A Characteristic Subgroup for Pushing Up in Finite Groups

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1 Introduction

2 The Kieler Lemma and Pointsstabilzers

An elementary abelian normal subgroup V of a finite group L is called p-reduced if any subnormal subgroup of L which acts unipotently on V has to act trivially. Note that this is equivalent to $O_p(L/C_L(V)) = 1$. Here are the basic properties of p-reduced normal subgroups.
Comment:due to Thompson? check history

Lemma 2.1 [**YL**] *Let L be a finite group of characteritic p and $T \in \mathrm{Syl}_p(L)$*

(a) [**a**] *There exists a unique maximal p-reduced normal subgroup Y_L of L.*

(b) [**b**] *Let $T \leq R \leq L$ and X a p-reduced normal subgroup of R. Then $\langle X^L \rangle$ is a p-reduced normal subgroup of L. In particular, $Y_R \leq Y_L$.*

(c) [**c**] *Let $T_L = C_T(Y_L)$ and $L^f = N_G(T_L)$. Then $L = L_f \mathbb{C}_L(Y_L)$, $T_L = O_p(L^f)$ and $Y_L = \Omega_1 \mathrm{Z}(T_L)$.*

(d) [**d**] *$Y_T = \Omega_1 \mathrm{Z}(T)$, $Z_L := \langle \Omega_1 \mathrm{Z}(T)^L \rangle$ is p-reduced for L and $\Omega_1 \mathrm{Z}(T) \leq Z_L \leq Y_L$.*

Now let L be any finite group and $T \in \mathrm{Syl}_p(L)$. definitionine $P_L(T) := O^{p'}(C_L(\Omega_1 \mathrm{Z}(T)))$. Then $P_L(T)$ is called a point stabilizer of L. The following lemma ist the principal tool for working with point stabilzers.

Lemma 2.2 [**kieler lemma**] *Let H be a finite group of local characteristic p, $T \in \mathrm{Syl}_p(H)$ and L a subnormal subgroup of H. Then* *(a)* [**a**] [Kieler Lemma] $C_L(\Omega_1 \mathrm{Z}(T)) = C_L(\Omega_1 \mathrm{Z}(T \cap L))$

(b) [**b**] $P_L(T \cap L) = O^{p'}(P_H(T) \cap L)$

(c) [**c**] $C_L(Y_L) = C_L(Y_H)$

(d) [**d**] *Suppose* $L = \langle L_1, L_2 \rangle$ *for some subnormal subgroups* L_1, L_2 *of* H*. Then*

- *(a)* [**da**] $P_L(T \cap L) = \langle P_{L_1}(T \cap L_1), P_{L_2}(T \cap L_2) \rangle$.
- *(b)* [**db**] *For* $i = 1, 2$ *let* P_i *be a point stabilizer of* L_i*. Then* $\langle P_1, P_2 \rangle$ *contains a point stabilizer of* L*.*

The proof of the above lemma is elementary and does not require any $\mathcal{K}$-group assumption assumption.
Comment: not all parts of this lemma are really needed

Lemma 2.3 [**minimal overgroups**] *Let* H *be a finite group and* $F < H$*.*

(a) [**a**] *Let* $\mathcal{I}_H(F)$ *be the set of all* I *with* $F < I \leq H$ *such that* F *lies in a unique maximal subgroup of* I*. Then* $H = \langle \mathcal{I}_H(F) \rangle$*.*

(b) [**b**] *Let* $\mathcal{J}_H(F) = \{I \in \mathcal{I}_G(F) \mid F \ntrianglelefteq I\}$*. Then* $H = \langle \mathcal{J}_H(F) \rangle N_H(F)$*.*

Proof: By induction on $|H|$. Suppose that F lies in two different maximal subgroups M_1, M_2 of H. By induction, $M_i = \langle \mathcal{I}_{M_i}(F) \rangle = \langle \mathcal{J}_{M_i}(F) \rangle N_{M_i}(F)$. Thus $H = \langle M_1, M_2 \rangle = \langle \mathcal{I}_H(F) \rangle = \langle \mathcal{J}_H(F) \rangle N_H(F)$.

So suppose F lies in a unique maximal subgroup of H. Then $H \in \mathcal{I}$ and $H = \langle \mathcal{I} \rangle$. Moreover either F is normal in H or $H \in \mathcal{J}$. In any case $H = \langle \mathcal{J} \rangle N_H(F)$. □

Lemma 2.4 (Schur multipliers) [**schur multipliers**]

Proof: [Schur] □

3 Modules

Lemma 3.1 (Point Stabilizer Theorem) [**the point stabilizer theorem**] *Let H be a finite group, V a $\mathbb{F}_pH$-module, L a point stabilizer for H on V and $A \leq O_p(L)$.*

(a) [**a**] *If V is p-reduced, then $|V/C_V(A)| \geq |A/C_A(V)|$.*

(b) [**b**] *If V is irreducible, $F^*(H)$ is quasi-simple, $H = \langle A^H \rangle$ and A is a non-trivial offender on V, then $M \cong SL_n(q)$, $Sp_{2n}(q)$, $G_2(q)$ or $\mathrm{Sym}(n)$, where $p = 2$ in the last two cases.*

Proof: [BBSM] □

Lemma 3.2 (FF-modules for miminal parabolics) [**ff-modules for miminal parabolics**]

Proof: [BBSM] □

Lemma 3.3 [**spin module**] *Let $H = Sp_{2n}(q)$, V a $\mathbb{F}_pH$-module, P a point stabilizer for H on the natural module, $T = O_p(P)$, $Z = Z(P)$ and W an $\mathbb{F}_pT$ submodule of V. Suppose that*

(i) [**i**] $V = \langle W^H \rangle$.

(ii) [**ii**] $[V, T, T] = 1$.

(iii) [**iii**] $[V, Z] \leq W \leq C_V(T)$.

Let $U = \bigcap_{h \in H} W^h[V,T]^h$ and $\overline{V} = V/U$. Let $h \in H$ with $Z \not\leq P^h$. Then

(a) [**a**] $V = [V, Z]C_V(T^h) = W[V,T]^h$, $\overline{W} = [\overline{W}, T] = C_{\overline{V}}(T) = C_{\overline{V}}(Z)$ *and* $\overline{V} = \overline{W} \times \overline{W}^h$.

(b) [**b**] *If $[W, H] \neq 1$, then $|\overline{V}| \geq q^{2^n}$ and $|V/C_V(T)| \geq q^{2^{n-1}}$.*

Proof: Let $Y = W[V,T]$. Then $Y \leq C_V(T)$. Note that $H = \langle Z, T^h \rangle$. Since $[V, Z] \leq W$ we conclude that H normalizes $W[V,T]^h$ and so by (i), $V = W[V,T]^h$. Also H also normalizes $[V,Z]Y^h$ and since $W^h \leq Y^h$ we conclude $V = [V,Z]Y^h = [V,Z]C_V(T^h)$. Let $X/U = C_{\overline{V}}(Z)$. Then $U \leq X \cap Y^h$. Thus $H = \langle Z, T^h \rangle$ normalizes $X \cap Y^h$ and so $X \cap Y^h = U$. Thus

$\overline{V} = \overline{X} \times \overline{Y}^h$. Since $V = [V, Z]Y^h$ we also get $\overline{V} = [\overline{V}, Z] \times \overline{Y}^h$. This implies $[\overline{V}, Z] = \overline{X} = C_{\overline{V}}(Z)$.

Note that

$$[\overline{V}, Z] \leq [\overline{V}, T] \leq \overline{Y} \leq C_{\overline{V}}(T) \leq C_{\overline{V}}(Z)$$

Now all the inequalities in the preceeding inequalities have to be equalities. So (a) is proved.

To prove (b) suppose that $[W, H] \neq 1$. By (a) also $[\overline{W}, H] = 1$ and so we may assume that $U = 1$.

Suppose first that $n = 1$ and $1 \neq z \in Z$. Since $H = \langle z, T^h \rangle$, $C_{Y^h}(z) \leq U = 1$. Let $1 \neq y \in Y^h$. We conclude that $|[y, Z]| \geq |Z| = q$ and so $|W| \geq q$ and $|V| \geq q^2$.

Suppose next that $n > 1$ and let $H^* = C_H(\langle Z, Z^h \rangle$. Then $H^* \cong Sp_{2n-2}(q)$ and $Z^* := Z^k \leq H^*$ for some $k \in H$. Then $P^* := P^k \cap H^*$ is a point stabilizer for H^* on its natural module, $T^* := T^k \cap H^* = O_p(P^*)$ and $Z^* = Z(P^*)$. Since $W = C_V(Z)$ and $H^* \leq C_G(Z)$, W is a $\mathbb{F}_p H^*$ submodule of W. Suppose that $[W, Z^*, H^*] = 1$. Let $h^* \in H^*$ with $Z^{*h^*} \not\leq P^*$. Then $[W, Z^*] \leq [V, Z^*] \cap [V, Z^{*h^*}] = 1$ and so $[W, Z^*] = 1$. Thus $C_V(Z) = W = C_W(Z^*)$ and so P and P^* normalize W, a contradcition since $H = \langle P, P^* \rangle$. Thus $[W, Z^*, H^*] \neq 1$. Let $V^* = \langle [W, Z^*]^{H^*} \rangle$. Then by induction $|V^*| \geq q^{2^{n-1}}$. Since $V^* \leq W$ and $|V| = |W|^2$ we get $|V| \geq q^{2^n}$.
□

We remark that (for example by [BBSM]), $\overline{V}$ from the preceeding lemma must be a direct sum of spin-modules for H.

Lemma 3.4 (H1 of natural modules) [**h1**]

Proof: [BBSM] □

4 The Baumann subgroup

For a p-group R we let $\mathcal{PU}_1(R)$ be the class of all finite groups L containing R such

(a) [**a**] L is of characteristic p,

(b) [**b**] $R = O_p(N_L(R))$

(c) [**c**] $N_L(R)$ contains a point stabilizer of L.

Let $\mathcal{PU}_2(R)$ be the class of all finite groups L containing R such that L is of characteristic p and

$$L = \langle N_L(R), H \mid R \leq H \leq L, H \in \mathcal{PU}_1(R) \rangle.$$

Let $\mathcal{PU}_3(R)$ be the class of all finite groups L such that

(a) [**a**] L is of characteristic p.

(b) [**b**] $R \leq L$ and $L = \langle R^L \rangle$

(c) [**c**] $L/C_L(Y_L) \cong SL_n(q), Sp_{2n}(q)$ or $G_2(q)$, where q is a power of p and $p = 2$ in the last case.

(d) [**d**] $Y_L/C_{Y_L}(L)$ is the corresponding natural module.

(e) [**e**] $O_p(L) \leq R$ and $N_L(R)$ contains a point stabilizer of L.

(f) [**f**] If $L/C_L(Y_L) \ncong G_2(q)$ then $R = O_p(N_L(R)$.

Let $\mathcal{PU}_4(R)$ be the class of all finite groups L containing R such that L is of characteristic p and

$$L = \langle N_L(R), H \mid R \leq H \leq L, H \in \mathcal{PU}_3(R) \rangle.$$

Let $\mathrm{B}(R) = C_R(\Omega_1 \mathrm{Z}(\mathrm{J}(R)))$, the Baumann subgroup of R. Recall that a finite group F is p-closed if $O'^F = O_p(F)$.

Lemma 4.1 (Baumann Argument) [**baumann argument**] *Let L be a finite group, R a p-sugroup of L, $V := \Omega_1\mathrm{Z}(O_p(L))$, $K := \langle \mathrm{B}(R)^L \rangle$, $\widetilde{V} = V/C_V(O^p(K))$, and suppose that each of the following holds:*

(i) [**i**] *$O_p(L) \leq R$ and $L = \langle \mathrm{J}(R)^L \rangle N_L(\mathrm{J}(R))$.*

(ii) [**ii**] *$C_K(\tilde{V})$ is p-closed.*

(iii) [**iii**] *$|\tilde{V}/C_V(A)| \geq |A/C_A(\tilde{V})|$ for all elementary abelian subgroups A of R.*

(iv) [**iv**] *If U is $L/O_p(L)$ module with $\tilde{V} \leq U$ and $U = C_U(B(R))\tilde{V}$, then $U = C_U(O^p(K))\tilde{V}$.*

Then $O_p(K) \leq B(R)$.

Proof: Let $T = O_p(L)$, $\overline{L} = L/C_L(V)$ and $Y = \Omega_1 Z\mathrm{J}(R)$. Let $A \in \mathcal{A}(R)$. Since $A \in \mathcal{A}(R)$ and $V \leq T \leq R$, $|V/C_V(A)| \leq |A/C_A(V)|$. By (ii) $C_A(\tilde{V}) = A \cap T$ and so also $C_A(V) = A \cap T$. Thus (iii) implies $|V/C_V(A)| = |\overline{A}| = |A/A \cap T|$ and so $V(A \cap T) \in \mathcal{A}(R) \cap \mathcal{A}(T)$. Thus $Y \leq V(A \cap T) \leq T$. Put $W = \langle Y^L \rangle V$. We conclude that $W \leq \Omega_1 \mathrm{Z}(\mathrm{J}(T))$ and so W is elementary abelian and $(A \cap T)V$ centralizes W. Hence $W \leq (A \cap T)V$ and $W = V(A \cap W) = VC_W(A)$. It follows that A centralizes W/V. Since A was arbitray in $\mathcal{A}(R)$, $\langle \mathrm{J}(R)^L \rangle$ centralizes W/V. Since $Y = \Omega_1 \mathrm{Z}(\mathrm{J}(R)$, $N_L(\mathrm{J}(R))$ normalizes Y. So by (i) also L normalizes YV. Thus $W = YU$ and $[W,T] = [Y,T] \leq Y$. Since L normalizes $[W,T]$ we get$[W,T] \leq C_W(K)$. Let $D = C_W(O^p(K))$ and $U = W/D$. Then T centralizes U. Since $\widetilde{V} \cong VD/D$ and $U = YV/D$, we can apply (iv) to conclude that $W = DV$ and $U \cong \widetilde{V}$. Since $A \in \mathcal{A}(R)$, $|W/W \cap A| \leq |A/C_A(W)| = |A/A \cap T|$. One the otherhand by (iii), $|A/A \cap T| \leq |\widetilde{V}/C_{\widetilde{V}}(A)| = |U/C_U(A)| \leq |W/C_W(A)D|$. Thus $|W/C_W(A)| \leq |W/C_W(A)D|$ and $D \leq C_W(A)$. Hence $[D,A] = 1$, $D \leq Y$ and $[D,K] = 1$. Therefore $[W, O_p(K)] \leq [D,K][V,T] = 1$ and so $O_p(K) \leq C_R(Y) = \mathrm{B}(R)$. □

Lemma 4.2 [**pu2(R) in pu4(B(R))**] *Let R be a p-group. Then $\mathcal{PU}_2(R) \subseteq \mathcal{PU}_4(\mathrm{B}(R))$.*

Proof: Let $L \in \mathcal{PU}_2(R)$. Since $N_L(R) \leq N_L(\mathrm{B}(R))$ we may assume that $L \in \mathcal{PU}_1(R)$. Set $P = N_L(R)$. If $P < H \leq L$, then clearly $H \in \mathcal{PU}_1(R)$. By 2.3(a) L is generated by the $H \leq L$ such that P is contained in a unique maximal subgroup of H. If $H \in \mathcal{PU}_4(\mathrm{B}(R))$ for all such H, then by the definition of $\mathcal{PU}_4$ also $L \in \mathcal{PU}_4(\mathrm{B}(R))$. Hence we may assume from now on that

1) [**1**] *P is contained in unique maximal subgroup H of L.*

Let D be the largest normal subgroup of L contained in P. Then $[D,R] \leq [P,R] \leq R$ and so $[D,R] \leq O_p(D) \leq O_p(L)$.

Choose $T \in Syl_p(L)$ with $P_L(T) \leq P$. Then $R \leq O_p(P_L(T) \leq O_p(C_L(\Omega_1 \mathrm{Z}(T))$ and $[R, C_L(Z_L)] \leq O_p(C_L(Z_L)) = O_p(L) \leq R$. Thus $C_L(Z_L) \leq N_L(R) \leq P$. We proved:

2) [**2**] *$[D, \langle R^L \rangle] \leq O_p(L)$ and $C_L(Z_L) \leq D$*

If $\mathrm{J}(R) \leq D$, then $J(R) = J(O_p(D))$ and so $J(R) \trianglelefteq H$. Thus $[Z_L, \mathrm{J}(R)] = 1$ and so also $[Z_L, \mathrm{B}(R)] = 1$. So by rr2, $\mathrm{B}(R) \leq D$ and $\mathrm{B}(R) = \mathrm{B}(O_p(D))$. Thus $\mathrm{B}(R) \trianglelefteq H$ and so $H \in \mathcal{PU}_4(\mathrm{B}(R))$.

So we may assume that $\mathrm{J}(R) \not\leq D$ and so by rr2 $[Z_L, J(R)] \neq 1$. Let $K = \langle \mathrm{J}(R)^L \rangle$, $\overline{L} = L/C_L(Z_L)$ and $\widetilde{Z_L} = Z_L/C_{Z_L}(O^p(K))$. By **??** there exists a L-invariant set of normal subgroups K_i, $1 \leq i \leq l$, in K such that

(3-i) $K_i = O^{p'}(K_i)$,

(3-ii) $\overline{K} = \overline{K_1} \times \overline{K_2} \times \ldots \times \overline{K_l}$,

(3-iii) $\widetilde{Z_L} = [\widetilde{Z_L}, K_1] \times [\widetilde{Z_L}, K_2] \times [\widetilde{Z_L}, K_l]$,

(3-iv) $\overline{K_i} \cong SL_n(q), Sp_{2n}(q), G_2(q)$ or $\mathrm{Sym}(n)$, where q is a power of p, $p = 2$ in the last two cases and $n \equiv 2, 3 \mod 4$ in the last case,

(3-v) $[\widetilde{Z_L}, K_i]$ is the natural module for K_i,

(3-vi) $\overline{\mathrm{J}(R)} = (\overline{\mathrm{J}(R)} \cap \overline{K_1}) \times \ldots \times (\overline{\mathrm{J}(R)} \cap \overline{K_l})$

It is now easy to see that $\overline{L} = \overline{K} N_{\overline{L}}(\overline{\mathrm{J}(R)}$

By rr2 $O_p(C_L(Z_L)\mathrm{J}(R)) = O_p(L)\mathrm{J}(R)$ and so $\mathrm{J}(R) = \mathrm{J}(O_p(C_L(Z_L)\mathrm{J}(R)))$. Thus $\overline{N_L(\mathrm{J}(R))} = N_{\overline{L}}(\overline{\mathrm{J}(R)}$ and so

3) [**4**] $\qquad L = KN_L(\mathrm{J}(R))$.

Suppose that $K \leq H$. Then by rr1 and rr4 $\mathrm{J}(R)$ is normal in L and $\mathrm{J}(R) \leq O_p(L) \leq D$, a contradiction to the assumptions.

Thus $K \not\leq H$. Pick j with $K_j \not\leq H$. Then by 1) $L = \langle K_j, P \rangle = \langle K_j^P \rangle P$. Thus $\langle K_j^P \rangle \mathrm{J}(R)$ is normal in L. So P acts transitively on $\{K_i \mid 1 \leq i \leq l\}$, and $L = KP$. By 2) $[C_L(Z_L), \mathrm{J}(R)] \leq O_p(L)$ and so $C_L(Z_L), K] \leq O_p(L)$. Hence $C_K(Z_L)$ is p-closed. Also $C_K(Z_L) = C_K(\widetilde{Z_L})$.

Note also that $\mathrm{B}(R) \leq KO_p(L)$ and so $\langle \mathrm{B}(R)^L \rangle = K\,\mathrm{B}(R)$.

Suppose that $\mathrm{B}(R)O_p(L) = O_p(P \cap KO_p(L))$ or that $\overline{K_j} \cong G_2(q)$. Then it is easy to see that the assumptions of 4.1 are fulfiled. We conclude that $O_p(K\,\mathrm{B}(R)) \leq \mathrm{B}(R)$. Moreover, either $\overline{K_j} \cong G_2(q)$ or $\mathrm{B}(R) = O_p(P \cap K\,\mathrm{B}(R))$. By 2.2(a)

$$C_{K_i}(\Omega_1 \mathrm{Z}(T \cap K_j\,\mathrm{B}(R))) = C_{K_j}(\Omega_1 \mathrm{Z}(T \cap K_i)) = C_{K_j}(\Omega_1 \mathrm{Z}(T))$$

and we conclude that $P \cap K_j\,\mathrm{B}(R)$ contains a point stabilizer of $K_i B(R)$. Suppose in addition that $\bar{K}_j \ncong \mathrm{Sym}(n)$, $n \geq 7$. Then $K_i\,\mathrm{B}(R) \in \mathcal{PU}_3(\mathrm{B}(R))$. Also $P \leq N_L(\mathrm{B}(R))$ and $L = \langle P, K_i\,\mathrm{B}(R) \mid 1 \leq i \leq l \rangle$ and so $L \in \mathcal{PU}_4(\mathrm{B}(R))$.

Suppose now that $\overline{K_j} \ncong G_2(q)$ and either $\mathrm{B}(R)O_p(L) \neq O_p(P \cap KO_p(L))$ or $\bar{K}_i \cong \mathrm{Sym}(n)$, $n \geq 7$. Put $q := 2$ in the second case. Then $\overline{K}_i \cong Sp_{2n}(q)$

or Sym(n) and $|B(R)/O_p(K_i\,\mathrm{B}(R)| = q$. Hence there exists a subgroup D_i of $K_i\,\mathrm{B}(R)$ with $\mathrm{B}(R) \le D_i$, $D_i = \langle \mathrm{B}(R)^{D_i}\rangle$ and $D_i/O_p(D_i) \cong SL_2(q)$. By 4.1 $\mathrm{B}(R) \in \mathrm{Syl}_p(D_i)$. Thus $D_i \in \mathcal{PU}_3(\mathrm{B}(R))$. Moreover, $K_i = \langle D_i, N_{K_i}(\mathrm{B}(R))\rangle$ and so $L = \langle D_i, N_L(\mathrm{B}(R)) \mid 1 \le i \le n$. Thus again $L \in \mathcal{PU}_4(\mathrm{B}(R)$. □

Lemma 4.3 [**P(T) in PU4(B(T))**] *Let P be a finite group of characteristic p. Let $T \in \mathrm{Syl}_p(T)$ and suppose that T lies in a unique maximal subgroup of P. Then either $Z_L = \Omega_1\mathrm{Z}(L)$ or $P \in \mathcal{P}_4(\mathrm{B}(T))$.*

Proof: Suppose that $[\mathrm{J}(T), Z_L] = 1$. Then also $[\mathrm{B}(T), Z_L] = 1$ and so by the Frattinargument $L = C_L(Z_L)N_L(\mathrm{B}(T))$. Since L is minimal parabolic, $L = C_L(Z_L)S$ or $\mathrm{B}(T)$ is normal in L. In the first case $Z_L = \Omega_1\mathrm{Z}(L)$ and in the second case $L \in \mathcal{PU}_4(T)$.

So we may assume that $[\mathrm{B}(T), Z_L] \ne 1$. Using 3.2 we can argue just as in 4.2. □

5 A solution to the principal amalgam problem

Let R be a group and Σ a set of groups containing R. Then

$$O_R(\Sigma) = \langle N \le R \mid N \trianglelefteq L\, \forall L \in \Sigma\rangle$$

So $O_R(\Sigma)$ is the largest subgroup of R which is normal in all the $L \in \Sigma$.

Theorem 5.1 [**simultanous pushing up**] *Let R be a finite p-group with $R = \mathrm{B}(R)$ and Σ a subset of $\mathcal{PU}_3(R)$. If $O_R(\Sigma) = 1$, then one of the following holds*

(a) [**a**] *who knows*

The proof will be achieved in a long sequence of lemmas. Let G^* be the free amalgameted product of the Σ over R. We view $L \in \Sigma$ as a subgroup of G^*. Let Γ be the graph with vertices G^* and edges (L_1g, L_2g), $g \in G^*$, $L_1 \ne L_2 \in \Sigma$. Note that G^* acts on Γ by right multiplication. For $\alpha \in \Gamma$ let $G_\alpha = \{g \in G^* \mid \alpha = \alpha^g\}$, $Q_\alpha = O_p(G_\alpha)$ and $Z_\alpha = Z_{G_\alpha}$ and $U_\alpha = [Z_\alpha, G_\alpha]$. For an edge (α, β) let $Q_{\alpha\beta} = G_\alpha \cap G_\beta$ and $Z_{\alpha\beta} = \Omega_1\mathrm{Z}(Q_{\alpha\beta}$. Let $\Delta(\alpha)$ be the set of neigbors of α and $G_\alpha^{(1)} = G_\alpha \cap \bigcap_{\beta\in\Delta(\alpha)} G_\beta$. Let $U_\alpha = [Z_\alpha, G_\alpha]$. Then by definition of Γ and of $\mathcal{PU}_3(R)$.

Lemma 5.2 [**basics of pushing up**]

(a) [**a**] *$G_\alpha = L^g$ for some $L \in \Sigma$ and $g \in G^*$, and G_α is of characteristic p.*

(b) [**b**] *$\overline{G_\alpha} := G_\alpha / C_{G_\alpha}(Z_\alpha) \cong SL_{n_\alpha}(q_\alpha), Sp_{2n}(q_\alpha)$ or $G_2(q_\alpha)$, q_α a power of p.*

(c) [**c**] *$\widetilde{Z_\alpha} := Z_\alpha / C_{Z_\alpha}(G_\alpha)$ is a natural module.*

(d) [**d**] *$Q_{\alpha\beta} = \mathrm{B}(Q_{\alpha\beta})$ and $G_\alpha = \langle Q_{\alpha\beta}^{G_\alpha} \rangle$*

(e) [**e**] *$P_{\alpha\beta} := N_{G_\alpha}(Q_{\alpha\beta})$ contains a point stabilzer of G_α.*

(f) [**f**] *If $\overline{G_\alpha} \not\cong G_2(q)$ then $Q_{\alpha\beta} = O_p(P_{\alpha\beta})$.*

Next we show

Lemma 5.3 [**more basics of pushing up**]

(a) [**a**] *$Z_{\alpha\beta} \le Z_\alpha = \Omega_1 \mathrm{Z}(Q_\alpha)$*

(b) [**b**] *$C_{G_\alpha}(Z_\alpha) = Q_\alpha$.*

(c) [**c**] *$Q_\alpha = G_\alpha^{(1)}$.*

(d) [**d**] *One of the following holds:*

1. [**1**] *$U_\alpha \cap \Omega_1 \mathrm{Z}(G_\alpha) = 1$, that is U_α is the natural module.*
2. [**2**] *$\overline{G_\alpha} \cong Sp_{2n}(q)$ or $G_2(q)$ and U_α is a quotient of the natural $O_{2n+1}(q)$-module for $\overline{G_\alpha}$, (where $n = 3$ in the $G_2(q)$-case).*

(e) [**e**] *For all $H \le G_\alpha$, $C_{\widetilde{Z_\alpha}}(H) = \widetilde{C_{Z_\alpha}(H)}$.*

(f) [**f**] *Let $T \in \mathrm{Syl}_p(P_{\alpha\beta})$ and $x \in \Omega_1 \mathrm{Z}(T)$ with $x \notin \Omega_1 \mathrm{Z}(G_\alpha)$. Then $C_{G_\alpha}(x) = O^{p'}(P_{\alpha\beta})$.*

(a) follows from 5.2(d),(e) and 3.1.

Let $T \in \mathrm{Syl}_p(P_{\alpha\beta})$. Since $C_{G_\alpha}(Z_\alpha) \le C_{G_\alpha}(\Omega_1 \mathrm{Z}(T)) \le P_{\alpha\beta} = N_{G_\alpha}(Q_{\alpha\beta})$ we get

$$[C_{G_\alpha}(Z_\alpha), Q_{\alpha\beta}] \le C_{G_\alpha}(Z_\alpha) \cap Q_{\alpha\beta}] \le O_p(C_{G_\alpha}(\Omega_1 \mathrm{Z}(T))) \le Q_\alpha$$

Thus 5.2(d), $[C_{G_\alpha}(Z_\alpha), G_\alpha] \leq Q_\alpha$. we proved this before, should have been recorded

Thus (b) follows from 2.4 and 5.2 (d).

By 5.2(f) $Q_\alpha \leq Q_{\alpha\beta} = G_\alpha \cap G_\beta$. So (c) holds.

(d) follows from 3.4, and (e) follows from (d). Finally (f) follows from (b),(e), and 5.2 (c),(e). □

We say that $\beta \in \Gamma$ is symplectic if $\overline{G_\beta} \cong Sp_{2n}(q)$ with $n \geq 2$, β is linear if $\overline{G_\beta} \cong SL_n(q)$ and β is a hex if $\overline{G_\beta} \cong G_2(q)$. Let $\alpha \in \Delta(\beta)$. definitionine

$$X_{\alpha\beta} := \begin{cases} [Z_\alpha, Q_{\alpha\beta}] & \text{if } \alpha \text{ is symplectic.} \\ Z_\alpha & \text{otherwise.} \end{cases}$$

Put

$$A_{\alpha\beta} = [X_{\alpha\beta}, Q_{\alpha\beta}]$$

Lemma 5.4 [**agammadelta**] *Let (α, β) be an edge in Γ. Then $A_{\alpha\beta} \leq \Omega_1 \mathrm{Z}(Q_{\alpha\beta}) \leq \Omega_1 \mathrm{Z}(Q_\beta) \leq Z_\beta$ and $A_{\alpha\beta} \not\leq Z(G_\alpha)$.*

Proof: Readily verfied. □

Lemma 5.5 [**offenders on xgammadelta**] *Let (α, β) be an edge in Γ, $D = X_{\alpha\beta}$ or $D = Z_\alpha$ and $B \leq Q_{\alpha\beta}$ be a non-trivial offender on D*

(a) [**a**] $|D/C_D(B)| = |B/C_B(D)|$.

(b) [**b**] *One of the following holds:*

1. [**1**] $[D, Q_{\alpha\beta}] \leq [D, B]$.
2. [**2**] *α is a symplectic, $D = Z_\alpha$ and $[D, C_{Q_{\alpha\beta}}(X_{\alpha\beta})] \leq [D, B]$.*

(c) [**c**] *One of the following holds*

1. [**1**] $[D, B, Q_{\alpha\beta}] = 1$.
2. [**2**] *α is symplectic, $D = Z_\alpha$, $[X_{\alpha\beta}, B] \neq 1$ and $[D, Q_{\alpha\beta}, Q_{\alpha\beta}] = A_{\alpha\beta}$.*

Proof: This follows easily from the action of $Q_{\alpha\beta}$ on D □

Lemma 5.6 [**agd in zgd**] *Let (α, β) be an edge in Γ and suppose that $Z_\beta \leq Q_\alpha$.*

(a) [**a**] *If $X_{\alpha\beta} \not\leq Z_\beta$ then $A_{\alpha\beta} \leq Z(G_\beta)$.*

(b) [**b**] *Suppose α is symplectic and that N is a normal p-subgroup of G_β with $[X_{\alpha\beta}, N] = 1$. Then $[Z_\alpha, N] \leq Z(G_\beta)$.*

Proof: For the proof of (b) we may assume (a) has been proved and that $[Z_\alpha, N] \neq 1$.

We prove (a) and (b) simultaneously. For the proof of (a) let $D_\alpha = X_{\alpha\beta}$ and $U = Q_\beta$. Note that D_α also depends on β but β will be fixed throughout the proof. For the proof of (b) let $D_\alpha = Z_\alpha$ and $U = N$. Let $A_\alpha = [D_\alpha, U]$. From the definition of A_α we obtain:

1) [**1**] $A_\alpha \leq Z_{\alpha\beta}$

Next we show:

2) [**2**] *Let $B \leq Q_{\alpha\beta}$ and suppose that B is a non-trivial offender on D_α. Then $A_\alpha \leq [D_\alpha, B] \cap Z_{\alpha\beta}$.*

By 1) we only need to show that $A_\alpha \leq [D_\alpha, B]$. We apply 5.5(b) with D_α. If 1. holds we have $A_\alpha = [D_\alpha, U] \leq [D_\alpha, Q_{\alpha\beta}] \leq [D_\alpha, B]$ and we are done. Suppose that 2.holds. Then $D_\alpha \neq X_{\alpha\beta}$ and so we must be in the proof of (b). So $U = N \leq C_{Q_{\alpha\beta}}(X_{\alpha\beta})$ and again $A_\alpha \leq [D_\alpha, B]$.

3) [**3**] *Let $B \leq Q_\beta$ and suppose that B is a non-trivial offender on D_α. $[D_\alpha, B, Q_{\alpha\beta}] \leq \Omega_1 \mathrm{Z}(G_\beta)$.*

We apply 5.5(c). If 1. holds we are done. So suppose 2. holds. Then we are in the proof of (b), $[X_{\alpha\beta}, B] \neq 1$ and $[D_\alpha, B, Q_{\alpha\beta}] = A_{\alpha\beta}$. Since $B \leq Q_\beta$, we get $X_{\alpha\beta} \not\leq Z_\beta$ and so by (a) $A_{\alpha\beta} \leq \Omega_1 \mathrm{Z}(G_\beta)$ and 3) is proved.

Since $Q_{\alpha\beta} = \mathrm{B}(Q_{\alpha\beta})$ and $C_{G_\beta}(Z_\beta) = Q_\beta$ we have $[Z_\beta, \mathrm{J}(Q_{\alpha\beta})] \neq 1$. Thus there exists $A \in \mathcal{A}(Q_{\alpha\beta})$ with $A \not\leq Q_\beta$. Let $a \in A$ with $a \notin Q_\beta$. If β is a hex we choose a such that in addition $C_{Z_\beta}(a) = Z_{\alpha\beta}$. Let $\gamma \in \alpha^{G_\beta}$ with $Z_{\alpha\beta} \cap Z_{\gamma\beta} = \Omega_1 \mathrm{Z}(G_\beta)$ and $a \notin P_{\beta\gamma}$. The choice of a implies

4) [**4**] $Z_{\gamma\beta} \cap Z^a_{\gamma\beta} = \Omega_1 \mathrm{Z}(G_\beta)$

Suppose first that

$$(*) \quad [D_\gamma, D^a_\gamma] \neq 1.$$

Then by 5.5 D_γ^a is an offender on D_γ and vice versa. So by 2) applied to (D_γ^a, γ) in place of (B, α)

$$A_\gamma \leq [D_\gamma, D_\gamma^a] \cap Z_{\gamma\beta}$$

By 3) applied to (D_γ, γ^a) in place of (B, α) we have $[[D_{\gamma^a}, D_\gamma], Q_{\gamma\beta}^a] \leq Z(G_\beta)$. Hence 5.3(f) implies $Z_\beta \cap [D_{\gamma^a}, D_\gamma] \leq Z_{\gamma\beta}^a$ and thus

$$A_\gamma \leq [D_{\gamma^a}, D_\gamma] \cap Z_{\gamma\beta}) \leq Z_{\gamma\beta} \cap Z_{\gamma\beta}^a \leq \Omega_1 \mathrm{Z}(G_\beta)$$

and we are done in this case.

Suppose next that

$$(**) \quad [D_\gamma, D_\gamma^a] = 1.$$

Set $B := A \cap Q_\beta$ and $C := C_B(D_\gamma)$. Then $Z_\beta B \in \mathcal{A}(Q_\beta) \subseteq \mathcal{A}(Q_{\alpha\beta})$. Since Z_β centralizes Z_γ, B is an offender on D_γ. Since A is abelian and $C \leq B \leq A$ we have $B = B^a$ and $C = C^a$. Thus $C = C_B(D_\gamma^a)$ and C centralizes D_γ^a. Since by assumption $Z_\beta \leq Q_\alpha$ we get $Z_\beta \leq Q_\gamma^a$. Thus by (**) $Z_\beta D_\gamma C$ centralizes D_γ^a. By 1) $Z_\beta D_\alpha C \in \mathcal{A}(Q_\beta)$ and we conclude that $D_\gamma^a \leq Z_\beta D_\gamma C$. By symmetry in γ and γ^a we conclude $Z_\beta D_\gamma C = Z_\beta D_\gamma^a C$. Thus

$$[D_\gamma, B] = [D_\gamma^a, B].$$

Suppose that B does not centralize D_γ. Then by 2) applied to γ in place of α, $A_\gamma \leq [D_\gamma, B] \cap Z_{\gamma\beta}$. From $[D_\gamma, B] = [D_\gamma^a, B]$ and 3) applied to γ^a in place of α we get $[D_\gamma, B, Q_{\gamma\beta}^a] \leq Z(G_\beta)$ Now as in the (*) case $A_\gamma \leq Z(G_\beta)$ and we are done.

Suppose next that B centralizes D_γ. Then also $Z_\beta B$ centralizes D_γ and so $D_\gamma \leq Z_\gamma B$. Since a centralizes B we conclude that $D_\gamma Z_\beta = D_\gamma^a Z_\beta$. Hence

$$A_\gamma = [D_\gamma, U] = [D_\gamma Z_\beta, U] = [D_\gamma^a, U] = A_{\gamma^a} \leq Z_{\gamma\beta} \cap Z_{\gamma\beta}^a \leq \Omega_1 \mathrm{Z}(G_\beta)$$

and we are also done in this final case. □

For adjacent vertices α, β let $V_\alpha^\beta = \langle Z_\beta^{G_\alpha} \rangle$.

Lemma 5.7 [**qgamma cap qdelta normal**] *Let (β, α) be an edge of Γ and suppose that V_α^β and V_β^α are abelian. Then $Q_\alpha \cap Q_\beta$ is normal in G_α.*

Proof: Choose A, a and γ as in the proof of 5.6. Assume that $Q_\alpha \cap Q_\beta$ is not normal in G_α. By conjugation $Q_\gamma \cap Q_\beta$ is not normal in G_γ and so $Q_\gamma \cap Q_\beta \neq Q_\delta \cap Q_\gamma$ for some $\delta \in \beta^{G_\gamma}$. Then $[Q_\gamma \cap Q_\beta, Z_\delta] \neq 1$.

If possible, choose δ such that $[Q_\gamma \cap Q_\beta, X_{\delta\gamma}] \neq 1$. In this case put $D_{\delta\gamma} = X_{\delta\gamma}$.

If not possible, put $N = \langle (Q_\alpha \cap Q_\beta)^{G_\gamma} \rangle$ and $D_{\delta\gamma} = Z_\delta$. Then $[X_{\beta\gamma}, N] = 1$.

Note that $Z_\gamma \leq V_\beta^\gamma$ and so $Z_\gamma \leq Q_\beta$. Thus we can apply 5.6 and to (β, γ) in place of (α, β). We conclude that $A_\gamma := [D_{\delta\gamma}, Q_\alpha \cap Q_\beta] \leq \Omega_1 Z(G_\gamma)$. Since $A_\gamma \not\leq \Omega_1 Z(G_\delta)$ and $\delta \in \beta^{G_\gamma}$ we get $A_\gamma \not\leq \Omega_1 Z(G_\beta)$. Since $Z_{\gamma\beta}^{a^{-1}} \cap Z_{\gamma\beta} \leq \Omega_1 Z(G_\beta)$ we have

1) [**1**] *$A_\gamma \leq \Omega_1 Z(G_\gamma)$ and $Z_{\gamma\beta}^{a^{-1}} \not\geq A_\gamma \not\leq Z_{\gamma\beta}^a$.*

From the definition of $D_{\delta\gamma}$ and 5.5(b) we deduce

2) [**2**] *Let $F \leq Q_{\delta\gamma}$ be an offender on $D_{\delta\gamma}$, then $A_\gamma \leq [D_{\delta\gamma}, F]$.*

Let $B = A \cap Q_\beta$ and $C = B \cap Q_\gamma$. Then $Z_\beta B$ and $Z_\beta Z_\gamma C$ are in $\mathcal{A}(Q_{\beta\gamma})$. Next we show

3) [**3**] *$D_{\delta\gamma} \leq Z_\beta Z_\gamma C$ for all $\delta \in \beta^{G_\gamma}$ with $[Q_\beta \cap Q_\gamma, D_{\gamma\delta}] \neq 1$.*

Assume that $[C, D_{\delta\gamma}] = 1$. Since V_γ^β is abelian, $Z_\gamma Z_\beta$ centralizes Z_δ and so also $D_{\delta\gamma}$. Since $Z_\beta Z_\gamma C \in \mathcal{A}(Q_{\beta\gamma})$ we conclude that 3) holds in this case. So assume for a contradiction that $[C, D_{\delta\gamma}] \neq 1$ and put $D = C_C(D_{\delta\gamma})$. Then by 2), $A_\gamma \leq [C, D_{\delta\gamma}]$ and by 5.5(a) $E := Z_\beta Z_\gamma D_{\delta\gamma} D \in \mathcal{A}(Q_\gamma)$.

We will show that $[E, D_{\delta\gamma}^a] = 1$. Since $V_{\gamma^a}^\beta$ is abelian, $D_{\delta\gamma}^a$ centralizes Z_β.

Suppose that $[D_{\delta\gamma}^a, Z_\gamma] \neq 1$. Since V_β^γ is abelian, $Z_\gamma \leq Q_\beta \cap Q_\gamma^a$. From 5.5(a) we conclude that Z_γ is an offender on $D_{\delta\gamma}$ and vice versa. By 2) $A_\gamma^a = [D_{\delta\gamma}^a, Z_\gamma] \leq Z_{\gamma\beta}$, a contradiction to 1).

Thus $[D_{\delta\gamma}^a, Z_\gamma] = 1$ and $D_{\delta\gamma}^a \leq Q_\beta \cap Q_\gamma$. By symmetry $D_{\delta\gamma} \leq Q_\beta \cap Q_\gamma^a$. Hence by 5.5(a) $D_{\delta\gamma}$ and $D_{\delta\gamma}^a$ are offenders on each other.

Suppose that $[D_{\delta\gamma}, D_{\delta\gamma}^a] \neq 1$. Then by 2) $A_\gamma \leq [D_{\delta\gamma}, D_{\delta\gamma}^a] \leq Z_{\gamma\beta}^a$, again a contradiction to 1).

Thus $[D_{\delta\gamma}, D_{\delta\gamma}^a] = 1$. Since D centralizes $D_{\delta\gamma}$ and since $D = D^a$, D centralizes $D_{\delta\gamma}^a$. Thus E centralizes $D_{\gamma\delta}^a$ and so $D_{\gamma\delta}^a \leq E$. Note that C is a non-trivial offender on $D_{\delta\gamma}$ and so by 2) $A_\gamma \leq [C, D_{\delta\gamma}$. Since a centralizes C we get

$$A_\gamma^a \leq [C, D_{\delta\gamma}^a] \leq [C, E] = [C, D_{\gamma\delta}] \leq Z_{\gamma\beta}$$

contradicting 1). This completes the proof of 3).

Suppose that $B \neq C$, that is $B \not\leq Q_\gamma$. By 3) $[B, D_{\delta\gamma}] \leq [B, Z_\gamma] \leq Z_\gamma$ and so $B \leq N_{G_\gamma}(D_{\delta\gamma} Z_\gamma)$. In particular, B normalizes $C_{Q_\gamma}(D_{\delta\gamma})$. Let $\rho \in \beta^{G_\gamma}$ with $[Q_\beta \cap Q_\gamma, D_{\rho\gamma}] = 1$. Then

$$[Q_\gamma, B] \leq [Q_\gamma, Q_\beta] \leq Q_\beta \cap Q_\gamma \leq C_{Q_\gamma}(D_{\rho\gamma})$$

So B normalizes $C_{Q_\gamma}(D_{\rho\gamma})$. It follows that B normalizes $C_{Q_\gamma}(D_{\tau\gamma})$ for all $\tau \in \beta^{G_\gamma}$. Since $B \not\leq Q_\gamma$ we conclude that $C_{Q_\gamma}(D_{\beta\gamma})$ is normal in $\langle B^{G_\gamma}\rangle Q_{\beta\gamma} = G_\gamma$. But then

$$Q_\beta \cap Q_\gamma \leq C_{Q_\gamma}(D_{\beta\gamma}) = C_{Q_\gamma}(D_{\beta\delta})$$

a contradiction.

Thus $B = C$. So B centralizes Z_γ, $Z_\gamma \leq Z_\beta B$ and by 2) $D_{\delta\gamma} \leq Z_\beta B$. Since A centralizes B, we conclude that A normalizes $Z_\gamma Z_\beta$ and $D_{\delta\gamma} Z_\beta$. But then A also normalizes $Q_\gamma \cap Q_\beta$ and $[Q_\gamma \cap Q_\beta, D_{\delta\gamma} Z_\beta]$. Since this latter group is A_γ we get a contradiction to 1). □

Lemma 5.8 [**zalpha offender**] *Let (α, β) and (γ, δ) be edges in Γ such that $Z_\alpha Z_\delta \leq Q_{\alpha\beta} \cap Q_{\delta\gamma}$ and $[Z_\alpha, Z_\delta] \neq 1$. Then*

(a) [**a**] *Z_α is an offender on Z_δ and vice versa.*

(b) [**b**] *$|Z_\alpha Q_\delta / Q_\delta| = |Z_\delta Q_\alpha / Q_\alpha|$.*

(c) [**c**] *$G_\alpha = \langle Z_\delta^{G_\alpha}\rangle Q_\alpha$.*

Proof: (a) and (b) follows from the fact that $Q_{\alpha\beta}$ contains no over-offender on Z_α.

Note that $O^p(G_\alpha) Q_\alpha = G_\alpha$ unless $\overline{G}_\alpha \cong SL_2(2), SL_2(3), Sp_4(2)$ or $G_2(2)$. In each of the four exceptionell case $O^p(G_\alpha) Q_\alpha$ has index p in G_α and $Q_{\alpha\beta} \cap O^p(G_\alpha) Q_\alpha$ contains no non-trivial offender on Z_α. Thus (c) follows from (a). □

Lemma 5.9 [**critical pairs**] *Let (α, β) and (γ, δ) be edges in Γ such that $Z_\alpha Z_\delta \leq Q_{\alpha\beta} \cap Q_{\delta\gamma}$ and $[Z_\alpha, Z_\delta] \neq 1$.*

Then $q := q_\alpha = q_\beta$ and one of the following holds.

1. [**1**] *$\overline{G}_\alpha \cong \overline{G}_\delta \cong G_2(q)$.*

2. [**2**]

(a) [**a**] $\overline{G}_\alpha \cong Sp_{2n_\alpha}(q)$ *and* $\overline{G}_\delta \cong Sp_{2n_\delta}(q)$

(b) [**b**] $|Z_\alpha Q_\delta/Q_\delta| = |Z_\delta Q_\alpha/Q_\alpha| = q$.

(c) [**c**] $[Z_\alpha, [Z_\delta, Q_{\gamma\delta}]] = 1$ *and* $[Z_\delta, [Z_\alpha, Q_{\alpha\beta}]] = 1$.

3. [**3**]

(a) [**a**] $\overline{G}_\alpha \cong Sp_{2n_\alpha}(q)$, $\overline{G}_\delta \cong Sp_{2n_\delta}(q)$, $n_\alpha, n_\delta \geq 2$,

(b) [**b**] $|Z_\alpha Q_\delta/Q_\delta| = |Z_\delta Q_\alpha/Q_\alpha| = q^2$,

(c) [**c**] $[X_{\alpha\beta}, X_{\delta\gamma}] = 1$.

(d) [**d**] *One of the following holds:*

1. [**1**] $[X_{\alpha\beta}, Z_\delta] = [X_{\delta\gamma}, Z_\alpha]$, U_α *is the natural module for* G_α *and* U_δ *is the natural module for* G_δ.
2. [**2**] $q = 2$, $[X_{\alpha\beta}, Z_\delta] \neq [X_{\delta\gamma}, Z_\alpha]$ *and* $U_\alpha \cap Z(G_\alpha) = U_\delta \cap Z(G_\delta)$

4. [**4**]

(a) [**a**] $\overline{G}_\alpha \cong SL_{n_\alpha}(q)$ *and* $\overline{G}_\delta \cong SL_{n_\delta}(q)$

(b) [**b**] $|[Z_\alpha, Z_\delta]| = q$.

5. [**5**] *After interchanging* (α, β) *with* (δ, γ) *if necessary:*

(a) [**a**] $\overline{G}_\alpha \cong SL_{n_\alpha}(q)$, $n_\alpha > 2$ *and* $\overline{G}_\delta \cong Sp_{2n_\delta}(q)$, $n_\beta > 1$

(b) [**b**] $|Z_\alpha Q_\delta/Q_\delta| = |Z_\delta Q_\alpha/Q_\alpha| = q$,

(c) [**c**] $[X_{\delta\gamma}, Z_\alpha] = 1$

(d) [**d**] $|[Z_\alpha, Z_\gamma]| = q$

Proof:
Let $I_{\alpha\delta} = \{|[Z_\alpha, y]| \mid 1 \neq y \in Z_\delta Q_\alpha/Q_\alpha$ and $J_{\alpha\delta} = \{|[x, Z_\delta]| \mid x \in Z_\alpha \setminus C_{Z_\alpha}(Z_\delta)$

By **??**(**??**) implies $|[Z_\alpha, y]| = |[\widetilde{Z_\alpha}, y]|$ and $|[\widetilde{x}, Z_\delta]|$, for all $y \in Z_\delta$ and $x \in Z_\alpha$. definitionine the positive integer $k_{\alpha\delta}$ by $|\widetilde{Z_\alpha}/C_{\widetilde{Z_\alpha}}(Z_\delta)| = q_\alpha^{k_\alpha\delta}$ and note that

$$q_\alpha^{k_{\alpha\delta}} = |Z_\alpha Q_\delta/Q_\delta| = Z_\delta Q_\alpha/Q_\alpha| = q_\delta^{k_{\delta\alpha}}$$

Also Z_δ is a quadratic offender on Z_α and the action of $\overline{G_\alpha}$ on $\widetilde{Z_\alpha}$ implies:

$\overline{G_\alpha}$	$I_{\alpha\delta}$	$J_{\alpha\delta}$
$G_2(q_\alpha)$	$\{q_\alpha^2, q_\alpha^3\}$	$\{q_\alpha^2, q_\alpha^3\}$
$SL_{n_\alpha}(q_\alpha)$	$\{q_\alpha\}$	$\{q_\alpha\}$
$Sp_{2n_\alpha}(q_\alpha), k_{\alpha\delta} = 1$	$\{q_\alpha\}$	$\{q_\alpha\}$
$Sp_{2n_\alpha}(q_\alpha), k_{\alpha\delta} > 1$	$\{q_\alpha, q_\alpha^2\}$	$\{q_\alpha, q_\alpha^{k_{\alpha\delta}}\}$

Note that the definitions of $I_{\alpha\delta}$ and $J_{\alpha\delta}$ imply $I_{\alpha\delta} = J_{\delta\alpha}$. This allows as to relate $\overline{G}_\alpha$ and $\overline{G}_\delta$. In particular we see that

$$q := q_\alpha = q_\delta \quad \text{ and } \quad k := k_{\alpha\delta} = k_{\delta\alpha}.$$

Furthermore, $\overline{G}_\alpha \cong G_2(q_\alpha)$ we conclude that also $\overline{G}_\delta \cong G_2(q_\delta)$ So (a) holds in this case.

If $\overline{G}_\alpha \cong SL_{n_\alpha}(q_\alpha)$ and $n_\alpha > 2$, we get $\overline{G}_\alpha \cong SL_{n_\delta}(q_\delta)$ or $Sp_{2n_\delta}(q_\delta)$. In the latter cae we get $k = 1$. In any case since $n_\alpha > 2$, $|[Z_\alpha, Z_\gamma]| = q$ and so (4) or (5) holds.

If $\overline{G}_\alpha \cong Sp_{2n_\alpha}(q)$ and $\overline{G}_\delta \cong Sp_{n_\alpha}(q)$ we get $k \in \{1, 2\}$. If $k = 1$, (2) holds.

So suppose that $k = 2$. Then clearly $n_\alpha, n_\delta > 2$. We will show that (3) holds. We already prived (3)(a) and (b). Also both $[X_{\alpha\beta}, Z_\delta]$ and $[X_{\delta\gamma}, Z_\alpha]$ have order q. It follows that $X_{\alpha\beta}Q_\delta/Q_\delta$ is the unique full transvection group in $Q_{\gamma\delta}/Q_\delta$ and thus (3)(c) holds.

If $q > 2$, then $|[X_{\delta\gamma}, Z_\alpha]| = q$ implies that U_α is a natural module and so also $[X_{\alpha\beta}, Z_\delta] = [Z_\alpha, X_{\delta\gamma}] = U_\alpha \cap Z_{\alpha\beta}$. Thus (3) holds in this case.

So suppose that $q = 2$. Note that $U_\alpha \cap Z_{\alpha\beta} = [X_{\alpha\beta}, Z_\delta][Z_\alpha, X_{\delta\gamma}]$. If $[X_{\alpha\beta}, Z_\delta] = [Z_\alpha, X_{\delta\gamma}]$ we conclude that U_α is a natural module and (3) holds. If $[X_{\alpha\beta}, Z_\delta] \neq [Z_\alpha, X_{\delta\gamma}]$ we get that $U_\alpha \cap Z(G_\alpha)$ is the unique subgroup of order two in $[X_{\alpha\beta}, Z_\delta][Z_\alpha, X_{\delta\gamma}]$ distinct from $[X_{\alpha\beta}, Z_\delta]$ and $[Z_\alpha, X_{\delta\gamma}]$. The same is true for $U_\delta \cap Z(G_\delta)$ and again (3) holds. □

Lemma 5.10 [**q=2 for g2(q)**] *Let $(\alpha, \beta, \gamma, \delta)$ be as in Case 1. of 5.9. Then $q = 2$ and $U_\alpha \cap Z(G_\alpha) = U_\delta \cap Z(G_\delta)$.*

Proof: The following argument is taken from [MS].

Let $R = [Z_\alpha, Z_\delta]$ and $X = R \setminus \{[x, y] \neq 1 \mid x \in Z_\alpha, y \in Z_\delta]\}$. Then it is not too difficult to see that $X = C_{U_\alpha}(G_\alpha) = C_{U_\delta}(G_\delta)$. We will compare the actions of U_α/X on U_δ/X as seen in G_δ with the action of U_δ/X on U_α/X

as seen in G_α. Let $\mathbb{F}_\alpha = \mathrm{End}_{G_\alpha}(U_\alpha/X)$. Then $\mathbb{F}_\alpha$ is a field isomorphic to $GF(q)$.

Let

$$K_{\delta\alpha} = \{C_{U_\delta}(y) \mid y \in Z_\alpha, U_\delta \cap Q_\alpha < C_{U_\delta}(y) < U_\delta\}.$$

and similarly define $K_{\alpha\delta}$. If $A \in K_{\delta\alpha}$ then $C_{U_\alpha} A) \neq U_\alpha \cap Q_\delta$ and $C_{U_\alpha}(A)/R$ is a 1-dim. $\mathbb{F}_\alpha$-subspace of U_α/R. Also $C_{U_\alpha}(A) = C_{U_\alpha}(a)$ for all $a \in A \setminus Q_\alpha$. So $C_{U_\alpha}(A) \in K_{\alpha\delta}$ and we obtained a bijection between $K_{\alpha\delta}$ and $K_{\delta\alpha}$. Moreover, $\overline{A}$ is a long root subgroup of $\overline{G}_\alpha$. Let $t \in Z_\alpha$ with $[t, A] \neq 1$.

We show next that

(*) $[t, A]X/X$ is a 1-dim. $\mathbb{F}_\alpha$ and $\mathbb{F}_\delta$ subspace of R/X and a

Clearly it is a 1-dim $\mathbb{F}_\delta$- subspace. Let $P = C_{G_\alpha}(\overline{A})$. Then $W := U_\alpha/C_{U_\alpha}(A)$ is a natural module for $P/O_p(P) \cong SL_2(q)$. Let t^* be the image of t in W. Then $S := C_P(\tilde{t}^*)$ is a Sylow p-subgroup of P and so of G_α. Since S centalizes $[t, A]$ we conclude that $[t, A]X/X = C_{U_\alpha/X}(S)$, which is a 1-dim. $\mathbb{F}_\alpha$-space.

The preceeding argument also shows that every 1-dim. $\mathbb{F}_\alpha$ subspace of $[U_\alpha, A]X/X$ is of the form $[t, A]$ for some $t \in Z_\alpha$. Moreover each 1-dim. $\mathbb{F}_\alpha$ subspace of R/X is contained in $[U_\alpha, A]X/X$ for some $A \in K_{\delta\alpha}$. Thus (*) implies

(**) The $\mathbb{F}_\alpha$ and $\mathbb{F}_\delta$ subspaces in R/X coincide.

Let $W_{\alpha\beta} = [U_\alpha, O_p(P_{\alpha\beta})]X$ and $U_{\alpha\beta} = C_{U_\alpha}(O_p(P_{\alpha\beta})$. Then $U_{\alpha\beta}/X$ is a 1-dim. $\mathbb{F}_\alpha$ subspace of R/X. Moreover, $U_{\alpha\delta} \leq [U_\alpha, A]X$ for all $A \in K_{\delta\alpha}$. Considering the action of $U_\alpha Q_\delta/Q_\delta$ on U_δ/X we conclude that $U_{\alpha\beta} = U_{\gamma\delta}$.

Fix $z \in U_\alpha \setminus W_{\alpha\beta}$ and define $Y/U_{\delta\gamma} := C_{U_\delta/U_{\delta\gamma}}(z)$. Then Y/R is 1-dimensional $\mathbb{F}_\delta$ subspace of U_δ/R. Since $[Y, z] \leq U_{\delta\gamma} = U_{\alpha\delta}$ we also have $[Y, \mathbb{F}_\alpha zX/X] \leq U_{\alpha\delta}$. Since $[z, Q_{\alpha\beta}]R = W_{\alpha\beta}$, the Frattin-argument shows that $L := C_{P_{\alpha\beta}}(zR/R)$ has a quotient $SL_2(q)$.. Since L normalizes Y, we conclude that YQ_α/Q_a is a short root subgroup of $\overline{G}_\alpha$.

Hence there exists a subgroup M of $\overline{G}_\alpha$ with $YQ_\alpha/Q_\alpha \leq M$ and $M \cong SL_2(q)$. Note that for all $t \in Y_\alpha$, $[t, Y]X/X$ is an $\mathbb{F}_\delta$-submodule of R/X. Hence $[t, Y]X/X$ is also an $\mathbb{F}_\alpha$-submodule of U_α/X. But this implies that U_α/X is as an $\mathbb{F}_\alpha M$-module the direct sum three isomorphic natural module. But this implies $q = 2$. (For example let P be a mimimal parabolic of G_α/Q_a with M as a Levi complement, $V_1 = C_{U_\alpha/X}(O_p(P))$ and $V_2 = [U_\alpha/X, O_p(P)]/V_1$. Then $O_p(P)/\Phi(O_p(P))$ is isomorphic to a $\mathbb{F}_p$-submodule

of $\mathrm{Hom}_{\mathbb{F}_\alpha}(V_2, V_1)$. Since V_2 and V_1 are isomorphic $\mathbb{F}_\alpha M$ modules, we conclude that every composition factor for M in $O_p(P)$ is either natural or trivial. Thus $q = 2$.
Comment: a quote from [BBSM] would be more appropriate □

Lemma 5.11 [**b=1 sigma=2**] *Suppose that* $|\Sigma| = 2$, $\Sigma = \{\alpha, \beta\}$ *and* $[Z_\alpha, Z_\beta] \neq 1$. *Then for* $\gamma \in \Sigma$ *there exists* $K_\gamma \leq \Omega_1 \mathrm{Z}(G_\gamma)$ *and* $L_\gamma \leq G_\gamma$ *such that* $G_\gamma = K_\gamma \times L_\gamma$ *and one of the follwing holds.*

1. [**1**] $L_\alpha \sim L_\beta \sim q^n SL_n(q)$ *and* $|K_\alpha| = K_\beta| \leq q$.
2. [**2**] $p = 2$ *and (after interchanging* α *and* β *if necessary),* $G_\alpha = L_\alpha \sim q^{1+2n} Sp_{2n}(q)$, $G_\beta = L_\beta \sim q^{1+2+2\cdot(2n-2)} SL_2(q)$.
3. [**3**] $p = 2$, $L_\alpha \sim L_\beta \sim 2^6 G_2(2)$ *and* $|K_\alpha| = |K_\beta| \leq 2^3$.
4. [**4**] $p = 2$ *and* $G_\alpha = L_\alpha \sim G_\beta = L_\beta \sim q^{1+6+8} Sp_6(q)$.
5. [**5**] $p \neq 2$, $L_\alpha \sim L_\beta \sim q^{2n} Sp_{2n}(q)$, $n \geq 2$ *and* $|K_\alpha| = |K_\beta| \leq q$.
6. [**6**] $q = 2$, $G_\alpha \sim 2^{1+2n} Sp_{2n}(2)$ *and* $G_\beta \sim 2^{1+2+1\cdot m+1\cdot m+2\cdot k} SL_2(2)$ *for some* m, k *with* $m + k = n - 2$ *and* k *even.*
7. [**7**] *who knows*

Proof:
By assumption, $[Z_\alpha, Z_\beta] \neq 1$. Clearly $Z_\alpha Z_\beta \leq Q_{\alpha\beta}$ and we can apply 5.9 with $(\delta, \gamma) = (\beta, \alpha)$.

For $\{\gamma, \delta\} = \{\alpha, \beta\}$ define $H_\gamma = \langle Z_\delta^{G_\gamma} \rangle$. Let $R = [Z_\alpha, Z_\beta]$, $I = \{1 \neq [x, y] \mid x \in Z_\alpha, y \in Z_b\}$ and $D_\gamma = C_{Q_\gamma}(O^p(G_\gamma))$.

We devide the proof in a series of Steps.

Step 1 [**da cap db**] $D_\alpha \cap D_\beta = 1$.

Proof: This holds since $D_\alpha \cap D_\beta$ is normalized by $G_\alpha = O^p(G_\alpha) Q_{\alpha\beta}$ and $G_\beta = O^p(G_\beta) Q_{\alpha\beta}$. □

We call α non-abelian if α is symplectic, $p \neq 2$ and $n_\alpha \geq 2$. Otherwise α is called abelian.

Step 2 [**abelian**]

(a) [**a**] *α is abelian if only if $Q_{\alpha\beta}/Q_\alpha$ is elementary abelian.*

(b) [**b**] *If α is abelian, then $\Phi(Q_\beta) \leq D_\beta$.*

(c) [**c**] *If α and β are abelian, then $Q_\alpha \cap Q_b$ is elementary abelian.*

Proof: (a) is obvious. If $\Phi(Q_\beta) \leq Q_\alpha$, then Z_α centralizes $\Phi(Q_\beta)$ and so $\Phi(Q_\beta) \leq D_\alpha$. Thus (b) holds.

Since $\Phi(Q_\alpha \cap Q_\beta) \leq \Phi(Q_\alpha) \cap \Phi(Q_\beta)$, Step 1 and (b) imply (c). □

Step 3 [**b=1 case 1**] *Suppose that 5.9(1) holds. Then 5.11(3) holds.*

Proof: Note first that $Q_\alpha \leq Q_{\alpha\beta} = Z_\alpha Q_\beta$. Thus $Q_\alpha = Z_\alpha(Q_\alpha \cap Q_\beta)$ and Step 2(c) implies that Q_α is elementary abelian. Thus by 5.3(a), $Q_\alpha = Z_\alpha$. By 5.10, $q = 2$ and

$$U_\alpha \cap Z(G_\alpha) = U_\beta \cap Z(G_\beta) \leq D_\alpha \cap D_\beta = 1.$$

Thus $|U_\alpha| = 2^6$.

By [Schur, Schur Multiplier] we get $O^2(G_\alpha)/U_a \cong G_2(2)'$. Since $G_\alpha = Q_\alpha Z_\beta O^2(G_\alpha)$ and $[Q_\alpha, Z_\beta] \leq [U_\alpha, Z_\beta] \leq U_\alpha \leq O^2 * G_\alpha)$ we get that $G_\alpha/O^2(G_\alpha)$ is elementary abelian. Hence there exists $L_\alpha \leq G_\alpha$ with $G_\alpha = D_\alpha \times L_\alpha$ and $L_\alpha \sim 2^6 G_2(2)$. Since $D_\alpha \leq Z_{\alpha\beta}$ and $D_\alpha \cap D_\beta = 1$ we have $|D_\alpha| \leq |Z_{\alpha\beta}/D_\beta| = 2^3$, a the proof of Step 3 is complete. □

Step 4 [**b=1 case 2**] *Suppose that 5.9(2) holds. Then*

Proof:

Let $D_{\alpha\beta} = [Z_\alpha, Q_{\alpha\beta}]$ and $A_{\alpha\beta} = [D_{\alpha\beta}, Q_{\alpha\beta}] \leq Z_{\alpha\beta}$.

We will show first

1) [**6**] $[D_{\beta\alpha}, Q_\alpha] \leq \Omega_1 \mathrm{Z}(G_\alpha)$. *In particular, either $D_{\beta\alpha} \leq Z_\alpha$ or $A_{\beta\alpha} \leq \Omega_1 \mathrm{Z}(G_\alpha)$.*

Choose $\delta \in \beta^{G_\alpha}$ with $[Z_{\delta\alpha}, Z_\beta] \neq 1$. If $[D_{\delta\alpha}, D_{\beta\alpha}] \neq 1$, then

$$[D_{\beta\alpha}, Q_\alpha] \leq A_{\beta\alpha} = [D_{\beta\alpha}, D_{\delta\alpha}] \leq Z_{\alpha\beta} \cap Z_{\alpha\delta} \leq \Omega_1 \mathrm{Z}(G_a)$$

So suppose that$[D_{\delta\alpha}, D_{\beta\alpha}] = 1$. Then $[D_{\delta\alpha}, Z_\beta \leq Z_{\alpha\beta} \leq Z_\alpha$ and so $D_{\beta\alpha} Z_\alpha$ is normal in $G_\alpha = \langle Q_{\alpha\delta}, Z_\beta \rangle$. Hence also $[D_{\beta\alpha}, Q_\alpha]$ is normal in G_α. Since $Q_{\alpha\beta}$ centralizes $D_{\beta\alpha}$ and $G_\alpha = \langle Q_{\alpha\beta}^{G_\alpha} \rangle$, the first statement in 1) hold. If $[D_{\beta\alpha}, Q_\alpha] = 1$ then since $\Omega_1 \mathrm{Z}(Q_\alpha) = 1$ we get $D_{b\alpha} \leq Z_\alpha$. If $\mathrm{D}_{\beta\alpha}, Q_\alpha] \neq 1$, then $A_{\beta\alpha} = [D_{\beta\alpha}, Q_\alpha] \leq \Omega_1 \mathrm{Z}(G_\alpha)$, completing the proof of 1).

Next we prove:

2) [**7**] *If* $[D_{\beta\alpha}, Q_\alpha] = 1$*, then* $D_{\beta\alpha} \leq Z_\alpha \cap Q_\beta = D_{\alpha\beta} Z_{\alpha\beta} 3.4$ *implies .*

By 5.3, $D_{\beta\alpha} \leq Z_\alpha$. Also $D_{\beta\alpha} \leq Z_\beta \leq Q_\beta$ and so 2) holds.

3) [**8**] *If p is odd, then 1. or 5 of 5.11 holds.*

If $[D_{\beta\alpha}, Q_\alpha] \neq 1$, then by 1), $R = A_{\beta\alpha} = [D_{\beta\alpha}, Q_\alpha] \leq Z(G_\alpha)$ a contradiction. Thus $[D_{\beta\alpha}, Q_\alpha] = 1$ and by 2) $D_{\beta\alpha} \leq D_{\alpha\beta} Z_{\alpha\beta}$. By symmetry $D_{\alpha\beta} \leq D_{\beta\alpha} Z_{\alpha\beta}$. Hence $Z_\alpha \cap Z_\beta = Z_\alpha \cap Q_\beta = Z_\beta \cap Q_\alpha$. Thus $Z_\alpha \cap Z_\beta / Z_{\alpha\beta} = q^{2n_\alpha - 2}$ and $n_\alpha = n_b$. Since $Q_\alpha \leq Z_\alpha Q_\beta$ we get that $Q_\alpha \cap Q_\beta$ is elementary abelian, $Q_\alpha = Z_\alpha$ and $Q_\beta = Z_b$. Also $D_\alpha \leq Z(G_\alpha)$, $D_\alpha \leq Z_{\alpha\beta}$ and $D_\alpha \cap D_\beta = 1$. Thus $|D_\alpha| \leq q$. Hence 5. holds and 3) is proved.

We may assume from now on that $p = 2$. Set $D = D_{\alpha\beta} D_{\beta\alpha}$ and $T = C_{Q_{\alpha\beta}}(D)$. By **??** $Q_\alpha \cap Q_\beta$ is elementary abelian. Since $C_{Q_{\alpha\beta}}(D_{\alpha\beta} = Z_\beta Q_\alpha$ we have $T = Z_\alpha Z_\beta (Q_\alpha \cap Q_\beta)$. Since $p = 2$ we conclude that

4) [**10**] $\mathcal{A}(T) = \{Z_\alpha(Q_\alpha \cap Q_\beta), Z_\beta(Q_\alpha \cap Q_\beta)\}$

Let $A \in \mathcal{A}(Q_{\alpha\beta})$. Then $C_A(D_{\alpha\beta}) D_{\alpha\beta}$ is in $\mathcal{A}(Q_{\alpha\beta}$. Then $C_A(D) \in \mathcal{A}(T)$ and so $C_A(D) D = Z_\gamma(Q_\alpha \cap Q_b)$ for some $\gamma \in \{\alpha, \beta\}$. In particular, $C_A(D) D \leq Q_\gamma$. Let $\{\alpha, \beta\} = \{\gamma, \delta\}$. Since $E := C_A(D_{\delta\gamma}) D_{\delta\gamma} \in \mathcal{A}(Q_{\alpha\beta}$, E is an offender on Z_γ. Moreover, $C_E(D) \leq C_A(D) D \leq Q_\gamma$, the action of $Q_{\gamma\delta}$ on Z_γ implies $E \leq Q_\gamma$. Since $E \in \mathcal{A}(Q_{\alpha\beta}$ we conclude, $Z_\gamma \leq E$. Thus $[Z_\gamma, A] \leq [E, A] \leq [D_{\delta\gamma}, A]$. Suppose that $[Z_\gamma, A] \neq 1$, then also $[Z_\gamma, A] \not\leq Z(G_\gamma)$ and 1) implies $[D_{\delta\gamma}, Q_\gamma] = 1$. By 2), we get $D_{\delta\gamma} \leq D_{\gamma\delta} Z_{\gamma\delta}$, so $Z_\gamma \leq A D_{\gamma\delta} Z_{\gamma\delta}$ and thus $Z_\gamma = C_{Z_\gamma}(A) D_{\gamma\delta}$. This implies $[Z_\gamma, A] = 1$. So $[Z_\gamma, A] = 1$ and $A \leq Q_\gamma$. Hence

5) [**11**] $\mathcal{A}(Q_{\alpha\beta}) = \mathcal{A}(Q_\alpha) \cup \mathcal{A}(Q_\beta)$.

Since $Q_{\alpha\beta} = \mathrm{J}(Q_{\alpha\beta})$ we conclude $Q_{\alpha\beta} = \mathrm{J}(Q_\alpha)\mathrm{J}(Q_\beta)$. In particular $Q_\alpha \leq \mathrm{J}(Q_\alpha) Q_\beta$ and so $Q_\alpha = J(Q_\alpha)(Q_\alpha \cap Q_\beta)$. Since $Z_\alpha(Q_\alpha \cap Q_\beta) \in \mathcal{A}(Q_{\alpha\beta})$ we get $Q_\alpha = \mathrm{J}(Q_\alpha)$. Thus

6) [**12**] $Q_\alpha = \mathrm{J}(Q_\alpha), Q_\beta = \mathrm{J}(Q_b)$ *and* $Q_{\alpha\beta} = Q_\alpha Q_\beta$.

Let $A \in \mathbb{A}(Q_\alpha)$. Then $Z_\alpha \leq A$ and $C_A(D_{\beta\alpha}) D_{\beta\alpha} = Z_\alpha(Q_\alpha \cap Q_\beta)$. Thus $Q_\alpha \cap Q_\beta = (A \cap Q_\beta) D_{\beta\alpha}$ and $[Q_\alpha \cap Q_\beta, A] = [D_{\beta\alpha}, A] \leq A_{\beta\alpha} \leq Z_\beta$.
So

7) [**13**] $[Q_\alpha \cap Q_\beta, Q_\beta] \leq A_{\alpha\beta}$ *and* $[Q_\alpha \cap Q_\beta, Q_{\alpha\beta} \leq A_{\alpha\beta} A\beta\alpha \leq Z_{\alpha\beta}$

Let $\widehat{Q_\beta} = Q_\beta/Z_\beta$. We conclude that

8) [**14**] $[(\widehat{Q_\alpha \cap Q_\beta})Z_\beta, Q_\alpha] = 1$ *and* $[\widehat{Q_\beta}, Q_\alpha] \leq \widehat{Q_\alpha \cap Q_\beta}$

We will now prove

9) [**9**] *Suppose* $p = 2$*, and* $D_{\beta\alpha}Z_\alpha$ *is normal in* G_α*, then 1. or 2, of 5.11 holds.*

Since $[Q_\alpha, Z_\beta] \leq D_{\beta\alpha}$ and $[D_{\beta\alpha}, Z_\beta] = 1$ we get $[Q_\alpha, O^p(G_\alpha)] \leq Z_\alpha$. Let $\overline{Q}_\alpha = Q_\alpha/D_\alpha$. Then Q_α centralizes $\overline{Q_\alpha}$, $C_{\overline{Q_\alpha}}(O^p(G_\alpha)) = 1$ and $[\overline{Q_\alpha}, O^p(G_\alpha)] = \overline{U_\alpha}$ is a natural module. Thus the stucture of $\overline{Q_\alpha}$ is determined by 3.4. From $[Q_\alpha \cap Q_\beta, Z_\beta] = 1$, $Q_\alpha Q_\beta = Q_{\alpha\beta}$ and (*) we get $\overline{Q_\alpha \cap Q_\beta} = \overline{D_{\alpha\beta}}$. Hence $Q_\alpha \cap Q_\beta \leq D_\alpha D_{\alpha\beta}$ and so

$$Q_\alpha \cap \beta = (D_\alpha \cap Q_\beta)D_{\alpha\beta}$$

Since $[D_\alpha \cap Q_\beta, Q_\beta] \leq D_\alpha \cap D_\beta = 1$ we have $D_\alpha \cap Q_\beta \leq Z_\beta$. As Z_α centralizes D_α, $D_\alpha \cap Q_\beta \leq Z_\beta \cap Q_\alpha = D_{\beta\alpha}Z_{\alpha\beta}$. We conclude

$$Q_\alpha \cap Q_\beta = D_{\alpha\beta}D_{\beta\alpha}Z_{\alpha\beta} \text{ and } T = Z_\alpha Z_\beta = U_\alpha Z_\beta$$

Since Q_β centralizes $D_{\beta\alpha}$, 3.4 implies $D_{\beta\alpha} \leq D_\alpha Z_{\alpha\beta}$ and so

$$D_{\beta\alpha}Z_{\alpha\beta} = (D_\alpha \cap (D_{\beta\alpha}Z_{\alpha\beta})Z_{\alpha\beta}.$$

. Note that $r := |Q_\alpha/D_\alpha U_\alpha| \leq q$. Let $F = O^p(G_\alpha) \cap Q_{\alpha\beta}$. Then $U_\alpha \leq F$ and $|Q_{\alpha\beta}/Q_\alpha F| = e$, where $e = 2$ if $(n_\alpha, q) = (2,2)$ or $(1,2)$ and $e = 1$ otherwise. Since $D_{\beta\alpha} \leq D_\alpha Z_\alpha$, F centralizes $D_{\beta\alpha}$ and so $F \leq U_\alpha Q_\beta$ and $F = U_\alpha(F \cap Q_\beta)$. Let $F_1 = C_F(D_{\alpha\beta})$. Since F centralizes $D_{\beta\alpha}$, $F_1 \leq T = U_\alpha Z_\beta$. Since $U_\alpha \leq F_1$, $F_1 = U_\alpha(F_1 \cap Z_\beta)$.

Suppose that $G_\alpha/Q_\alpha \cong Sp_2(2)$. Then $Q_\alpha = D_a \times U_\alpha$. Moreover $Q_\beta \leq Z_\beta Q_\alpha$ and $Q_\beta = Z_\beta(Q_\alpha \cap Q_\beta) = Z_\beta D_{\alpha\beta} = Z_\beta$. Since $[D_\alpha, Z_\beta] \leq R \cap D_\alpha = 1$, $D_\alpha \leq Z_\beta$. Thus D_α is abelian and D_α is centralized by $D_\alpha U_\alpha Z_\beta = Q_{\alpha\beta}$. Thus $D_\alpha \leq Z_{\alpha\beta}$ and $Q_\alpha = Z_\alpha$. Hence $Z_\beta \cap Q_\alpha = Z_{\alpha\beta}$ and so $G_\beta/Q_\beta \cong Sl_2(2)$. Thus 1. or 2. of **??** holds in this case.

Suppose that $G_\alpha/Q_\alpha \notin \{Sp_2(2), Sp_4(2)\}$. Then $F_1 \cap Z_\beta \not\leq Q_\alpha$. Since D_α centralizes $F_1 \cap Z_\beta$ we conclude that $D_\alpha \leq Q_\beta$. Since $|Q_{\alpha\beta}/D_\alpha(F \cap Q_\beta)Z_\beta \leq rq \leq q^2$ we get $|Q_{\alpha\beta}/Q_\beta| \leq q^2$ and so $n_\beta = 1$. Thus $D_{\beta\alpha} \leq Z_{\alpha\beta}$ and so $Q_\alpha \cap Q_\beta = D_{\alpha\beta}Z_{\alpha\beta} = Z_\alpha \cap Q_\beta)$. Moreover, $Q_\alpha \leq U_\alpha Q_\beta$ and so $Q_\alpha = U_\alpha(Q_\alpha \cap Q_\beta) = Z_\alpha$. Assume that $(Z_\alpha \cap Q_\beta)Z_\beta$ is normal in

G_β. If $G_\beta/Q_\beta \cong SL(2)$, the preceeding paragraph gives a contradiction. If $G_\beta/Q_b \cong Sp_4(2)$??? And if $G_\beta/Q_\beta \notin \{Sp_2(2), Sp_4(2)\}$, the first half of this paragraph applied with the roles of α and β reversed, gives $n_\alpha = 1$. But then case (1) or (2) holds. Assume now that $(Z_\alpha \cap Q_\beta)$ is not normal in G_β. Let $W = (Z_\alpha \cap Q_\beta)Z_\beta$, $V = \langle W^{G_\beta}\rangle$ and $U = \bigcap_{g\in G_\beta} W^g$. Since $[W, Q_\beta] \leq Z_\beta \leq U$ and $[V, Q_\alpha] \leq Q_\alpha \cap Q_\beta \leq W$ we have $[V, Q_{\alpha\beta} \leq W$ and $[W, Q_{\alpha\beta} \leq U$. Thus we can apply 3.3 to V/U and conclude that $W = [Z_\alpha, V]U$. Hence

$$Z_\alpha \cap Q_\beta = [Z_\alpha, V](Z_\alpha \cap U$$

We claim that $Z_\alpha \cap U = C_{Z_\alpha}(V)$. Indeed, $U \leq Z(V)$ and so $Z_\alpha \cap V \leq C_{Z_\alpha}(V)$. For the converse let $g \in G_\beta$. Then $[C_{Z_\alpha}(V), Z_\alpha^g] \leq R^g \leq Z_\alpha$ and so $C_{Z_\alpha}(V)Z_\beta$ is normal in G_β. Thus $C_{Z_\alpha}(V) \leq U$. This proves the claim and so

$$Z_\alpha \cap Q_\beta = [Z_\alpha, V]C_{Z_\alpha}(V).$$

The action of $Q_{\alpha\beta}$ on Z_α implies $[Z_\alpha, V] \cap C_{Z_\alpha}(V) \leq Z_{\alpha\beta}$. Let $V^* = [V, H_\beta]$. Since H_β is generated by two conjugates of Z_α we derive

$$V/Z_\beta = V^*/Z_\beta \times U/Z_\beta$$

.

$U \leq X \leq Z(V)$ with $[X, Q_{\alpha\beta} \leq U$. Then $X \leq W$ and so $X = Z_\beta(X \cap Z_\alpha)$. Since $Z(V) \cap Z_\alpha \leq U$ we conclude that $X \leq Z(V)$. Since $Q_{\alpha\beta}$ normalizes $Z(V)/U$ we get $U = Z(V)$. Since $[W, Q_\beta] = A_{\alpha\beta}$ and $\Phi(Q_\beta \leq D_\beta$ we get that $A_\beta := A_{\alpha\beta} \leq Z(G_\beta)$ and $A_\beta = [V, Q_\beta]$. Hence also $[V^*, Q_\beta] = A_\alpha$. Put $D^* = C_{Q_\beta}(V^*)$. Then Q_β/D^* is dual to V^*/Z_β as G_β module. Hence $Q_\beta = V^*D^*$. Note that $[D^*, O^p(G_\beta) \leq Z_\beta$. Suppose that $q \neq 2$. Then

$$[Z_\alpha, Q_\beta] \leq ([D^*V^*O^p(G_\beta), D^*] \cap Z_\alpha)[Z_\alpha, V] \leq (D_\beta \cap Z_\alpha)[Z_\alpha, V]$$

But $D_\beta \cap Z_\alpha$ is □ □

For $\alpha \in \Sigma$ let

$$\Sigma_1(\alpha) = \{\beta \in \Sigma \mid [Z_\alpha, Z_\beta] \neq 1\}$$

and

$$\Sigma_2(\alpha) = \{\beta \in \Sigma \mid [Z_\alpha, Z_\beta] = 1 \neq [Z_\alpha, V_\beta^\alpha]\}$$

Lemma 5.12 *Let $\alpha \in \Sigma$ and $\beta \in \Sigma_1(\alpha)$. definitionine $L := \langle G_\alpha, G_\beta \rangle$, $L^* := \langle \Omega_1 \mathrm{Z}(R)^L \rangle$, $K := O_R(\{G_\alpha, G_\beta\})$ and $\widetilde{L} := L/K$. For $\{\alpha, \beta\} = \{\gamma, \delta\}$, put $K_\gamma = C_{Q_\gamma}(\langle Z_\delta^{G_\gamma} \rangle)$. Then for $\gamma \in \{\alpha, \beta\}$ there exists a normal subgroup L_γ of G_γ such that*

(a) [**a**] $[K, L^*] = 1$.

(b) [**b**] $K = K_\alpha \cap K_\beta$ *and* $\Phi(K_\alpha K_\beta) \leq K$.

(c) [**c**] $G_\alpha = K_\alpha L_\alpha$ *and* $G_\beta = K_\beta L_\beta$.

(d) [**d**] *Interchanging α and β if necessary one of the following holds (where q is a power of p.*

1. [**1**] $\widetilde{L}_\alpha \sim \widetilde{L}_b \sim q^n SL_n(q)$.
2. [**2**] $p = 2$, $\widetilde{L}_\alpha \sim q^{1+2n} Sp_{2n}(q)$, *and* $\widetilde{L}_\beta \sim q^{1+2+2\cdot(2n-2)} SL_2(q)$.
3. [**3**] $p = 2$ *and* $\widetilde{L}_\alpha \sim \widetilde{L}_\beta \sim 2^6 G_2(2)$
4. [**4**] $p = 2$ *and* $\widetilde{L}_\alpha \sim \widetilde{L}_\beta \sim q^{1+6+8} Sp_6(q)$.
5. [**5**] *Who knows.*

Proof: Note that K is normal in L and $K \leq R$, indeed K is the largest normal subgroup of L contained in R. Let $g \in K$ then

$$[\Omega_1 \mathrm{Z}(R)^g, K] = [\Omega_1 \mathrm{Z}(R)^g, K^g = [\Omega_1 \mathrm{Z}(R), K]^g = 1.$$

Thus (a) holds.

Let $H_\gamma = \langle Z_\delta^{G_\gamma} \rangle$, $R = [Z_\alpha, Z_\beta]$ and $D_{\beta\alpha} = [Z_\beta, Q_{\alpha\beta}$.

Note that by (a), $K \leq K_\alpha \cap K_\beta$ also $K_\alpha \cap K_\beta$ is normalized by

$$\langle O^2(G_\alpha), O^2(G_\beta), Q_{\alpha\beta} \rangle = L.$$

Thus $K = K_\alpha \cap K_b$. So the first part of (b) holds. By definition $[K_\alpha, Z_\beta] = 1$ and so $K_a \leq Q_\beta$. Thus $\Phi(K_\alpha) \leq \Phi(Q_\beta) \cap K_\alpha$. Note that $\Phi(Q_\beta) \leq \Phi(Q_{\alpha\beta})$. Since $Q_{\alpha\beta}/Q_\alpha$ is elementary abelian, unless α is symplectic, $n_\alpha > 1$ and $p \neq 2$, we get

(*) $\Phi(K_\alpha) \leq K$ and $[\Phi(Q_\beta), H_\beta] = 1$, unless α is symplectic, $n_\alpha > 1$ and $p \neq 2$.

Note that by definition of $\Sigma_1(\alpha)$, $[Z_\alpha, Z_\beta] \neq 1$. Clearly $Z_\alpha Z_b \leq Q_{\alpha\beta}$ and we can apply 5.9 with $(\delta, \gamma) = (\beta, \alpha)$.

Suppose that Case c.1 of 5.9 holds. Then $Q_\alpha \leq Q_{\alpha\beta} = Z_\alpha Q_\beta$. Since Q_α normalizes Z_β, H_α is generated by two conjugates of Z_β. Thus $|Q_\alpha/K_\alpha| \leq q^6$ and so $Q_\alpha = K_\alpha U_\alpha$. By 5.10, $q = 2$ and $U_\alpha \cap Z(G_\alpha) = U_\beta \cap Z(G_\beta)$. Thus $U_\alpha \cap Z(G_\alpha) \leq K$ and $|\widetilde{U}_\alpha| = 2^6$. Using [Schur, Schur Multiplier] we get $O^2(G_\alpha)/U_a \cong G_2(2)'$ also by (*) $G_\alpha/O^2(G_\alpha)K$ is elementary abelian. Hence there exists $L_\alpha \leq G_\alpha$ with $O^2(G_\alpha)K \leq L$, $G_\alpha = K_\alpha L_\alpha$ and $L_\alpha \cap K_\alpha = K$. Thus d.3 holds in this case.

Suppose next that Case c.2 of 5.9 holds.

Suppose that $n_\beta = 1$. Then $[Q_\alpha, Z_\beta] \leq [Z_\alpha, Z_\beta] \leq U_\alpha$ and so $[Q_\alpha, H_\alpha] \leq U_\alpha$. Also $\Phi(Q_\alpha) \leq Q_\beta$ and so $[\Phi(Q_\alpha), H_\alpha] = 1$. Suppose that also $n_\alpha = 1$. Then H_α is generated by two conjugates of Z_β and we conclude that $|Q_\alpha/K_\alpha| = q^2$ and $Q_\alpha = K_\alpha U_\alpha$. Let $I = \{1 \neq [x,y] \mid x \in Z_\alpha, y \in Z_b\}$. If $q \leq |[Z_\alpha, Z_\beta]| < q^2$ then $U_\alpha \cap Z(G_\alpha) = [Z_\alpha, Z_b] \setminus I = U_\beta \cap Z(G_\beta)$ and thus d.1 holds. If $|[Z_\alpha, Z_\beta]| = q^2$, then $[Z_\alpha, Z_b] \setminus I$ contains exactly two subgroups of order q and these two subgroups have trivial intersection. Hence either $U_\alpha \cap Z(G_\alpha) = U_\beta \cap Z(G_\beta)$ and d.1 holds; or $U_\alpha \cap Z(G_\alpha) \cap U_\beta \cap Z(G_b) = 1$ and d.2 holds.

Suppose next that $n_\beta > 1$ and that $D_{\beta\alpha} Z_\alpha$ is normal in G_α. Then $A_\alpha := [D_{\beta\alpha}, Q_{\alpha\beta} = [D_{\beta\alpha} Z_\alpha, Q_\alpha]$ is normal in G_α. Since $Q_{\alpha\beta}$ centralizes A_α we get $A_\alpha \leq Z(G_\alpha)$. Let $D_\alpha := C_{Q_\alpha}(O^p(G_\alpha))$. We conclude that $D_{\alpha\beta} \leq U_\alpha D_\alpha$ and $D_{\alpha\beta} \leq D_\alpha Z_{\alpha\beta}$. Note that $[Q_\alpha, Z_\beta] \leq D_{\beta\alpha}$ and so $[Q_\alpha, H_\alpha] \leq U_\alpha D_\alpha$.

Note that $|RA_\alpha/A_a| \geq q$ and so $p = 2$ and $|U_\beta \cap Z(G_\beta)|$ =q. By (*) $[\Phi(Q_\alpha), H_\alpha] = 1$. Thus $|Q_\alpha/U_\alpha D_\alpha| \leq q$. Note that $O^2(G_\alpha) \cap Q_{\alpha\beta}$ centralizes $D_\alpha Z_{\alpha\beta}$and so we have $O^2(G_\alpha) \cap Q_{\alpha\beta} \leq C_{Q_{\alpha\beta}}(D_\beta \alpha) = Z_\alpha Q_b$. Note also that $Z_\beta \leq Q_\beta$, $G_\alpha = O^2(G_a) Z_\beta$ and $Z_\alpha \leq Q_\alpha$. Thus $Q_{\alpha\beta} = Q_\alpha Q_\beta$.

If $q > 2$, then $A_a \leq R$ and we conclude that $A_\alpha = U_\alpha \cap Z(G_\alpha)$.

Let $\gamma \in \beta^{G_\alpha}$ with $[Z_{\gamma\alpha}, Z_\beta] \neq 1$.

Lemma 5.13 [**sigma symmetric**] *Let $\alpha, \beta \in \Sigma$ and $i \in \{1, 2\}$. Then $\alpha \in \Sigma_i(\beta)$ if and only if $\beta \in \Sigma_i(\alpha)$.*

Proof: For $i = 1$ this is obvious. Suppose now that $\beta \in \Sigma_2(\alpha)$ but $\alpha \notin \Sigma_2(\beta)$. The $Z_a Z_\beta \leq Q_\alpha \cap Q_b$, $V_\beta^\alpha \not\leq Q_\alpha$ and $V_\alpha^\beta \leq Q_\beta$.

Lemma 5.14 [**vdelta non abelian**] *There exists an edge (γ, δ) in Γ such that $\langle Z_\delta^{G_\gamma} \rangle$ is not abelian.*

Proof: Suppose not. Let $V = \langle Z_L \rangle L \in \Sigma$ and $Q = \bigcap O_p(L) \mid L \in \Sigma$. Then $V \leq Q$ and so $Q \neq 1$. Let $L \in \Sigma$. Then $Q = \bigcap (O_p(L) \cap O_p(H) \mid L \neq H \in \Sigma$ and so by 5.7 Q is normal in L. Hence Q is a non-trivial subgroup of R which is normal in all the $L\Sigma$, a contradiction. $\square$

Some ideas on the rest of the proof. definitionine a relation $\approx$ on Σ by $L \approx H$ if $\langle Z_L^H \rangle$ is not abelian or if $Z_L = Z_H$. This should be an equivalence relation and $L \approx H$ if and only if $O_p(L) \cap O_p(H)$ is not normal in L. If $L \not\approx H$ we should have $[(R \cap O^p(L), O^p(H)] = 1$. $b = 2$ (that is $L \approx H$ and $Z_L \leq O_p(H)$) seems to occur only for the $G_2(3^k)$ situation, and $2^{1+4+6}L_4(2)$

What still needs to be discussed in this section is the consequences of 5.1 for the sets $\mathcal{PU}_i$, $i = 1, 2, 4$. There are some interesting cases: for example an amalgam if Z_L is the 6-dimensional module for $L/O_2(L) \cong 3Alt(6)$ then $L \in \mathcal{PU}_4(R)$. Same for $Alt(6)$ or $Alt(7)$ on the four dimensional module.

Also it seems possile to enlarge the set $\mathcal{PU}_3$ without having to change the "$b < 3$" part of the proof of 5.1. Namely can drop the assumption on $N_L(R)$ containing a point stabilizor one can allow $[Z_L, L]$ to be the four dimensionnal module for $SL_3(2)$, This would be usefull for the $\neg E!$ case. Other exceptional FF-modules could be included to. The properties one really needs is: no over-offenders and good commutator control. For example Alt(n) on the natural module should be o.k. This also would be o.k for $D_10(q)$ on the 16-dimensional spinmodule and $L_n(q), n \geq 5$ on the exterior square. But the choice of $a \in A$ will cause some problems. Might not be so important though, maybe we only need $\bigcap_{a \in A} Z_\gamma^a \leq \Omega_1 \mathrm{Z}(G_\delta)$.

6 The C(G,T)-Theroem

Suppose that G fullfills CGT. Then S is contained in unique maximal subgroup M of G, but there exists $L \in \mathcal{L}(S)$ such that $L \not\leq M$ and $|L \cap M|_p \neq 1$. Choose such an L such that $|H \cap L|_p$ is maximal. Let T be a Sylow p-subgroup of $H \cap T$. Without loss $T \leq S$. If $T = S$ we get that $L \in \mathcal{L}(S)$ contradicting our assumption M is the unique maximal p-local subgroup of M. Thus $T \neq S$. Let C be a non-trivial characteristic subgroup of S. Then $N_S(T) \leq N_G(C)$ and so $|M \cap N_G(C)|_p > |M \cap L|$ Hence the maximal choice of $|M \cap L|_p$ implies $N_G(C) \leq M$. In particular, $N_L(C) \leq M \cap L$. For $C = S$ we conclude that $T \in \mathrm{Syl}_p(T)$. Then we can apply the

Theorem 6.1 (Local C(G,T)-Theorem) [**local CGT**] *Let L be a finite $\mathcal{K}_p$ group of characteristicp, T a Sylow p-subgroup of L, and suppose that*

$$C(L,T) := \langle N_L(C) \mid 1 \neq C \text{ a characteristic subgroup of } S \rangle$$

is a proper subgroup of L. Then there exists a L-invariant set $\mathcal{D}$ of subnormal subgroup of L such that

(a) [**a**] $L = \langle \mathcal{D} \rangle C(L,T)$

(b) [**b**] $[D_1, D_2] = 1$ *for all* $D_1 \neq D_2 \in \mathcal{D}$.

(c) [**c**] *Let* $D \in \mathcal{D}$*, then* $D \not\leq C(L,T)$ *and one of the following holds:*

1. [**1**] $D/Z(D)$ *is the semidirect product of* $SL_2(p^k)$ *with a natural module for* $SD_2(p^k)$*. Moreover* $O_p(D) = [O_p(D), D]$ *is elementary abelian.*
2. [**2**] $p = 2$ *and* D *is the the semidirect product of* $Sym(2^k + 1)$ *with a natural module for* $\mathrm{Sym}(2^k + 1)$.
3. [**3**] $p = 3$*,* D *is the semidirect product of* $O_3(D)$ *and* $SD_2(3^k)$*,* $Z(D) = O_p(D)$ *has order* 3^k *and both* $[Z(O_3(D)), D]$ *and* $O_3(D)/Z(O_3(D))$ *are natural* $SL_2(3^k)$ *modules for* D.

For $p = 2$ the local $C(G,T)$-theorem was proved by Aschbacher in [Asch]. For general p by GLS?. For us it will be consequence of the **??**.

Back to G. Case 3 can be rules out using that $N_S(T)/T$ is odd. Let $m = |\mathcal{D}|$ and suppose that $m > 1$. Let $g \in N_S(T) \setminus T$. Then there exists $X, Y \in \mathcal{D}$ such that $R := [[V, X], [V, Y]^g] \neq 1$. Let $H = N_G(R)$. Then for all $Z \in \mathcal{D}$ with $D \neq D$, $D \leq N_G(R)$ and since $[[V, D], V^g] \neq 1$, $[V, D] \not\leq O_p(N_{L^g}(R)$. Thus $[V, D] \not\leq O_p(H)$. Let $U = O_p(H)$. We conclude that $[Q \cap T, D] = 1$. Since H is of characteristic p, D acts non-trivially on $Q/Q \cap T$.

Let $T^* \in \mathrm{Syl}_p(H)$ with $N_T(R) \leq T^*$. The maximal choice of $|T|$ implies $|T^*/N_T(R)| \leq |T/N_T(R)| = T/N_T(X)$. In particular $|U/U \cap T| \leq |T/N_T(X)$. Thus T does not normalize X. Let $e := |T/N_T(X)|$. Then there are at least $e - 1$ choices for D, each two of whcih commute and each acting non-trivialy on $U/U \cap T$ whci has order at most e. This is impossible.

Hence there exists a unique $D \in \mathcal{D}$.

Suppose that case 2. holds and $n \geq 3$. Then $O_2(M \cap L) = O_2(L)$. Let $Q = O_2(M)$. Then $T \cap Q \leq O_2(M \cap L) \leq O_2(L)$. On the otherhand the maximality of $|T|$ implies $N_Q(O_2(L)) \leq T$. Thus $N_Q(O_2(L)) \leq O_2(L)$ and so $Q \leq O_2(L)$.

If Q is not elementary abelian that $[\Phi(Q), D] = 1$ implies $D \leq M$, a contradiction. Hence Q is elementary abelian.

Since $[Q, O_2(D)] = 1$ and M is of characteritic p we conclude $O_2(D) \leq Q$. Thus $[Q, D] \leq [O_2(L), D] \leq O_2(D) \leq Q$ and so $D \leq N_G(Q) \leq M$. Thus also $L = D(M \cap L) \leq M$, a contradiction.

Suppose that case 2 holds and $n = 2$. Then we can choose $x \in [V, D]$ so that $R := [V^g, x]$ has order two. Also $C_D(x)$ is divisible by 3 and $[V, O^2(C_D(x))], C_{D^g}(x)]$ is not a 2-group. Argue as above we get $C_D(x)$ acts non trivially on $Q/Q \cap T$. But $|Q/Q \cap T$ has order 2 a contradiction.

Thus Case 1. holds. We have proved:

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