{L}(S)$ are alternating groups, rank one group of Lie type over $GF(q)$, $G_2(q)$'s or classical groups over $GF(q)$, where q is a power of two.*

Proof: By 2.10 we may assume that $H \leq L^* \in \mathcal{L}(S)$. Hence the claim follows from 2.12.

O2Ln1

Lemma$_{QT}$ 13.2 *Put $L = \langle U_1, U_2 \rangle$ and suppose that $L \in \mathcal{L}(S)$. Then the L_0 and L_1 in 12.3 and $\{i, j\} = \{0, 1\}$ can be chosen so that one of the following holds*

1. $[O^2(U_0), O^2(U_1) \leq Q$.
2. *L is not solvable and $L \in \mathcal{N}(S)$.*
3. *$O^2(L)O_2(L)/O_2(L)$ is a p-group for some prime odd p.*
4. *L_i is a $\{2, p\}$-group for some prime p, $O^2(P_i) \leq L_1$ and L_1/L_0 is an elementary abelian p-group. Moreover, there exists an odd prime $q \neq p$ so that the image of $O^2(P_j)$ in $\mathrm{Aut}(L_1/L_0)$ has one of the following shapes: cyclic q group with $q \mid p^4 - 1$; homocyclic q group of rank 2 with $q \mid p - 1$; $Ext(3^{1+2})$ with $p \neq 3$; $Ext_-(2^{1+4}).5$; $Ext_-(2^{1+4}).Alt(5)$; $Alt(4)$, $2^{\cdot}Alt(n), n = 4, 5$; $2^{\cdot}Alt(4) \times 2^{\cdot}Alt(4)$; $2^{\cdot}Alt(5) \times 2^{\cdot}Alt(5)$, $p \equiv 0, 1, 4(5)$; $2^{\cdot}Alt(6)$; $2^{\cdot}Alt(7)$ (with $p = 7$); $Alt(5)$; $L_3(2)$ or $3^{\cdot}Alt(6)$.*
5. *U_i induces $Sym(3)$ on the set of components of L_1/L_0, U_j is the product of one or two 2-components of L_1 and $U_i/O_2(U_i) \cong Dih_{2 \cdot 3^l}$.*
6. *$O^2(U_i)$ acts trivially on the set of components of L_1/L_0, $U_i/O_2(P_i)$ is a dihedral group, U_i normalizes $O^2(U_j)$, and $O^2(U_j) = E_2(L_1)$. Moreover, $O_2(U_j) = O_2(L)$.*

Remark: The case that $O^2(U_i) \leq L_1$ for $i = 0$ and 1 and L_1/L_0 is a direct product of perfect simple groups still needs some attention: one needs to show that L_1/L_O is "central" (and this should be possible) and also things $L/O_2(L) \cong C_3 \times Alt(5).2$ arise here, this is covered by case 6. But $O^2(U_i)$ induces inner automorphism on $O^2(U_j)$. So this probably should be listed as a seperate case, but it is also kind of the same as 1.

Proof: Remark: numbering and notation needs to be updated

We use the results and notation of 12.3. As $m_{2'}(L) \leq 3$, case d.2 in 12.3 is not possible. Put $D = C_L(L_1/L_0)$.

Suppose first that L_1/L_0 is not solvable. Then $O^2(U) \leq L_1$. If $D \neq L_0$ we get $D \cap L_1 = L_0$ and by maximality L_O, $O^2(P) \leq D$. Thus $O^2(U), O^2(P) \leq L_O$. In this case we replace L_1 by $O^2(P)L_O$. So we may assume that $D = L_0$. As $m_{2'}(L) \leq 3$, $r \leq 3$

Assume in addition that $O^2(P) \leq L_1$. As P is solvable, d.1 is impossible. Thus d.3 holds. Moreover, $L = L_1 S$ and so $O^2(L) \leq L_1$ thus 4. holds in this case.

So assume that $O^2(P) \not\leq L_1$.

If $O^2(P)$ does not act trivially on the set of components of L_1/L_0 we conclude that $r = 3$ and P induces $Sym(3)$ on the set of components of L_1/L_0. As $e(G) \leq 3$ and L_1/L_0 has three components, $[L_1^\infty, L_0] \leq O_2(L)]$. Thus 5. holds.

So suppose that $O^2(P)$ acts trivially on the set of components of L_1/L_0. The S acts transitively thereon and $r \leq 2$. If $r = 2$, then $O^2(U) = E_2(L_1)$. Since $e(G) \leq 3$ we have E_1 is $L_2(q), Sz(q), L_3(4), L_3(2), Alt(6), Alt(7)$. But in the last three cases $\mathrm{Out}(E_1$ is a 2-group, a contradiction. In the first two cases, $\mathrm{Out}(E_1)$ is cyclic and so PL_1/L_1 is a dihedral group. If $E_1 \cong L_3(4)$, then $O^2(U)O_2(L)/O_2(L) \cong SL_3(4) * SL_3(4)$. Since the action of $Aut(L_1/L_O)$ on $\mathrm{Out}(L_1/L_0)$ on the 3-part of the Schur multiplier respectively the outer automorphisms of L_1/L_O are isomorphic we conclude that S does not act irreducibly on $O_3(\mathrm{Out}(L_1/L_0)$ and so $O^2(P)L_1/L_1 \cong C_3$ and so again $P/O_2(L)$ is a dihedral group. Thus 6. holds

If $r = 1$ we conclude that PL_1/L_1 is isomorphic to a subgroup of $\mathrm{Out}(E_1)$ and so $Out(E_1)$ is not abelian. Hence $E_1 \cong U_3(q), U_4(q), L_3(q)$ and $P/O_2(P)$ is a dihedral group and 6. holds.

Assume now that L_1 is solvable.

Suppose that L_2/L_O is a minimal normal subgroup of L/L_0 different from L_1/L_O. Then we may choose notation so that $O^2(P) \leq L_1$ and $O^2(U) \leq L_2$. Then $[O^2(P), O^2(U)] \leq L_0$, $L_1 = O^2(P)L_O$ and $L_2 = O^2(U)L_0$.

Suppose that $O^2(U) \leq L_1$. Then by assumption L_1/L_O is an elementary abelian 3-group.

TO BE CONTINUED

Corollary 13.3 *Assume that*

(i) $U_0 \in \mathcal{P}(S)$

(ii) *If* $U_1 \in \mathcal{P}(S)$ *and* U_1 *is solvable then* U_1 *is a* $\{2,3\}$*-group.*

(iii) $L \overset{def}{=} \langle U_0, U_1 \rangle \in \mathcal{L}(S)$.

Then one of the following holds

TO BE CONTINUED

$O2L1$

Lemma$_{QT}$ **13.4** *Suppose that*

(i) $E \in \mathcal{E}(S) \setminus \mathcal{P}(S)$.

(ii) $O_2(\langle U_1, E\rangle) = 1$.

(iii) *For all* $U^* \in \mathcal{N}(E,S)$ *with* $U^* \neq E$, $\langle U_1, U^*\rangle \in \mathcal{L}(S)$

(iv) *There exists a maximal element* $U_1 \in \mathcal{N}(E,S)$ *so that one of the cases 3-6 in 13.2 holds.*

Then one of the following holds for $L(1) = \langle U_0, U_1\rangle$.

1. U_1 *is solvable.*
2. $\mathrm{Head}(U_1) \cong L_2(q)^r$, $r \leq 2, q \geq 4$; $U_O/O_2(U_O) \cong D_{2\cdot 3^k}$, $\mathrm{Head}(L_1(1) \cong L_2(q)^3$ *and* $O^2(U_O)$ *transitively permutes the three 2-components of* $L(1)$
3. $O^2(U_1)/O_2(U_1) \cong Alt(5)$, $\mathrm{Head}(E) \cong U_4(2)$ *and* $O^2(U_0) \leq O_{2,p}(L(1))$, *p a prime with* $p > 3$. *Moreover, if* **TO BE CONTINUED**
4. *Put* $R_1 = O^2(U_1)O_2(U_1)$. *Then*

   (a) U_O *normalizes* R_1 *and no non-trivial characteristic subgroup of* R_1 *is normal in* E.

   (b) *One of the following holds*

      1. $\mathrm{Head}(E) \cong U_4(2)$, $U_O/O_2(U_O) \cong D_{2\cdot 3^k}$ *and* $\mathrm{Head}(U_1) \cong Alt(5)$.
      2. *There exists a maximal element* U_2 *of* $\mathcal{N}(E,S)$ *which fulfils 3. with* U_2 *in place of* U_1.

Remark: Case 4b1 is impossible by a trivial pushing up argument (or by quoting pushing up)

Proof: Let $\mathcal{N}$ be the set of proper maximal elements $U^* \in \mathcal{N}(E,S)$. We assume without loss that U_1 is not solvable.

By 8.2 there exists U_2 in $\mathcal{N}$ so that $\langle U_1, U_2\rangle = E$. Under all these U_2's with pick one which (possibly trivial) 2-component K with $K/O_2(K)|$ maximal.

In particular $O^2(E) = \langle O^2(U_1), O^2(U_2)\rangle$. For $i = 1,2$ let $L(t) = \langle U_O, U_i$. We will apply 13.2 to $L(1)$ and $L(2)$. We write Case t(i) if Case t in 13.2 holds for $L(1)$. For $i = 0,1,2$ put $Q_i^* = [O_2(U_i), O^2(U_i)]$. The next two statement follow immediately from 13.2 applied to $L(1)$.

(1) U_O is solvable and $O^2(U_1)/O_2(O^2(U_1)) \not\cong Alt(n)$ for $8 \le n \le 11$. $O2L1-1$

(2) One of the following holds: $O2L1-2$

- Case 4(1) with $(i,j) = (O,1)$ and $\mathrm{Head}(U_1) \cong Alt(5), Alt(6), 3\cdot Alt(6), Alt(7)$ or $L_3(2)$
- Case 5(1) with $(i,j) = (O,1)$ and $\mathrm{Head}(U_1) \cong L_2(k)^r$ or $L_3(2)^r$, with $r \le 2$.
- Case 6(1) with $(i,j) = (O,1)$

By 8.2, the second statement in (1) and as U_1 is not solvable we can choose U_2 so that $U_1 \cap U_2$ is a maximal parabolic of U_1.

Remark: this needs to be proved very carefully for the the symmetric groups

Next we prove

(3) In Case 1(2), 5(1) holds. $O2L1-3$

As we are in case 1(2), $[O^2(U_O), O^2(U_2)]$ is a 2-group. Hence also $[O^2(U_O), U_1 \cap O^2(U_2)]$ is a 2-group. On the other hand in case 4(1), $U_1 \cap O^2(U_2)$ acts fixed point freely on $L_1(1)/L_0(1)$, a contradiction. In case 6(1) $O^2(U_0)$ normalizes $O^2(U_1)$ and $O^2(U_2)$, again a contradiction. Thus case 5(1) holds.

(4) In Case 4(1), Case 4(2) holds. $O2L1-4$

By (3) we may assume that Case 2(2),3(2), 5(2) or 6(2) holds. As P_O is solvable, we get in case 2(2), 3(2) and 5(2) that P_0 is a $2,3$-group a contradiction. Hence Case 6(2) holds, $\mathrm{Head}(U_O)$ is cyclic and $O^2(P_0)$ induces field or diagonal automorphism of odd order larger than 3 on $O^2(U_2)/O_2(O^2(U_2)$. But this contradicts the structure of U_1 and E.

(5) If Case 4(1) and Case 4(2) holds, 3. holds $O2L1-5$

Considering the action of Q_2^* on $L_1(1)/L_0(1)$ we see that $[O^2(U_O), Q_2^*] = O^2(U_0)$ **Remark: more details please** . Hence $O^2(U_2) \not\le L_1(2)$ and so $O^2(U_O) \le L_1(2)$. Moreover, $Q_2^* \not\le O_2(L(2)$. Hence either U_2 is solvable or acts as $Ext2^{1+4}.A_5$ on $L_1(2)/L_0(2)$. In the latter we get $L_1(2) \le P_0 \le L(2)$ and then $L_1(1) = L_2(1)$, a contradiction. Thus U_2 is solvable and so $U_2/O_2(U_2) \cong Sym(3)$ or $Sym(3) \wr C_2$.

In the latter case, $[L_1(2)/L_0(2), Q_2^* \neq 1$ implies that S acts irreducible on $[L_1(2)/L_0(2)]$. But then $L_1(2) \le P_0 \le L_1(1)$, a contradcition.

Thus $U_2/O_2(U_2) \cong Sym(3)$ and as U_1 is not solvable we conclude that $\mathrm{Head}(E) \cong U_4(2)$. Hence 3. holds.

(6) In case 5(1), 2.holds. $O2L1-6$

We may assume that Head(U_1) $\cong L_3(2)^r$, $r = 1,2$. If $r = 1$ and U_1 induces no graph automorphism on Head(U_1), then Head(E) $\cong L_4(2), Sp_6(2), \Omega_8^-(2)$ or $(3^{\cdot})Alt(7)$. If $r = 1$ and U_1 induces a graph automorphism on Head(U_1), then Head(E) $\cong L_5(2)$. If $r = 2$ then now element of U_1 induces a graph automorphism on Head(U_1) and Head(E) $\cong L_6(2), L_7(2)$ or $3^{\cdot}(Alt(7) \times Alt(7)$. Let K be the normaliser in U_1 of some 2-component of U_1 and $P \in \mathcal{P}(K, S \cap T)$. Then $|S/S \cap P| \leq 2$. Let $H_O = N_{L(1)}(O^2(P)$, $H_1 = N_E(O^2(P))$ and $H = \langle H_1, H_2$. Then Head($H_O/O^2(P)$) $\cong L_3(2) \times L_3(2)$. Moreover we can and do choose P so that $H_1 \not\leq L(1)$ and so $H \neq H_1$. As $m_3(H) \leq 3$ and $O^2(P)O_2(H)/O_2(H)$ is a normal subgroup of order three in H. By 4.10 we conclude that $H^\infty/O_{2,2'}(H^\infty) \cong L_3(2) \times L_3(2)$ or $L_3(2) \times Alt(7)$. In the first case each minimal parabolics of H is either contained in H_0 or is solvable and not a $\{2,3\}$-group, a contradition to $H_1 \not\leq H_O$. In the second case H has a 2-component R with Head(R) $\cong 3^{\cdot}Alt(7)$, $O_{2,3}(R) \leq P$ and Head($R \cap H_1$) $\cong C_3 \times L_3(2)$. It follows that $P \cap K$ induces a group of automorphisms on $3^{\cdot}Alt(7)$(= Head(R)) which inverts the central three but centralizes an $L_3(2)$ subgroup, a contradiction.

$O2L1-7$

(7) In case 6(1),4. holds.

By case 6(1) $O_2(U_1) = O_2(L(1))$ and U_0 normalizes $O^2(U_1)$. Thus the first statement in 4. holds. As U_1 induces diagonal or field automorphism of odd order on Head(U_1), E is not a group of Lie type in over the field of 2-elements, except maybe $U_4(2)$.

Suppose first that U_2 is solvable. Then Head(E) $\cong U_4(2)$, Head(U_1) $\cong Alt(5)$ and so 4b1 holds.

Suppose next that U_2 is not solvable.In case 1(2) or 6(2), P_O normalizes $O^2(U_2)$, a contradiction as P_0 already normalizes $O^2(U_1)$. Suppose Case 2(2) holds. As U_O is solvable, we conclude that Head($L_1(2)$) $\cong U_4(2)$. Let $Q = O_2(U_2)$. In E we see that Q induces inner automorphism on Head(U_1), in $L(2)$ we see that Q inverts Head(U_0) and in $L(1)$ we see that every element that inverts Head(U_0) induces an outer automorphism on Head(U_1), a contradiction.

Hence we may assume that one of 4(2) or 5(2) holds. In particular, U_2 in place of U_1 fulfils the assumption of this lemma and so by (4) and (6) applied with U_1 and U_2 interchanged 5(2) we get that case 5(2) holds and Head(U_2) $\cong L_2(q)^r$, $r \leq 2$. Thus 4b2 holds. **Remark: I forget to think about** $3^{\cdot}Alt(6)$ **for** Head(U_1)**. This might arise for** Head(E) $= 3^{\cdot}Alt(7)$

$nl2ws3$

Lemma$_{QT}$ **13.5** *Retain the assumptions of 13.4 and assume that 13.4.2 holds. Then one of the following holds:*

a. *troet*

Proof:

$nl2ws3-1$

(1) (a) If $r = 1$, then Head(H) $\cong (3^{\cdot})Alt(7)$ (with $q = 4$); $Alt(10)$ (with $q = 4$); $(S)L_3(q)$; $Sp_4(q)$; $G_2(q)$; $U_4(q)$; $U_4(\sqrt{q})$; or $L_4(q)$ (with S inducing a graph automorphismus).

(b) If $r = 2$ then Head$(H) \cong 3^{\cdot}(L_3(4) \times L_3(4))$ (with $q = 4$), $3^{\cdot}(Alt(7) \times Alt(7)$ with $q = 4$) or $L_4(q)$(with S inducing a graph automorphism).

(c) Let $H \in \mathcal{L}(S)$ with $L_1(1)S \leq H$. Then Head$(H^\infty) = H_1 \cdot H_2 \cdot H_3$, where S normalizes H_1 and interchanges H_2 and H_3, for $1 \leq i \leq 2$, $H_i/O(H_i) \cong (2^{\cdot})Alt(5)$ and $O(H_0)$ and $O(H_1)$ have coprime order.

This follows easily from 4.10

Let K_1, K_2, K_3 be three different 2-components of $L(1)$ with $K_1 \leq U_1$. Put $K = K_1K_2K_3$. Let $\{i, j, k\} = \{1, 2, 3\}$. Put $H^i = N_G(K_i)$ and $K^i_j = \langle K_j^{H^{i\infty}}$. As $L(1) \leq H^i$ and H^i contains a Sylow 2-subgroup of G we can apply (1)c and conclude that K^i_k normalizes K^i_j and K_j). Hence $K^i_k \leq H^j$ and $K^i_k \leq K^j_k$. By symmetry $K^j_k \leq K^i_k$ and so $K^*_k \stackrel{=}{def} K^j_k = K^i_k$. In particular K^*_i normalizes K^*_j and the K^*_j's are pairwise isomorphic. By (1)c applied to $K^*_1K^*_2K^*_2S$ we conclude that $O_22'(K^*_i) = O_2(K_i)$ and so $K^*_i = K_i$. It follows that

$nl2ws3-2$

(2) Put $L = N_G(K)$. Then L is the unique maximal 2-local of G containg KS. Moreover, $C_L(K/O_2(K)/O_2(L)$ is coprime to $|L_2(q)|$

Remark: the same argument works for any group with three 2-componets which are conjugate in G so we should make an extra lemma and use it in the $L_3(2) \wr Sym(3)$ case

Suppose that Head$(E) \cong Alt(7) or Alt(10)$. Then $Head(U_2) \cong Alt(6)$ or $Alt(8)$ respectively and $U_1 \cap U_2/O_2(U_1 \cap U_2) \cong Sym(3)$. Hence we see in $L(1)$ that U_0 does not normalize $U_1 \cap U_2$ and Head$(\langle U_O, U_1 \cap U_2\rangle \cong C_3 \wr C_3$. Hence U_O does not normalize U_2. It follows that case 2(2) holds and Head$(L(2)) \cong Alt(7), Sp_6(2), L_6(2), Alt(9), Alt(10)$ or $Alt(11)$. But this contadicts the stucture of $\langle U_O, U_1 \cap U_2\rangle$.

Suppose that $q = 4$ and Head$(E) \cong {}^{\cdot}(L_3(4) \times L_3(4))$ or $3^{\cdot}(Alt(7) \times Alt(7)$ and let K_1 be a 2-component of U_1. Then $N_G(K_1)$ involves $L_3(4)$ respectively $Alt(7)$, a contradiction to (1).

Let $L = KO_2(L(1)$, $T = L \cap S$ and $B = N_L(T)$. Note that B normalizes K_1. Let $F = \langle B, E\rangle$.

Suppose that $F \notin \mathcal{L}(S)$. **TO BE CONTINUED**

14 Large Alternating Groups

In this section we assume that G is a quasi thin group, and that there exists an amalgam (P, E) so that $P \in \mathcal{P}(S)$, $E \in \mathcal{E}(S)$, Head$(E) \cong Alt(n)$, $n = 10, 11$ **Remark: we should at least also allow $E/O_2(E) \cong Sym(9)$**

$notA11$

Lemma$_{QT}$ 14.1 *Suppose $n = 11$ and let $U \leq calL(E, S)$ with* Head$(U) \cong Alt(10)$. *Then (P, U) is an amalgam.*

Proof: Let $L = \langle P, U \rangle$ and suppose that $L \in \mathcal{L}(S)$. Then by 13.2, $[O^2(P), O^2(U)]$ is a 2-group or $L \in \mathcal{N}(S)$. In the second case we get that $\mathrm{Head}(L) \cong Alt(11)$ and so $O_2(L) = O_2(U) = O_2(E)$ a contradiction. Thus $[O^2(P), O^2(U)]$ is a 2-group. As $m_3(O^2(U)) = 3$ we conclude that P is a $3'$ group. Let T be a Sylow 2-subgroup of $O^2(P)$. Then clearly U normalizes T and so $T \leq O_2(U)$ and $O_2(U)$ is a Sylow 2-subgroup of $O_2(U)O^2(P)$. As $O_2(U) = O_2(E)$, no non-trivial characteristic subgroup of $O_2(U)$ is normal in $O_2(U)O^2(P)$. Hence $O_2(U)O^2(P)$ has a non-trivial irreducible FF-module and so is not a $3'$ group, a contradiction.

$notA10$

Lemma$_{QT}$ 14.2 *Suppose $E/O_2(U) \cong Sym(9), Alt(10)$ or $Sym(10)$ and let $U \leq calL(E, S)$ with $U/O_2(U) \cong Sym(8)$. Then (P, U) is an amalgam.*

Proof: Let $L = \langle P, U \rangle$ and suppose that $L \in \mathcal{L}(S)$. Then by 13.2, $[O^2(P), O^2(U)]$ is a 2-group or $L \in \mathcal{N}(S)$.

Suppose that $O_2(E) \leq O_2(L)$. Then $O_2(U) \neq O_2(E)$ and $E/O_2(E) \cong Sym(10)$. Let $R \leq E$ with $O_2(L) \in \mathrm{Syl}_2(R)$ and $R/O_2(E) \cong Sym(3)$. Let C be a characteristics subgroup of $O_2(L)$ normal in R. Then C is normal in L and in $\langle U, R \rangle = E$. Hence $C = 1$ and so by 8.12 $O^2(P)$ normalizes $\Omega_1(Z(O_2(E))$, a contradiction.

$notA10-1$

(1) $O_2(E) \not\leq O_2(L)$.

Let $U^* \in \mathcal{L}(U, S)$ with $U^*/O_2(U^*) \cong Sym(3)$ and Let $Q/O_2(U)$ be the unique elementary abelian, normal subgroup of order 16 in $U^*/O_2(U)$. Then $N_E(Q)/Q \cong \mathrm{Sym}(5)$. Let C be a characteristic subgroup of Q normal in L. Then C is normal in $\langle U, N_E(U) \rangle = E$ and so $C = 1$. We proved

$notA10-2$

(2) $O_2(L) < Q$ and no non trivial characteristic subgroup of Q is normal in L.

Remark: (2) and its set up makes no sense for the Sym(9) case, some fixing necessary

Suppose that $L \in calN(S)$. Then $\mathrm{Head}(L) \cong Alt(m), 9 \leq m \leq 11$ or $L/O_2(L) \sim L_6(2).2$.

If $\mathrm{Head}(L) \cong Alt(m), m = 9$ or 11, L cannot be generated by U and a minimal parabolic unless $m = 9$ and $P = L$. We conclude $P/O_2(P) \cong Sym(9)$ and $O_2(E) \leq O_2(U) \leq O_2(L)$, a contradiction

If $\mathrm{Head}(L) \cong Alt(10)$, the situation is symmetric in E and L. $L(1) = \langle N_E(Q), N_L(Q)$. Then $Q = O_2(L(1))$ and 13.4 provides a contradiction. **Remark: One has to make sure that the possibility of two different complements $Sym(5)$ to a group of odd order was really ruled out**

If $L/O_2(L) \cong L_6(2).2$,

$$O_2(U) = [O_2(U), U]O_2(L) \leq O_2(E)O_2(L) \leq O_2(U)$$

and so $O_2(U) = O_2(E)O_2(L)$. If $E/O_2(E) \cong Sym(9)$ or $Alt(10)$, then $O_2(L) \leq O_2(U)$. Hence no non-trivial characteristic subgroup of $O_2(U)$ is normal in L and we conclude that $[J(U), \langle \Omega_1(Z(O_2(U))^L\rangle = 1$, a contradiction. Thus $E/O_2(E) \cong Sym(10)$. Let $V = \Omega_1(Z(O_2(L))$. Then by (2), $C_S(V) = O_2(L)$. On the otherhand, $L/O_2(L)$ has no faithful module with respect to it $O_2(U)/O_2(L)$ contains an offending subgroup. Hence $J(O_2(U) \leq O_2(L)$ and so $J(O_2(U) \not\leq O_2(E)$. It follows that there exists a conjugate of $J(O_2(U))$ under E which is contained in U but not in $U'O_2(U)$. Hence by 2.11 there exists an offender for L on V which is not contained in $L'O_2(L)$, a contradiction.

We have proved that $[O^2(U), O^2(P)] \leq O_2(U)$. Put $P^0 = O^2(P)Q$. As $O^2(P) \cap S \leq O_2(U) \leq Q$, $S \cap P^0 = Q$. Put $U_1 = N_E(Q)$ and $L(1) \overset{def}{=} \langle P, U_1\rangle$ By 8.12 we conclude that

(3) $[O_2(P), O^2(P)] \leq O_2(L(1))$. $notA10-3$

By a similar argument $O_2(L) = O_2(U)$ leads to a contradiction and so $O_2(L) \neq O_2(U)$. In particular, $E/O_2(E) \cong Sym(10)$. As U normalizes $O^2(P)$, U_1 does not. So by 13.4, $L(1) \in \mathcal{N}(S)$. By (3), the compoents of Head($L(1)$) cannot be groups of Lie type in characteristic 2 and thus are alternating groups. Furthermore, as $m_3(L) \leq 3$ and $m_3(U) = 2$, $m_3(P) \leq 1$. This leads to Head($L(1) \cong (3^\cdot)Alt(7)$ or $Alt(11)$. In particular $P/O_2(P) \cong Sym(3)$. In the second case $N_{(L(1)}(O^2(P))$ involves $Sym(8)$ and we obtain a contradiction by considering $\langle N_{(L(1)}(O^2(P)), U\rangle$ (note here that $U \not\leq L(1)$ as already $U_1 \leq L(1)$. Thus Head($L(1)$) $\cong (3^\cdot)Alt(7)$. By (1), $O_2(E)$ inverts Head(P). Thus $L/O_2(L) \cong Sym(3) \times Sym(8)$. As $U^* \leq U_1 \leq L(1)$ we get $L(1)/O_2(L(1)) \cong (3^\cdot)Sym(7)$. The $3^\cdot Sym(7)$ case is exclude by considering $N_G(O^2(P))$. Thus $L(1)/O_2(L(1)) \cong Sym(7)$.

In L we see that $O_2(L) = O_2(U) \cap O_2(P)$, in $L(1)$ that $O_2(L(1)) = O_2(U_1) \cap O_2(P)$ and in E that $O_2(U) \leq O_2(U_1)$. Hence $O_2(L) \leq O_2(L(1))$. Moreover, in L we see that $|O_2(E)O_2(L)/O_2(L)| = 2$ and in $L(1)$ that $|O_2(E)O_2(L(1))/O_2(L(1))| = 2$. It follows that $F \overset{def}{==} O_2(E) \cap O_2(L) = O_2(E) \cap O_2(L(1))$. Thus F is normalized U and U_1 and so F is normal in E. Note that $O^2(U) \cap O_2(E) \leq O_2(O^2(U)) \leq O_2(L)$ and so $O^2(U) \cap O_2(E) \leq F$. Hence by the " Satz von Gaschütz, $O^2(E) \cap O_2(E) \leq F$. Put $E^* = O^2(E)O_2(L)$. Since $O_2(L) \cap O^2(E)O_2(E) = F$ we conclude that $O_2(E^*) = F \leq O_2(L)$. Now the same argument as in the proof of (1) gives a contradiction, which completes the proof of the lemma.

We remark that $Sym(14)$ has parabolics $C_2 \wr Sym(7)$, $Sym(8) \times C_2 \wr Sym(3)$ and $Sym(10) \times C_2 \wr C_2$, intersecting in the same way has the groups in the last case we ruled out. But of course these parabolics in $Sym(14)$ are not of 2-type and so do not furnish a counter example.

$notAlt9$

Lemma$_{QT}$ 14.3 *Suppose $E/O_2(E) \cong Alt(9)$ and let $U \leq calL(E, S)$ with $U/O_2(U) \cong Alt(10)$. Then one of the following holds*

1. *(P, U) is an amalgam.*
2. *Let $L = \langle P, U\rangle$. Then $L/O_2(L) \cong L_5(2)$, $[O_2(L), O^2(L)$ is a natural module and $[Z, E] = 1$.*

Proof: We may assume that $L \in \mathcal{L}(S)$.As above $[O^2(U), O^2(P)$ is not a 2-group and $L/O_2(L) \ncong Alt(9)$. This leaves the possiblity $L/O_2(L) \cong L_5(2)$. Note that $O_2(L) \leq O_2(E)$

and so no non-trivial characteristic subgroup of $O_2(E)$ is normal in L. Let $Z_1 = \Omega_1(Z(O_2(L)$ and $Z_2 = \Omega_1(O_2(E)$ and note that $Z_2 = C_{Z_1}(O_2(E))$. Suppose that $[Z_2, E] \neq 1$. Then $[Z_2, E] \neq 1$. As Z_1 is an FF-module, all non-trivial composition non-trivial factors of L in Z_1 are isomorphic natural modules. Hence Z_2 is as U module the direct sum of isomorphic natural modules and trivial modules. Let d be an element of order three in U acting fixed point freely on the natural module for U, then it is easu to see that $C_{Z_2}(d) = C_{Z_2}(U) = C_{Z_1}(E)$ and so d acts fixed point freely on $Z_2/C_{Z_2}(E)$. It follows that Z_2 involves a spinmodule for E and so also two non-isomorphic natural modules for U, a contradiction.
Remark: u se the easier alt 7 argument

Hence $[Z_2, E] = 1$. It follows that Z_1 is a natural module for L and so by 8.14 and as $C_{Z_1}(E)$ we get $[O_2(E), O^2(E) = Z_1$ and so (2) holds

15 Tits Chamber Systems

In this section we us the following assumptions and notations:

(i) I is a finite set with $|I| \geq 3$,

(ii) For $i \in I$, $P_i \in \mathcal{P}(S)$.

(iii) For $J \subset I$ put $J' = I \setminus J$, $P_J = \langle P_j \mid j \in J\rangle$ and $M_J = P_{I'}$

(iv) Define a graph on I by considering i and j to be adjacent if and only if $[O^2(P_i), O^2(P_j)]$ is not a 2-group.

(v) If $J \subset I$ is connected with $|J| \geq 2$, then $P_J \in \mathcal{E}(S)$ and for all $j \in J$, $S \cap P_J' \ntrianglelefteq P_j$.

(vi) Let $i \in I$. Then Head$(M)i$ is a central extension of a groups of Lie type in characteristic two.

(vii) Let J be a proper subset of J. $Q_J = O_2(P_J)$ and $Z_J = \langle Z^{P_J}$. Then $C_{P_J}(Q_J) \leq Q_J)$.

(viii) $\langle P_i | \in I \nleq \mathcal{L}(S)$.

zni2p

Lemma 15.1 *Suppose there exists two distinct i, j in I with $Z \ntrianglelefteq P_i$ and $Z \ntrianglelefteq P_j$. Then one of the following holds:* **TO BE CONTINUED**

Proof: Suppose first that there exists $k \in I \setminus \{i, j\}$ so that k' is connected. Apply 8.6 with to $G_\alpha = M_k$ and $G_\beta = P_k$. As P_i does not centralize Z, 8.61 does not hold. By the stucture of M_k, 8.61 implies $C \trianglelefteq M_k$ and P_k, a contradiction.

In case (6) 8.12 implies that $[Q_k.O^2(P_k)] \leq Z_k$. let $k \neq r$ so that r is connected. Then $[Q_k, O^2(P_k)] \leq Z_k \leq Z_{r'} \leq Q_{r'}$ a contradiction to (v) and (vi).

Hence we mau assume that $q(M_k, Z_{k'} \leq 2$. As two parabolics of M_k act non-trivially on Z we get from 6.12 that M_k is of type $L_n(q)$, k' is a string with i and j as endpoints and M_k has exactly two non-central composition factors on $Z_{k'}$. Moreover these composition factors are natural modules dual to each other. Is is easy to see that $Z \trianglelefteq P_k$. Let $J = I \setminus \{i, j, k\}$. Assume that k is adjacent to some element of J. Then we can apply 8.22 to $G_\alpha = M_i$, $G_\beta = M_j$ and $G_{\alpha\beta} = M_{ij}$. Thus **TO BE CONTINUED**Assume that k is not adjacent to an element of J and without loss that k is adjacent to i. Then we can apply 8.22 to $G_\alpha = M_i$, $G_b = M_k$ and $G_{\alpha\beta}$ and we conclude that $J = \emptyset$. Thus **TO BE CONTINUED**

Remark: the effect of graph automorphisms needs to be worked in, $Z_\alpha Z_\beta \trianglelefteq G_\beta$ **needs to be ruled out**

Suppose next that no such k exists. Then clearly I is a string with i and j as the end notes. Then we can apply 8.22 to $G_\alpha = M_i$, $G_\beta = M_j$ and $G_{\alpha\beta} = M_{ij}$. Thus **TO BE CONTINUED**

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