The P!-Theorem

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Let H be a finite group and p be a prime dividing the order of H. Then H is of **characteristic p** if $C_H(O_p(H)) \leq O_p(H)$; and H is of **local characteristic p** if every p-local subgroup of H is of characteristic p. Moreover, H is a $\mathcal{K}_p$**-group** if the simple sections of the p-local subgroups are "known" simple groups[1].

Every group with a self-centralizing cyclic Sylow p-subgroup, as for example the alternating group A_p, is of local characteristic p, and these groups are particular examples of groups with a strongly p-embedded subgroup. Apart from such groups, all groups of Lie type in characteristic p of rank at least 2 and some sporadic groups (for suitably chosen p) have local characteristic p. Therefore it would be a major contribution to a revision of the classification of the finite simple groups to give a classification of all finite groups of local characteristic p that do not have a strongly p-embedded subgroup. This is the goal of a project initiated by U. Meierfrankenfeld. For an overview of this project see [MSS1].

The part of the project our paper deals with uses the following hypothesis:

Q!-Hypothesis. H is a finite $\mathcal{K}_p$-group of local characteristic p, $S \in Syl_p(H)$ and $Z := \Omega_1(Z(S))$. There exists a maximal p-local subgroup $\widetilde{C}$ of H with $N_H(Z) \leq \widetilde{C}$ such that for $Q := O_p(\widetilde{C})$

$$C_H(x) \leq \widetilde{C} \text{ for every } 1 \neq x \in Z(Q). \quad \textbf{(Q-Uniqueness)}$$

In the subdivision given in [MSS1] this hypothesis refers to the E!-case, see [MSS1, Lemma 2.4.2], and we will prove the P!-Theorem, as it was announced in section 2.4.2 of [MSS1]. To state this result we need some further notation.

[1] Which means, they are groups of prime order, groups of Lie type, alternating groups or one of the 26 sporadic groups.

Throughout this paper $S \in Syl_p(H)$, and Z, $\widetilde{C}$ and Q are as in the above hypothesis. Moreover

$$C := C_H(Z),\ B(T) := \Omega_1(Z(J(T)))\ (T \text{ a } p\text{-subgroup}),\ X^0 := \langle Q^X \rangle\ (X \text{ a subgroup}).$$

A subgroup $P \leq H$ is called **minimal parabolic** (with respect to p), if P is not p-closed and every Sylow p-subgroup of P is contained in a unique maximal subgroup of P.

Let X and M be subgroups of H, and let T be a p-subgroup of H:

$Loc_M(X) := \{U \leq M \mid X \leq U \text{ and } C_M(O_p(U)) \leq O_p(U)\}$,

$\mathcal{M}_M(X)$ is the set of maximal elements of $Loc_M(X)$.

$\mathcal{L}_M(T) := \{U \in Loc_M(T) \mid T \in Syl_p(U)\}$,

$\mathcal{P}_M(T) := \{P \in \mathcal{L}_M(T) \mid \ P \text{ is minimal parabolic}\}$,

According to (1.2) below every element $U \in Loc_M(X)$ contains a unique maximal elementary abelian normal subgroup Y_U satisfying $O_p(U/C_U(Y_U)) = 1$.

Let $P \in \mathcal{P}_H(S)$ and $B(P) := \langle B(S)^P \rangle$. Then P is said to be of **type** L_3, if p is odd, $O_p(P) = Y_P \leq B(S)$, $B(P)/Y_P \cong SL_2(p^m)$, and Y_P is a natural $SL_2(p^m)$-module for $B(P)/Y_P$.

Hypothesis I. The Q!-Hypothesis holds, and there exists $P \in \mathcal{P}_H(S)$ such that $P \not\leq \widetilde{C}$ and $Y_M \leq Q$ for every $M \in \mathcal{M}_H(P)$.

In this paper we prove:

P!-Theorem. Assume Hypothesis I. Let $P^* := P^0 O_p(P)$ and $Z_0 := \Omega_1(Z(S \cap P^*))$. Then the following hold:

(a) $P^*/O_p(P) \cong SL_2(p^m)$ and Y_P is a natural $SL_2(p^m)$-module for $P^*/O_p(P)$.

(b) Z_0 is normal in $\widetilde{C}$; in particular $P \cap \widetilde{C}$ is the unique maximal subgroup of P containing S.

(c) Then either P is the unique element of $\mathcal{P}_H(S)$ not in $\widetilde{C}$, or every element of $\mathcal{P}_H(S) \setminus \mathcal{P}_{\widetilde{C}}(S)$ is of type L_3.

The proof of the P!-Theorem uses the Structure Theorem, which was proved in [MSS2]. To state this result we need some further notation. Let

$$\overline{\mathcal{L}}_H(S) := \{U \in \mathcal{L}_H(S) \mid C_H(Y_U) \leq U\}.$$

For $U, \widetilde{U} \in \overline{\mathcal{L}}_H(S)$ define

$$U << \widetilde{U} \iff U = (U \cap \widetilde{U})C_U(Y_U).$$

Then (1.5) below shows that $<<$ is a partial order on $\overline{\mathcal{L}}_H(S)$. Let

$$\mathcal{L}^*_H(S) = \{L \in \overline{\mathcal{L}}_H(S) \mid \ L \text{ is maximal with respect to } <<\}.$$

Note that $\mathcal{M}_H(S) \subseteq \overline{\mathcal{L}}_H(S)$ and $\mathcal{L}^*_H(S) \subseteq \mathcal{M}_H(S)$, if H has local characteristic p.

Structure-Theorem. Assume the Q!-Hypothesis. Suppose that there exists $M \in \mathcal{L}^*_H(S) \setminus \{\widetilde{C}\}$ such that $Y_M \le Q$. Then for $M_0 := M^0 C_S(Y_M)$ and $\overline{M} := M/C_M(Y_M)$ one of the following holds:

(a) $F^*(\overline{M}) = \overline{M}_0'$, $\overline{M}_0 \cong SL_n(p^m)$, $n \ge 2$, $Sp_{2n}(p^m)$, $n \ge 2$, or $Sp_4(2)'$ (and $p = 2$), and $[Y_M, M_0]$ is the corresponding natural module for $\overline{M}_0$. Moreover, either $C_{M_0}(Y_M) = O_p(M_0)$ or $p = 2$ and $M_0/O_p(M_0) \cong 3Sp_4(2)'$.

(b) $P_1 := M_0 S \in \mathcal{P}_H(S)$, $Y_M = Y_{P_1}$, and there exists a a normal subgroup $P_1^* \le P_1$ containing $C_{P_1}(Y_{P_1})$ but not Q such that

(i) $\overline{P}_1^* = K_1 \times \cdots \times K_r$, $K_i \cong SL_2(p^m)$, $Y_M = V_1 \times \cdots \times V_r$, where $V_i := [Y_M, K_i]$ is a natural K_i-module,

(ii) Q permutes the components K_i of (i) transitively,

(iii) $O^p(P_1^*) = O^p(M_0)$, and $P_1^* C_M(Y_M)$ is normal in M,

(iv) $C_{P_1}(Y_{P_1}) = O_p(P_1)$, or $r > 1$, $K_i \cong SL_2(2)$ (and $p = 2$) and $C_{P_1}(Y_{P_1})/O_2(P_1)$ is a 3-group.

We will refer to property (b) (ii) of the Structure Theorem as **Q-transitivity**. As a corollary of the Structure- and the P!-Theorem we get:

Corollary. Assume Hypothesis I. Then for every $L \in Loc_H(P)$ the following hold, where $\overline{L} := L/C_L(Y_L)$ and $L_0 = L^0 C_S(Y_L)$:

(a) $F^*(\overline{L}) = \overline{L}_0'$, $\overline{L}_0 \cong SL_n(p^m)$, $Sp_{2n}(p^m)$ or $Sp_4(2)'$ (and $p = 2$), and $[Y_L, L_0]$ is the corresponding natural module.

(b) Either $C_{L_0}(Y_L) = O_p(L_0)$, or $p = 2$, $L_0/O_p(L_0) \cong 3Sp_4(2)'$ and $LC_H(Y_L) \in \mathcal{L}^*_H(S)$.

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1. Elementary Properties.

(1.1) Let $X = S_k$ and V be the non-central irreducible constituent of the $GF(2)$-permutation module for X.

(a) Let $k = 2m+1$ and set $t_i := (2i-1, 2i)$ and $d_i = (2i-1, 2i, k)$, $i = 1, ..., m$. Then $X = \langle t_i, d_i \mid i = 1, ..., m \rangle$.

(b) Let t be a transposition of X and $x \in X$ such that $[V, t, x] = 0$. Then $k = 4$ or $t^x = t$.

(c) Let $k \neq 4$, $t_1, ..., t_m$ be a maximal set of commuting transpositions and $V_0 = C_V(t_1, ..., t_m)$. Then $C_X(V_0) = \langle t_1, ..., t_m \rangle$.

Proof. (a): It is well known that $\Omega := \{(k, k+1) \mid k = 1, ..., 2m\}$ is a generating set for X. Thus the claim follows from the fact that

$$t_m^{d_m} = (2m, 2m+1) \text{ and } t_i^{d_i d_{i+1}} = (2i, 2i+1),\ i = 1, ..., m-1.$$

(b): Let $W = \langle v_1, ..., v_k \rangle$ be the $GF(2)$-permutation module for X with basis $\{v_1, ..., v_k\}$, where $v_i x := v_{ix}$ for $x \in X$. Set

$$W_0 := \langle \sum_{i=1}^{k} v_i \rangle,\ W_1 := \langle v_i + v_j \mid i, j \in \{1, ..., k\} \rangle \text{ and } \overline{W}_1 := (W_1 + W_0)/W_0.$$

Then $V = \overline{W}_1$. Let $t = (i, j)$ and $t^x = (r, s)$, so

$$\langle \overline{v}_i + \overline{v}_j \rangle = [\overline{W}_1, t] = [\overline{W}_1, t^x] = \langle \overline{v}_r + \overline{v}_s \rangle.$$

It follows that $v_i + v_j + v_r + v_s \in W_0$, and either $\{i, j\} = \{r, s\}$ and $t = t^x$, or $k = 4$.

(c): This is a direct consequence of (b).

(1.2) Let U be a finite group of characteristic p, $T \in Syl_p(U)$ and $T \leq \widetilde{U} \leq U$. Then the following hold:

(a) There exists a unique maximal elementary abelian normal p-subgroup Y_U of U such that $O_p(U/C_U(Y_U)) = 1$.

(b) $Y_{\widetilde{U}} \leq Y_U$.

(c) $\Omega_1(Z(T)) \leq Y_U$.

(d) If $U = \widetilde{U}C_U(Y_U)$ then $Y_U = Y_{\widetilde{U}}$.

(e) If $O_p(U) = C_T(Y_U)$ then $Y_U = \Omega_1(Z(O_p(U)))$.

Proof. (a): Let Ω be the set of all elementary abelian normal p-subgroups X of U satisfying $O_p(U/C_U(X)) = 1$. For the existence of a unique maximal element in Ω it suffices to show that the product of two elements of Ω is again in Ω.

Let $A_1, A_2 \in \Omega$ and $A = A_1A_2$. Then $A \leq C_U(A_1) \cap C_U(A_2)$ and thus A is elementary abelian. Let $C_U(A) \leq D \leq U$ such that $D/C_U(A) = O_p(U/C_U(A))$. Then $DC_U(A_i)/C_U(A_i)$ is a p-group since $C_U(A) \leq C_U(A_i)$. Hence $D \leq C_U(A_1) \cap C_U(A_2) = C_U(A)$.

(b): Set $V = \langle (Y_{\widetilde{U}})^U \rangle$. By the definition of $Y_{\widetilde{U}}$, $O_p(U) \leq C_U(Y_{\widetilde{U}})$ and so $Y_{\widetilde{U}} \leq \Omega_1(Z(O_p(U)))$ as U is of characteristic p. Hence also V is in $\Omega_1(Z(O_p(U)))$; i.e. V is elementary abelian.

Let $C_U(V) \leq D \leq U$ such that $D/C_U(V) = O_p(U/C_U(V))$. Then

$$D = (D \cap T)C_U(V) \leq (D \cap T)C_U(Y_{\widetilde{U}}).$$

Hence $O_p(\widetilde{U}/C_{\widetilde{U}}(Y_{\widetilde{U}})) = 1$ gives $T \cap D \leq C_U(Y_{\widetilde{U}})$ and thus $D = C_U(V)$. Since V is elementary abelian we conclude that $V \in \Omega$ and thus $Y_{\widetilde{U}} \leq V \leq Y_U$.

(c): This follows from (b) with $\widetilde{U} := T$.

(d): According to (b) it suffices to show that $Y_U \leq Y_{\widetilde{U}}$. But this is clear since $U/C_U(Y_U) \cong \widetilde{U}/C_{\widetilde{U}}(Y_U)$ and thus $O_p(\widetilde{U}/C_{\widetilde{U}}(Y_U)) = 1$.

(e): Let $Y := \Omega_1(Z(O_p(U)))$. Then $Y_U \leq Y$ by the definition of Y_U. Let $C_U(Y) \leq D \leq U$ such that $D/C_U(Y) = O_p(U/C_U(Y))$. Since $C_U(Y) \leq C_U(Y_U)$ we get $DC_U(Y_U)/C_U(Y_U) \leq O_p(U/C_U(Y_U)) = 1$, and so $D \leq C_U(Y_U)$. It follows that $D/O_p(U)$ is a p'-group and $O_p(U/C_U(Y)) = 1$, so $Y \leq Y_U$.

(1.3) Let U be a finite group of characteristic p, $T \in Syl_p(U)$ and $P \in \mathcal{P}_U(T)$. Then the following hold:

(a) $U = \langle \mathcal{P}_U(T) \rangle N_U(T)$.

(b) For every normal subgroup N of P either $O^p(P) \leq N$ or $T \cap N \leq O_p(P)$.

(c) For every normal subgroup T_0 of T either $T_0 \leq O_p(P)$ or $O^p(P) = [O^p(P), T_0]$.

(d) $Y_P = \Omega_1(Z(O_p(P)))$ or $[\Omega_1(Z(O_p(P))), O^p(P)] = 1$.

Proof. (a): We proceed by induction on $|U|$. Set $U_0 = \langle \mathcal{P}_U(T) \rangle N_U(T)$, and note that $N_U(T)$ normalizes $\langle \mathcal{P}_U(T) \rangle$, so U_0 is a subgroup of U. By induction all proper subgroups of U containing T are in U_0. If $U \neq U_0$, then U_0 is the unique maximal subgroup of U containing T. But then $U \in \mathcal{P}_U(T)$ and thus $U = U_0$, a contradiction.

(b): By the Frattini argument $P = N_P(N \cap T)N$. As T is in a unique maximal subgroup of P at least one of NT and $N_P(N \cap T)$ is not a proper subgroup of P. This gives (b).

(c): Let $P_0 = [O^p(P), T_0]$ and $P_1 = [O^p(P), T_0]T_0$. Then P_1 is normal in P. Hence, by (b) either $O^p(P) \leq P_1$ and thus $P_0 \leq O^p(P_1) = O^p(P) \leq P_0$, or $T_0 \leq O_p(P)$.

(d): If $C_T(Y_P) = O_p(P)$, then $Y_P = \Omega_1(Z(O_p(P))$ follows from (1.2)(e). In the other case (c) gives $[\Omega_1(Z(O_p(P))), O^p(P)] = 1$.

Hypothesis and Notation. For the rest of this section the Q!-Hypothesis holds. We use the notation given in the introduction. For $L_1, L_2 \in \mathcal{L}_H(S)$ we define

$$L_1 << L_2 \iff L_1 = (L_1 \cap L_2) C_{L_1}(Y_{L_1}).$$

(1.4) Let $L, \widetilde{L} \in \mathcal{L}_H(S)$ such that $L << \widetilde{L}$. Then $L^0 \leq \widetilde{L}^0$.

Proof. Note that $C_L(Y_L) \leq \widetilde{C}$. Hence $C_L(Y_L)$ normalizes Q and $Q^L = Q^{L \cap \widetilde{L}}$.

(1.5) $<<$ is a partial ordering on $\overline{\mathcal{L}}_H(S)$.

Proof. By (1.2) $L_1 = (L_1 \cap L_2) C_{L_1}(Y_{L_1})$ implies that $Y_{L_1} = Y_{L_1 \cap L_2} \leq Y_{L_2}$. This gives the reflexivity and anti-symmetry. Assume now that $L_1 << L_2$ and $L_2 << L_3$. Then

$$L_1 \cap L_2 \leq (L_2 \cap L_3) C_{L_2}(Y_{L_2}) \text{ and } Y_{L_1} \leq Y_{L_2}.$$

It follows that $C_{L_2}(Y_{L_2}) \leq C_H(Y_{L_1}) = C_{L_1}(Y_{L_1})$ and thus $L_2 = (L_2 \cap L_3) C_{L_2}(Y_{L_1})$. Hence

$$L_1 \cap L_2 = (L_1 \cap L_2 \cap L_3) C_{L_2}(Y_{L_1}).$$

This shows $L_1 = (L_1 \cap L_3) C_{L_1}(Y_{L_1})$ and the transitivity of $<<$.

(1.6) Every p-subgroup of H contains at most one conjugate of Q; in particular Q is the only conjugate in $\widetilde{C}$.

Proof. Let $g \in H$ and $Q^g \leq S$. It suffices to show that $Q^g = Q$. As $Z \leq C_{\widetilde{C}^g}(Q^g) = Z(Q^g)$, Q-Uniqueness shows that $S \leq \widetilde{C}^g$, so $S \leq \widetilde{C} \cap \widetilde{C}^g$. Now Sylow's Theorem shows that $\widetilde{C}$ and $\widetilde{C}^g$ are conjugate by an element of $N_H(S)$. As by the definition of $\widetilde{C}$, $N_H(S) \leq N_H(Z) \leq \widetilde{C}$ we conclude that $\widetilde{C} = \widetilde{C}^g$ and thus also $Q = Q^g$.

(1.7) Let P be a subgroup of H with $Q \leq O_p(P)$. Then $P \leq \widetilde{C}$.

Proof. This is a direct consequence of (1.6).

2. Pushing Up

Hypothesis and Notation. In this section the Q!-Hypothesis holds. In addition, $P \leq H$ is a minimal parabolic subgroup of characteristic p and $T \in Syl_p(P)$. We set $\overline{P} := P/C_P(Y_P)$ and

$$B(T) := C_T(\Omega_1(Z(J(T)))) \text{ and } Z_0 := \Omega_1(Z(J(T))),$$

$$\mathcal{U}(P) := \{A \mid A \leq P, \overline{A} \text{ an elem. abelian } p\text{-group, and } |A/C_A(Y_P)| \geq |Y_P/C_{Y_P}(A)|\},$$

$$U(P) := \langle A \mid A \in \mathcal{U}(P)\rangle \text{ and } B(P) := \langle B(T)^P\rangle.$$

Moreover $\mathcal{K}(P)$ denotes the set of all $B(T)$-invariant subgroups $K \leq P$ satisfying:

(i) $\overline{K}$ is normal in $\overline{U(P)}$,

(ii) $L := KB(T)$ is minimal parabolic of characteristic p and $O_p(P) \leq T \cap L \in Syl_p(L)$,

(iii) $\overline{K} \cong SL_2(p^m)$ and $[Y_P, K]/C_{[Y_P,K]}(K)$ is a natural $SL_2(p^m)$-module for $\overline{K}$, or $p = 2$, $\overline{K} \cong S_{2^n+1}$ and $[Y_P, K]$ is a natural S_{2^n+1}-module for $\overline{K}$.

Note that trivially $C_P(Y_P) \in \mathcal{U}(P)$ and so $C_P(Y_P) \leq U(P)$. Then recall from (1.3) that either $U(P) = C_P(Y_P)$ or $P = U(P)T$ and similarly $B(P) = B(T) \leq O_p(P)$ or $P = B(P)T$.

Let $K = SL_2(p^m)$ and V be an irreducible $GF(p)K$-module. Set $F := End_K(V)$. By Schur's Lemma, F is a finite field, so V is an FK-module. We say that V is a **natural** $SL_2(p^m)$**-module** for K if $dim_F(V) = 2$.

(2.1) Suppose that $\overline{U(P)} \neq 1$ and $A \in \mathcal{U}(P)$. Then there exist subgroups $U_1, ..., U_r$ of $U(P)$ such that the following hold:

(a) $\overline{U(P)} = \overline{U}_1 \times \cdots \times \overline{U}_r$, $\overline{U}_i \cong SL_2(p^m)$ or S_{2^n+1} (and $p = 2$).

(b) Either $[Y_P, U_i]/C_{[Y_P,U_i]}(U(P))$ is a natural $SL_2(p^m)$-module for $\overline{U}_i$, or $[Y_P, U_i]$ is a natural S_{2^n+1}-module for $\overline{U}_i$, $i = 1, ..., r$.

(c) $Y_P = C_{Y_P}(U(P)) \prod_{i=1}^r [Y_P, U_i]$ and $[Y_P, U_i, U_j] = 1$ for $i \neq j$.

(d) $\overline{T}$ acts transitively on $\{\overline{U}_1, ..., \overline{U}_r\}$.

(e) $[Y_P, A, A] = 1$ and $|\overline{A}| = |Y_P/C_{Y_P}(A)|$. In particular $|E| \leq |Y_P/C_{Y_P}(E)|$ for every elementary abelian p-group $E \leq \overline{P}$.

(f) $\overline{A} = \overline{A \cap U_1} \times \cdots \times \overline{A \cap U_r}$ and $A \cap U_iC_P(Y_P) \in \mathcal{U}(P)$, $i = 1, ..., r$.

(g) $\overline{A \cap U_i} \in Syl_p(\overline{U}_i)$ if $\overline{U}_i \cong SL_2(p^m)$ and $\overline{A \cap U_i} \neq 1$.

(h) $\overline{A \cap U_i}$ is generated by a set of commuting transpositions if $\overline{U}_i \cong S_{2^n+1}$.

Proof. See [Cher].

(2.2) $\mathcal{A}(T) \subseteq \mathcal{U}(P)$ and $\overline{J(T)} = \overline{B(T)} \leq \overline{U(P)}$.

Proof. Assume that $J(T) \leq C_P(Y_P)$. Then clearly $\mathcal{A}(T) \subseteq \mathcal{U}(P)$ and $Y_P \leq Z_0$; in particular $B(T) \leq C_P(Y_P)$ and $1 = \overline{J(T)} = \overline{B(T)} \leq \overline{U(P)}$.

Assume now that $J(T) \not\leq C_P(Y_P)$. Let $A \in \mathcal{A}(T)$ such that $\overline{A} \neq 1$. The maximality of A gives $C_{Y_P}(A) = A \cap Y_P$. Hence

$$|C_A(Y_P)||Y_P||C_{Y_P}(A)|^{-1} = |C_A(Y_P)||Y_P||A \cap Y_P|^{-1} = |C_A(Y_P)Y_P| \leq |A|$$

and $A \in \mathcal{U}(P)$; in particular $\overline{J(T)} \leq \overline{U(P)} \neq 1$.

We now use the notation given in (2.1). In addition we set $Y_i := [Y_P, U_i]$ and $\widetilde{Y}_P := Y_P/C_{Y_P}(U(P))$. Then (2.1)(c) implies

$$(*) \quad \widetilde{Y}_P = \widetilde{Y}_1 \times \cdots \times \widetilde{Y}_r \text{ and } [Y_i, U_j] = 1 \text{ for } i \neq j.$$

Assume first that $\overline{U}_i \cong SL_2(p^m)$. Then (2.1)(f) and (g) show that $\overline{J(T)} \in Syl_p(\overline{U(P)})$, and (2.1)(b), (e) and (f) that $[Y_i, J(T)] \leq Y_i \cap Z_0$ and $|Y_i/Y_i \cap Z_0| = p^m$; in particular $Y_i \cap Z_0 \not\leq C_{Y_i}(U_i)$. As $B(T)$ centralizes $Y_i \cap Z_0$, we get from $(*)$ that $\overline{B(T)} \leq N_{\overline{P}}(\overline{U}_i)$.

Let $F := End_{\overline{U}_i}(\widetilde{Y}_i)$. Then the elements of $N_{\overline{P}}(\overline{U}_i)$ induce field automorphisms on F and semi-linear transformations on $\widetilde{Y}_i$. As $\widetilde{Y_i \cap Z_0}$ is a 1-dimensional F-subspace centralized by $B(T)$, we conclude that the elements of $B(T)$ act F-linear on $\widetilde{Y}_i$, so $\overline{B(T)} \leq (\overline{J(T)} \cap \overline{U}_i) C_{\overline{P}}(\overline{U}_i)$ by (2.1)(g). It follows that $\overline{B(T)} \leq \overline{J(T)}$ since $C_{\overline{P}}(\overline{U(P)}) \leq \overline{U(P)}$, whence $\overline{B(T)} = \overline{J(T)}$.

Assume now that $\overline{U}_i \cong S_{2^n+1}$. Recall that any two transpositions of S_m commute if they generate a 2-group. Hence, by (2.1)(h) $\overline{J(T)} \cap \overline{U}_i$ is generated by a maximal set of commuting transpositions, and as above, by (2.1)(e) and (f) $[Y_i, J(T)] \leq Y_i \cap Z_0$ and $\overline{B(T)} \leq N_{\overline{P}}(\overline{U}_i)$. Now (1.1)(c) shows that $\overline{B(T)} \leq (\overline{J(T)} \cap \overline{U}_i) C_{\overline{P}}(\overline{U}_i)$ and, again as above, $\overline{B(T)} = \overline{J(T)}$.

(2.3) Suppose that $\overline{U(P)} \neq 1$. Then $\mathcal{K}(P) \neq \emptyset$, and for every $K \in \mathcal{K}(P)$ and $L := KB(T)$:

(a) $U(L)/C_L(Y_L) \neq 1$; i.e. L satisfies the hypothesis of (2.1).

(b) $Y_L \leq Y_P$ and $[Y_L, K] = [Y_P, K]$.

(c) $B(T) \leq O_p(P)$ or $L = [K, B(T)](T \cap L)$.

(d) There exists U_i as in (2.1) such that $\overline{K} = \overline{U}_i$.

Proof. We first show that $\mathcal{K}(P) \neq \emptyset$. Let $U_1, ..., U_r$ be as in (2.1) and fix $U \in \{U_1, ..., U_r\}$. By (2.1) and (2.2) $\overline{J(T)} = \overline{B(T)} \leq N_{\overline{P}}(\overline{U})$ and $B(T) \leq N_P(UC_P(Y_P))$. Among all subgroups $K_0 \leq UC_P(Y_P)$, which are $B(T)$-invariant and satisfy

(∗) $\overline{K}_0 = \overline{U}$ and $O_p(P) \leq T \cap K_0B(T) \in Syl_p(K_0B(T))$,

we choose K minimal and set $L = KB(T)$. According to (2.1)(a) there exists $C_L(Y_P)(T \cap L) \leq L_0 \leq L$ such that $\overline{L}_0$ is the unique maximal subgroup of $\overline{L}$ containing $\overline{T \cap L}$. Hence, the minimality of K implies that L_0 is the unique maximal subgroup of L containing $T \cap L$, so L is minimal parabolic. Moreover, L is of characteristic p since $O_p(P) \leq O_p(L)$. This shows that $K \in \mathcal{K}(P)$.

Now let $K \in \mathcal{K}(P)$. Then (d) follows from (2.1)(a). Let $L = KB(T)$. From (1.3)(d) we get $\Omega_1(Z(O_p(L))) = Y_L \leq \Omega_1(Z(O_p(P))) = Y_P$, so $Y_L = C_{Y_P}(O_p(L))$. Since $[\overline{K}, \overline{O_p(L)}] = 1$ the $P \times Q$-Lemma gives $[Y_L, K] \neq 1$ and thus by (2.1)(b) $[Y_L, K] = [Y_P, K]$. This is (b).

From (1.3)(c) we get either $L = [K, B(T)](T \cap L)$ or $B(T) \leq O_p(L)$. In the latter case $[\overline{K}, \overline{B(T)}] = 1$, and (2.1)(d) implies $B(T) \leq C_T(Y_P)$. This shows (c) since $C_T(Y_P) = O_p(P)$ by (1.3)(c).

According to (2.1)(d) and (f) there exists $A \in \mathcal{U}(P)$ such that $\overline{A} \neq 1$ and $\overline{A} \leq \overline{T \cap K}$. Since $C_T(Y_P) = O_p(P) \leq L$ and $\overline{A}$ is a p-group we may assume that $A \leq T \cap L$. Set $A_0 = C_A(Y_L)$. By (2.1)(e)

$$|\overline{A}_0| \leq |Y_P/C_{Y_P}(A_0)| \leq |Y_P/Y_LC_{Y_P}(A)| = |Y_P/C_{Y_P}(A)||Y_L/C_{Y_L}(A)|^{-1} = |\overline{A}||Y_L/C_{Y_L}(A)|^{-1}$$

and $|Y_L/C_{Y_L}(A)| \leq |A/A_0|$. It follows that $U(L) \neq C_L(Y_L)$, and (a) holds.

(2.4) Suppose that $\overline{U(P)} \neq 1$. Let $A \in \mathcal{U}(P)$ and $A_1 \leq P$ such that $[Y_P, A, A_1] = 1$. Then

$$[Y_P, A_1] \leq [Y_P, A][C_{Y_P}(A), A_1].$$

Proof. We apply (2.1) and choose the subgroups $U_1, ..., U_r$ as in (2.1). Let $V_i := [Y_P, U_i]$. By (2.1)(c)

$$[Y_P, A_1] = [C_{Y_P}(A), A_1] \prod_{i=1}^{r} [V_i, A_1].$$

Hence, it suffices to show that

$(*)$ $[V_i, A_1] \leq [V_i, A][C_{Y_P}(A), A_1]$.

If $\overline{A \cap U_i} = 1$, then by (2.1)(c) and (f) $V_i \leq C_{Y_P}(A)$, and $(*)$ is obvious. Hence, we may assume that $\overline{A \cap U_i} \neq 1$. Then $[V_i, A, A_1] = 1$ shows that A_1 normalizes U_i and V_i.

Assume first that $\overline{U}_i \cong SL_2(p^m)$. By (2.1)(g) $\overline{A \cap U_i} \in Syl_p(\overline{U}_i)$, so $[V_i, A, A_1] = 1$ implies $A_1 \leq AC_P(V_i)$, and $(*)$ follows.

Assume now that $\overline{U}_i \cong S_{2^n+1}$. By (2.1)(h) $\overline{A \cap U_i} = \langle t_1, ..., t_s \rangle$, $t_1, ..., t_s$ commuting transpositions of S_{2^n+1}; in particular

$$C_{\overline{U}_i}(\overline{A}) = C_{\overline{U}_i}(\overline{A \cap U_i}) = \langle t_1, ..., t_s \rangle \times X,\ \ X \cong S_{2^n+1-2s} \text{ and } [V_i, X] = [C_{V_i}(A), X].$$

Since $[V_i, t_j, A_1] = 1$ for $j = 1, ..., s$ we get $\overline{A}_1 \leq C_{\overline{U}_i}(\overline{A})C_{\overline{P}}(V_i)$. Hence,

$$[V_i, A_1] \leq [V_i, A][C_{V_i}(A), A_1] \leq [V_i, A][C_{Y_P}(A), A_1],$$

and again $(*)$ follows.

(2.5) Suppose that $T = S$, $\overline{U(P)} \neq 1$ and $P \not\leq \widetilde{C}$. Let $K \in \mathcal{K}(P)$. Then the following hold:

(a) $Z(P) = Z(U(P)) = 1$.

(b) $Y_P = \times_{\{\overline{K} | K \in \mathcal{K}(P)\}} [Y_P, \overline{K}]$, and $[Y_P, \overline{K}]$ is an natural $\overline{K}$-module.

(c) Q acts transitively on $\{\overline{K} \mid K \in \mathcal{K}(P)\}$.

(d) $\overline{K} \cong SL_2(p^m)$ or $p = 2$ and $\overline{K} = \overline{U(P)} \cong S_5$.

(e) If $\overline{K} \cong SL_2(p^m)$ and $A \leq P$ with $[Y_P, A, A] = 1$, then $[Y_P, K, A] = [Y_P, K, a]$ for all $a \in A \setminus C_P([Y_P, K])$. Moreover, either $|A/C_A([Y_P, K])| = 2\,(= p)$ or $\overline{A} \leq \overline{K}C_{\overline{A}}(\overline{K})$.

Proof. (a): It suffices to show that $C_{Y_P}(U(P)) = 1$ since $\Omega_1(Z(P)) \leq Y_P$. If $C_{Y_P}(U(P)) \neq 1$, then there exists $1 \neq x \in C_{Y_P}(U(P)) \cap Z(Q)$, and by Q-Uniqueness $U(P) \leq C_H(x) \leq \widetilde{C}$. Since also $S \leq \widetilde{C}$ we get that $P = U(P)S \leq \widetilde{C}$, a contradiction.

(b): This follows from (a) and (2.1)(c).

(c): By (b) and (2.1)(c),(d) together with (2.3)(d)

$$Y_P = [Y_P, K_1] \times \cdots \times [Y_P, K_r],$$

where $K_i \in \mathcal{K}(P)$ and $\Omega := \{\overline{K} \mid K \in \mathcal{K}(P)\} = \{\overline{K}_1, ..., \overline{K}_r\}$. Assume that Q is not transitive on Ω. Then there exist $1 \neq x \in Z(Q) \cap Y_P$ and $K_i \in \{K_1, ..., K_r\}$ such that $[K_i, x] = 1$. Again by Q-Uniqueness $K_i \leq \widetilde{C}$ and thus $P = \langle K_i, S\rangle \leq \widetilde{C}$, a contradiction.

(d): We use (2.1) and (2.3)(d). Assume that $\overline{K} \cong S_{2^n+1}$, $n \geq 2$ (and $p = 2$). The action of $U(P)$ on Y_P shows that there exists $1 \neq x \in Z(Q) \cap Y_P$ such that $\overline{C_K(x)} \cong S_{2^n}$. On the other hand by Q-Uniqueness $C_H(x) \leq \widetilde{C}$ and thus $[C_K(x), Q] \leq Q$. Since S_{2^n} is not a 2-group we get $\overline{K}^{Q} = \overline{K}$, and $\overline{P} \cong S_{2^n+1}$ follows with (c). Moreover $\overline{Q}$ is a normal 2-subgroup of $\overline{C_K(x)}$.

If $n = 2$, then (d) follows. In the other cases $\overline{Q} = 1$ and thus $Q \leq C_S(Y_P) = O_2(P)$. But this contradicts (1.7).

(e): By (b) $V := [Y_P, K]$ is a natural $SL_2(p^m)$-module for $\overline{K}$. Assume first that $V^A = V$. Then again (b) implies that $\overline{K}^{\overline{A}} = \overline{K}$. Since V is a faithful irreducible $\overline{K}$-module we conclude that $C_{\overline{A}}(\overline{K}) = C_{\overline{A}}(V)$.

Let $V_0 := [V, A]$ and $F := End_{\overline{K}}(V)$. Recall that the elements of $\overline{A}$ induce semi-linear transformations on the F-vector space V. Thus, if V_0 contains a 1-dimensional F-subspace, then $\overline{A} \leq \overline{K}C_{\overline{P}}(\overline{K})$. In the other case no element of $\overline{A}^\sharp$ induces an F-linear transformation on V. As $\Gamma L(V)/GL(V)$ has cyclic Sylow p-subgroups, we get in this case that $|A/C_A(V)| = p$. Moreover, the quadratic action of A on V shows that the elements of $A^\sharp$ induce field automorphisms of order 2 in F, so $p = 2$.

Assume now that $V^A \neq V$. Then the quadratic action of A gives

$$\langle V^A\rangle = V \times V^a \text{ for } a \in A \setminus N_A(V);$$

in particular $|A/N_A(K)| = p$ $(= 2)$. Since

$$[V, N_A(K)] \leq C_V(A) \leq C_V(a) = 1$$

we get $N_A(K) \leq C_A(V)$ and $|A/C_A(V)| = p$. Now again (e) is obvious.

(2.6) Suppose that neither $\Omega_1(Z(T))$ nor $B(T)$ is normal in P. Then $\overline{B(P)} = \overline{U(P)} \neq 1$ and $\overline{B(T)} = \overline{J(T)} \neq 1$.

Proof. According to (1.3) $C_T(Y_P) = O_p(P)$ since $\Omega_1(Z(T))$ is not normal in P. Hence $B(T) \not\leq C_P(Y_P)$ since also $B(T)$ is not normal in P. It follows with (2.2) that $\overline{B(T)} = \overline{J(T)} \leq \overline{U(P)} \neq 1$, and (2.1) gives $\overline{B(P)} = \overline{U(P)}$.

(2.7) Suppose that neither $\Omega_1(Z(T))$ nor $B(T)$ is normal in P. Then $Z_0 \leq \Omega_1(Z(J(O_p(P))))$ and

$$[\Omega_1(Z(J(O_p(P)))), J(T)] \leq Z_0 \cap Y_P;$$

in particular $[\Omega_1(Z(J(O_p(P)))), O^2(P)] \leq Y_P$. Moreover, if in addition $\overline{K} \cong SL_2(p^m)$ for $K \in \mathcal{K}(P)$, then $B(T) \in Syl_p(O^p(K)B(T))$.

Proof. By (2.6) $\overline{U(P)} \neq 1$ and $\overline{J(T)} = \overline{B(T)} \neq 1$. Let $A \in \mathcal{A}(T)$ such that $\overline{A} \neq 1$ and $Z_1 := \Omega_1(Z(J(O_p(P))))$. Then by (2.1) $[Y_P, A] \leq C_{Y_P}(J(T)) \leq Z_0$, and (2.1)(e) gives $Y_P C_A(Y_P) \in \mathcal{A}(T)$. This shows that

$$Y_P C_A(Y_P) \in \mathcal{A}(O_p(P)) \subseteq \mathcal{A}(T).$$

Hence $Z_1 \leq Y_P C_A(Y_P)$ and $Z_0 \leq Z_1$. It follows that $[Z_1, A] \leq Y_P \cap Z_0$ and thus $[Z_1, J(T)] \leq Y_P \cap Z_0$. Since $O^p(P) \leq \langle J(T)^P \rangle$ by (1.3) we get $[Z_1, O^p(P)] \leq Y_P$.

Assume now that $\overline{K} \cong SL_2(p^m)$, where $K \in \mathcal{K}(P)$. By (2.2) and (2.1)(d), (g) we can choose A such that $\overline{A} \cap \overline{K} \in Syl_p(\overline{K})$; in particular

$$\langle \overline{A} \cap \overline{K}, (\overline{A} \cap \overline{K})^{\overline{g}} \rangle = \overline{K} \text{ for some } g \in K.$$

Set $L = KB(T)$, $W = [Y_L, K]$, $Z_0^* := Z_0 \cap Z_0^g$ and $L_0 = C_L(Z_0^*)$. Then $B(T) \leq L_0$ and $L = L_0 C_L(Y_P)$. Since L is minimal parabolic and by (1.3) $C_T(Y_P) = O_p(P)$ we get

(1) $L = L_0 O_p(P)$, and L_0 is normal in L.

By (2.3) L satisfies the hypothesis of (2.1), and $W = [Y_P, K]$. As $[Z_0, K] = [Z_0, K, K] \leq W$, $Z_0 W$ is normal in L, and (2.1)(b),(g), applied to L, gives $Z_0 W = Z_0 Z_0^g$, $C_W(T \cap L) = W \cap Z_0$ and $|WZ_0/Z_0| = p^m$; in particular $Z_0^* \cap W = C_W(L)$. It follows that

$$|Z_0^* W/Z_0^*| = |W/W \cap Z_0^*| = p^{2m} \text{ and } |Z_0 W/Z_0^*| = |Z_0 Z_0^g/Z_0^*| \leq p^{2m}.$$

This shows that $Z_0^* W = Z_0 W$ and $Z_0 = Z_0^* C_W(T \cap L)$; in particular

(2) $B(T) = C_{T \cap L}(Z_0) = C_{T \cap L}(Z_0^*)$.

By (1) and (2) $B(T) \in Syl_p(L_0)$ and $O^p(K) \leq O^p(L) \leq L_0$, so $B(T) \in Syl_p(O^p(K)B(T))$.

(2.8) Suppose that neither $B(T)$ nor $\Omega_1(Z(T))$ is normal in P and $Z(P) = 1$. Then $O_p(P) \leq B(T)$.

Proof. By (2.7) Z_0Y_P is normal in P. Hence, $R := [Z_0Y_P, O_p(P)]$ is a normal subgroup of P in Z_0. But then by (2.6) and (1.3) $O^p(P)$ centralizes R, and $Z(P) = 1$ implies $R = 1$. This gives $O_p(P) \leq B(T)$.

(2.9) Suppose that neither $B(T)$ nor $\Omega_1(Z(T))$ is normal in P. Then there exist subgroups $L_1, ..., L_k \leq P$ such that for $i = 1, ..., k$ and $\hat{L}_i = L_i/C_{L_i}(Y_{L_i})$:

(a) L_i is minimal parabolic of characteristic p and $O_p(P)B(T) \in Syl_p(L_i)$.

(b) $\hat{L}_i \cong SL_2(p^m)$, and $Y_{L_i}/C_{Y_{L_i}}(L_i)$ is a natural $SL_2(p^m)$-module for $\hat{L}_i$.

(c) $[Y_{L_i}, O^p(L_i)] = [Y_P, O^p(L_i)]$.

(d) $L_1, ..., L_k$ are conjugate under T, $\langle L_1, ..., L_k\rangle T = P$, and $\cap_{i=1}^k O_p(L_i) = O_p(P)$.

(e) $[Y_P, B(P)] \cap Z_0 = \prod_{i=1}^k [Y_{L_i}, B(T)]$ and $[Y_{L_i}, B(T), L_j] = 1$ for $i \neq j$.

Proof. By (2.6) $\overline{U(P)} \neq 1$, and we are allowed to apply (2.1) and (2.3) to P. Let $K \in \mathcal{K}(P)$, and set $L = KB(T)$ and $\hat{L} = L/C_L(Y_L)$. Then (2.3) shows that L satisfies (2.1) and $[Y_L, O^p(L)] = [Y_P, O^p(L)]$.

Assume first that $\overline{K} \cong SL_2(p^m)$. Then (2.1)(f),(g) gives

$$\overline{L} = \overline{K} \times C_{\overline{B(T)}}(\overline{K}) \text{ and } \overline{B(T)} \cap \overline{K} \in Syl_p(\overline{K});$$

in particular $O_p(P)B(T) \in Syl_p(L)$ and $[O_p(L), O^p(L)] \leq O_p(P)$. Now (a) – (d) follow for $k = 1$, and (e) is a consequence of (2.1)(b).

Assume now that $\overline{K} \cong S_{2^n+1}$ (and $p = 2$). Then $\overline{K \cap B(T)}$ is generated by a maximal set $\{\bar{t}_1, ..., \bar{t}_{2^{n-1}}\}$ of transpositions, where $t_1, ..., t_{2^{n-1}} \in K$. For every t_i there exists $d_i \in K$ such that $\overline{d}_i$ has order 3 and

$$\langle \overline{d}_i, \overline{K \cap B(T)}\rangle = \langle \overline{d}_i, \bar{t}_i\rangle \times \langle \bar{t}_j \mid i \neq j\rangle \text{ and } \langle \overline{d}_i, \bar{t}_i\rangle \cong SL_2(2).$$

Note that the subgroups $\langle \overline{d}_i, \bar{t}_i\rangle$, $i = 1, ..., 2^{n-1}$, are conjugate under $\overline{T \cap K}$ and that by (1.1)

$$\langle \overline{d}_i, \bar{t}_i \mid i = 1, ..., 2^{n-1}\rangle = \overline{K}.$$

Note further that by (2.1)(b)

$$(*) \quad [Y_P, K] \cap Z_0 = [Y_P, \langle t_1, ..., t_{2^{n-1}}\rangle] \text{ and } [Y_P, t_i, d_j] = 1 \text{ for } i \neq j.$$

We now choose $L_1 \leq \langle d_1, B(T)\rangle$ minimal with respect to

$$O_2(P)B(T) \leq T_1 := T \cap L_1 \in Syl_2(L_1) \text{ and } \overline{L}_1 = \langle \overline{d}_1, \overline{B(T)}\rangle.$$

Then L_1 is a minimal parabolic subgroup of characteristic 2. Moreover $O_2(P)B(T) = T_1$ and $[O_2(L_1), O^2(L_2)] \leq O_2(P)$, and (a) follows for L_1. Since $Y_{L_1} \leq \Omega_1(Z(O_2(L_1))) \leq \Omega_1(Z(O_2(P)))$ we get from (1.3)(d) that $Y_{L_1} \leq Y_P$. It follows that $|[Y_{L_1}, L_1]| = 4$, and (b) and (c) hold for L_1 since $O^2(\overline{L}_1) \cong C_3$.

Finally, for every $i \in \{1, ..., 2^{n-1}\}$ there exists a T-conjugate L_i of L_1 with $d_i \in L_i$, and $\langle \overline{L}_1, ..., \overline{L}_{2^{n-1}}\rangle\overline{B(T)} = \overline{L}$. Since L is minimal parabolic we get $\langle L_1, ..., L_{2^{n-1}}\rangle B(T) = L$. Similarly, since P is minimal parabolic (2.1)(d) and (2.3)(d) imply (d); and (e) follows from (d) and $(*)$.

Notation. Let

$$\mathcal{P}_0 := \mathcal{P}_H(S) \setminus (\mathcal{P}_{N_H(B(S))}(S) \cup \mathcal{P}_{\widetilde{C}}(S)) \text{ and } \mathcal{P}_0^* := \{P^g \mid P \in \mathcal{P}_0,\ g \in N_H(B(S))\},$$

and let $\mathcal{P}$ be the set of all subgroups $X \leq H$ satisfying:

(i) X is minimal parabolic of characteristic p and $B(S) \in Syl_p(X)$,

(ii) $\langle X, S\rangle = P$ for some $P \in \mathcal{P}_0$,

(iii) $X/C_X(Y_X) \cong SL_2(p^m)$ and $Y_X/C_{Y_X}(X)$ is a natural $SL_2(p^m)$-module for $X/C_X(Y_X)$.

Let $\mathcal{P}^* := \{X^g \mid X \in \mathcal{P},\ g \in N_H(B(S))\}$, $G := \langle X \mid X \in \mathcal{P}^*\rangle$ and $L := GN_H(B(S))$.

Theorem 1. One of the following holds:

(a) $L \in \mathcal{L}_H(S)$ and $\mathcal{P}_H(S) = \mathcal{P}_L(S) \cup \mathcal{P}_{\widetilde{C}}(S)$.

(b) $\mathcal{P}_H(S) = \mathcal{P}_{N_H(B(S))}(S) \cup \mathcal{P}_{\widetilde{C}}(S)$.

(c) $O_p(P) = Y_P$ and $Z(P) = 1$ for every $P \in \mathcal{P}^*$.

Proof. We may assume that neither (a) nor (b) holds. Then $\mathcal{P}_0 \neq \emptyset \neq \mathcal{P}_0^*$. Let $P^* \in \mathcal{P}_0^*$ and set $Z_0 := \Omega_1(Z(B(S)))$.

(1) P^* satisfies the hypotheses of (2.1), (2.8) and (2.9), and, after a suitable conjugation, also that of (2.5).

By the definition of $\mathcal{P}_0^*$ there is $P_0 \in \mathcal{P}_0$ and $g \in N_H(B(S))$ such that $P_0^g = P^*$. Hence, it suffices to show the claim for P_0.

From the choice of P_0 and the definition of $\widetilde{C}$ follows that neither $B(S)$ nor Z is normal in P_0. Hence, P_0 satisfies the hypotheses of (2.6) and (2.9), and by (2.6) also those of (2.1) and (2.5). Finally, by (2.5) P_0 satisfies the hypothesis of (2.8).

(2) $Z(P^*) = 1$ and $O_p(P^*) \leq B(S)$.

This follows from (1), (2.5) and (2.8).

Let $P_0 \in \mathcal{P}_0$. According to (2.9) and (2) there exists a subset

$$\Omega(P_0) := \{L_1, ..., L_k\} \subseteq \mathcal{P}$$

such that the subgroups $L_1, ..., L_k$ satisfy (2.9)(a) – (e) (with respect to P_0 and S). We fix this notation. From (2), (2.1)(c) and (2.9)(e) we get

(3) $Z_0 = \prod_{i=1}^{k} [Y_{L_i}, B(S)]$.

Next we prove:

(4) $L = \langle N_H(B(S)), P_0 \mid P_0 \in \mathcal{P}_0 \rangle$.

Let $\widetilde{L} := \langle N_H(B(S)), P_0 \mid P_0 \in \mathcal{P}_0 \rangle$. By the definition of $\mathcal{P}^*$ we have $L \leq \widetilde{L}$. On the other hand, for $P_0 \in \mathcal{P}_0$ by (2.9)(d) $P_0 \leq GS$ and so also $\widetilde{L} \leq L$.

(5) $O_p(G) = 1 = O_p(L)$.

From (4) we get

$$\mathcal{P}_H(S) = \mathcal{P}_L(S) \cup \mathcal{P}_{\widetilde{C}}(S).$$

Hence, $O_p(L) = 1$ since (a) does not hold. As G is normal in L we also have $O_p(G) = 1$.

In the following let

$$\Delta^* := \cup_{P_0 \in \mathcal{P}_0} \Omega(P_0).$$

We now apply the amalgam method to G with respect to the subgroups in $\mathcal{P}^*$ and use the standard notation, see for example [DS] or [KS]. For the convenience of the reader we repeat some of the notation:

$\Gamma = \{Px \mid x \in G,\ P \in \mathcal{P}^*\}$ is the set of vertices, and two vertices are adjacent, if they are different and have non-empty intersection. $\mathcal{P}^*$ is a (maximal) set of pairwise adjacent vertices (where the elements of $\mathcal{P}^*$ are understood as cosets), and every pair of adjacent vertices is conjugate (under G) to a pair of vertices from $\mathcal{P}^*$. For a vertex $\delta \in \Gamma$ the stabilizer of δ in G is denoted by G_δ. Moreover

$$Q_\delta = O_p(G_\delta) \text{ and } Z_\delta = \langle \Omega_1(Z(X)) \mid X \in Syl_p(G_\delta) \rangle.$$

A critical pair (δ, δ') of vertices satisfies $Z_\delta \not\leq Q_\delta$ with the distance $d(\delta, \delta')$ being minimal. This distance is denoted by b.

Note that by (2.9)(b) $Z_\delta = Y_{G_\delta}$ for every $\delta \in \Gamma$. Since by (1.3)(b) $C_{B(S)}(Y_P) = O_p(P)$ for every $P \in \mathcal{P}^*$ we get from (2.1)(g):

(6) $Z_\alpha Q_{\alpha'} \in Syl_p(G_{\alpha'} \cap G_{\alpha'-1})$ and $Z_{\alpha'} Q_\alpha \in Syl_p(G_\alpha \cap G_{\alpha+1})$ for every critical pair (α, α').

Let (α, α') be a critical pair with $G_\alpha \in \mathcal{P}^*$. Then there exists $T_1 \in Syl_p(G_\alpha)$ such that $G_\alpha = \langle T_1, Z_{\alpha'} \rangle$. Thus, possibly after conjugation in G_α, we may assume

(∗) (α, α') is a critical pair such that $G_\alpha \in \mathcal{P}^*$ and $G_\alpha = \langle B(S), Z_{\alpha'} \rangle$.

In the steps (7), (8) and (9) below (α, α') is a critical pair satisfying (∗). Further we set $R_\rho := [Z_\rho, Q_\alpha]$ for every $\rho \in \mathcal{P}^*$. Note that by (2.1)(e) and (g) $R_\rho \leq Z(B(S))$. We first show:

(7) Let $\rho \in \mathcal{P}^*$ and $b > 1$ or $Z_\rho \leq Q_{\alpha'-1}$. Then $R_\rho \leq Z(G_\alpha)$.

Assume first that $Z_\rho \leq Q_{\alpha'-1}$. Then by (6) $Z_\rho \leq Z_\alpha Q_{\alpha'}$ and

$$[Z_\rho, Z_{\alpha'}] \leq [Z_\alpha, Z_{\alpha'}] \leq Z_\alpha.$$

Hence, $Z_\rho Z_\alpha$ is normal in $\langle B(S), Z_{\alpha'} \rangle = G_\alpha$; so also $[Z_\rho, Q_\alpha] = R_\rho$ is normal in G_α. Since $R_\rho \leq Z(B(S))$ we get $R_\rho \leq Z(G_\alpha)$.

Assume now that $Z_\rho \not\leq Q_{\alpha'-1}$. Then $(\rho, \alpha'-1)$ is a critical pair, and (6) gives $[Z_\rho, Z_{\alpha'-1}] = [Z_\rho, Q_\alpha] = R_\rho$. If $b > 1$, then R_ρ is centralized by $\langle B(S), Z_{\alpha'} \rangle = G_\alpha$.

Next we show:

(8) Let $\rho \in \mathcal{P}^*$. Suppose that $b > 1$ or $Z_\rho \leq Q_{\alpha'-1}$. Then either $Q_\alpha = Q_\rho$ or $Q_\alpha Q_\rho = B(S)$.

Let $T := Q_\alpha Q_\rho$. Assume that $Q_\alpha \leq Q_\rho$ but $Q_\alpha \neq Q_\rho$. Then the action of G_α on Z_α shows that

$$Z_\rho \leq C_{Z_\alpha}(T) = Z_0,$$

so $B(S) \leq Q_\rho$, a contradiction. Hence, we may assume now that $Q_\rho < T < B(S)$.

There exists $x \in G_\alpha$ such that $(\alpha+1)^x \in \mathcal{P}^*$ and $(\alpha, {\alpha'}^x)$ is a critical pair; so by (6) $B(S) = Z_{\alpha'}^x Q_\alpha$. If $(\rho, {\alpha'}^x)$ is not a critical pair, we get $Z_{\alpha'}^x \leq Q_\rho$ and thus $T = B(S)$, a contradiction. Hence, also $(\rho, {\alpha'}^x)$ is a critical pair, and by (6) $B(S) = Z_{\alpha'}^x Q_\rho$ and $T = Q_\rho(Z_{\alpha'}^x \cap T)$.

Let $t \in Z_{\alpha'}^x$ such that $t \in T \setminus Q_\rho$. Then there exists $y \in Z_\rho$ such that $[t, y] \neq 1$, and by (7) $[t, y] \in Z(G_\alpha)$. On the other hand, according to (6) (applied to $(\rho, {\alpha'}^x)$ and $(\alpha, {\alpha'}^x)$) there exists $y' \in Z_\alpha$ such that $[t, y] = [t, y']$. The action of $Z_{\alpha'}^x$ on Z_α gives $[t, y'] \notin Z(G_\alpha)$, a contradiction.

We now let $N_H(B(S))$ act on Γ in the following way: Let $g \in N_H(B(S))$ and $\delta \in \Gamma$, so $\delta = Py$ for some $P \in \mathcal{P}^*$ and $y \in G$. Then

$$g : \delta \mapsto \delta^g := P^g y^g.$$

(9) For every $P \in \Delta^*$ there exists a critical pair (δ, δ') satisfying $(*)$ such that $G_\delta = P$.

There exists $P_0 \in \mathcal{P}_0$ such that $P \in \Omega(P_0) \subseteq \Delta^*$. Hence, there exist $\delta_1, ..., \delta_k \in \mathcal{P}^*$ such that

$$\Omega(P_0) = \{G_{\delta_1}, ..., G_{\delta_k}\}.$$

Note that by (2.9)(d) the subgroups in $\Omega(P_0)$ are conjugate under S. We will show that there exists a critical pair (δ_i, δ_i') for some $i \in \{1, ..., k\}$. The $(*)$-property then can be achieved by a suitable conjugation in G_{δ_i} and the claim for the other δ_j by the action of S.

Hence, we may assume that $Z_{\delta_i} \leq Q_{\alpha'-1}$ for all $i = 1, ..., k$. If there exists $j \in \{1, ..., k\}$ such that $Q_{\delta_j} = Q_\alpha$, then $(\delta_j, {\alpha'}^x)$ is a critical pair, where $x \in G_\alpha$ such that $B(S)^{x^{-1}} \leq G_{\alpha+1}$. Thus, we may also assume that $Q_\alpha \neq Q_{\delta_i}$ for all $i = 1, ..., k$. Now (7) and (8) give

$$R_{\delta_i} = [Z_{\delta_i}, B(S)] \leq Z(G_\alpha),\ i = 1, ..., k,$$

and by (3)

$$Z_0 = \prod_{i=1}^{k} [Z_{\delta_i}, B(S)] = \prod_{i=1}^{k} R_{\delta_i} \leq Z(G_\alpha),$$

a contradiction.

(10) There exists $\rho \in \mathcal{P}^*$ and $P \in \Delta^*$ such that $Q_\rho^q \neq O_p(P)$ for all $q \in Q$.

Assume that (10) does not hold. Let $P_0 \in \mathcal{P}_0$ and $\Omega(P_0) = \{L_1, ..., L_k\}$. By (2.9)(d)

$$\cap_{i=1}^{k} O_p(L_i) = O_p(P_0).$$

Now let $\rho \in \mathcal{P}^*$ and $L_i \in \Omega(P_0)$. Then there exists $q \in Q$ such that $Q_\rho^q = O_p(L_i)$; in particular $O_p(P_0) \leq Q_\rho^q$. Since $O_p(P_0)$ is Q-invariant we get

$$O_p(P_0) \leq Q_\rho \text{ for all } \rho \in \mathcal{P}^* \text{ and all } P_0 \in \mathcal{P}_0.$$

Note that $\mathcal{P}^*$ is invariant under $N_H(B(S))$. Hence also

$$O_p(P^*) \leq Q_\rho \text{ for all } \rho \in \mathcal{P}^* \text{ and all } P^* \in \mathcal{P}_0^*.$$

It follows that

$$O_p(P^*) \le \cap_{L_i \in \Omega(P_0)} O_p(L_i) = O_p(P_0) \text{ for all } P_0 \in \mathcal{P}_0 \text{ and all } P^* \in \mathcal{P}_0^*.$$

This shows that $O_p(P^*) = O_p(P_0)$ for all $P^* \in \mathcal{P}_0^*$ and all $P_0 \in \mathcal{P}_0$, and by (4) $O_p(P_0)$ is normal in L, a contradiction to (5).

By (10) there exists $\rho \in \mathcal{P}^*$ and $P \in \Delta^*$ such that $Q_\rho^q \neq O_p(P)$ for all $q \in Q$, and by (9) there exists a critical pair (α, α') satisfying $(*)$ such that $G_\alpha = P$. We fix this notation with the additional property that $P_0 := \langle P, S \rangle \in \mathcal{P}_0$ and $P \in \Omega(P_0)$.

(11) There exists $q \in Q$ such that (ρ^q, α) is a critical pair; in particular $b = 1$.

Suppose that $b > 1$ or $Z_{\rho^q} \le Q_{\alpha'-1}$ for all $q \in Q$. Then (8) shows that $B(S) = Q_\alpha Q_\rho^q$ for all $q \in Q$. Hence $[Z_\rho^q, Q_\alpha] = [Z_\rho^q, B(S)]$ and by (7)

$$R := \prod_{q \in Q} [Z_\rho^q, B(S)] \le Z(G_\alpha);$$

in particular R is a Q-invariant and non-trivial subgroup of $Z(G_\alpha)$. Hence, Q-Uniqueness gives $G_\alpha = P \le \widetilde{C}$. But then also $P_0 \le \widetilde{C}$, which contradicts $P_0 \in \mathcal{P}_0$. This shows that $b = 1$ and there exists $q \in Q$ such that (ρ^q, α) is a critical pair.

(12) Let $\gamma \in \mathcal{P}^*$ such that $G_\gamma \le P_0$. Then $Y_{G_\gamma} \le Y_{P_0}$; in particular $Z_\alpha \le Y_{P_0}$ and no Q-conjugate of G_ρ is contained in P_0.

Since by (2) $O_p(P_0) \le B(S)$, we have $\Omega_1(Z(Q_\gamma)) \le \Omega_1(Z(O_p(P_0)))$. Hence (1.3)(d) and (2) yield $Y_{G_\gamma} \le Y_{P_0}$. This gives, together with (11), that there exists $q \in Q$ such that G_ρ^q is not contained in P_0, and, since $Q \le S \le P_0$, no Q-conjugate of G_ρ is contained in P_0.

Let $\mu := \rho^q$ be as in (11). Then (6) and $b = 1$ give

$$B(S) = Z_\mu Z_\alpha (Q_\alpha \cap Q_\mu);$$

in particular

$$\Phi(Q_\alpha) = \Phi(Q_\alpha \cap Q_\mu) = \Phi(Q_\mu).$$

This gives $[Q_\alpha, Z_\mu] = [Z_\alpha, Z_\mu] \le Z_\alpha$. Hence (2), (1.3)(b) and (12) yield

$$[O_p(P_0), O^p(G_\alpha)] \le [Q_\alpha, O^p(G_\alpha)] \le [Q_\alpha, \langle Z_\mu^{G_\alpha} \rangle] \le Z_\alpha \le Y_{P_0}.$$

From $G_\alpha \in \Omega(P_0)$ and (2.9)(d) we get $[O_p(P_0), O^p(P_0)] \le Y_{P_0}$. Now $Z(P_0) = 1$ yields $Y_{P_0} = O_p(P_0)$, and (2.1) and (2.9) applied to P_0 give $B(S) = Y_{P_0}\langle Z_\mu^S \rangle$. From (2.1) and (3) it follows that $\Phi(B(S)) = Z_0$; in particular

$$\Phi(Q_\alpha) = \Phi(Q_\mu) \le Z(G_\alpha) \cap Z(G_\mu).$$

Assume that $\Omega(P_0) = \{P\}$. Then $Z(G_\alpha) = 1$ and $Z_\alpha = Q_\alpha$ is a natural G_α/Q_α-module. In particular

$$B(S) = Z_\alpha Z_\mu \text{ and } Z_\alpha \cap Z_\mu = Z_0.$$

Thus, also $Q_\mu = Z_\mu$, and the action of Z_α on Z_μ also shows that $Z(G_\mu) = 1$.

Let $\lambda \in \mathcal{P}^*$. If $Q_\lambda^q \ne Q_\alpha$ for all $q \in Q$, then, as for ρ and μ, $Q_\lambda = Z_\lambda$ and $Z(G_\lambda) = 1$. If $Q_\lambda^q = Q_\alpha$ for some $q \in Q$, then $Z_\alpha = Z_\lambda^q = Q_\lambda^q$, and the action of Z_μ shows that $Z(G_\lambda^q) = Z(G_\lambda) = 1$. Hence, (c) holds in the case $\Omega(P_0) = \{P\}$.

Assume now that $\Omega(P_0) \ne \{P\}$ and choose $L_i \in \Omega(P_0)\backslash\{P\}$; i.e. $L_i = G_\nu$ for some $\alpha \ne \nu \in \mathcal{P}^*$. Since $[Z_\mu, Q_\alpha] = [Y_P, B(S)]$ and by (2.9)(e) $[Y_{L_i}, B(S)] \ne [Y_P, B(S)]$ we get from $b = 1$ and (6) that $Z_\nu \le Q_\mu \cap Q_\alpha$. Hence,

$$R_0 := [Z_\nu, B(S)] = [Z_\nu, Q_\alpha \cap Q_\mu] \le Z(G_\alpha) \cap Z(G_\mu).$$

Let $U = N_H(R_0)$. Then U is of characteristic p and $\langle G_\alpha, G_\mu \rangle \le C_H(R_0)$. Thus

$$O_p(U) \cap Q_\mu = O_p(U) \cap B(S) = O_p(U) \cap Q_\alpha,$$

so $O_p(U) \cap B(S)$ is normal in G_α and $[O_p(U) \cap B(S), Z_\mu] = 1$. Note that $[O_p(U), Z_\mu] \le O_p(U) \cap B(S)$. Since $O^p(G_\alpha) \le \langle Z_\mu^{G_\alpha} \rangle$ we get that $[O_p(U), O^p(G_\alpha), O^p(G_\alpha)] = 1$. This contradicts the fact that U is of characteristic p.

Corollary 1. Suppose that the cases (a) and (b) of Theorem 1 do not hold. Let $P \in \mathcal{P}_H(S) \setminus \mathcal{P}_{\widetilde{C}}(S)$ such that $\Omega_1(Z(B(S)))$ is not normal in P. Then $\overline{B(P)} \cong SL_2(p^m)$, and $O_p(P)$ is a natural $SL_2(p^m)$-module for $\overline{B(P)}$. Moreover, either $N_H(B(S)) \le N_H(O_p(P))$, or P is of type L_3.

Proof. By the choice of P and the definition of $\widetilde{C}$, P satisfies the hypothesis of (2.6). Hence $\overline{U(P)} \ne 1$ and by (2.5)(a) $Z(P) = 1$. Thus (2.8) gives $O_p(P) \le B(S)$. Applying (2.9) and Theorem

1 (c) we get that $\overline{B(P)} \cong SL_2(p^m)$ and that $O_p(P) = Y_P$ is a natural $SL_2(p^m)$-module for $\overline{B(P)}$. Hence either P is of type L_3 or $p = 2$.

Assume that $p = 2$. Suppose that $N_H(B(S))$ is not contained in $N_H(Y_P)$ and pick $x \in N_H(B(S)) \setminus N_H(Y_P)$. Then $B(S) = Y_P Y_P^x$ and $\mathcal{A}(S) = \{Y_P, Y_P^x\}$. Since $N_H(B(S))$ acts on $\mathcal{A}(S)$ we get $O^2(N_H(B(S))) \leq N_H(Y_P)$ and thus also $N_H(B(S)) \leq N_H(Y_P)$, a contradiction.

3. P-Uniqueness

Throughout this section we assume Hypothesis I. In particular, the Structure Theorem applies to all $M \in \mathcal{L}^*_H(S)$ with $P \leq M$. In addition, among all P satisfying Hypothesis I we choose P maximal (with respect to inclusion).

Local P!-Theorem. Let $P^* = U(P)$ and $P \leq M \in \mathcal{L}^*_H(S)$. Then one of the following holds:

(a) Case (a) of the Structure Theorem holds for M, $P^* = P \cap M_0$ and

(i) $P^*/O_p(P) \cong SL_2(p^m)$ and Y_P is a natural $SL_2(p^m)$-module,

(ii) $\mathcal{P}_M(S) = \{P\} \cup \mathcal{P}_{M\cap\widetilde{C}}(S)$,

(iii) $M \cap \widetilde{C} = N_M(\Omega_1(Z(S \cap P^*)))$.

(b) Case (b) of the Structure Theorem holds for M, and

(i) $\mathcal{P}_M(S) = \mathcal{P}_P(S) \cup \mathcal{P}_{M\cap\widetilde{C}}(S)$, in particular $P = O^p(M_0)S$,

(ii) $M \cap \widetilde{C} \leq N_M(\Omega_1(Z(S \cap P^*)))$,

(iii) $\mathcal{M}_H(P) = \{M\}$.

Proof. We discuss the two cases of the Structure Theorem separately. Assume first that case (a) of the Structure Theorem holds for M. Let $\overline{M} := M/C_M(Y_M)$, $S_0 := S \cap M_0$ and $Z_0 := \Omega_1(Z(S_0))$. The p-local structure of $M_0/O_p(M_0)$ shows:

(+) There exists a unique $U \in \mathcal{P}_{M_0}(S_0)$ such that $[Z_0, U] \neq 1$; in particular $\mathcal{P}_{M_0}(S_0) = \{U\} \cup \mathcal{P}_{M_0\cap\widetilde{C}}(S_0)$.

(++) $U/O_p(U) \cong SL_2(p^m)$, and $Y := C_{Y_M}(O_p(U))$ is a natural $SL_2(p^m)$-module for $U/O_p(U)$.

Since $Q \leq S_0$ from (1.7) it follows $N_H(S_0) \leq \widetilde{C}$, hence (+) gives $N_H(S_0) \leq N_H(U)$, in particular S normalizes U.

Let $P_1 \in \mathcal{P}_M(S)$ such that $P_1 \not\leq \widetilde{C}$. By (1.7) $Q \not\leq O_p(P_1)$, and so by (1.3)(b) $P_1 = (P_1)^0 S$ and $(P_1)^0 S_0 \leq M_0$. Since $O_p(M) \leq O_p((P_1)^0 S_0)$, $(P_1)^0 S_0$ has characteristic p, whence (1.3)(a) and the uniqueness of U give

$$(P_1)^0 S_0 = \langle U, (P_1)^0 S_0 \cap \widetilde{C} \rangle.$$

Since P_1 is a minimal parabolic subgroup not contained in $\widetilde{C}$ we get that $P_1 = US$; in particular $P = US$, and (a)(ii) follows.

From $O_p(U) \le O_p(P)$ and (1.2)(b) we get $Y_P \le Y_M$, thus $Y_P \le Y$ and (++) yields $Y_P = Y$. Now (2.1) gives $P^* = UO_p(P) \le M_0$, whence (a)(i) and $P \cap M_0 = P^*$ follow.

Note that $M_0 C_M(Y_M)$ is a normal subgroup of M. It follows that

$$M \cap \widetilde{C} = C_M(Y_M)(M_0 \cap \widetilde{C}) N_{M \cap \widetilde{C}}(S_0) \le (M_0 \cap \widetilde{C}) N_M(Z_0),$$

so (+) and (1.3)(a) yield $M \cap \widetilde{C} \le N_M(Z_0)$. On the other hand by Q-Uniqueness $C_M(Z_0) \le \widetilde{C}$, so by (1.6) Q is the unique conjugate of Q in $C_M(Z_0)$. Hence $N_M(Z_0) \le N_M(Q) = M \cap \widetilde{C}$.

By the Structure Theorem $C_S(Y_M) = O_p(M_0) \in Syl_p(C_{M_0}(Y_M))$, whence by (1.2)(e) $Y_{M_0} = \Omega_1(Z(C_S(Y_M))) \le Y_M$. This gives $Z_0 \le Y_M$ and thus $Z_0 = C_{Y_M}(S_0)$. From (++) it follows that $Z_0 \le Y = Y \cap Z(O_p(P))$, therefore $S \cap P^* = O_p(P)S_0$ yields $Z_0 = \Omega_1(Z(S \cap P^*))$. This shows (a)(iii).

Assume now that case (b) of the Structure Theorem holds. Let P_1 and P_1^* be as given there and set $S_0 := P_1^* \cap S$ and $Z_0 := \Omega_1(Z(S_0))$. Then $P_1 = M^0 S$ and by (2.1) $P_1^* = U(P_1)$; moreover, by (1.3)(c) and (1.7) $\mathcal{P}_M(S) = \mathcal{P}_{P_1}(S) \cup \mathcal{P}_{M \cap \widetilde{C}}(S)$. The maximality of P gives $P = P_1$ and $P^* = P_1^*$, and (b)(i)holds.

Since $P^* C_M(Y_M)$ is normal in M we get as above

$$M \cap \widetilde{C} = C_M(Y_M)(P^* \cap \widetilde{C}) N_{M \cap \widetilde{C}}(S_0).$$

As P is a minimal parabolic subgroup, the structure of P^* and its action on Y_P show that $N_P(Z_0)$ is the unique maximal subgroup containing S. It follows that $P^* \cap \widetilde{C} \le N_P(Z_0)$ and thus $M \cap \widetilde{C} \le N_M(Z_0)$. This is (b)(ii).

Let $P \le L \in \mathcal{M}_H(S)$ and $L << \widetilde{L} \in \mathcal{L}_H^*(S)$. Then $L = (L \cap \widetilde{L}) C_L(Y_L)$ and thus

$$P^0 \le L^0 = (L \cap \widetilde{L})^0 \le \widetilde{L}^0.$$

It follows that $P = P^0 S \le \widetilde{L}$, and we are allowed to apply the Structure Theorem to $\widetilde{L}$.

If case (a) of the Structure Theorem holds for $\widetilde{L}$, then by case (a) of the Local P! Theorem $P \cap \widetilde{L}_0 = P^* = U(P)$. But then $Q \le P^*$, a contradiction.

If case (b) of the Structure Theorem holds for $\widetilde{L}$, then the maximality of P gives $Y_P = Y_{\widetilde{L}}$ and thus $Y_{\widetilde{L}} = Y_M$; in particular $M = \widetilde{L}$. This shows (b)(iii).

Notation. We fix M, P and P^* as in the Local P!-Theorem. (Observe that in case (b) of the Local P!-Theorem the definition of P^* differs from that given in the P!-Theorem. But it will be shown in section 4 that this case does not occur.) Furthermore, we set $\overline{P} := P/C_P(Y_P)$, $S_0 := S \cap P^*$ and $Z_0 := \Omega_1(Z(S_0))$. Recall that P satisfies the hypotheses of (2.1) – (2.5) and if $B(S) \not\leq O_p(P)$ also those of (2.6) – (2.9). Later in the course of the amalgam method we will apply these Lemmata not only to P but also to conjugates of P.

(3.1) P admits the decompositions

$$(\mathcal{D}_1) \quad \overline{P}^* = K_1 \times \cdots \times K_r, \quad K_i \cong SL_2(p^m), \text{ and}$$

$$(\mathcal{D}_2) \quad Y_P = V_1 \times \cdots \times V_r, \quad V_i \text{ a natural } SL_2(p^m)\text{-module for } K_i.$$

Moreover, $[Y_P, Q \cap S_0] = Z_0$ and either $S_0 = B(S)$ or $B(S) \leq O_p(P)$.

Proof. The decompositions $\mathcal{D}_1$ and $\mathcal{D}_2$ are from the Local P!-Theorem. Assume that $B(S) \not\leq O_p(P)$. Since $P^* = U(P)$ (2.6), (2.1) and (2.8) show that $S_0 = B(S)$.

Remark. The next result, Theorem 2, establishes part (a) and (c) of the P!-Theorem if case (a) of the Local P!-Theorem holds. We then embark on the proof of the main result of this section, Theorem 3, where we show that Z_0 is normal in $\widetilde{C}$. This establishes part (b) of the P!-Theorem in all cases. It then remains to treat case (b) of the Local P!-Theorem. This is done in the next section, where the F!-Theorem eliminates this case.

Theorem 2. Assume Hypothesis I. Then either $\mathcal{P}_H(S) = \mathcal{P}_P(S) \cup \mathcal{P}_{\widetilde{C}}(S)$, or the following hold:

(a) Z_0 is normal in $\widetilde{C}$.

(b) $Q = B(S) = S_0$.

(c) $\widetilde{P}$ is of type L_3 for every $\widetilde{P} \in \mathcal{P}_H(S) \setminus \mathcal{P}_{\widetilde{C}}(S)$.

Proof. Assume first that P is of type L_3. Then by (2.2) $Y_P \in \mathcal{A}(S)$, $B(S) = S_0$, and for every $A \in \mathcal{A}(S)$ either

$$S_0 = AY_P \text{ or } A = Y_P.$$

Moreover, $Y_P \leq Q$ by (1.2)(b) and Hypothesis I. It follows that also $J(S) = S_0 \leq Q$ since Y_P is not normal in $\widetilde{C}$. But then $J(S) = J(Q)$ and $Z_0 = \Omega_1(Z(S_0)) = \Omega_1(Z(J(S)))$ is normal in $\widetilde{C}$. On the other hand, $N_P(S_0)$ is transitive on Z_0 and by (1.7) contained in $\widetilde{C}$, so $Z_0 \le Z(Q)$ and $Q \le S_0$. We conclude that $Q = S_0$; in particular $N_H(B(S)) = \widetilde{C}$.

Let $\widetilde{P} \in \mathcal{P}_H(S) \setminus \mathcal{P}_{\widetilde{C}}(S)$. Then $\widetilde{P} \not\le N_H(\Omega_1(Z(B(S))))$, and Corollary 1 shows that also $\widetilde{P}$ is of type L_3. Hence, Theorem 2 holds if P is of type L_3.

We may assume now:

(1) P is not of type L_3 and $\mathcal{P}_H(S) \neq \mathcal{P}_P(S) \cup \mathcal{P}_{\widetilde{C}}(S)$.

By (1) there exists $\widetilde{P} \in \mathcal{P}_H(S)$ such that

(2) $\widetilde{P} \not\le P$ and $\widetilde{P} \not\le \widetilde{C}$.

Assume that $O_p(\langle P, \widetilde{P}\rangle) \neq 1$. Then there exists $L \in \overline{\mathcal{L}}_H(S)$ such that $\langle P, \widetilde{P}\rangle := R \le L$. Since $P = P^0 S$ and $\widetilde{P} = \widetilde{P}^0 S$, we also get $R \le L^0 S$. Now (1.4) shows that there exists $\widetilde{M} \in \mathcal{L}^*_H(S)$ such that $R \le \widetilde{M}$. The Local P!-Theorem applied to $\widetilde{M}$, together with the maximal choice of P, gives $\widetilde{P} \le P$, which contradicts (2). We have shown:

(3) $O_p(\langle P, \widetilde{P}\rangle) = 1$.

We now apply Theorem 1. Then (3) shows that the cases (a) and (b) of Theorem 1 do not hold. Assume that $B(S)$ is not normal in P, so by (1.3) also $\Omega_1(Z(B(S)))$ is not normal in P. Hence by Corollary 1 $O_p(P) = Y_P$ and $P^*/Y_P \cong SL_2(p^m)$, and Corollary 1 and (1) show that $N_H(B(S)) \le N_H(Y_P)$. On the other hand, as above, $Y_P \le Q$ implies $B(S) = S_0 = Q$ since Y_P is not normal in $\widetilde{C}$. Hence $\widetilde{C} = N_H(B(S)) \le N_H(Y_P)$, and Y_P is normal in $\widetilde{C}$, a contradiction. We have shown:

(4) $P \le N_H(B(S))$.

By (3) and (4) $\Omega_1(Z(B(S)))$ is not normal in $\widetilde{P}$. Hence again (3) and Corollary 1 show that $\widetilde{P}$ is of type L_3. In particular $p \neq 2$, and there exists an involution $t \in N_{\widetilde{P}}(S)$ such that $[S,t] = Y_{\widetilde{P}}$. Since $Y_{\widetilde{P}} \le B(S)$ and $Y_{\widetilde{P}} = O_p(\widetilde{P})$ we get $Y_P \le \Omega_1(Z(B(S)) \le Y_{\widetilde{P}}$. Hence $Y_P = [Y_P, t]$, and t inverts Y_P. This shows that $[t, P] \le C_H(Y_P) \cap N_H(O_p(P)) =: X$, and P^0 normalizes $\langle t\rangle X$. Since

$$[\langle t\rangle X, Q] \le Q \cap \langle t\rangle X \le C_S(Y_P) = O_p(P)$$

we conclude that $[t, P^0] \le O_p(P)$ and thus also $[t, P] \le O_p(P)$. Hence, P normalizes $\langle t\rangle O_p(P)$ and thus also $O^p(\langle t\rangle O_p(P)) = \langle t\rangle Y_{\widetilde{P}}$. It follows that P normalizes $Y_{\widetilde{P}}$, which contradicts (3). This completes the proof of Theorem 2.

(3.2) Suppose that $O^p(\overline{P}) \leq \langle \overline{x}, \overline{A} \rangle$, where x is a p-element in P and A a normal subgroup of S in Q. Then $O^p(P) \leq \langle x, A \rangle$.

Proof. Let $P_0 = \langle x, A \rangle$ and $P_1 = O^p(P)$. Note that $P_1 \not\leq C_P(Y_P)$ by our choice of P, so $P_1 \leq \langle A^P \rangle$ by (1.3)(b). Note further that $[C_P(Y_P), A] \leq O_p(P)$ since $A \leq Q$ and that $P_1 \leq P_0 C_P(Y_P)$. It follows that

$$P_1 \leq \langle A^P \rangle = \langle A^{P_1} \rangle \leq \langle A^{P_0} \rangle O_p(P).$$

Since $\langle A^{P_0} \rangle$ is normal in $P_0 O_p(P)$ we get that

$$P_1 = O^p(\langle A^P \rangle) = O^p(P_0 O_p(P)) = O^p(P_0).$$

Hypothesis II. Assume Hypothesis I and $\mathcal{P}_H(S) = \mathcal{P}_P(S) \cup \mathcal{P}_{\widetilde{C}}(S)$. Further assume that there exists $\widetilde{P} \in \mathcal{P}_{\widetilde{C}}(S)$ such that $(P, \widetilde{P})$ is an amalgam and $N_{\widetilde{P}}(Z_0)$ is a maximal subgroup of $\widetilde{P}$.

Our goal, which we will achieve in (3.9), is to prove that no group H satisfies Hypothesis II.

(3.3) Assume Hypothesis II. Let $x \in \widetilde{P}$ and $O_p(P) \leq N_{\widetilde{P}}(Z_0^x)$. Then $x \in N_{\widetilde{P}}(Z_0)$.

Proof. Assume first that $J(S) \leq O_p(\widetilde{P})$. Then $J(S)$ is normal in $\widetilde{P}$ and thus not normal in P since $(P, \widetilde{P})$ is an amalgam. Hence, by (3.1) $S_0 = B(S)$ and $Z_0 = \Omega_1(Z(J(S)))$. But then Z_0 is normal in $\widetilde{P}$, a contradiction. Thus, $J(S)$ is not normal in $\widetilde{P}$. Since $\widetilde{P}$ is minimal parabolic we get that $N_{\widetilde{P}}(J(S)) \leq N_{\widetilde{P}}(Z_0)$ and that $N_{\widetilde{P}}(Z_0)$ is self-normalizing.

Assume now that $x \notin N_{\widetilde{P}}(Z_0)$ but $O_p(P) \leq N_{\widetilde{P}}(Z_0^x)$, so $N_{\widetilde{P}}(Z_0) \neq N_{\widetilde{P}}(Z_0^x)$. We choose x in addition such that $|T|$ is maximal, where

$$O_p(P) \leq T \in Syl_p(N_{\widetilde{P}}(Z_0) \cap N_{\widetilde{P}}(Z_0^x)).$$

Note that $O_p(\widetilde{P}) \leq T \cap S$.

After conjugation in $N_{\widetilde{P}}(O_p(P))$ we may assume that $T_1 := N_T(O_p(P)) \leq S$, so $T_1 = T \cap S$. Note that $T \notin Syl_p(\widetilde{P})$ since $\widetilde{P}$ is minimal parabolic; in particular T is not a Sylow p-subgroup of $N_{\widetilde{P}}(Z_0^x)$. Hence, the maximality of T yields

(1) $N_{\widetilde{P}}(T) \not\leq N_{\widetilde{P}}(Z_0)$.

From (1) and $N_{\widetilde{P}}(J(S)) \leq N_{\widetilde{P}}(Z_0)$ we get:

(2) $J(S) \neq J(T)$ and $J(S) \not\leq T$.

In particular $J(S) \not\leq O_p(P)$, and (3.1) and (2.7) yield

(3) $S_0 = B(S) = J(S)O_p(P)$ and $[\Omega_1(Z(J_0)), J(S)] = Z_0 = \Omega_1(Z(J(S)))$, where $J_0 := J(O_p(P))$.

Assume that $J(T_1) \neq J_0$. Since $Q \leq T_1$ the Q-transitivity and (2.1) imply

$$S_0 = J(T_1)O_p(P) \leq T.$$

This contradicts (2) and (3). We have shown:

(4) $J_0 = J(T_1)$.

Since $(P, \widetilde{P})$ is an amalgam and $O_p(\widetilde{P}) \leq T_1$ we get from (4) $J_0 \not\leq O_p(\widetilde{P})$ and thus $N_{\widetilde{P}}(J_0) \leq N_{\widetilde{P}}(Z_0)$.

Set $T_2 := N_T(J_0)$ and note that $J_0 \neq J(T_2)$ by (1). There exists $y \in N_{\widetilde{P}}(J_0)$ such that $T_2 \leq S^y$. From (3) we get

$$[\Omega_1(Z(J_0)), J(S)^y] = Z_0,$$

in particular $J(T_2) \leq N_H(Y_P)$ since $Y_P \leq \Omega_1(Z(J_0))$. Hence also $T_3 := \langle O_p(P), J(T_2)\rangle \leq N_H(Y_P)$, and $O_p(P) = C_{T_3}(Y_P)$ is normal in T_3 since $O_p(P) \in Syl_p(N_H(Y_P))$. It follows that $T_3 \leq T_1$ and thus by (4) $J_0 = J(T_2)$, a contradiction.

(3.4) Assume Hypothesis II. Let $V = \langle Y_P^{\widetilde{P}}\rangle$. Then V is abelian.

Proof. Set $V_0 = \langle Z_0^{\widetilde{P}}\rangle$. By Hypothesis I and (1.2)(b) $Y_P \leq Q$ and thus $V \leq O_p(\widetilde{P}) \leq S$. Assume that V is not abelian. Then there exists $x \in \widetilde{P}$ such that $A := Y_P^x \not\leq O_p(P)$. Then (2.1) and the Q-invariance of A show that $[V, Y_P] = [A, Y_P] = Z_0$ and $AO_p(P) = VO_p(P) = S_0$. Moreover $V_0 \leq Z(V) \leq O_p(P)$, and $O^{p'}(C_{\widetilde{P}}(V_0)) \leq O_p(\widetilde{P})$ since Z_0 is not normal in $\widetilde{P}$.

There exists $y \in P$ such that $\langle V, V^y\rangle C_{P^*}(Y_P) = P^*$. Since V is contained in Q and normal in S (3.2) implies $O^p(P) \leq \langle V, V^y\rangle$. Hence $Z(P) = 1$ gives $Z(\langle V, V^y\rangle) = 1$.

Note that $V_0 \leq O_p(P) \leq S^y$ and thus

$$[V_0, V_0^y] \leq V_0 \cap V_0^y \leq Z(\langle V, V^y\rangle) = 1.$$

Let $z', z \in \widetilde{P}$ such that for $A_1 = Y_P^z$ and $A_2 = Y_P^{z'}$

$$[A_1, A_2] = Z_0^z \neq Z_0.$$

It follows that $U := \langle A_1, A_2 \rangle \leq O_p(P)$. In addition $V_0^y \leq O^{p'}(C_{\widetilde{P}}(V_0)) \leq C_{O_p(\widetilde{P})}(Z_0^z)$ and thus $[V_0^y, U] \leq V_0^y \cap V_0 = 1$. Hence $U \leq C_{O_p(\widetilde{P}^y)}(V_0^y)$ and thus $[A_1, A_2, V^y] = 1$. It follows that Z_0^z centralizes V^y and

$$Z_0^z \leq Z(\langle V, V^y \rangle) = 1,$$

a contradiction.

Notation. From now through (3.9) we will apply the amalgam method to the amalgam $(P, \widetilde{P})$. With one exception we will use the standard terminology (see [DS], [KS] and the proof of Theorem 1). In particular we choose $\alpha, \beta, \alpha' \in \Gamma$ so that (α, α') is a critical pair and so that $\{G_\alpha, G_\beta\} = \{P, \widetilde{P}\}$. The exception to standard notation is the definition of Z_δ. For $\delta \in \Gamma$ we define

$$Z_\delta := Y_{G_\delta}.$$

In addition, we define for $g \in G$, $\delta = \alpha^g$ and $\lambda = \beta^g$

$$Z_\lambda^* = C_{Z_\delta}(O^p(G_\lambda)),\ \widetilde{Q}_\lambda = Q^g,\ Z(\delta, \lambda) = Z_0^g,\ \widetilde{C}_\lambda = \widetilde{C}^g,$$

$$V_\lambda^* = \langle x^h \mid h \in G_\lambda,\ x \in Z_\delta \text{ and } [x, S^g] \leq Z_\lambda^* \rangle.$$

Note that Z_λ^* is normal in G_λ and thus $[V_\lambda^*, Q_\lambda] \leq Z_\lambda^*$. Note further that

$$V_\lambda^* = \langle (Z_\delta \cap V_\lambda^*)^{G_\lambda} \rangle.$$

(3.5) Assume Hypothesis II. Then $Z = Y_{\widetilde{P}}$ and $\widetilde{P} = G_\beta$.

Proof. Clearly $Z = Y_{\widetilde{P}}$ implies $\widetilde{P} = G_\beta$. Thus, we may assume that $Z \neq Y_{\widetilde{P}}$. Then by (1.3)(b) $C_S(Y_{\widetilde{P}}) = O_p(\widetilde{P})$ and $[Z_\alpha, Z_{\alpha'}] \neq 1$. Let $1 \neq x \in [Z_\alpha, Z_{\alpha'}]$.

Assume that $G_\alpha = \widetilde{P}$. Then $Z_\alpha \leq Y_{\widetilde{C}} \leq Z(Q)$ by (1.2)(b) and $C_H(x) \leq \widetilde{C}$ by Q-Uniqueness. Since $Z_\alpha \not\leq Q_{\alpha'}$ we get $G_{\alpha'} \not\leq \widetilde{C}$. It follows that $G_{\alpha'}$ is conjugate to P.

Hence, after switching to another critical pair we may assume that $G_\alpha = P$. (3.4) shows that $b > 2$. Let $\alpha - 1 \in \Delta(\alpha)$ such that $\langle G_{\alpha-1} \cap G_\alpha, Z_{\alpha'} \rangle = G_\alpha$ and set $A := Z_{\alpha'-1}(Z_{\alpha'} \cap Q_\alpha)$. Since $b > 2$ we have

$$(*) \quad [Z_{\alpha-1}, A, Z_{\alpha'}] = 1.$$

Assume first that $[Z_{\alpha-1}, A] =: R \neq 1$. As above $C_H(R) \leq \widetilde{C}_{\alpha-1}$ since $G_{\alpha-1}$ is conjugate to $\widetilde{P}$. Hence $(*)$ gives

$$\langle G_{\alpha-1}, Z_{\alpha'} \rangle = \langle G_{\alpha-1}, G_\alpha \rangle \leq \widetilde{C}_{\alpha-1},$$

a contradiction.

Assume now that $[Z_{\alpha-1}, A] = 1$. Then $Z_{\alpha-1} \leq G_{\alpha'}$ and

$$Z_{\alpha'} \cap Q_\alpha = C_{Z_{\alpha'}}(Z_\alpha) \leq C_{Z_{\alpha'}}(Z_{\alpha-1}),$$

while (2.1) gives

$$|Z_\alpha / C_{Z_\alpha}(Z_{\alpha'})| = |Z_{\alpha'} / C_{Z_{\alpha'}}(Z_\alpha)|.$$

It follows that

$$(**) \quad |Z_{\alpha'}/C_{Z_{\alpha'}}(Z_\alpha Z_{\alpha-1})| = |Z_{\alpha'}/C_{Z_{\alpha'}}(Z_\alpha)| = |Z_\alpha/C_{Z_\alpha}(Z_{\alpha'})| \leq |Z_\alpha Z_{\alpha-1}/C_{Z_\alpha Z_{\alpha-1}}(Z_{\alpha'})|.$$

According to (2.1)(e), this time applied to $G_{\alpha'}$, equality holds in $(**)$, so $Z_{\alpha-1} \leq Z_\alpha Q_{\alpha'}$ and $[Z_{\alpha-1}, Z_{\alpha'}] \leq [Z_\alpha, Z_{\alpha'}] \leq Z_\alpha$. Hence, $Z_{\alpha-1}Z_\alpha$ and thus also $[Z_{\alpha-1}, Q_\alpha]$ is normal in G_α. Now the irreducibility of Z_α and (1.2)(e) yield $Z_{\alpha-1} \leq Z_\alpha$. But then $Q_\alpha \leq Q_{\alpha-1}$ and thus also $Q_\alpha \leq Q_\beta$. Since $Z_{\alpha'} \leq Q_\beta$ (2.1) and (3.1) give $S_0 \leq Q_\beta$ and $S_0 = B(S)$. Hence, Z_0 is normal in $\widetilde{P}$, which contradicts Hypothesis II.

(3.6) Assume Hypothesis II. Then $[Z_\alpha, Z_{\alpha'}] = 1$.

Proof. Asssume that $[Z_\alpha, Z_{\alpha'}] \neq 1$. From (3.5) we get that $\widetilde{P} = G_\beta$ and $Z_\beta = Z$. In particular b is even, and $G_{\alpha'}$ is conjugate to G_α. Moreover, (3.4) gives:

(1) V_β is an elementary abelian subgroup of Q_β, and $b \geq 4$.

Pick $\alpha'+1 \in \Delta(\alpha')$ such that $Z(\alpha', \alpha'+1) \neq Z(\alpha', \alpha'-1)$. The $\widetilde{Q}_{\alpha'+1}$-transitivity shows that $O^p(G_{\alpha'}) \leq \langle Z_\alpha, \widetilde{Q}_{\alpha'+1} \rangle C_{G_{\alpha'}}(Z_{\alpha'})$. So (3.2) yields $O^p(G_{\alpha'}) \leq \langle Z_\alpha, \widetilde{Q}_{\alpha'+1} \rangle$.

(2) $Z_\alpha \cap V_\beta^* \leq Z(\alpha, \beta)$.

Note that $S_0 = Q_\alpha \langle Z_{\alpha'}^{\widetilde{Q}_\beta} \rangle$ by (2.1) and Q-transitivity since $Z_{\alpha'} \in \mathcal{U}(P)$, so $[Z_\beta^*, S_0] = 1$. Hence $Z_\beta^* \leq Z(\alpha, \beta)$. Moreover

$$D := [Z_\alpha \cap V_\beta^*, S_0] \leq [Z_\alpha \cap V_\beta^*, Q_\alpha Q_\beta] \leq [V_\beta^*, Q_\beta] \leq Z_\beta^*.$$

Note that D is Q-invariant. Hence, the action of S_0 on Z_α and the Q-transitivity either give $D = 1$, or $D = Z(\alpha, \beta)$. The first case implies (2). In the second case $Z(\alpha, \beta) = Z^*_\beta$ is normal in G_β, which contradicts Hypothesis II.

(3) $V^*_{\alpha'+1} \leq Q_{\alpha+2}$.

This follows from (2) since $Z_{\alpha+2}$ centralizes $Z(\mu, \alpha'+1)$ for all $\mu \in \Delta(\alpha'+1)$.

(4) Let $A \leq V^*_{\alpha'+1}$ such that $O^p(G_{\alpha'}) \leq N_G(AZ_{\alpha'})$. Then $A \leq Z(\alpha', \alpha'+1)$.

Since $\langle A^{G_{\alpha'} \cap G_{\alpha'+1}} \rangle$ satisfies the hypothesis of (4) we may assume that A is $(G_{\alpha'} \cap G_{\alpha'+1})$-invariant; i.e. $AZ_{\alpha'}$ is normal in $G_{\alpha'}$. Then also $Y := [AZ_{\alpha'}, Q_{\alpha'}]$ is normal in $G_{\alpha'}$ and $Y \leq V^*_{\alpha'+1}$.

If $Y = 1$, then (1.2)(e) shows that $A \leq \Omega_1(Z(Q_{\alpha'})) = Z_{\alpha'}$ since $G_{\alpha'}$ is conjugate to P. Now (2) yields $A \leq Z(\alpha', \alpha'+1)$. If $Y \neq 1$, then the irreducibility of $Z_{\alpha'}$ gives $Z_{\alpha'} \leq Y$, which contradicts (2).

(5) $V^*_{\alpha'+1} \not\leq G_\alpha$.

Assume that $V^*_{\alpha'+1} \leq G_\alpha$. As $b > 2$ and thus $V^*_{\alpha'+1} \leq Q_{\alpha'}$, (2.4) gives

$$[Z_\alpha, V^*_{\alpha'+1}] \leq [Z_\alpha, Z_{\alpha'}][Z_\alpha \cap Q_{\alpha'}, V^*_{\alpha'+1}] \leq Z_{\alpha'} V^*_{\alpha'+1},$$

so $Z_{\alpha'} V^*_{\alpha'+1}$ is normal in $\langle Z_\alpha, G_{\alpha'} \cap G_{\alpha'+1} \rangle = G_{\alpha'}$. Now (4) shows that $V^*_{\alpha'+1} = Z(\alpha', \alpha'+1)$, which contradicts Hypothesis II.

By (1.3)(b) and Hypothesis II Q_β is the unique Sylow p-subgroup of $\cap_{\rho \in \Delta(\beta)} N_{G_\beta}(Z(\rho, \beta))$. Hence, by (5) there exists $\rho \in \Delta(\beta)$ such that $V^*_{\alpha'+1} \not\leq N_{G_\beta}(Z(\rho, \beta))$. Note that by (3) and (3.3) also $\langle Q_\rho^{V^*_{\alpha'+1}} \rangle \not\leq N_{G_\beta}(Z(\rho, \beta))$.

(6) $Z_\rho \leq Q_{\alpha'}$.

Assume that $Z_\rho \not\leq Q_{\alpha'}$. Then (ρ, α') is a critical pair, and $Z(\rho, \beta) = \langle [Z_\rho, Z_{\alpha'}]^{\widetilde{Q}_\beta} \rangle$ centralizes $\langle Q_\rho^{V^*_{\alpha'+1}} \rangle$, a contradiction.

(7) Set $R := [Z_\rho, V^*_{\alpha'+1}]$. Then $|R| < |Z(\rho, \beta)|$.

Note that by (3) and (6) $R \leq V^*_{\alpha'+1} \cap V_\beta$ and by (1) $[R, Z_\alpha] = 1$. Since $[V^*_{\alpha'+1}, Q_{\alpha'+1}] \leq Z^*_{\alpha'+1} \leq Z_{\alpha'}$ we get that $RZ_{\alpha'}$ is normalized by $\langle Z_\alpha, Q_{\alpha'+1} \rangle$ anf thus by $O^p(G_{\alpha'})$. Now (4) shows that $R \leq Z(\alpha', \alpha'+1)$; and equality does not hold since Z_α centralizes R but not $Z(\alpha', \alpha'+1)$.

We now derive a final contradiction. Let $t \in V^*_{\alpha'+1} \setminus N_{G_\beta}(Z(\rho, \beta))$, $U = \langle Q_\rho, t \rangle$ and $Y_0 = C_{Z_\rho}(t)$. Note that

$$|Z_\rho / Y_0| \leq |[Z_\rho, t]| \leq |[Z_\rho, V^*_{\alpha'+1}]|,$$

so by (7) $|Z_\rho/Y_0| < |Z(\rho,\beta)|$. On the other hand, by (3.1) $|Z_\rho| = |Z(\rho,\beta)|^2$ and so $|Y_0| > |Z(\rho,\beta)|$.

Set $U_0 := \langle Q_\rho^U \rangle$ and $Y_1 = C_{Z_\rho}(U_0)$. By (3.3) $U_0 \not\leq N_{G_\beta}(Z(\rho,\beta))$. Since $Y_0 \leq Y_1$ we also have $|Y_1| > |Z(\rho,\beta)|$. Moreover, Y_1 and U_0 are Q_β-invariant.

Let $x \in G_\beta$ such that $\alpha^x = \rho$. As seen above $S_0^x \leq Q_\beta Q_\rho$, so Y_1 is S_0^x-invariant. Moreover, since $|Y_1| > |Z(\rho,\beta)|$ we also have $[Y_1, S_0^x] \neq 1$. Now (3.1), applied to P^x $(= G_\rho)$, and the Q-transitivity yield

$$Z(\rho,\beta) = \langle [Y_1, S_0^x]^Q \rangle \leq Y_1.$$

This contradicts $U_0 \not\leq N_{G_\beta}(Z(\rho,\beta))$.

(3.7) Assume Hypothesis II. Let $A \leq \widetilde{P}$ and $Y_0 := [Y_P, A \cap P]$. Suppose that $A \not\leq N_{\widetilde{P}}(Z_0)$ and $[Y_0, A] = 1$. Then either $Y_0 = 1$, or the following hold:

(a) $p = 2$ and $\overline{P} \cong S_3 \ wr \ C_2$ or S_5.

(b) $|A \cap P / A \cap O_2(P)| = 2$, $|Y_0| = |Z_0| = 4$ and $C_{P^*}(Y_0) = O_2(P)$.

Proof. Set $A_0 := A \cap P$, $U := \langle O_p(P), A \rangle$, $U_0 := \langle O_p(P)^U \rangle$ and $Y_1 := C_{Y_P}(U_0)$. Note that

(1) $Y_0 \leq Y_1$, and U_0 is Q-invariant.

Hence Y_1 is the largest Q-invariant subgroup of Y_P centralized by U_0. By (3.3) $U_0 \not\leq N_{\widetilde{P}}(Z_0)$ and thus

(2) $Z_0 \not\leq Y_1$.

From now on we assume that $Y_0 \neq 1$ and use the notation of (3.1); in addition we set $q := p^m$ and $R_i := [V_i, A_0]$, $i = 1, ..., r$. It is convenient to treat the following two cases separately:

$(*)$ There exists $i \in \{1, ..., r\}$ such that $1 \neq R_i \leq V_i$.

$(**)$ $R_i \not\leq V_i$ for all $i \in \{1, ..., r\}$ with $R_i \neq 1$.

Case $(*)$: We have $A_0 \leq N_H(V_i)$ and thus $\overline{A}_0 \leq N_{\overline{P}}(K_i)$. If $\overline{A}_0 \leq K_i C_{\overline{P}}(V_i)$, then $R_i = Z_0 \cap V_i \leq Y_0$, and (1) and the Q-transitivity give $\langle R_i^Q \rangle = Z_0 \leq Y_1$, which contradicts (2). Hence, by (2.5)(e) $|A_0/C_{A_0}(V_i)| = 2 = p$.

Assume that $r > 1$. Then there exists $x \in Q$ such that $K_i^{\overline{x}} = K_j \neq K_i$ and

$$[K_i \cap \overline{S}, \overline{x}] C_{\overline{P}}(V_i) = (K_i \cap \overline{S}) C_{\overline{P}}(V_i).$$

It follows that

$$[R_i, K_i \cap \overline{S}] = [R_i, [K_i \cap \overline{S}, \overline{x}]] \leq [R_i, \overline{Q}],$$

so by (1) $[R_i, K_i \cap \overline{S}] = Z_0 \cap V_i \leq Y_1$. Now as above the Q-transitivity yields $Z_0 \leq Y_1$, which contradicts (2). Hence $r = 1$. Thus $|A_0/A_0 \cap O_2(P)| = 2$; moreover $|Y_P/C_{Y_P}(A_0)| = q$ and $C_{Y_P}(A_0) = Y_0$ since $\overline{A}_0$ acts as a field automorphism on $\overline{P}^*$.

We have proved:

(3) In case $(*)$ $r = 1$, $p = 2$, $C_{P^*}(Y_0) = O_2(P)$, $|A_0/A_0 \cap O_2(P)| = 2$ and $|Y_P/Y_0| = q$.

Case $(**)$: Fix $i \in \{1, .., r\}$ such that $R_i \neq 1$. Then $A_0 \not\leq N_P(V_i)$ since $R_i \not\leq V_i$, and from (2.5)(e) we get that $|A_0/C_{A_0}(V_i)| = 2(= p)$ and there exists $j \neq i$ such that $\langle V_i^{A_0} \rangle = V_i \times V_j$. Note that

$$V_i V_j = V_i(Y_1 \cap V_i V_j) = V_j(Y_1 \cap V_i V_j).$$

Assume that $r > 2$. Then by the Q-transitivity there exists $x \in Q$ such that $V_i^x \notin \{V_i, V_j\}$. In particular, there exists $\overline{b} \in (K_i \times K_i^{\overline{x}}) \cap \overline{Q}$ such that

$$V_i \cap Z_0 = [V_i, b] \leq [V_i V_j, b] = [V_j Y_1, b] = [Y_1, b].$$

As above, (1) and the Q-transitivity give $Z_0 \leq Y_1$, which contradicts (2). We have shown that $r = 2$, so $N_{A_0}(V_i) = C_{A_0}(V_i)$ implies $|A_0/A_0 \cap O_2(P)| = 2$.

For every $c \in P^* \setminus O_2(P)$ we have $[Y_0, c] \neq 1$ since $Y_P = Y_0 V_i$ for $i = 1, 2$. It follows that $C_{P^*}(Y_0) = O_2(P)$. Moreover $V_i \cap Y_0 = 1$ implies $|Y_0| = |V_i| = |Y_P/Y_0| = q^2$. We have shown:

(4) In case $(**)$ $r = 2 = p$, $C_{P^*}(Y_0) = O_2(P)$, $|A_0/A_0 \cap O_2(P)| = 2$ and $|Y_P/Y_0| = q^2$.

Assume that case (a) of the Local P!-Theorem holds for P. Then $r = 1$, $QO_2(P) = S_0$ and $[y, Q] = Z_0$ for every $y \in Y_P \setminus Z_0$. As $Y_0 \not\leq Z_0$ by (3), this gives $Z_0 \leq Y_1$, which contradicts (2). We have shown:

(5) Case (b) of the Local P!-Theorem holds for P; in particular $\mathcal{M}_H(P) = \{M\}$.

As a trivial consequence of (5) we get:

(6) $N_H(J(O_2(P))) \leq M$.

Let $O_2(P) \leq T \in Syl_2(U_0)$ and $T_0 = N_T(J(O_2(P)))$. Note that $T_0 \leq M$ by (6). By (3.1) $J(S) \leq S_0$ and by (2.1)(e)

$$\mathcal{A}(O_2(P)) \subseteq \mathcal{A}(S),$$

so $J(T_0) \leq S_0^x$ for some $x \in M$. According to (5) $P^* C_M(Y_P)$ is normal in M, hence $J(T_0) \leq P^* C_M(Y_P)$. Now by (1), (3) and (4) imply

$$J(T_0) \leq C_M(Y_0) \cap P^* C_M(Y_P) = C_{P^*}(Y_0) C_M(Y_P) = O_2(P) C_M(Y_P) = C_M(Y_P).$$

Since $O_2(P)$ is a Sylow 2-subgroup of $C_M(Y_P)$ we conclude that $J(T_0) = J(O_2(P))$ and thus also $J(T) = J(O_2(P))$; in particular $T = N_T(J(O_2(P))) = T_0 \le M$. In addition, (3.3) implies $T \le N_{\widetilde{P}}(Z_0)$ and (5) implies $Y_P = Y_M$. We have shown:

(7) $J(T) = J(O_2(P))$, and T normalizes Y_P and Z_0.

According to (5), (6), (7) and (b)(ii) of the Local P!-Theorem $N_{U_0}(T) \le M \cap \widetilde{C} \le N_M(Z_0)$. Since $U_0 \not\le N_{\widetilde{P}}(Z_0)$ there exists $F \in \mathcal{P}_{U_0}(T)$ such that $F \not\le N_H(Z_0)$; see (1.3)(a). As $O_2(\widetilde{P}) \le N_H(U_0)$ we get $[U_0, O_2(\widetilde{P})] \le O_2(U_0)$; in particular, F is $O_2(\widetilde{P})$-invariant and $[F, O_2(\widetilde{P})] \le O_2(F)$. In addition, (3.3) and (7) show $O_2(P) \not\le O_2(F)$ and thus by (1.3)(c)

(8) $O^2(F) = [O^2(F), O_2(P)] \le \langle O_2(P)^F \rangle$.

Set $W = \langle Y_P^F \rangle$. Clearly $[W, O^2(F)] \ne 1$ since by (7) $O^2(F) \not\le N_H(Z_0)$. Moreover, (3.4) shows that W is elementary abelian. Assume that $O_2(P) \cap O_2(F)$ is normal in F. Then by (8)

$$[O^2(F), O_2(\widetilde{P})] \le [\langle O_2(P)^F \rangle, O_2(\widetilde{P})] \le O_2(P) \cap O_2(F)$$

and $W = \langle Y_P^{O^2(F)} \rangle \le Z(O_2(P) \cap O_2(F))$ since $Y_P \le Z(O_2(P) \cap O_2(\widetilde{P}))$ by Hypothesis I and (1.2)(b). The $P \times Q$-Lemma implies that $[C_W(O_2(\widetilde{P})), O^2(F)] \ne 1$; in particular $[Y_{\widetilde{P}}, O^2(\widetilde{P})] \ne 1$, which contradicts (3.5). We have shown:

(9) $O_2(P) \cap O_2(F)$ is not normal in F.

Note that $F \not\le M$ since $M \cap \widetilde{C} \le N_M(Z_0)$, so by (6) and (7) $J(O_2(P)) = J(T) \not\le O_2(F)$. Assume that there exists only one non-central F-chief factor (in a given F-chief series) of W. As by (9)

$$[O^2(F), O_2(F)] \not\le O_2(F) \cap O_2(P) \text{ and } C_{O_2(F)}(W) \le O_2(F) \cap O_2(P),$$

we get $[O^2(F), O_2(F), W] \ne 1$. Thus by [Ste2, 3.3] there exists $B \le O_2(F)$ such that

$$[Y_P, B, B] = 1 \ne [Y_P, B] \text{ and } |[Y_P, B]| \le |B/C_B(Y_P)|.$$

The structure of P given in (3.1) shows that $B \le P^*$. But then (1), (3) and (4) imply $B \le C_{P^*}(Y_0) = O_2(P) = C_{P^*}(Y_P)$, a contradiction.

We have shown that there are at least two non-central F-chief factors in W. Let $B_1 \in \mathcal{A}(O_2(P))$ with $B_1 \not\le O_2(F)$. From (2.1) we get that

$$|B_1/C_{B_1}(W^*)| \le |W^*/C_{W^*}(B_1)|$$

for all non-central F-chief factors W^* of W.

We now apply the qrc-Lemma [Ste2, 3.1(c)] to F and B_1 and get $(q-1)(rc-1) \leq 1$ (where q, r and c are the parameters defined in [Ste]). Since by [Cher] $r \geq 1$ it follows that $q \leq 2$. Hence, there exists $B \leq O_2(F)$ such that

$$(+) \quad |B/C_B(Y_P)|^2 \geq |Y_P/C_{Y_P}(B)|.$$

Again by (3) and (4) $C_{P^*}(Y_0) = O_2(P)$ and thus $B \cap P^* \leq O_2(P)$.

As above, we now treat the two cases $(*)$ and $(**)$ separately. It remains to prove the isomorphism type of $\overline{P}$.

Assume case $(*)$. Then $\overline{B}$ induces a field automorphism of order 2 on $\overline{P}^*$. Hence $(+)$ gives $|Y_P| = 4^2$ and $\overline{P} \cong S_5$.

Assume case $(**)$. Then $Y_P = Y_0V_i$, $i = 1, 2$, and again $|\overline{B}| = 2$ and $|Y_P| = 4^2$, so $\overline{P} \cong S_3 \; wr \; S_2$.

L-Lemma. Let $X \in \mathcal{P}_H(S)$ and $A \leq S$ such that $A \not\leq O_p(X)$, and let M be the unique maximal subgroup of X containing S. Then there exists a subgroup $O_p(X) \leq L \leq X$ with $A \leq L$ satisfying:

(i) $AO_p(L)$ is contained in a unique maximal subgroup L_0 of L, and $L_0 = L \cap M^g$ for some $g \in X$.

(ii) $L = \langle A, A^x \rangle O_p(L)$ for every $x \in L \setminus L_0$.

(iii) L is not contained in any X-conjugate of M.

Proof. For $U \leq X$ set

$$U^* := \langle A^g \mid g \in X, A^g \leq U \rangle.$$

Note that $N_X(U) \leq N_X(U^*)$; in particular $N_X(S^*) \leq M$. Choose Y among all X-conjugates of M such that $Y \neq M$ and for $T \in Syl_p(Y \cap M)$

$$|T^*| \text{ is maximal.}$$

Without loss of generality we may assume that $T \leq S$. Let $h \in X$ such that $T \leq S^h \leq Y$ and set $N := N_X(T^*)$ and $S_1 := S \cap N$. Then $T \neq S^h$ since $Y \neq M$, so also $T < N_{S^h}(T) \leq N \cap S^h$. As $T \in Syl_p(Y \cap M)$ this gives $N \not\leq M$. Since $N_X(S^*) \leq M$ this implies that $T^* \neq S^*$ and thus also

$T^* \neq S_1^*$. Hence, there exists a conjugate $B = A^g$, $g \in X$, such that $B \leq S_1$ and $B \not\leq T$. Choose $z \in N \setminus M$ such that $L_1 := \langle B, z\rangle T^*$ is minimal, and set $L := L_1^{g^{-1}} O_p(X)$.

Since $T^* \neq BT^* = (BT^*)^*$ the maximality of T^* shows that M is the unique conjugate containing BT^*. In particular, (iii) holds since $L_1 \not\leq M$. Moreover, the minimality of L_1 gives (i). Let $x \in L_1 \setminus M$. Then M^x is the unique conjugate of M containing $B^x T^*$ and $M \neq M^x$, so $B^x \not\leq M$ and $\langle B, B^x\rangle T^* = L_1$. This gives (ii).

(3.8) Assume Hypothesis II. Let $A \leq S$ such that $[V_\beta, A, A] = 1$ and $A \not\leq Q_\beta$. Then there exist $\tau \in \Delta(\beta)$, $T \in Syl_p(G_\beta \cap G_\tau)$ and $L \leq G_\beta$ such that for $L(\tau) := N_L(Z(\tau, \beta))$, $W := \langle Z_\tau^L \rangle$ and $W^* := \langle v^h \mid v \in Z_\tau,\ h \in L,\ [v, T] \leq Z_\beta^* \rangle$ the following hold:

(a) $Q_\beta \leq AO_p(L) \leq T \cap L \in Syl_p(L(\tau))$, and $L(\tau)$ is a maximal subgroup of L.

(b) $L = \langle y, A^x\rangle O_p(L)$ for every $x \in L$ and every $y \in L \setminus L(\tau)^x$.

(c) $[W^*, O^p(L)] \neq 1$ and $[W, O^p(L)] \not\leq W^*$.

(d) Let U be a non-central L-chief factor of W. Then $C_U(A) = C_U(a)$ for every $a \in A \setminus O_p(L)$, and $|U/C_U(A)| \geq |A/A \cap O_p(L)|$.

Proof. According to (3.1), (3.4), (3.5) and (3.6) $b \geq 3$ and $\alpha' \in \beta^G$; in particular $Q_\tau \not\leq Q_\beta$ for all $\tau \in \Delta(\beta)$ since $Z_\alpha \leq Q_{\alpha'-1}$ and $Z_\alpha \not\leq Q_{\alpha'}$. We apply the L-Lemma with G_β in place of X. Then there exists $Q_\beta \leq L \leq G_\beta$ and $\tau \in \Delta(\beta)$ such that

(i) $L(\tau)$ is the unique maximal subgroup of L containing $AO_p(L)$, and $AO_p(L) \leq T \cap L \in Syl_p(L(\tau))$ for some $T \in Syl_p(G_\beta \cap G_\tau)$.

(ii) $L = \langle A, A^x\rangle O_p(L)$ for every $x \in L \setminus L(\tau)$.

(iii) $\langle L, T_0\rangle = G_\beta$ for every $T_0 \in Syl_p(G_\beta)$.

Claim (a) follows directly from (i).

Let y and x be as in (b). Then $y' := y^{x^{-1}} \in L \setminus L(\tau)$ and by (ii)

$$L = \langle A, A^{y'}\rangle O_p(L) = \langle A, y'\rangle O_p(L).$$

This implies (b).

For the proof of (c) assume first that $[W^*, O^p(L)] = 1$. Then $W^* \leq Z_\tau$ and $[W^*, T] \leq Z_\beta^* \leq W^*$ since $L = O^p(L)(T \cap L)$. By (iii) W^* is normal in $\langle L, T\rangle = G_\beta$. But this implies that $W^* = Z_\beta^* = Z_\tau$, a contradiction.

Assume now that $[W, O^p(L)] \leq W^*$. Then $W = W^* Z_\tau$ and

$$Z_\beta^*[W, \widetilde{Q}_\beta] = Z_\beta^*[Z_\tau, \widetilde{Q}_\beta] \leq Z_\tau.$$

Hence $Z_\beta^*[Z_\tau, \widetilde{Q}_\beta]$ is normal in $\langle T, L\rangle = G_\beta$. On the other hand $Q_\tau \not\leq Q_\beta$ and thus by (1.3)(b) $[Z_\tau, \widetilde{Q}_\beta] \leq Z_\beta^*$. Let $g \in G_\beta$ such that $\tau = \alpha^g$. Then the action of P^g on Z_τ, as described in (3.1), shows that

$$[Z_\tau, \widetilde{Q}_\beta \cap S_0^g] = Z(\tau, \beta) \leq Z_\beta^*,$$

which contradicts Hypothesis II. Hence, (c) is proved.

Note that L is minimal parabolic (with respect to $T \cap L$ and $L(\tau)$). Hence by (1.3)(b) $C_{T\cap L}(U) = O_p(L)$ for every non-central L-chief factor U in W. (2.1)(e) shows that

$$|U/C_U(A)| \geq |A/A \cap O_p(L)|.$$

Let $a \in A \setminus O_p(L)$. Then by (1.3)(b) there exists $x \in L$ such that $a \notin L(\tau)^x$. By (b) $L = \langle a, A^x\rangle O_p(L)$ and thus, together with the quadratic action of A on U,

$$U = [U, a] \times [U, A^x] = C_U(a) \times C_U(A^x);$$

in particular $C_U(a) = [U, a] \leq C_U(A)$ and equality holds. This is (d).

(3.9) No group satisfies Hypothesis II.

Proof. Assume Hypothesis II. By (3.1), (3.4), (3.5) and (3.6) $[Z_\alpha, Z_{\alpha'}] = 1$ and $b \geq 3$. In particular, $\alpha' \in \beta^G$ and V_β acts quadratically on $V_{\alpha'}$, and vice versa. We apply (3.8) with $(G_{\alpha'}, V_\beta)$ in place of (G_β, A) and choose the notation τ, L, T, W, W^* as there.

(1) $Z_\mu \not\leq G_\rho$ for every $\rho \in \Delta(\beta)$ and $\mu \in \tau^L$ such that $Z_\rho \not\leq L(\mu)$.

Assume that there exist $\rho \in \Delta(\beta)$ and $\mu \in \tau^L$ such that $Z_\rho \not\leq L(\mu)$ but $Z_\mu \leq G_\rho$. Let $x \in L$ such that $\mu = \tau^x$. Then, with the notation of (3.1) applied to G_ρ, there exists a submodule $V_i \leq Z_\rho$ such that $V_i \not\leq L(\mu)$. By (3.8)(b) $\langle V_i, V_\beta^x\rangle O_p(L) = L$. On the other hand $Z_\mu \leq G_\rho$, and (3.1) together with the quadratic action of Z_μ on Z_ρ gives either

$$[V_i, Z_\mu \cap W^*] = 1 \text{ or } [V_i, Z_\mu] = [V_i, Z_\mu \cap W^*].$$

In the first case $Z_\mu \cap W^*$ is normal in L. Hence $W^* = Z_\mu \cap W^*$, and by (1.3)(b) $[W^*, O^p(L)] = 1$ since $V_i \not\leq O_p(L)$. In the second case $[W, O^p(L)] \leq W^*$ since $O^p(L) \leq \langle V_i^L \rangle$, so both cases contradict (3.8)(c), and (1) is proved.

In particular, (1) together with $V_\beta \not\leq O_p(L)$ gives $W \not\leq Q_\beta$. Hence, we are allowed to apply (3.8) to (G_β, W) in place of (G_β, A). Again we use the notation of (3.8), but this time indicated by $\sim$ to distinguish from the above notation, so $\widetilde{\tau}$, $\widetilde{L}$, $\widetilde{T}$, $\widetilde{W}$, $\widetilde{W}^*$ are given as there. With the same argument as above we get

(2) $Z_{\widetilde{\mu}} \not\leq G_{\widetilde{\rho}}$ for every $\widetilde{\rho} \in \Delta(\alpha')$ and $\widetilde{\mu} \in \tau^{\widetilde{L}}$ such that $Z_{\widetilde{\rho}} \not\leq \widetilde{L}(\widetilde{\mu})$.

As above, (2) implies $\widetilde{W} \not\leq O_p(L)$. We now choose $\mu \in \tau^L$ and $\widetilde{\mu} \in \widetilde{\tau}^{\widetilde{L}}$ such that $\widetilde{W} \not\leq L(\mu)$ and $W \not\leq \widetilde{L}(\widetilde{\mu})$. From (1) and (2) we get that $Z_{\widetilde{\mu}} \not\leq O_p(L)$ and $Z_\mu \not\leq O_p(\widetilde{L})$. Moreover, we may assume that $|W/W \cap O_p(\widetilde{L})| \leq |\widetilde{W}/\widetilde{W} \cap O_p(L)|$, since the other case follows by the same argument with the roles of W and $\widetilde{W}$ reversed.

From (3.8)(c) we get that there exist two non-central L- chief factors U_1 and U_2 in W. As $Z_{\widetilde{\mu}} \not\leq O_p(L)$ (3.8)(d) implies that $C_{U_i}(V_\beta) = C_{U_i}(Z_{\widetilde{\mu}})$, so, again by (3.8)(d),

$$|\widetilde{W}/\widetilde{W} \cap O_p(L)| \leq |V_\beta/V_\beta \cap O_p(L)| \leq |U_i/C_{U_i}(V_\beta)| = |U_i/C_{U_i}(Z_{\widetilde{\mu}})|.$$

Hence

$$\begin{aligned} |\widetilde{W}/\widetilde{W} \cap O_p(L)|^2 &\leq |U_1/C_{U_1}(Z_{\widetilde{\mu}})||U_2/C_{U_2}(Z_{\widetilde{\mu}})| \leq |W/C_W(Z_{\widetilde{\mu}})| \\ &\leq |W/W \cap Q_{\widetilde{\mu}}| \leq |W/W \cap O_p(\widetilde{L})||W \cap G_{\widetilde{\mu}}/W \cap Q_{\widetilde{\mu}}| \\ &\leq |\widetilde{W}/\widetilde{W} \cap O_p(L)||W \cap G_{\widetilde{\mu}}/W \cap Q_{\widetilde{\mu}}|. \end{aligned}$$

On the other hand by (3.7), applied to $G_{\widetilde{\mu}}$ with $A = W$, we get $|W \cap G_{\widetilde{\mu}}/W \cap Q_{\widetilde{\mu}}| \leq 2$. It follows that

(3) $|W/W \cap O_p(\widetilde{L})| = |\widetilde{W}/\widetilde{W} \cap O_p(L)| = 2 = p$ and $|Z_\mu| = |Z_{\widetilde{\mu}}| = 16$.

(4) $|W/C_W(Z_{\widetilde{\mu}})| = |\widetilde{W}/C_{\widetilde{W}}(Z_\mu)| = 4$.

As a consequence we get from (3)

(5) $Z_{\widetilde{\mu}} \not\leq L(\mu)$ and $Z_\mu \not\leq \widetilde{L}(\widetilde{\mu})$.

Next we prove:

(6) $L/C_L(W) \cong \widetilde{L}/C_{\widetilde{L}}(\widetilde{W}) \cong S_3$.

Let $t \in Z_{\widetilde{\mu}} \setminus O_2(L)$ and $x \in L$ such that $\mu = \tau^x$. Then by (3) and (5) $L = \langle t, t^x \rangle O_2(L)$ and thus $O^2(L) \leq \langle t^L \rangle$. Hence, (3.8)(c) gives $W^* \not\leq C_W(t)$ and $W^* C_W(t) \neq W$, and (6) follows for L

from $|W/C_W(Z_{\widetilde{\mu}})| = 4$. A similar argument gives the claim for $\widetilde{L}$.

Set $W_0 := W$ and $W_i := [W_{i-1}, \widetilde{Q}_{\alpha'}]$ for $i \geq 1$, and note that $W_i = \langle (W_i \cap Z_\mu)^L \rangle$.

(7) Assume that $(W_i \cap Z_\mu)W_{i+1} = W_i$. Then $W_i \leq Z_\mu$.

Note that $W_{i+1} = [W_i, \widetilde{Q}_{\alpha'}] \leq [Z_\mu W_{i+1}, \widetilde{Q}_{\alpha'}] \leq Z_\mu W_{i+2}$. It follows that $W_i = (W_i \cap Z_\mu)W_k$ for all $k \geq i+1$ and thus $W_i \leq Z_\mu$.

(8) $[Z_{\widetilde{\mu}}, Z_\mu \cap O_2(\widetilde{L})] \neq 1$.

Let $A_1 := Z_\mu \cap O_2(\widetilde{L})$,and assume that $[Z_{\widetilde{\mu}}, A_1] = 1$. By (6) $L(\mu) = (L(\mu) \cap G_\mu)C_L(W)$. Suppose that $Z_\mu = A_1(Z_\mu \cap W_1)$. Then $W = Z_\mu W_1$ and by (7) $W = Z_\mu$. But then Z_μ is normal in $\langle L, G_{\alpha'} \cap G_\mu \rangle = G_{\alpha'}$, a contradiction. We have shown that $Z_\mu \cap W_1 \leq A_1$. It follows that $Z_\mu \cap W_1$ is centralized by $Z_{\widetilde{\mu}}$ and thus normalized by L, so $W_1 \leq A_1$ and $[W_1, O^2(L)] = 1$. In particular $[Z_\mu, \widetilde{Q}_{\alpha'}]$ is normalized by L and centralized by $O^2(L)$. Hence, by the L-Lemma (iii) it is also normalized by $G_{\alpha'}$ and centralized by $O^2(G_{\alpha'})$. Since $Z(\mu, \alpha') \leq [Z_\mu, Q_{\alpha'}]$ we get that $Z(\mu, \alpha')$ is normal in $G_{\alpha'}$, a contradiction to Hypothesis II.

(9) $R := [Z_{\widetilde{\mu}} \cap O_2(L), Z_\mu \cap O_2(\widetilde{L})] \neq 1$, and R is centralized by a Sylow 2-subgroup of $G_{\widetilde{\mu}}$ and G_μ.

Let $A := Z_\mu$ and $A_0 := A \cap G_{\widetilde{\mu}}$. By (8) $Y_0 := [Z_{\widetilde{\mu}}, A_0] \neq 1$, and by (5) A and $G_{\widetilde{\mu}}$ satisfy the hypothesis of (3.7). Then (3.7) shows that $|Y_0| = 4$ and $|A_0/A_0 \cap Q_{\widetilde{\mu}}| = 2$; in particular $A_0 = A \cap O_2(\widetilde{L})$. Moreover, (3.7) gives $|Z_{\widetilde{\mu}}/C_{Z_{\widetilde{\mu}}}(A_0)| = 4$ and thus $R \neq 1$ since $|Z_{\widetilde{\mu}}/Z_{\widetilde{\mu}} \cap O_2(L)| = 2$.

The action of $G_{\widetilde{\mu}}$ on $Z_{\widetilde{\mu}}$ also shows that all elements of Y_0 are centralized by a Syolw 2-subgroup of $G_{\widetilde{\mu}}$. This and the symmetric argument in G_μ yields the additional claim of (9).

We now derive a final contradiction. According to (9) there exist $y \in G_{\widetilde{\mu}}$ and $z \in G_\mu$ such that $R = Z_\beta^y = Z_{\alpha'}^z$. Then by (1.6) $\widetilde{C}_\beta^y = \widetilde{C}_{\alpha'}^z$ and thus $\widetilde{Q}_\beta^y = \widetilde{Q}_{\alpha'}^z$. On the other hand, Hypothesis I and (1.2)(b) yield $Z_{\widetilde{\mu}} \leq \widetilde{Q}_\beta^y$, so $Z_{\widetilde{\mu}} \leq \widetilde{Q}_{\alpha'}^z \leq G_\mu$, which contradicts (2) and (5).

Theorem 3. Assume Hypothesis I. Then Z_0 is normal in $\widetilde{C}$.

Proof. Assume that Z_0 is not normal in $\widetilde{C}$. By the definition of $\widetilde{C}$ $N_H(S) \leq \widetilde{C}$. Hence, $N_H(S)$ acts on $\mathcal{P}_H(S) \setminus \mathcal{P}_{\widetilde{C}}(S)$, and Theorem 2 implies that $N_H(S) \leq N_H(P)$ and thus also $N_H(S) \leq N_H(P^*)$ since $P^* = U(P)$. It follows that $N_H(S) \leq N_H(S_0) \leq N_H(Z_0)$.

According to (1.3)(a) there exists $\widetilde{P} \in \mathcal{P}_{\widetilde{C}}(S)$ such that Z_0 is not normal in $\widetilde{P}$. We choose $|\widetilde{P}|$

minimal with this property. If $(P,\widetilde{P})$ is an amalgam, then $(P,\widetilde{P})$ satisfies Hypothesis II, which is impossible by (3.9).

Thus, $(P,\widetilde{P})$ is not an amalgam, and there exists $L \in \mathcal{L}_H(S)$ such that $\langle P,\widetilde{P}\rangle \leq L$. Let $L << \widetilde{M} \in \mathcal{L}_H^*(S)$. Then by (1.2) $Y_L \leq Y_{\widetilde{M}}$ and by (1.4) $P^0 \leq L^0 \leq \widetilde{M}^0 \leq \widetilde{M}$.

We now apply the Local P!-Theorem to $\widetilde{M}$. Assume that also $\widetilde{P} \leq \widetilde{M}$. Then $\widetilde{P} \leq \widetilde{M} \cap \widetilde{C} \leq N_{\widetilde{M}}(Z_0)$, a contradiction. Thus, we have $\widetilde{P} \not\leq \widetilde{M}$.

Assume first that case (a) of the Local P!-Theorem holds. Then $Q \leq S_0$, so $Z_0 \leq Z(Q)$ and thus also $W := \langle Z_0^{\widetilde{P}}\rangle \leq Z(Q)$. Note that

$$Z_0 \leq Y_P = [Y_P, P^0] \leq [Y_L, L^0] \text{ and } W \leq [Y_L, L^0]$$

by (1.2). It follows that $W \leq [Y_{\widetilde{M}}, \widetilde{M}_0]$ since $Y_L \leq Y_{\widetilde{M}}$ and $L^0 \leq \widetilde{M}^0$. In case (a) $[Y_{\widetilde{M}}, \widetilde{M}_0]$ is a natural $SL_n(p^m)$- or $Sp_{2n}(p^m)'$-module. In particular, $C_{[Y_{\widetilde{M}},\widetilde{M}_0]}(Q) = Z_0$ and so $Z_0 = W$ and $\widetilde{P} \leq N_H(Z_0)$, a contradiction.

Assume finally that case (b) of the Local P!-Theorem holds for $\widetilde{M}$. Then $P^0 = L^0 = \widetilde{M}^0$ and $\widetilde{P} \leq N_H(\widetilde{M}^0) = \widetilde{M}$, which contradicts $\widetilde{P} \not\leq \widetilde{M}$.

Corollary 2. Assume Hypothesis I and $p = 2$. Then $\mathcal{P}_H(S) = \{P\} \cup \mathcal{P}_{\widetilde{C}}(S)$.

Proof. We apply Theorem 2. Then $\mathcal{P}_H(S) = \mathcal{P}_P(S) \cup \mathcal{P}_{\widetilde{C}}(S)$, and the structure of P, see (3.1), implies $\mathcal{P}_P(S) = \{P\} \cup \mathcal{P}_{N_P(Z_0)}(S)$. Now Theorem 3 yields the assertion.

Corollary 3. Assume Hypothesis I. Suppose that case (b) of the Local P!-Theorem holds for $P \leq M \in \mathcal{L}_H^*(S)$. Then the following holds:

(a) $p = 2$ and $\mathcal{M}_H(P) = \{M\}$,

(b) $\overline{P}^* = K_1 \times \cdots \times K_r$, $K_i \cong SL_2(2)$,

(c) $Y_P = V_1 \times \cdots \times V_r$, where $V_i = [Y_P, K_i]$ is a natural $SL_2(2)$-module for K_i,

(d) $r \geq 4$.

(e) Q is transitive on $K_1, ..., K_r$.

Proof. We are in case (b) of the Structure Theorem. According to Theorem 3 Z_0 is normal in $\widetilde{C}$. Hence

$$(*) \quad [N_P(Z_0), Q] \leq O_p(N_P(Z_0)).$$

We apply (3.1). Then either $\overline{P}^* \cong SL_2(p^m)$, or the Q-transitivity and $(*)$ show that $N_{K_i}(Z_0)$ is a p-group and $r \geq 2$.

In the first case Y_P is a natural $SL_2(p^m)$-module for P^*. Thus, Y_P is an F-vector space for $F := End_{\overline{P}^*}(Y_P)$, and P induces semi-linear transformations on Y_P. As $N_{P^*}(Z_0)$ is irreducible on Z_0, we get from $(*)$ that $[Z_0, Q] = 1$, so Q centralizes a 1-dimensional F-subspace of Y_P. Hence, Q induces F-linear transformations on Y_P, and $Q \leq P^*$. But this contradicts case (b) of the Structure Theorem.

In the second case (a) – (c) and (e) are clear. For the proof of (d) note that Q-transitivity yields $r = 2$ or (d). Assume $r = 2$, so $P/C_P(Y_P) \cong O_4^+(2)$ and $|Z_0| = 4$. Hence, Theorem 3 shows that $\widetilde{C}/C_{\widetilde{C}}(Z_0)$ is a subgroup of S_3. If all involutions in Z_0 are conjugate in $\widetilde{C}$, then Q-Uniqueness implies that $P \leq \widetilde{C}$, which is not the case. It follows that $\widetilde{C} = C_{\widetilde{C}}(Z_0)S$, in particular $C_{\widetilde{C}}(Z_0) \not\leq M$. We conclude that $C_H(x) \not\leq M$ for all $1 \neq x \in Y_P$. Now Theorem 3 of [MSS2] shows that $Y_M \not\leq Q$, a contradiction.

4. F-Uniqueness

In this section we treat the exceptional case described in Corollary 3, so in this section we assume:

Hypothesis III. Hypothesis I and case (b) of the Local P!-Theorem holds for $P \leq M \in \mathcal{L}^*_H(S)$; in particular $\mathcal{M}_H(P) = \{M\}$.

Notation. We use the notation given in Corollary 3 (and (3.1)). Set

$$F := C_{\widetilde{C}}(Z_0) \text{ and } \Omega := \{K_1, ..., K_r\}.$$

Recall that by Theorem 3 F is normal in $\widetilde{C}$, and by Corollary 3

(∗) $p = 2$, $K_i \cong SL_2(2)$, $r \geq 4$, and Q is transitive on Ω.

We will use these facts without further reference.

(4.1) $P^* \cap \widetilde{C} = S_0 C_{P^*}(Y_P)$ and $\widetilde{C} = C$.

Proof. Assume that $U := C_{P^*}(Y_P)S_0 < P^* \cap \widetilde{C}$. Then by Corollary 3 (b) $K_i \leq [\overline{S}_0, \overline{P^* \cap \widetilde{C}}]\overline{S}_0$ for some i, and the Q-transitivity yields $P^* \leq \widetilde{C}$, which is not the case.

Let $Z^* = \langle Z^{\widetilde{C}} \rangle$. By Theorem 3 $Z^* \leq Z_0 \cap Z(Q)$, and by Q-uniqueness $C_{P^*}(z) \leq P^* \cap \widetilde{C} = S_0 C_{P^*}(Y_P)$ for all $1 \neq z \in Z^*$. Now Corollary 3 (c) yields $|Z^*| = 2$, so $C = \widetilde{C}$.

(4.2) $N_H(B(S)) \leq M$.

Proof. It suffices to show that P and $N_H(B(S))$ are contained in a 2-local subgroup of H since $\mathcal{M}_H(P) = \{M\}$. Assume that this is not the case; i.e. $O_2(\langle P, N_H(B(S)) \rangle) = 1$. Then $B(S)$ is not normal in P and by (3.1) $B(S) = S_0$. Hence, $N_H(B(S)) = N_H(S_0) \leq N_H(Z_0) = \widetilde{C}$. For every $i = 1, ..., r$ we choose $X_i \leq P^*$ minimal with respect to

$$B(S) \leq X_i \text{ and } \overline{X}_i = K_i\overline{B(S)}.$$

Then $X_i \in \mathcal{P}_H(B(S))$ and $\langle X_i, S \rangle = P$. Moreover $V_i = [Y_{X_i}, O^2(X_i)] = [Y_P, O^2(X_i)]$ since $Y_{X_i} \leq \Omega_1(Z(O_2(P))) = Y_P$.

Suppose that there is a non-trivial characteristic subgroup A of $B(S)$, which is normal in X_1. Then $\langle S, X_1, N_H(B(S))\rangle = \langle P, N_H(B(S))\rangle \leq N_H(A)$, which contradicts $O_2(\langle P, N_H(B(S))\rangle = 1$.

Hence, no non-trivial characteristic subgroup of $B(S)$ is normal in X_1. Now [Ste1] gives $[O_2(X_1), O^2(X_1)] = V_1 \leq Y_P$. Hence also $[O_2(P), O^2(P)] \leq Y_P$, and $Z(P) = 1$ yields

$$Y_P = O_2(P) = V_1 \times \cdots \times V_r.$$

Since Q is transitive on $\{V_1, ..., V_r\}$ and $N_H(B(S))$ does not normalize Y_P there exists $t \in N_H(B(S))$ such that $R := [V_1, V_1^t] \neq 1$. It follows that also $[V_1^t, V_1^{t^2}] \neq 1$, so

$$R^t = [V_1^t, V_1^{t^2}] = [V_1^t, V_1] = R$$

since $\langle V_1, V_1^{t^2}\rangle \leq B(S) \leq N_P(V_1^t)$. As $t \in \widetilde{C}$ and Y_P is normal in Q the Q-transitivity gives

$$(*) \quad S_0 = Y_P Y_P^t \text{ and } Y_P \cap Y_P^t = Z_0.$$

Let $U = N_H(R)$ and $W = O_2(U)$. Then $\langle t, X_2, ..., X_r\rangle \leq U$, and $V_i \cap W$ is X_i-invariant for every $i \geq 2$. It follows that either there exists an $i \geq 2$ such that $V_i \leq W$, or $V_i \cap W = 1$ for every $i \geq 2$. The first case gives $V_i^t \leq W$ and so $V_i^t \leq O_2(X_2 \cdots X_r)$. On the other hand, by $(*)$ $[Y_P, V_i^t] \neq 1$, so we get that $[V_i^t, V_1] = R$. But this implies that $R \leq V_i^t$ and $R = R^t \leq V_i$, which is impossible since $V_1 \cap V_i = 1$ for $i > 1$.

We have shown that $V_i \cap W = 1$ for $i > 1$. It follows that $[S_0 \cap W, O^2(X_2)] = 1$. Since $S_0 \cap W$ is normalized by X_2 and W we get $[(S_0 \cap X_2)^x, W] \leq S_0 \cap W$ for every $x \in X_2$. Hence $[W, O^2(X_2)] \leq S_0 \cap W$ and $[W, O^2(X_2), O^2(X_2)] = 1$. But then U is not of characteristic 2, a contradiction.

(4.3) Let $S_0 \leq T$, T a 2-subgroup of H. Then S_0 is normal in $N_H(T)$ and $N_H(T) \leq M \cap \widetilde{C}$.

Proof. Note that $N_H(T) \leq N_H(B(S)) \leq M$ by (3.1) and (4.2). Moreover, by the Structure Theorem, case (b), $Y_P = Y_M$ and $P^* C_M(Y_M)$ is normal in M, so $T \cap P^* C_M(Y_M) = S_0$. Hence Theorem 3 gives $N_M(T) \leq N_M(S_0) \leq M \cap \widetilde{C}$.

(4.4) Let $\widetilde{L} \in \mathcal{L}_H(S)$. Then either $\widetilde{L} \leq \widetilde{C}$, or $P \leq \widetilde{L} \leq M$ and $F \not\leq \widetilde{L}$.

Proof. Assume that $\widetilde{L} \not\leq \widetilde{C}$. Then (1.3)(a) and the Corollaries 2 and 3 show that $P \leq \widetilde{L} \leq M$. If in addition $F \leq M$, then the Frattini argument and (4.3) imply that $\widetilde{C} = FN_H(S_0) \leq M$, a contradiction.

(4.5) Suppose that $S_0 \leq T \leq S$ such that $|S/T| = 2$ and $S = TQ$. Let $T \leq L \leq H$ and $O_2(L) \neq 1$. Then one of the following holds:

(a) $L \leq M$.

(b) $L \leq \widetilde{C}$.

(c) $L \in \mathcal{L}_H(T)$.

Proof. Let $U = N_H(O_2(L))$ and $T \leq T_0 \in Syl_2(U)$. By (4.3) $T_0 \leq M \cap \widetilde{C}$ and thus either $T = T_0$ or $T_0 \in Syl_2(\widetilde{C})$ and $Q \leq T_0$. In the second case $T_0 = TQ = S$, and (4.4) yields $L \leq U \leq M$ or $L \leq U \leq \widetilde{C}$. In the first case $U \in \mathcal{L}_H(T)$ and thus also $L \in \mathcal{L}_H(T)$.

Notation. From now on we fix a maximal subgroup T of S containing $N_S(K_1)$. Recall that $B(S) \leq S_0 \leq T$. Let $Q_0 := T \cap Q$ and

$$\mathcal{L}_0(T) := \{U \in \mathcal{L}_H(T) \mid U \not\leq \widetilde{C} \text{ and } U \cap \widetilde{C} \not\leq M\}.$$

By $\mathcal{L}_0(T)_*$ we denote the set of minimal elements of $\mathcal{L}_0(T)$.

(4.6) Let $\overline{P}^* := P^*/C_{P^*}(Y_P)$ and $1 \neq K \leq O^2(\overline{P}^*)$. Suppose that K is Q_0-invariant. Then $K = O^2(\overline{P}^*)$ or $K = \times_{X \in \Omega_i} X'$ for some T-orbit Ω_i of Ω; in particular $[K, \overline{Q}_0] \neq 1$.

Proof. Since $K \neq 1$ there exist $K_i \in \Omega$ and $t \in S_0 \cap K_i$ such that $[K, \bar{t}] = K_i'$. Let $q \in Q$ such that $K_i^q \neq K_i$, and let $q_0 := [t, q]$ and $R := [K, \overline{q}_0]$. Then $q_0 \in S_0 \cap Q \leq Q_0$ and $R \leq (K_i \times K_i^q) \cap K$ with $[R, \bar{t}] = K_i'$.

Since $r > 2$ there exists $x \in Q$ such that $K_i^x \notin \{K_i, K_i^q\}$. Let $x_0 = [t, x]$. Then as above $x_0 \in Q_0 \cap S_0$, while $\overline{x}_0 C_{\overline{S}_0}(K_i \times K_i^q) = \bar{t} C_{\overline{S}_0}(K_i \times K_i^q)$. It follows that $[R, \overline{x}_0] = K_i' \leq K$.

We have shown that $K_i' \leq K$ for every $K_i \in \Omega$ such that $[K, K_i] \neq 1$. Now the action of Q_0 on K and Ω gives the desired structure of K. Moreover, $r > 2$ implies that $[K, \overline{Q}_0] \neq 1$.

(4.7) $|S/T| = 2$, $S = TQ$, and T has two orbits Ω_1 and Ω_2 on Ω such that for $Z_i := C_{\Omega_1(Z(T))}(\Omega_i)$ the following hold:

(a) $|\Omega_i| = \frac{r}{2}$ and $|Z_i| = 2$, $i = 1, 2$, and

(b) $\Omega_1(Z(T)) = Z_1 \times Z_2$.

Proof. This is a direct consequence of the choice of T.

(4.8) $\mathcal{L}_0(T) \neq \emptyset$.

Proof. Let $L := C_H(Z_1)$, Z_1 as in (4.7). Then $L \not\leq \widetilde{C}$, and by (4.4) $L \cap \widetilde{C} \not\leq M$ since $F \leq L \cap \widetilde{C}$. Now (4.5) shows that $L \in \mathcal{L}_0(T)$.

(4.9) Let $L \in \mathcal{L}_0(T)$. Then $O_2(\langle O^2(P^*), L \cap \widetilde{C} \rangle) = 1$.

Proof. Let $L_0 := \langle O^2(P^*), L \cap \widetilde{C} \rangle$ and assume that $O_2(L_0) \neq 1$. Let $t \in Q \setminus T$. Then $T\langle t \rangle = S$ since T has index 2 in S. Moreover, $[t, L \cap \widetilde{C}] \leq Q_0 \leq O_2(L \cap \widetilde{C})$. It follows that t normalizes L_0. Hence $S \leq L_0\langle t \rangle$ and $1 \neq O_2(L_0) \leq O_2(L_0\langle t \rangle)$. This contradicts (4.4) since $L_0 \not\leq M$ as $L \cap \widetilde{C} \not\leq M$ and $L_0 \not\leq \widetilde{C}$ as $O^2(P^*) \not\leq \widetilde{C}$.

Theorem 4. Suppose that $L \in \mathcal{L}_0(T)$. Then

$$\mathcal{P}_L(T) = \mathcal{P}_{L \cap M}(T) \cup \mathcal{P}_{L \cap \widetilde{C}}(T).$$

Proof. Assume that there exists $X \in \mathcal{P}_L(T)$ such that $X \not\leq M$ and $X \not\leq \widetilde{C}$. By (4.2) and (1.3)(b) neither $B(S)$ nor $\Omega_1(Z(T))$ is normal in X. Hence, (2.9) implies that there exists a minimal parabolic subgroup X_0 of characteristic 2 in X such that X_0 satisfies (2.9)(a) – (e) (in place of L_i); in particular $X = \langle T, X_0 \rangle$, $O_2(X)B(S) \in Syl_2(X_0)$ and $X_0/C_{X_0}(Y_{X_0}) \cong SL_2(2^k)$. We choose $X^* \leq X_0$ minimal with respect to

$$B(S) \leq X^* \text{ and } X_0 = X^* C_{X_0}(Y_{X_0}).$$

Then X^* is a minimal parabolic subgroup and $X = \langle X^*, T \rangle$. Moreover $B(S) \in Syl_2(X^*)$ by (2.7) applied to X^*.

Assume that there exists a non-trivial characteristic subgroup A of $B(S)$ which is normal in X^*. As A is also characteristic in S we get

$$(*) \quad X = \langle T, X^* \rangle \leq N_H(A) \text{ and } S \leq N_H(A).$$

Hence by (4.4) $N_H(A) \le \widetilde{C}$ or $N_H(A) \le M$, which contradicts $X \le N_H(A)$.

Thus, no non-trivial characteristic subgroup of $B(S)$ is normal in X^*. As X^* is a minimal parabolic subgroup the hypothesis of [Ste1] is satisfied. We get $[O^2(X^*), O_2(X^*)] = [Y_{X^*}, O^2(X^*)]$ and $Y_{X^*}/C_{Y_{X^*}}(X^*)$ is a natural $SL_2(2^k)$-module for $X^*/C_{X^*}(Y_{X^*})$, so $[O^2(X^*), O_2(X^*)] \le Y_X$. Since $[O_2(X), B(S)] \le B(S) \cap O_2(X) \le O_2(X^*)$ we also get

$$[O_2(X), O^2(X^*)] \le Y_X \text{ and } [O_2(X), O^2(X)] \le Y_X.$$

As in the proof of (4.9) pick $t \in Q \setminus T$. Then

$$(**) \quad [L \cap \widetilde{C}, t] \le Q \cap T \le O_2(L \cap \widetilde{C}).$$

Assume first that $Y_X^t \le O_2(X)$. Then

$$S \le \langle X, t \rangle \le N_H(Y_X Y_X^t) \in \mathcal{L}_H(S),$$

and by (4.4) $N_H(Y_X Y_X^t) \le M$ or $N_H(Y_X Y_X^t) \le \widetilde{C}$. But this contradicts $X \le N_H(Y_X Y_X^t)$.

We have shown that $Y_X^t \not\le O_2(X)$. As $|Y_X/C_{Y_X}(Y_X^t)| = |Y_X^t/C_{Y_X^t}(Y_X)|$ we get $Y_X^t \in \mathcal{U}(X)$ (for the definition see section 2). Since Y_X^t is normal in T we conclude with (2.1) that $Y_X^t O_2(X) = B(S)O_2(X)$. In addition, (2.1) shows that $B(S)C_X(Y_X)/C_X(Y_X)$ is self-centralizing in $X/C_X(Y_X)$. It follows that $O_2(X^t) \le Y_X^t O_2(X)$. Hence, for $D := O_2(X) \cap O_2(X^t)$ we get $O_2(X^t) = Y_X^t D$ and similarly $O_2(X) = Y_X D$. This gives

$$\Phi(O_2(X^t)) = \Phi(D) = \Phi(O_2(X));$$

in particular $\langle X, S \rangle \le N_H(\Phi(D))$. Now as above (4.4) implies that $\Phi(D) = 1$, so $O_2(X) = Y_X$ and $B(S) = Y_X Y_X^t$.

The action of T on $B(S)$ shows that Y_X and Y_X^t are the only maximal T-invariant elementary abelian normal subgroups of $B(S)$; in particular $Y_X = Y_L$, and by $(**)$ $L \cap \widetilde{C}$ normalizes $B(S)$. Now (4.2) yields $L \cap \widetilde{C} \le M$, which contradicts $L \in \mathcal{L}_0(T)$.

(4.10) Let $L \in \mathcal{L}_0(T)_*$ and N be a normal subgroup of L that is minimal with respect to $N \not\le \widetilde{C}$. Then $N = [N, Q_0] = O^2(L)$.

Proof. As $N(L\cap \widetilde{C})\in \mathcal{L}_H(T)$ the minimality of L yields $L=N(L\cap \widetilde{C})$. Hence $N_0:=[N,Q_0]$ is a normal subgroup of L. Assume that $N\neq N_0$. The the minimal choice of N gives $N_0\leq \widetilde{C}$, so N_0Q_0 is a normal subgroup of L in $\widetilde{C}$. It follows that $Q_0\leq O_2(N_0Q_0)\leq O_2(L)$. But then $[Q,O_2(L)]\leq Q_0\leq O_2(L)$ and $S=TQ\leq N_H(O_2(L))$, so (4.4) implies that $L\leq \widetilde{C}$ or $L\leq M$. This contradicts the definition of $\mathcal{L}_0(T)$.

We have shown that $N=N_0$. The minimality of N also gives that $N=O^2(N)$. Thus, it remains to prove that $L=NT$. Assume now that $L\neq NT$. By Theorem 4

$$\mathcal{P}_{NT}(T)\subseteq \mathcal{P}_M(T)\cup \mathcal{P}_{\widetilde{C}}(T).$$

Since $NT\not\leq \widetilde{C}$ the minimality of L shows that $NT\cap\widetilde{C}\leq M$. Thus $\mathcal{P}_{NT}(T)\subseteq \mathcal{P}_M(T)$. As by (4.3) also $N_L(T)\leq M$ we conclude from (1.3)(a) that $NT\leq M$.

Now $N=[N,Q_0]\leq P$, and $N=O^2(N)$ implies $N\leq O^2(P^*)$. Since N is also S_0-invariant we get from (4.1) that $[Z,N]$ is normal in P^*. On the other hand by (4.6) $[Z,N]=[Z,L]$, so $[Z,L]$ is normalized by L and P^*. But this contradicts (4.9).

Corollary 4. Let $L\in\mathcal{L}_0(T)_*$. There exists a unique $P_1\in\mathcal{P}_L(T)$ such that $P_1\not\leq \widetilde{C}$. Moreover, the following hold:

(a) $Q_0\not\leq O_2(P_1)$,

(b) $O^2(P_1)\leq O^2(P^*)$, and

(c) $O^2(P_1)C_{P^*}(Y_P)/C_{P^*}(Y_P)=K_1'\times\cdots\times K_s'$, where $\{K_1,...,K_s\}$ is a T-orbit of Ω.

Proof. By (4.3) $N_L(T)\leq \widetilde{C}$, so by (1.3)(a) there exists $P_1\in\mathcal{P}_L(T)$ such that $P_1\not\leq\widetilde{C}$. Now Theorem 4 gives $P_1\leq M$ and again by (4.3) $S_0\not\leq O_2(P_1)$. Since $P^*C_M(Y_P)$ is normal in M we get from (1.3)(c) that $O^2(P_1)=[O^2(P_1),S_0]\leq P^*C_M(Y_P)$.

Let $\overline{M}:=M/C_M(Y_P)$. Note that $O^2(\overline{P}_1)\neq 1$ and by (4.1) (a) and (c) hold. By (a) and (1.3)(c) $O^2(P_1)=[O^2(P_1),Q_0]\leq [M,Q]\leq M^0\leq P$, so also (b) holds.

Let P_0 be another minimal parabolic in $\mathcal{P}_L(T)$, which is not in $\widetilde{C}$. Then (a) – (c) hold for P_0 in place of P_1. By (4.6) either

$$O^2(P_0)O^2(P_1)C_{P^*}(Y_P)=O^2(P^*)C_{P^*}(Y_P) \text{ or } O^2(P_0)C_{P^*}(Y_P)=O^2(P_1)C_{P^*}(Y_P).$$

Note that $[C_{P^*}(Y_P), Q_0] \le O_2(P) \le T$ and Q_0 is normal in S. Hence, in the first case (1.3)(c) implies that $O^2(P^*) = [O^2(P^*), Q_0] \le O^2(P_0)O^2(P_1)O_2(P^*) \le L$, which contradicts (4.9). In the second case we conclude that $O^2(P_0)O_2(P) = O^2(P_1)O_2(P)$ and thus $O^2(O^2(P_0)O_2(P)) = O^2(P_0) = O^2(P_1)$. Hence $P_0 = P_1$.

(4.11) Let X be a finite group and V a faithful $GF(2)X$-module, and let $S \in Syl_2(X)$ and $V_0 = C_V(S)$. Suppose that $F^*(X)$ is simple, $V = \langle V_0^X \rangle \neq V_0$, and

(∗) there exists an elementary abelian subgroup $1 \neq A \le S$ such that $|V/C_V(A)| \le |A|$.

Then there exists a minimal parabolic subgroup P_1 containing S such that $P_1 \not\le C_X(V_0)$ and $(P_1 \cap F^*(X))/O_2(P_1 \cap F^*(X)) \cong SL_2(2^k)$ or S_ℓ.

Proof. A theorem of Gaschütz (see for example [Hu, I.17.4]), applied to the semidirect product of V with X, shows that $V = C_V(X)[V, X]$. Hence, there exists a X-submodule W such that $\overline{V} := V/W$ is a faithful irreducible X-module. Moreover, property (∗) implies that$|\overline{V}/C_{\overline{V}}(A)| \le |A|$. Thus, the F-Module Theorem for $\mathcal{K}$-groups, see [GM1] and [GM2], gives the conclusion.

F!-Theorem. No group satisfies the hypothesis of this section.

Proof. We will derive a contradiction using the previous results of this chapter. According to (4.8) there exists $L \in \mathcal{L}_0(T)_*$. We fix the following additional notation:

$$C_L = L \cap \widetilde{C},\ V = \langle Z^L \rangle,\ \overline{L} = L/C_L(V).$$

As in Corollary 4 let P_1 be the unique element of $\mathcal{P}_L(T)$ with $P_1 \not\le \widetilde{C}$. Then

(1) $O^2(P_1) \le O^2(P^*)$ and $O^2(P_1)C_{P^*}(Y_P)/C_{P^*}(Y_P) = K_1' \times \cdots \times K_s'$,

where $\Omega_1 := \{K_1, ..., K_s\}$ is one of the two T-orbits of Ω. From (1.3)(b) and (1) we get

$$O^2(P_1) \cap C_{P^*}(Y_P) = O_2(O^2(P_1)) \ge O^2(P_1) \cap C_L(V),$$

in particular

(∗) $O^2(\overline{P}_1)/O_2(O^2(\overline{P}_1)) = K_1' \times \cdots \times K_s'$.

As in (4.10) let N be a normal subgroup of L that is minimal with respect to $N \not\le C_L$. Then by (4.10)

(2) $N = [N, Q_0] = O^2(L)$.

Moreover, since by (4.1) every normal subgroup of L in C_L centralizes V we get

(3) $\overline{N}$ is a minimal normal subgroup of $\overline{L}$, and $O_2(\overline{L}) = 1$.

Next we show:

(4) $C_L(V) \leq M$, in particular $L \neq (L \cap M)C_L(V)$.

Assume that $C_L(V) \not\leq M$. Then the minimality of L yields $L = C_L(V)P_1$. It follows from (2) that

$$N = N \cap (O^2(P_1)C_L(V)) = O^2(P_1)(N \cap C_L(V))$$

and

$$L = NT = [N, Q_0]T = O^2(P_1)T = P_1 \leq M,$$

which contradicts the choice of L in $\mathcal{L}_0(T)$.

(5) $\overline{N} \cap \overline{T} \neq 1$; in particular $\overline{N}$ is not abelian.

Assume that $\overline{N} \cap \overline{T} = 1$. For every prime q the Frattini argument gives a $\overline{Y}_q \in Syl_q(\overline{N})$ such that $\overline{T} \leq N_{\overline{U}}(\overline{Y}_q)$ and $\overline{N} = \langle \overline{Y}_q \mid q \in \pi(\overline{N}) \rangle$.

Let Y_q be the inverse image of $\overline{Y}_q$ in L. From (1), $(*)$ and (1.3)(a) we get that $Y_q \leq C_L$ for every $q \neq 3$. Hence $\overline{N} = \overline{Y}_3 C_{\overline{N}}(\overline{Q}_0)$, so by (2)

$$\overline{N} = [\overline{N}, \overline{Q}_0] = [\overline{Y}_3 C_{\overline{N}}(\overline{Q}_0), \overline{Q}_0] = [\overline{Y}_3, \overline{Q}_0] \leq \overline{Y}_3.$$

Now (3) shows that $\overline{N}$ is elementary abelian, moreover $\overline{N} = O^2(\overline{P}_1)$. Thus (4) gives $L \leq M$, a contradiction. Hence, (5) is proved.

Let Ω_2 be the T-orbit of Ω different from $\Omega_1 = \{K_1, ..., K_s\}$. Then by (4.7)

$$\Omega_1(Z(T)) = Z_1 \times Z_2,\ Z_i := C_{Y_P}(\Omega_i),$$

and $P_1 \leq L_1 := C_L(Z_2)$.

Assume that $L_1 \cap \widetilde{C} \leq M$. Then $L \cap F \leq L_1 \cap \widetilde{C} \leq M$ since $\Omega_1(Z(T)) \leq Z_0$. Now (4.3) and the Frattini argument imply $C_L \leq N_{C_L}(S_0)(L \cap F) \leq M$, which contradicts the choice of $L \in \mathcal{L}_0(T)$. Thus $L_1 \cap \widetilde{C} \not\leq M$, and the minimality of L yields:

(6) $Z_2 \leq Z(L)$, in particular $O^2(P^*) \not\leq L$.

Next we show:

(7) $\overline{N}$ is simple.

According to (3) and (5) there exist subgroups $C_L(V) \leq N_i \leq NC_L(V)$, $i = 1, ..., k$ such that $\overline{N} = \overline{N}_1 \times \cdots \times \overline{N}_k$, and $\overline{N}_1, ..., \overline{N}_k$ are simple groups conjugate under $\overline{T}$.

Assume first that $\overline{N}_i \cap \overline{C}_L \leq \overline{T}$, $i = 1, ..., k$. The projection $\overline{C}_i$ of $\overline{N} \cap \overline{C}_L$ in $\overline{N}_i$ is a subgroup of $\overline{N}_i$ that normalizes $\overline{N}_i \cap \overline{T}$. Hence by (5) $\overline{C}_L(\overline{C}_1 \times \cdots \times \overline{C}_k)$ is a proper subgroup of $\overline{L}$, and the minimality of L implies that $\overline{C}_i \leq \overline{C}_L \cap \overline{N}_i$, so $\overline{N} \cap \overline{T} = \overline{N} \cap \overline{C}_L$. Now (4) yields $C_L \leq M$, which contradicts the choice of $L \in \mathcal{L}_0(T)$.

Assume now that there exists a component $\overline{N}_1$ such that $\overline{N}_1 \cap \overline{C}_L$ is not a 2-group. Then $O^2(\overline{N}_1 \cap \overline{C}_L) = O^2((\overline{N}_1 \cap \overline{C}_L)O_2(\overline{N} \cap \overline{C}_L)) \neq 1$ and

$$[\overline{N}_1 \cap \overline{C}_L, \overline{Q}_0] \leq O_2(\overline{C}_L) \cap \overline{N} \leq O_2(\overline{N} \cap \overline{C}_L),$$

so $\overline{Q}_0$ normalizes $O^2(\overline{N} \cap \overline{C}_L)$ and thus also $\overline{N}_1$. It follows:

$(**)$ $\overline{Q}_0$ normalizes every component of $\overline{N}$.

Among all T-invariant subgroups $U \leq N$ satisfying

(i) $\overline{U} = \overline{U}_1 \times \cdots \times \overline{U}_k$, $U_i \leq N_i$, and

(ii) $O^2(P_1) \leq U$

we choose U to be minimal. Then $\overline{U \cap N_i}$ is the projection of $O^2(\overline{P}_1)$ into $\overline{N}_i$. From $(*)$ and (3) we conclude that $UT \neq L$. The minimality of L implies that $UT \cap \widetilde{C} \leq M$ and thus by (1.3)(a) and (4.3) $UT \leq M$. On the other hand the minimality of U yields $U = [U, Q_0] = O^2(U)$. It follows that U is a Q_0-invariant subgroup of $O^2(P^*)$. Now (4.6) and (6) show that

$$\overline{U} = [\overline{U}, \overline{Q}_0] = O^2(\overline{P}_1) = \overline{U}_i \times \cdots \times \overline{U}_k.$$

By $(**)$ $\overline{U}_1$ is Q_0-invariant. Hence, another application of (4.6) shows that $O^2(\overline{P}_1) \leq \overline{N}_1$. As $O^2(\overline{P}_1)$ is $\overline{T}$-invariant, also $\overline{N}_1$ is. Since the groups $N_1, ..., N_k$ are conjugate under T we conclude that $k = 1$.

(6) $J(S) \not\leq C_L(V)$.

Assume that $J(S) \leq C_L(V)$. Then $V \leq \Omega_1(Z(J(S)))$ and thus also $B(S) \leq C_L(V)$. Now the Frattini argument and (4.2) yield $L = N_L(B(S))C_L(V) = (L \cap M)C_L(V)$, which contradicts (4).

We now derive a final contradiction. According to (8) there exists $A \in \mathcal{A}(S)$ such that $\overline{A} \neq 1$. Hence, the maximality of A implies that $|V/C_V(\overline{A})| \leq |\overline{A}|$, so by (7) we can apply (4.11) to $\overline{L}$. Thus, there exists $C_L(V)T \le P_0 \le L$ such that $\overline{P}_0$ is a minimal parabolic subgroup of $\overline{L}$, $\overline{P}_0 \not\le C_{\overline{L}}(V_0)$, where $V_0 := C_V(T) = \Omega_1(Z(T))$, and

$$(***) \quad (\overline{P}_0 \cap \overline{N})/O_2(\overline{P}_0 \cap \overline{N}) \cong SL_2(2^k) \text{ or } S_\ell.$$

Since by (6) $V_0 = Z_2 \times Z \le Z(L)Z$ we get $C_L = C_L(V_0)$, $P_0 \not\le C_L$ and $P_1 \le P_0$. Now $(*)$ and $(***)$ show that $s = 1$ and $r = 2$, which contradicts $r \ge 4$.

The proof of the $P!$-Theorem and the Corollary. Let $P \le M \in \mathcal{L}^*_H(S)$. Then the $F!$-Theorem and Corollary 3 show that case (a) of the Local $P!$-Theorem and case (a) of the Structure Theorem hold for M. The $P!$-Theorem now follows from Theorem 2 and Theorem 3.

For the proof of the Corollary let $L \in Loc_H(P)$. We may assume that $C_H(Y_L) \le L$. By (1.5) there exists $M \in \mathcal{L}^*_H(S)$ such that

$$P = P^0S \le L^0S \le M.$$

Hence, M satisfies case (a) of the Structure Theorem. In particular, we get from the structure of $M/C_M(Y_M)$ and its action on Y_M:

(i) $(L \cap M_0)/C_{L\cap M_0}(Y_L) \cong SL_k(p^m)$ or $Sp_{2k}(p^m)$, and $[Y_L, L \cap M_0]$ is the corresponding natural module.

(ii) $L_0 = (L \cap M_0)C_S(Y_L)$ and $C_{L_0}(Y_L) = C_S(Y_L)C_{L_0}(Y_M)$.

This gives claim (a) of the Corollary.

Assume that $C_{L_0}(Y_L) \ne O_p(L_0)$. Then $C_{L_0}(Y_M) \ne O_p(M_0)$, and we get $M_0/O_2(M_0) \cong Sp_4(2)'$ (and $p = 2$). But then $L_0 = M_0$ since otherwise $L^0/O_2(L^0) \cong SL_2(2)$ and $C_{L_0}(Y_L) = O_2(L_0)$.

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