Finite groups of local characteristic p
An Overview

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Introduction

Let p be a fixed prime and H be a finite group whose order is divisible by p. A *p-local subgroup* of H is a subgroup of the form $N_H(U)$, where U is a non-trivial p-subgroup of H.

H has *characteristic* p if $C_H(O_p(H)) \leq O_p(H)$, where $O_p(H)$ is the largest normal p-subgroup of H. If all the p-local subgroup of H have characteristic p, we say that H has *local characteristic* p.

In this paper we describe the current status of a project whose goals are

– to understand the p-local structure of finite simple groups of local characteristic p, and

– to classify the finite simple groups of local characteristic 2.

The generic examples of groups of local characteristic p are the groups of Lie type defined over fields of characteristic p. Also some of the sporadic groups have local characteristic p, for example J_4, M_{24} and Th for $p = 2$, McL for $p = 3$, Ly for $p = 5$, and $O'N$ for $p = 7$.

But also every group with a self-centralizing Sylow p-subgroup of order p, like $Alt(p)$, is of local characteristic p. These latter groups are particular examples of groups with a strongly p-embedded subgroup. Because of such groups we used the word "understand" rather than "classify" in the first item.

We hope to obtain information that allows to understand why, apart from groups with a strongly p-embedded subgroup, p-local subgroups of groups of local characteristic p look like those in the above examples.

For $p = 2$ Bender's fundamental classification of groups with a strongly 2-embedded subgroup puts us in a much better situation. In this case the information collected about the 2-local structure actually suffice to classify the finite simple groups of local characteristic 2. This then can be seen as part of a third generation proof of the classification of the finite simple groups.

At this point we also should justify another technical hypothesis we have not mentioned yet. We will assume that the simple sections (i.e., the composition factors of subgroups) of p-local subgroups are "known" simple groups, a property that surely holds in a minimal counterexample to the Classification Theorem of the finite simple groups.

One final word about a possible third generation proof of that classification and its relation to existing proofs. In 1954 R. Brauer [Br] suggested to classify the finite simple groups by the structure of the centralizers of their involutions. In principle the classification went this way, based on the epoch-making Theorem of Feit-Thompson [FeTh] which shows that every non-abelian finite simple group possesses involutions. Of course, a priori, there are as many centralizers as there are finite groups, so one of the main steps in the proof is to give additional information about the possible structure of centralizers of involutions in finite simple groups (this corresponds to the first item of our project).

In a given simple group the centralizers of involutions are particular 2-local subgroups, and there are basically two cases: Either every such centralizer has characteristic 2, in which case the group is of local characteristic 2, or this is not the case.

In the latter case, with a great amount of work, one can prove that there exists a centralizer of an involution that has a certain standard form. There is a well established machinery that then can be used to classify the corresponding groups.

The situation is more complicated if the simple group has local characteristic 2. The actual classification then works with a suitably chosen odd prime p and centralizers of elements of order p rather than involutions. For example, in the groups $L_n(2^m)$, which are of local characteristic 2, one would choose an element of order p in a standard torus, or an element of order 3 if $m = 1$. The idea is then to prove that there exists a p-element whose centralizer is again in some standard form. This needs very delicate signalizer functor and uniqueness group arguments, moreover, the classification of quasi-thin groups has to be done separately.

If successful, our classification of groups of local characteristic 2 would give an alternative proof that does not need the above described switch to another prime and also does not need the separate treatment of quasi-thin groups.

In fact, in view of the part of the classification that deals with groups that are not of local characteristic 2, it might be desirable to classify groups of parabolic characteristic 2 rather than of local characteristic 2. Here a *parabolic subgroup* of H is a subgroup of H which contains a Sylow p-subgroup of H. And H has *parabolic characteristic* p if all p-local, parabolic subgroups of H have characteristic p. The remaining simple groups would then have a 2-central involution whose centralizer is not of characteristic 2, a condition which seems to be fairly strong. We hope that our methods also work in the more general situation of groups of parabolic characteristic p, but have not spent much time on it.

Notation and Hypothesis

Let p be a fixed prime and H be a finite group whose order is divisible by p.

The largest normal p-subgroup of H, $O_p(H)$, is called the *p-radical* of H.

H is *p-minimal* if every Sylow p-subgroup S of H is contained in a unique maximal subgroup of H and $S \neq O_p(H)$. The p-minimal parabolic subgroups of H are called *minimal parabolic subgroups.*

If every simple section of H is a known finite simple group, then H is a $\mathcal{K}$-group. If every p-local subgroup of H is a $\mathcal{K}$-group, then H is a $\mathcal{K}_p$-group.

A proper subgroup M of H is called *strongly p-embedded* if p divides $|H|$, but does not divide $|H \cap H^g|$ for any $g \in G \setminus H$.

$F_p^*(H)$ is defined by $F_p^*(H)/O_p(H) = F^*(H/O_p(H))$.

For any set $\mathcal{T}$ of subgroups of H and $U \leq H$ we set

$$\mathcal{T}_U := \{T \in \mathcal{T} \mid T \leq U\} \text{ and } \mathcal{T}(U) := \{T \in \mathcal{T} \mid U \leq T\}.$$

We further set

$$\mathcal{L} := \{L \leq H \mid C_H(O_p(L)) \leq O_p(L)\} \text{ and } \mathcal{P} := \{P \in \mathcal{L} \mid P \text{ is } p\text{-minimal}\},$$

and denote the set of maximal elements of $\mathcal{L}$ by $\mathcal{M}$. Observe that in the case when H has local characteristic p and $S \in Syl_p(H)$

> $\mathcal{L}$ contains every p-local subgroup of H,
>
> $\mathcal{M}$ is the set of maximal p-local subgroups of H,
>
> $\mathcal{L}(S)$ is the set of parabolic subgroups containing S with a non-trivial p-radical,
>
> $\mathcal{P}(S)$ is the set of p-minimal parabolic subgroups containing S with a non-trivial p-radical.

Let $1 = D_0 < D_2 < \ldots D_{n-1} < D_n = H$ be a chief-series for H and put $V_i = D_i/D_{i-1}$. The *shape* of H is define to be the ordered tuple $(H/C_H(V_i), V_i)_{1 \leq i \leq n}$. Isomorphisms between the shapes of two groups are defined in the canonical way. Note that by the Jordan Hölder Theorem the shape of H is unique up to isomorphism. Abusing language we will say that two groups have the same shape if they have isomorphic shapes.

From now on we assume

Main Hypothesis *G is a finite $\mathcal{K}_p$-group of local characteristic p with trivial p-radical.*

In the following we will discuss the principal steps and subdivisions in the investigation of G. It splits into three major parts:

- Modules
- Local Analysis
- Global Analysis.

In the first part we collect information about pairs (H, V), where H is a finite $\mathcal{K}$-group and V is a faithful $\mathbb{F}_pH$-module fulfilling certain assumptions, like quadratic action or $2F$. The results of this part serve as an invaluable background for the local analysis.

The local analysis generates information about the structure of the p-local subgroups of G, and in the global analysis this information is used to identify G up to isomorphism.

1 The Modules

In this part we collect some theorems about finite groups and their $\mathbb{F}_p$-modules that are needed in the local analysis of groups of local characteristic p. Some of these theorems are known, others are not. Proofs for the theorems in this section will appear in [BBSM].

Let H be a finite group, V an $\mathbb{F}_pH$ module and $A \leq H$. We say that A acts *quadratically* on V if $[V, A, A] = 0$. Let i be a positive real number. We say that A is an *iF-offender* provided that $|V/C_V(A)| \leq |A/C_A(V)|^i$. A is an *offender* if A is an $1F$-offender. If in addition $[V, A] \neq 0$, A is called a *non-trivial iF-offender*. If there exists a non-trivial iF-offender in H then V is called an *iF-module*. An *FF-module* is a $1F$-module.

We say that H is a $\mathcal{CK}$-group if every composition-factor of H is isomorphic to one of the known finite simple groups.

1.1 Results

Theorem 1.1.1 (Quadratic Module Theorem) *Let H be a finite $\mathcal{CK}$-group with $F^*(H)$ quasi-simple, V be a faithful irreducible $\mathbb{F}_pH$-module and $A \leq G$ such that*

(i) $[V, A, A] = 0$.

(ii) $H = \langle A^H \rangle$.

(iii) $|A| > 2$.

Then one the following holds:

1. *$p = 2$, $H \cong Alt(n)$ or $Sym(n)$ and V is the natural module.*
2. *$p = 2$, $H \cong Alt(n)$ and V is the spin-module.*
3. *$p = 2$, $H \cong 3.Alt(6)$, $Alt(7)$, $3.U_4(3)$, M_{12}, $\mathrm{Aut}(M_{12})$, $\mathrm{Aut}(M_{22})$, $3.M_{22}$, M_{24}, J_2, Co_1, Co_2 or $3.Sz$ and V is known.*
4. *$p = 2$, $H \cong O^{\pm}_{2n}(2)$ and V is the natural module.*
5. *$p = 3$, $H \cong 2.Alt(n)$ and V is the spin module.*
6. *$p = 3$, $H \cong PGU_n(2)$ and V is the Weil-module.*
7. *$p = 3$, $H \cong 2.Sp_6(2)$, $2.\Omega_8(2)$, $2.J_2$, $2.G_2(4)$, $2.Sz$, $2.Co_1$ and V is known.*
8. *$G = F^*(H) \cong {}^{\sigma}G_{\Phi}(\mathbb{F})$ is a group of Lie type over the field $\mathbb{F}$ with $\mathrm{char}\,\mathbb{F} = p$. Moreover, if $|A| > |\mathbb{F}|$ or if there exists a root subgroup R of H with $A \cap R \neq 1$ and $A \not\leq R$, then $V = V(\lambda_i)$ where λ_i is a fundamental weight with $\lambda_i(\alpha) = 1$ for the highest long root $\alpha \in \Phi$.*

Theorem 1.1.2 (FF-Module Theorem) *Let H be a finite $\mathcal{CK}$-group and V a faithful, irreducible FF-module for H over $\mathbb{F}_p$. Suppose that $F^*(H)$ is quasi-simple and that H is generated by the quadratic offenders on V. Then one of the following holds (where q is a power of p):*

1. *$H \cong SL_n(q)$, $n \geq 2$; $Sp_{2n}(q)$, $n \geq 2$; $SU_n(q)$, $n \geq 4$; $\Omega^+_{2n}(q)$, $n \geq 3$; $\Omega^-_{2n}(q)$, $n \geq 4$; or $\Omega_n(q)$, $n \geq 7$, n and q odd; and V is the corresponding natural module.*
2. *$H \cong SL_n(q)$, $n \geq 3$ and V is the exterior square of a natural module.*
3. *$H \cong \Omega_7(q)$, and V is the spin-module.*
4. *$H \cong \Omega^+_{10}(q)$ and V is one of the two half-spin modules.*
5. *$H \cong O^{\pm}_{2n}(q)$, $p = 2$, $n \geq 3$ and V is the natural module.*
6. *$H \cong G_2(q)$, $p = 2$ and $|V| = q^6$.*
7. *$H \cong Alt(n)$ or $Sym(n)$, $p = 2$ and V is the natural module.*
8. *$H \cong Alt(7)$, $p = 2$ and $|V| = 2^4$.*
9. *$H \cong 3.Alt(6)$, $p = 2$ and $|V| = 2^6$.*

Let V be an $\mathbb{F}_pH$-module and $S \in Syl_p(H)$. The group $O^{p'}(C_H(C_V(S)))$ is called a point stabilizer for H on V. V is called p-reduced if $O_p(H/C_H(V)) = 1$.

Lemma 1.1.3 (Point Stabilizer Theorem) *Let H be a finite $\mathcal{CK}$- group, V a $\mathbb{F}_pH$-module, L a point stabilizer for H on V and $A \leq O_p(L)$.*

(a) *If V is p-reduced, then $|V/C_V(A)| \geq |A/C_A(V)|$.*

(b) *Suppose V is faithful and irreducible for H, $F^*(H)$ is quasi-simple, $H = \langle A^H \rangle$ and A is a non-trivial offender on V. Then $H \cong SL_n(q)$, $Sp_{2n}(q)$, $G_2(q)$ or $Sym(n)$, where $p = 2$ in the last two cases, $n = 2, 3 \mod (4)$ in the last case, and q is a power of p. Moreover, V is the corresponding natural module.*

Theorem 1.1.4 *Let H be a finite $\mathcal{CK}$-group with $F^*(H)$ quasi-simple. Let V be a faithful irreducible $\mathbb{F}_pH$ module. Suppose there exists $1 \neq A \leq T \in \mathrm{Syl}_p(H)$ such that $|A| > 2$ and $\langle A^L \rangle$ acts quadratically on V for all $S \leq L < H$. Then*

(a) *$F^*(H)A \cong SL_n(q), Sp_{2n}(q), SU_n(q), G_2(q)'$ or $Sz(q)$, where $p = 2$ in the last two cases.*

(b) *Let I be an irreducible $F^*(H)A$ submodule of V. Then one of the following holds:*

    1. *I is a natural module for $F^*(H)A$.*

2. $p = 2$, $F^*(H)A \cong L_3(q)$, *H induces a graph automorphism on* $F^*(H)$ *and I is the adjoint module.*

3. $p = 2$, $F^*(H)(A) \cong Sp_6(q)$ *and I is the spin-module.*

(c) *Either A is contained in a long root subgroup of* $F^*(H)A$*, or* $p = 2$, $F^*(H)A \cong Sp_4(q)$, $A \leq Z(S \cap F^*(H)A)$ *and H induces a graph automorphism on* $F^*(H)$*.*

The information given in the above theorem can be used to prove the following corollary, which is of great help in the local analysis.

Corollary 1.1.5 (Strong L-Lemma) *Let L be a finite* $\mathcal{CK}$*-group with* $O_p(L) = 1$ *and V a faithful* $\mathbb{F}_pL$*-module. Suppose that there exists* $1 \neq A \leq S \in Syl_p(L)$ *such that*

(∗) $\langle A^P \rangle$ *acts quadratically on V for every proper subgroup* $P < L$ *satisfying* $A \leq P$ *and* $S \cap P \in Syl_p(P)$*.*

Then

(a) $L \cong SL_2(p^m)$, $Sz(2^m)$ *or* D_{2r}*, where* $p = 2$ *in the last two cases and r is an odd prime.*

(b) $[V, L]C_V(L)/C_V(L)$ *is a direct sum of natural modules for L.*

Let H be a finite group, V a $\mathbb{F}_pH$-module and $A \leq H$. We say that A is *cubic* on V if $[V, A, A, A] = 0$. We say that V is a *cubic 2F-module* if H contains a non-trivial cubic $2F$-offender. The following theorem is due to R. Guralnick and G. Malle [GM]:

Theorem 1.1.6 (The 2F-Module Theorem,I) *Let H be a finite* $\mathcal{CK}$*-group and V a faithful irreducible cubic 2F-module for H. Suppose that* $F^*(H)$ *is quasi-simple, but* $F^*(H)$ *is not a group of Lie-type in characteristic p. Then one of the following holds:*

1. $F^*(H)/Z(F^*(H)) \cong Alt(n)$, $p = 2$ *or* 3 *and one of the following holds.*

   1. *V is the natural module.*
   2. $H \cong Alt(n)$, $p = 2$, $n = 7$ *or* 9 *and V is a half-spin module.*
   3. $H \cong Sym(7)$, $p = 2$ *and V is the spin-module.*
   4. $F^*(H) \cong 2.Alt(5)$, $p = 3$ *and V is the half spin module.*
   5. $F^*(H) \cong 3.Alt(6)$ *and* $|V| = 2^6$*.*

2. $F^*(H) \cong G_2(2)'$, $p = 2$ *and* $|V| = 2^6$*.*

3. $F^*(H) \cong 3.U_4(3)$, $p = 2$ *and* $|V| = 2^{12}$*.*

4. $F^*(H) \cong 2.L_3(4)$, $p = 3$ *and* $|V| = 3^6$*.*

5. *$F^*(H) \cong Sp_6(2)$, $p = 3$ and $|V| = 3^7$.*

6. *$F^*(H) \cong 2.Sp_6(2)$, $p = 3$ and $|V| = 3^8$.*

7. *$F^*(H) \cong 2.\Omega_8^+(2)$, $p = 3$ and $|V| = 3^8$.*

8. *$F^*(H) \cong M_{12}, M_{22}, M_{23}, M_{24}$, $p = 2$ and V is a non-trivial composition factor of dimension $10, 10, 11, 11$ resp. of the natural permutation module.*

9. *$F^*(H) = 3.M_{22}$, $p = 2$ and $|V| = 2^{12}$.*

10. *$F^*(H) = J_2$, $p = 2$ and V is the 12-dimensional module which arises from the embedding into $G_2(4)$.*

11. *$F^*(H) \cong Co_2$ or Co_1, $p = 2$ and V is 22– resp. 24–dimensional module arising from the Leech Lattice*

12. *$F^*(H) \cong M_{11}$ or $2.M_{12}$, $p = 3$ and $|V| = 3^5$ and 3^6 respectively.*

It is not known whether Case 11 in the preceding theorem really occurs. We tend to believe it does not.

1.2 An Example

To get an idea how these theorems are used in the local analysis we now discuss briefly a particular but fairly general situation.

Let G be as in the Main Hypothesis, that is a finite $\mathcal{K}_p$-group of local characteristic p with trivial p-radical. Fix $S \in Syl_p(G)$ and put $Z := \Omega_1 Z(S)$. Let $M_1, M_2 \in \mathcal{L}(S)$ and put $F_i = F_p^*(M_i)$. Suppose that

(i) $F_i/O_p(F_i)$ is quasisimple, $i = 1, 2$,

(ii) $O_p(\langle F_1, F_2 \rangle) = 1$,

(iii) $M_i = SF_i$, $i = 1, 2$.

Let $Z_i := \langle Z^{M_i} \rangle$ and $V_i := \langle Z_j^{M_i} \rangle$ for $i \neq j$. Note first that $Z_j \leq Z(O_p(M_j)) \leq S \leq M_i$, so Z_i and V_i are normal subgroups of M_i.

As an elementary consequence of (i) we get:

(1) Let $U \leq M_i$ and $F_i \leq N_{M_i}(U)$. Then either $F_i \leq U$, or $U \cap O_p(M_i) \in Syl_p(U)$.

This property (1) together with (iii) applied to $U = C_{M_i}(F_i/O_p(M_i))$ and $U = C_{M_i}(V)$, V a non-central M_i-chief factor in $O_p(M_i)$, gives:

(2) Suppose that V is a non-central M_i-chief factor in $O_p(M_i)$. Then

$$C_S(V) = O_p(M_i) = C_S(F_i/O_p(M_i)).$$

Next we show that one of the following cases holds:

(I) There exist $g \in G$ and $i \in \{1,2\}$, say $i = 1$, such that

$$[Z_1, Z_1^g] \neq 1,\ [Z_1, Z_1^g] \leq Z_1 \cap Z_1^g \text{ and } Z_1 Z_1^g \leq M_1 \cap M_1^g.$$

(II) There exists an $i \in \{1,2\}$, say $i = 1$, such that $Z_1 \not\leq O_p(M_2)$, and (I) does not hold.

(III) V_1 and V_2 are elementary abelian, and (I) does not hold.

To see this, assume that (I) and (II) do not hold. Then $V_1V_2 \leq O_p(M_1) \cap O_p(M_2)$, and either (III) holds, or for some $i \in \{1,2\}$, say $i = 2$, V_2 is not abelian. In the latter case, there exists $g \in M_2$ such that $[Z_1, Z_1^g] \neq 1$. Since $\langle Z_1, Z_1^g \rangle \leq O_p(M_2) \leq M_1 \cap M_1^g$ we also have $[Z_1, Z_1^g] \leq Z_1 \cap Z_1^g$. This gives (I) contrary to our assumption. We now discuss these three cases separately.

Assume case (I). We can choose the notation such that

$$|Z_1^g / C_{Z_1^g}(Z_1)| \geq |Z_1 / C_{Z_1}(Z_1^g)|,$$

so Z_1^g is a quadratic offender on Z_1.

Clearly $[Z_1, M_1] \neq 1$ since $[Z_1, Z_1^g] \neq 1$, so the definition of Z_1 implies that $M_1 \neq C_{M_1}(Z_1)S$. Thus $[Z_1, O^p(M_1)] \neq 1$ and there exists a non-central M_1-chief factor $V = U/W$ of Z_1. From (2) we conclude that $C_{Z_1^g}(Z_1) = C_{Z_1^g}(V) = Z_1^g \cap O_p(M_1)$. It follows that

$$|V/C_V(Z_1^g)| \leq |U/C_U(Z_1^g)| \leq |Z_1/C_{Z_1}(Z_1^g)| \leq |Z_1^g/C_{Z_1^g}(Z_1)| = |Z_1^g/C_{Z_1^g}(V)|.$$

Hence Z_1^g is a non-trivial quadratic offender on V, and the FF-Module Theorem gives the structure of $F_1/O_p(M_1)$ and V.

Assume case (II). Then

$$[O_p(M_2), Z_1, Z_1] \leq [O_p(M_2) \cap Z_1, Z_1] = 1,$$

so Z_1 is quadratic on every M_2-chief factor V of $O_p(M_2)$. Hence (unless $|Z_1 O_p(M_1)/O_p(M_1)| = 2$) the Quadratic Module Theorem applies to $M_2/C_{M_2}(V)$ and $A = Z_1 C_{M_2}(V)/C_{M_2}(V)$. But in this case one also gets information about M_1:

Among all subgroups $U \leq M_2$ with $Z_1 \leq U$, $U \cap S \in Syl_p(U)$ and $Z_1 \not\leq O_p(U)$ choose U minimal and set $\overline{U} = U/O_p(U)$. Then for every proper subgroup $O_p(U) \leq P < U$ with $S \cap P \in Syl_p(P)$ and $Z_1 \leq P$ we get that $Z_1 \leq O_p(P)$. But this implies, since we are not in case (I), that $X := \langle Z_1^P \rangle$ is abelian. Hence as above, since X is normal in $S \cap P$, $[O_p(P), X, X] = 1$. This shows that the Strong L-Lemma 1.1.5 applies with $L = \overline{U}$, $V = O_p(U)/\Phi(O_p(U))$ and $A = \overline{Z}_1$.

Set $\widetilde{B} := Z_1 \cap O_p(U)$ and $B := \widetilde{B}^x$ for some $x \in U \setminus N_U(S \cap U)$. Note that $U = \langle Z_1, Z_1^x \rangle$ and so U normalizes $\widetilde{B}B$ and $\widetilde{B} \cap B \leq Z(U)$. By 1.1.5, $|Z_1/\widetilde{B}| \leq |B/\widetilde{B} \cap B|$ and $C_B(y) = \widetilde{B} \cap B$ for every $y \in Z_1 \setminus \widetilde{B}$. It follows that

$$|Z_1/C_{Z_1}(B) \leq |Z_1/\widetilde{B} \cap B| = |Z_1/\widetilde{B}||\widetilde{B}/\widetilde{B} \cap B| = |Z_1/\widetilde{B}||B/\widetilde{B} \cap B| \leq |B/\widetilde{B} \cap B|^2,$$

so $B/\widetilde{B} \cap B$ is a $2F$-offender on Z_1. Using (ii) we see that the $2F$-Module Theorem applies to $M_1/C_{M_1}(Z_1)$ and a non-central M_1-chief factor of Z_1.

Assume case (III). Note that $\langle C_{M_i}(V_i), M_j\rangle \leq N_G(Z_j)$ and so by condition (ii)

$$F_i \not\leq C_{M_i}(V_i) \text{ for every } i \in I.$$

In particular by (1) $C_S(V_i) \leq O_p(M_i)$ for every $i \in I$. Since by property (ii) $\mathrm{J}(S) \not\leq O_p(M_1) \cap O_p(M_2)$ we may assume that

(∗∗) $F_1 \not\leq C_{M_1}(V_1)$ and $\mathrm{J}(S) \not\leq C_{M_1}(V_1)$.

Let D be the inverse image of $O_p(M_1/C_{M_1}(V_1))$. Pick $A \in \mathcal{A}(S)$ such that $A \not\leq C_{M_1}(V_1)$. According to the Thompson Replacement Theorem we may assume that A acts quadratically on V_1. The maximality of A gives

$$|V_1||C_A(V_1)||V_1 \cap A|^{-1} = |V_1 C_A(V_1)| \leq |A|$$

and thus $|V_1/C_{V_1}(A)| = |V_1/V_1 \cap A| \leq |A/C_A(V_1)|$, so A is a quadratic offender on V_1. This looks promising, but $A_0 := A \cap D$ might not centralize V_1. This is an obstacle for the application of the FF-Module Theorem to F_1A and non-central F_1A-chief factors of V_1.

Evidently $|A_0/C_A(V_1)| \leq |A/A_0|$ or $|A/A_0| \leq |A_0/C_A(V_1)|$. In the first case

$$|V_1/C_{V_1}(A_0)| \leq |V_1/V_1 \cap A| \leq |A/C_A(V_1)| = |A/A_0||A_0/C_A(V_1)| \leq |A/A_0|^2,$$

so in this case, using again (2), A/A_0 is a quadratic 2F-offender on the non-central F_1A-chief factors of V_1.

In the second case

$$|V_1/C_{V_1}(A_0)| \leq |V_1/V_1 \cap A| \leq |A/C_A(V_1)| = |A/A_0||A_0/C_{A_0}(V_1)| \leq |A_0/C_{A_0}(V_1)|^2,$$

so A_0 is a quadratic 2F-offender on V_1. An elementary calculation then shows that there exists a quadratic 2F-offender on Z_2.

This concludes the discussion of the cases (I) – (III). In all cases the module theorems from 1.1 reveal the structure of $F_1/O_p(F_1)$ or $F_2/O_p(F_2)$.

2 The Local Analysis

In this part we discuss the p-local structure of G, where G is, according to our Main Hypothesis, a finite $\mathcal{K}_p$-group of local characteristic p with trivial p-radical. We fix

$$S \in Syl_p(G),\ Z := \Omega_1 \mathrm{Z}(S).$$

For further notation see the introduction.

The basic idea is to study the structure of $L \in \mathcal{L}$ by its action on elementary abelian normal subgroups contained in $Z(O_p(L))$ and by its interaction with other elements of $\mathcal{L}$ having a common Sylow p-subgroup. It is here where the module results of Part 1 are used.

The appropriate candidates for such normal subgroups in $Z(O_p(L))$ are the *p-reduced* normal subgroups, i.e. elementary abelian normal subgroups V of L with $O_p(L/C_L(V)) = 1$. Note that an elementary abelian normal subgroup V is p-reduced iff any subnormal subgroup of L that acts unipotently on V already centralizes V. Here are the basic properties of p-reduced normal subgroups. They include the fact that there exists a unique maximal p-reduced normal subgroup of L which we always denote by Y_L.

Lemma 2.0.1 *Let L be a finite group of characteristic p and $T \in \mathrm{Syl}_p(L)$. Then*

(a) *There exists a unique maximal p-reduced normal subgroup Y_L of L.*

(b) *Let $T \leq R \leq L$ and X a p-reduced normal subgroup of R. Then $\langle X^L \rangle$ is a p-reduced normal subgroup of L. In particular, $Y_R \leq Y_L$.*

(c) *Let $T_L = C_T(Y_L)$ and $L_T = N_L(T_L)$. Then $L = L_T C_L(Y_L)$, $T_L = O_p(L_T)$ and $Y_L = \Omega_1 Z(T_L)$.*

(d) *$Y_T = \Omega_1\mathrm{Z}(T)$, $Z_L := \langle \Omega_1\mathrm{Z}(T)^L \rangle$ is p-reduced for L and $\Omega_1\mathrm{Z}(T) \leq Z_L \leq Y_L$.*

(e) *Let V be p-reduced normal subgroup of L and K a subnormal subgroup of L. Then $[V, O^p(K)]$ is a p-reduced normal subgroup of K.*

Of course, the action of L on Y_L might be trivial, whence $Y_L = \Omega_1 Z(T)$, $T \in Syl_p(L)$. This leads to another notation. Let H be any finite group and $T \in Syl_p(H)$. Then $P_H(T) := O^{p'}(C_H(\Omega_1 Z(T)))$ is called a *point stabilizer* of H. In the above situation trivial action on Y_L implies that $O^{p'}(L) = P_L(T)$. Here are some basic (but not entirely elementary) facts about point stabilizers.

Lemma 2.0.2 *Let H be a finite group of local characteristic p, $T \in \mathrm{Syl}_p(H)$ and L a subnormal subgroup of H. Then*

(a) (Kieler Lemma) $C_L(\Omega_1\mathrm{Z}(T)) = C_L(\Omega_1\mathrm{Z}(T \cap L))$

(b) $P_L(T \cap L) = O^{p'}(P_H(T) \cap L)$

(c) $C_L(Y_L) = C_L(Y_H)$

(d) *Suppose $L = \langle L_1, L_2 \rangle$ for some subnormal subgroups L_1, L_2 of H. Then*

  (da) $P_L(T \cap L) = \langle P_{L_1}(T \cap L_1), P_{L_2}(T \cap L_2) \rangle$.

  (db) *For $i = 1, 2$ let P_i be a point stabilizer of L_i. Then $\langle P_1, P_2 \rangle$ contains a point stabilizer of L.*

It is evident that all elements of $\mathcal{L}(S)$ having a normal point stabilizer are contained in $N_G(Z)$. Therefore, controlling $N_G(Z)$, or better a maximal p-local subgroup containing $N_G(Z)$, means controlling all elements of $L \in \mathcal{L}(S)$ with trivial action on Y_L. This point of view leads to the next definition and subdivision.

Let $\widetilde{C}$ be a fixed maximal p-local subgroup of G containing $N_G(Z)$. Put

$$E := O^p(F_p^*(C_{\widetilde{C}}(Y_{\widetilde{C}}))), \quad Q := O_p(\widetilde{C})$$

The major subdivision is:

Non-E-Uniqueness $(\neg E!)$: E is contained in at least two maximal p-local subgroups of G.

E-Uniqueness $(E!)$: $\widetilde{C}$ is the unique maximal p-local subgroup containing E.

Another subdivision refers to the rank of G. Define the *rank* of G to be the minimal size of a non-empty subset Σ of $\mathcal{P}(S)$ with $\langle \Sigma \rangle \notin \mathcal{L}$. If no such subset exists we define the rank to be 1. Note that $\operatorname{rank} G = 1$ if and only if $|\mathcal{M}(S)| = 1$. The cases $\operatorname{rank} G = 1$ and $\operatorname{rank} G \geq 2$ are treated separately, so in the $E!$-case we will assume, in addition, that G has rank at least 2.

The subgroup $\langle \mathcal{M}(S) \rangle$ is called the *p-core* of G (with respect to S). Note that G has a proper p-core if G has rank 1, so the rank 1 case can be treated in this more general context.

2.1 Pushing Up

Various times in the local analysis we will encounter a p-local subgroup L of G and a parabolic subgroup H of L such that $N_G(O_p(H))$ and L are not contained in a common p-local subgroup of G. In other words $O_p(\langle L, N_G(O_p(H))\rangle) = 1$. In this section we provide theorems that allow, under additional hypotheses, to determine the shape of L.

For a p-group R we let $\mathcal{PU}_1(R)$ be the class of all finite $\mathcal{CK}$-groups L containing R such

(a) L is of characteristic p,

(b) $R = O_p(N_L(R))$

(c) $N_L(R)$ contains a point stabilizer of L.

Let $\mathcal{PU}_2(R)$ be the class of all finite $\mathcal{CK}$- groups L containing R such that L is of characteristic p and

$$L = \langle N_L(R), H \mid R \leq H \leq L, H \in \mathcal{PU}_1(R)\rangle.$$

Let $\mathcal{PU}_3(R)$ be the class of all finite $\mathcal{CK}$-groups L such that

(a) L is of characteristic p.

(b) $R \leq L$ and $L = \langle R^L \rangle$

(c) $L/C_L(Y_L) \cong SL_n(q), Sp_{2n}(q)$ or $G_2(q)$, where q is a power of p and $p = 2$ in the last case.

(d) $Y_L/C_{Y_L}(L)$ is the corresponding natural module.

(e) $O_p(L) < R$ and $N_L(R)$ contains a point stabilizer of L.

(f) If $L/C_L(Y_L) \not\cong G_2(q)$ then $R = O_p(N_L(R))$.

Let $\mathcal{PU}_4(R)$ be the class of all finite $\mathcal{CK}$- groups L containing R such that L is of characteristic p and

$$L = \langle N_L(R), H \mid R \leq H \leq L, H \in \mathcal{PU}_3(R) \rangle.$$

For a finite p-group T let $\mathcal{A}(T)$ be the set of elementary abelian subgroups of maximal order in T, $\mathrm{J}(T) = \langle \mathcal{A}(T) \rangle$, the *Thompson subgroup* of T, and $\mathrm{B}(T) = C_R(\Omega_1 \mathrm{Z}(\mathrm{J}(T)))$, the *Baumann subgroup* of T. Recall that a finite group F is p-closed if $O^{p'}(F) = O_p(F)$. The following lemma is a generalization of a well known lemma of Baumann, also the proof is similar to Baumann's.

Lemma 2.1.1 (Baumann Argument) *Let L be a finite group, R a p-subgroup of L, $V := \Omega_1 \mathrm{Z}(O_p(L))$, $K := \langle \mathrm{B}(R)^L \rangle$, $\widetilde{V} = V/C_V(O^p(K))$, and suppose that each of the following holds:*

(i) *$O_p(L) \leq R$ and $L = \langle \mathrm{J}(R)^L \rangle N_L(\mathrm{J}(R))$.*

(ii) *$C_K(\tilde{V})$ is p-closed.*

(iii) *$|\tilde{V}/C_{\tilde{V}}(A)| \geq |A/C_A(\tilde{V})|$ for all elementary abelian subgroups A of R.*

(iv) *If U is an FF-module for $L/O_p(L)$ with $\tilde{V} \leq U$ and $U = C_U(B(R))\tilde{V}$, then $U = C_U(O^p(K))\tilde{V}$.*

Then $O_p(K) \leq \mathrm{B}(R)$.

Using the Point Stabilizer Theorem 1.1.3 and the Baumann Argument 2.1.1 one can prove

Lemma 2.1.2 *Let R be a p-group. Then $\mathcal{PU}_2(R) \subseteq \mathcal{PU}_4(\mathrm{B}(R))$.*

Similarly,

Lemma 2.1.3 *Let L be a finite p-minimal $\mathcal{CK}$- group of characteristic p. Let $T \in \mathrm{Syl}_p(L)$. Then either L centralizes $\Omega_1 \mathrm{Z}(T)$ (and so $P_L(T)$ is normal in L) or $L \in \mathcal{PU}_4(\mathrm{B}(T))$.*

If R is a group and Σ is a set of groups containing R we define

$$O_R(\Sigma) = \langle T \leq R \mid T \trianglelefteq L, \forall L \in \Sigma \rangle$$

So $O_R(\Sigma)$ is the largest subgroup of R which is normal in all $L \in \Sigma$.

Theorem 2.1.4 *Let R be a finite p-group with $R = \mathrm{B}(R)$ and Σ a subset of $\mathcal{PU}_3(R)$. Suppose $O_R(\Sigma) = 1$. Then there exists $L \in \Sigma$ such that $O^p(L)$ has one of the following shapes: (where q is a power of p.)*

1. $q^n SL_n(q)'$;
2. $q^{2n} Sp_{2n}(q)', p$ *odd;*
3. $q^{1+2n} Sp_{2n}(q)', p = 2$;
3. $2^6 G_2(2)', p = 2$;
4. $q^{1+6+8} Sp_6(q), p = 2$;
5. $2^{1+4+6} L_4(2), p = 2$; *or*
6. $q^{1+2+2} SL_2(q)', p = 3$.

Examples for above configurations can be found in $SL_{n+1}(q)$, $L_{p^n}(r)($ with $q = p \mid r-1)$, $Sp_{2n+2}(q)$, Ru, $F_4(q)$, Co_2 and $G_2(q)$, respectively.

We are currently working on determining the shapes of all $L \in \Sigma$, not only of one. We expect all elements $L \in \Sigma$ to have one of the structures of the previous theorem, accept for one additional possibility namely $L/O_p(L) \cong SL_2(q)$ and all non-central chief-factors for L on $O_p(L)$ are natural. For a given R, the number of such chief-factors is bounded. But as R varies it cannot be bounded.

About the proof: Using elements $A \in \mathcal{A}(R)$ and their interaction with the Y_L's, $L \in \Sigma$ one shows that there exist $L, M \in \Sigma$ such that $\langle Y_L^M \rangle$ is not abelian. The fact that $\langle Y_L^M \rangle$ is not abelian allows us to pin down the structure of L and M. (Compare this with the cases (I) and (II) in 1.2).

Theorem 2.1.5 (The Pushing Up Theorem) *Let R be a finite p-group, $1 \leq i \leq 4$, and Σ a subset of $\mathcal{PU}_i(R)$ with $O_R(\Sigma) = 1$. If $i = 3$ or 4 suppose that $R = \mathrm{B}(R)$. Then the shape of $\langle \mathrm{B}(R)^L \rangle$ will be known for all $L \in \Sigma$.*

Given 2.1.2, the Pushing Up Theorem should be a straight forward but tedious consequence of 2.1.4. The details still need to be worked out.

2.2 Groups with a Proper p-Core

Recall from the introduction that a proper subgroup $M < G$ is strongly p-embedded if M is not a p'-group but $M \cap M^g$ is a p'-group for every $g \in G \setminus M$. The following lemma is well known and elementary to prove:

Lemma 2.2.1 *Let H be a finite group, T a Sylow p-subgroup of H and M a proper subgroup of H with p dividing $|M|$. Put $K := \langle N_G(A) \mid 1 \neq A \leq T\rangle$. Then M is strongly p-embedded iff $N_G(A) \leq M$ for all non-trivial p-subgroups A of M and iff M contains a conjugate of K. In particular, H has a strongly p-embedded subgroup if and only if p divides $|H|$ and K is a proper subgroup of M.*

Note that the group K from the proceeding lemma contains the p-core of H with respect to T. Thus if our G has a strongly p-embedded subgroup then G also has a proper p-core. We say that G satisfies CGT if G has proper p-core but no strongly p-embedded subgroups.

2.2.1 Strongly p-embedded subgroups

Suppose that G has a strongly p-embedded subgroup. If $p = 2$, we can apply Bender's theorem [Be]:

Theorem 2.2.2 (Bender) *Let H be a finite group with a strongly 2-embedded subgroup. Then one of the following holds:*

1. *Let t be an involution in H. Then $H = O(H)C_H(t)$ and t is the unique involution in $C_H(t)$.*
2. *$O^{2'}(H/O(H)) \cong L_2(2^k), U_3(2^k)$ or $Sz(2^k)$.*

If $p \neq 2$ we end our analysis without a clue.

2.2.2 CGT

Suppose that G satisfies CGT. Let $M := \langle \mathcal{M}(S)\rangle$ be the p-core with respect to S. According to CGT, M is a proper subgroup of G, but M is not strongly p-embedded. Thus there exists $g \in G \setminus M$ such that $|M \cap M^g|_p \neq 1$. Evidently we can choose g such that $M \cap S^g$ is a Sylow p-subgroup of $M \cap M^g$. Thus $|S^g \cap M|_p \neq 1$. If $S^g \leq M$, then $S^{gm} = S$ for some $m \in M$. Since $N_G(S) \leq M$ we obtain the contradiction to $g \notin M$. Thus $S^g \not\leq M$. Also $S^g \in \mathcal{L}$.

Among all $L \in \mathcal{L}$ satisfying $L \not\leq M$ we choose L such that $|L \cap M|_p$ is maximal. Then $|L \cap M|_p \geq |S^g \cap M|_p \neq 1$. Let $T \in Syl_p(L \cap M)$ and without loss $T \leq S$.

If $T = S$ we get that $L \in \mathcal{L}(S)$ and so by the definition of M, $L \leq M$, a contradiction. Thus $T \neq S$. Let C be a non-trivial characteristic subgroup of T. Then $N_S(T) \leq N_G(C)$ and so $|M \cap N_G(C)|_p > |M \cap L|$. Hence the maximal choice of $|M \cap L|_p$ implies $N_G(C) \leq M$. In particular, $N_L(C) \leq M \cap L$. For $C = T$ we conclude that $T \in \mathrm{Syl}_p(L)$. We can now apply the following theorem with L in place of H:

Theorem 2.2.3 (Local C(G,T)-Theorem) *Let H be a finite $\mathcal{K}_p$-group of characteristic p, T a Sylow p-subgroup of H, and suppose that*

$$C(H,T) := \langle N_H(C) \mid 1 \neq C \text{ a characteristic subgroup of } T\rangle$$

is a proper subgroup of H. Then there exists an H-invariant set $\mathcal{D}$ of subnormal subgroups of H such that

(a) $H = \langle \mathcal{D} \rangle C(H,T)$

(b) $[D_1, D_2] = 1$ *for all* $D_1 \neq D_2 \in \mathcal{D}$.

(c) *Let $D \in \mathcal{D}$, then $D \not\leq C(H,T)$ and one of the following holds:*

1. *$D/Z(D)$ is the semi-direct product of $SL_2(p^k)$ with a natural module for $SL_2(p^k)$. Moreover $O_p(D) = [O_p(D), D]$ is elementary abelian.*
2. *$p = 2$ and D is the semi-direct product of $Sym(2^k+1)$ with a natural module for $Sym(2^k+1)$.*
3. *$p = 3$, D is the semi-direct product of $O_3(D)$ and $SL_2(3^k)$, $\Phi(D) = Z(D) \leq O_3(D)$ has order 3^k, and both $[Z(O_3(D)), D]$ and $O_3(D)/Z(O_3(D))$ are natural modules for $D/O_p(D)$.*

For $p = 2$ the local $C(G,T)$-theorem was proved by Aschbacher in [Asch] without using the $\mathcal{K}_2$-hypothesis. For us it will be consequence of the 2.1.5. Using the local $C(G,T)$ theorem and that G is of local characteristic p it is not difficult to show:

Theorem 2.2.4 *Suppose that G fulfills CGT. Let M be a p-core for G and $L \in \mathcal{L}$ such that $|L \cap M|_p$ is maximal with respect to $L \not\leq M$. Then there exists a normal subgroup D of L such that $D/Z(D) \cong q^2SL_2(q)$ and $C_L(D) \leq O_p(L)$.*

Using the preceding theorem, A. Hirn is currently trying to show that for $p = 2$, G cannot fulfill CGT.

2.3 ¬E!

In this section we assume that we are in the ¬E!-case, so E is contained in $\widetilde{C}$ and at least one other maximal p-local subgroup of G. To illustrate this situation we look at a few examples.

Let $p = 2$, $q = 2^k$ and $G = F_4(q)\langle\sigma\rangle$ where σ induces a graph automorphism of order 2. (Yes, G is not of local characteristic 2, only of parabolic characteristic 2. But as we mostly look at subgroups containing a Sylow p-subgroup, or at least a large part of the Sylow p-subgroup, it is difficult for us to detect that G is not of local characteristic p.)

Note that G' is a group of Lie-type with Dynkin-diagram

.

Also S is only contained in two parabolic subgroups, namely the σ-invariant B_2- and $A_1 \times A_1$-parabolic. Trying to treat this amalgam would not be easy. To determine E, note that $Z(S)$ has order q and is contained in the product of the highest long root group and the highest short root group. It follows that

$E \leq G'$ and E is essentially the B_2-parabolic. So E is contained in the B_3- and C_3-parabolic, and G will be identified by the $(Sp_6(q), Sp_6(q))$-amalgam.

As a second example consider $G = E_8(q) \wr Sym(p^k)$ (Again this is a group of parabolic characteristic p, but not of local characteristic p.) Here E helps us to find p-local subgroups which are not of characteristic p. Let H be the normalizer of a root subgroup in $E_8(q)$, i.e. the E_7-parabolic. Then $\tilde{C}$ is $H \wr Sym(p^k)$, and E is essentially a direct product of p^k copies of H. Hence, E is contained in a p-local subgroup L which is a direct product of $p^k - 1$ copies of H and $E_8(q)$, so L is not of characteristic p.

As a final example consider $p = 2$ and $G = M_{24}$. Then $\widetilde{C} = 2^4 L_4(2)$ and so $C_{\widetilde{C}}(Y_{\widetilde{C}}) = O_2(\widetilde{C})$ and $E = 1$. It seems that E is not of much use in this case, but $E = 1$ can only occur if $\widetilde{C}/O_p(\widetilde{C})$ acts faithfully on $Y_{\widetilde{C}}$. Together with the fact that $\widetilde{C}$ contains $N_G(Z)$, the $E = 1$-situation can be handled with the amalgam method.

To summarize, the $\neg E!$-case detects situations which allow a treatment via the amalgam method. The general idea is to find a p-subgroup R and a set Σ of subgroups of G containing R such that we can apply the Pushing Up Theorem 2.1.5 to (R, Σ).

To get started we choose a subgroup X of $\widetilde{C}$ such that X is the point stabilizer of some subnormal subgroup $\widetilde{X}$ of $\widetilde{C}$ and such that X is maximal with respect to $\mathcal{M}(EX) \neq \{\widetilde{C}\}$. By assumption $\mathcal{M}(E) \neq \widetilde{C}$ so such a choice is possible. For $L \in \mathcal{L}(EX)$ let $S_{\widetilde{C}}(L)$ be the largest subnormal subgroup of $\widetilde{C}$ contained in L. We choose L such that in consecutive order

$L1$. $L \in \mathcal{L}(EX)$ with $L \not\leq \widetilde{C}$.

$L2$. $|\widetilde{C} \cap L|_p$ is maximal.

$L3$. $S_{\widetilde{C}}(L)$ is maximal.

$L4$. $\widetilde{C} \cap L$ is maximal.

$L5$. L is minimal.

Let $R = O_p(\widetilde{C} \cap L)$. Consider the following two conditions:

(PU-L) $\quad N_{\widetilde{C}}(R) \not\leq L \cap \widetilde{C}$.

$\neg$ (PU-L) $\quad N_{\widetilde{C}}(R) = L \cap \widetilde{C}$

If (PU-L) holds we define $H := N_{\widetilde{C}}(R)$. Note here that $L \cap \widetilde{C} < H$.

If $\neg$ (PU-L) holds we choose a $\widetilde{C} \cap L$-invariant subnormal subgroup N of $\widetilde{C}$ minimal with respect to $N \not\leq L$ and put $H = N(L \cap \widetilde{C})$.

Note that in both cases $H \not\leq L$ and $H \cap L = \widetilde{C} \cap L$, since $\widetilde{C} \cap L \leq H \leq \widetilde{C}$. Let T be a Sylow p-subgroup of $H \cap L$ such that $T \cap X$ is a Sylow p-subgroup of X. Without loss $T \leq S$.

Lemma 2.3.1 (a) $O_p(\langle H, L \rangle) = 1$.

(b) $N_G(\Omega_1 \mathrm{Z}(T)) \leq \widetilde{C}$.

(c) *T is a Sylow p-subgroup of L and $H \cap L$ contains a point stabilizer of L.*

(d) *If $\neg$(PU-L) holds, then $O_p(N_H(R)) = R$ and $Q \leq R$. In particular, H is of characteristic p.*

Proof: Suppose (a) is false. Then there exists a p-local subgroup L^* of G with $\langle H, L\rangle \leq L^*$. Since $L \leq L^*$, L^* fulfills all the assumptions on L (except for the minimality of L, our last choice). But $H \leq L^*$ and so $\tilde{C} \cap L < \widetilde{C} \cap L^*$ contradicting (L4). This proves (a).

We claim that $EX \leq N_G(\Omega_1 \mathrm{Z}(T))$. Since $\tilde{X}$ is subnormal in $\widetilde{C}$, and T contains a Sylow p-subgroup of X and so of $\widetilde{X}$, we conclude that T is a Sylow p-subgroup of $\langle \widetilde{X}, T\rangle$. Thus by the Kieler Lemma 2.0.2, $X \leq C_{\widetilde{X}}(\Omega_1 \mathrm{Z}(T \cap X)) \leq C_G(\Omega_1 \mathrm{Z}(T)) \leq N_G(\Omega_1 \mathrm{Z}(T))$. Similarly $E \leq N_G(\Omega_1 \mathrm{Z}(T))$.

By the choice of $\widetilde{C}$, $N_G(\Omega_1 \mathrm{Z}(S)) \leq \widetilde{C}$. Thus to prove (b) we may assume $T \neq S$. Since $N_S(T) \leq N_{\widetilde{C}}(\Omega_1 \mathrm{Z}(T))$ we conclude that $|\widetilde{C} \cap N_G(\Omega_1 \mathrm{Z}(T))|_p > |\widetilde{C} \cap L|_p$. If (L1) holds for $N_G(\Omega_1 \mathrm{Z}(T))$ we obtain a contradiction to (L2). Thus $N_G(\Omega_1 \mathrm{Z}(T)) \leq \widetilde{C}$ and (b) holds.

By (b) $N_L(T) \leq H$ and so T is a Sylow p-subgroup of L. Hence (c) follows from (b).

Suppose $\neg$ (PU-L) holds. Then

$$L \cap \widetilde{C} \leq N_H(R) \leq N_{\widetilde{C}}(R) = L \cap \widetilde{C}, \text{ and}$$

$$N_Q(R)) \leq O_p(N_{\widetilde{C}}(R)) = O_p(\widetilde{C} \cap L) = R,$$

so $R = O_p(N_H(R))$ and $Q \leq R$. This is (d). □

Proposition 2.3.2 *Suppose $\neg E!$ and that (PU-L) holds. Set $\Sigma = L^H$. Then $\Sigma \subseteq \mathcal{PU}_1(R)$ and $O_R(\Sigma) = 1$.*

Proof: By 2.3.1(c) and since R is normal in H, $\Sigma \subseteq \mathcal{PU}_1(R)$. As H and L both normalize $O_R(\Sigma)$ we get from 2.3.1(a) that $O_R(\Sigma) = 1$. □

In view of the preceding proposition the (PU-L)-case can be dealt with via the Pushing Up Theorem 2.1.5. The $\neg$ (PU-L)-case is more complicated. As a first step we show

Lemma 2.3.3 *Suppose $\neg E!$ and $\neg$ (PU-L). Put $\Sigma = \{H, L\}$. Then $O_R(\Sigma) = 1 = O_{\mathrm{B}(R)}(\Sigma)$ and $L \in \mathcal{PU}_1(R) \subseteq \mathcal{PU}_4(\mathrm{B}(R))$. If $H \cap L$ contains a point stabilizer of H, then $\Sigma \subseteq \mathcal{PU}_1(R) \subseteq \mathcal{PU}_4(\mathrm{B}(R))$.*

Proof: By 2.3.1(a) $O_R(\Sigma) = 1$ and by 2.3.1(c), $L \in \mathcal{PU}_1(R)$. If $H \cap L$ contains a point stabilizer of H, then by 2.3.1(d) $H \in \mathcal{PU}_1(R)$. By 2.1.2, $\mathcal{PU}_1(R) \subseteq \mathcal{PU}_4(\mathrm{B}(R))$. Also $O_{\mathrm{B}(R)}(\Sigma) \leq O_R(\Sigma) = 1$ and all parts of the lemma have been verified. □

The preceding lemma is the main tool in the proof of:

Proposition 2.3.4 *Suppose $\neg E!$, $\neg$ (PU-L) and $Y_H \leq O_p(L)$. Then $H \in \mathcal{PU}_4(\mathrm{B}(R))$.*

Outline of a Proof: Suppose that $H \notin \mathcal{PU}_4(\mathrm{B}(R))$. Note that by 2.3.1(d) $Q \leq O_p(H) \leq R$, so $Y_H \leq Q$. Since $H \notin \mathcal{PU}_4(\mathrm{B}(R))$, $\mathrm{B}(R)$ is not normal in H and so $\mathrm{B}(R) \not\leq O_p(H)$. The definition of H (and the minimal choice of N) shows that $N = [N, B(R)]$. As $H \notin \mathcal{PU}_1(R)$, we get $[Y_H, N] \neq 1$ and thus also $[Y_H, B(R)] \neq 1$. It follows that Y_H is an FF-module for H. Let $\overline{H} := H/C_H(Y_H)$. Then there exist subnormal subgroups $K_1, \ldots, K_m$ of N such that $\overline{K}_i$ is quasi-simple and

$$\overline{N} = \overline{K}_1 \times \cdots \times \overline{K}_m \text{ and } [Y_H, N] = \bigoplus_{i=1}^{m} [Y_H, K_i].$$

(Note here that $H \notin \mathcal{PU}_1(R)$ rules out the case where $\overline{N}$ is solvable.) We show next:

($*$) Let $1 \neq x \in C_S((S \cap K_i)C_S(K_i/O_p(K_i))$. Then $\mathcal{M}(C_G(x)) = \{\widetilde{C}\}$.

Since K_i and $\widetilde{X}$ are subnormal in $\widetilde{C}$ and $K_i \not\leq \widetilde{X}$ we get $[K_i, \widetilde{X}] \leq O_p(K_i)$. By 2.0.2 and the choice of x, $C_G(x)$ contains a point stabilizer of $\langle K_i, \widetilde{X} \rangle$. Suppose that $C_G(x) \leq L^*$ for some $L^* \in \mathcal{L}$ with $L^* \not\leq \widetilde{C}$. Then the maximal choice of X implies that X contains a point stabilizer of K_i. But then by 2.0.2(d), $H \cap L$ contains a points stabilizer of H, which contradicts 2.3.3 and $H \notin \mathcal{PU}_1(R)$. So ($*$) holds.

We apply the amalgam method to (H, L) using the standard notation as it is given in [DS]. For $\alpha = Hg$ put $K_{\alpha i} = K_i^g$ and $\widetilde{C}_\alpha = \widetilde{C}^g$.

Suppose that b is even and (α, α') is a critical pair with $\alpha = H$. Typically we will find $1 \neq x \in [Y_\alpha, Y_{\alpha'}]$ such that x is centralized by a Sylow p-subgroup of $K_{\alpha i}$ and $K_{\alpha' j}$. Thus ($*$) implies $\widetilde{C}_\alpha = \widetilde{C}_{\alpha'}$. But this contradicts $Y_\alpha \leq O_p(\widetilde{C}_\alpha)$ and $Y_\alpha \not\leq O_p(G_{\alpha'})$.

A typical case where one cannot find such an x is, when $\overline{K}_i \cong \Omega_{2n}^{\pm}(2)$ and $A := [[Y_\alpha, K_{\alpha i}], Y_{\alpha'}]$ has order 2 (for some i) and $\mathcal{M}(C_G(A)) \neq \{\widetilde{C}\}$. Then there exists $C_G(A) \leq L^* \in \mathcal{L}$ with $L^* \not\leq \widetilde{C}$. It is easy to see that $K_i R^* \in \mathcal{PU}_4(R^*)$. Using the Pushing Up Theorem 2.1.5 one derives a contradiction.

Suppose that b is even, but $\alpha \neq H$ for every critical pair (α, α'). One then proves that $Y_H Y_L$ is normal in L and $O_p(\langle O_p(H)^L \rangle) \leq O_p(H)$. Another application of 2.1.5 gives a contradiction.

So b is odd and without loss $\alpha = H$. Let $\alpha' + 1 \in \triangle(\alpha')$ with $Y_\alpha \not\leq G_{\alpha'+1}$. One usually gets that $Y_\alpha \cap Q_{\alpha'} \cap Q_{\alpha'+1}$ contains an element x as in ($*$). This forces $Y_{\alpha'+1} \leq \widetilde{C}_{\alpha'} \cap G_{\alpha+1} \leq G_\alpha$. This allows us to find $y \in Y_{\alpha'+1} \cap Q_\alpha$ with $C_G(y) \leq \widetilde{C}_{\alpha'+1}$. Hence $Y_\alpha \leq G_{\alpha'+1}$, a contradiction. □

The propositions in this section together with the Pushing Up Theorem leave us with the following open problem:

2.3.1 The open "$\neg E!$, $b = 1$"-Problem

Suppose $\neg E!$, $\neg(PU\text{-}L)$, $Y_H \not\leq O_p(L)$ and $H \notin \mathcal{PU}_4(\mathrm{B}(R))$. Determine the shapes of H and L.

2.4 E!

The way we usually use $E!$ is through an intermediate property called *Q-Uniqueness.*

$$(Q!) \quad C_G(x) \leq \widetilde{C} \text{ for all } 1 \neq x \in C_G(Q)$$

Lemma 2.4.1 *$E!$ implies $Q!$.*

Proof: Since $\widetilde{C}$ is a maximal p-local subgroup, $N_G(Q) = \widetilde{C}$. Thus $x \in C_G(Q) = C_{\widetilde{C}}(Q)$. Since $\widetilde{C}$ is of characteristic p we conclude $x \in Z(Q)$. Without loss $|x|$ has order p and thus $x \in \Omega_1\mathrm{Z}(Q)$. Note that EQ/Q has no p-chief-factors and so $\Omega_1\mathrm{Z}(Q) = Y_{EQ}$. By 2.0.2(c)

$$C_E(Y_{EQ}) = C_E(Y_E) = C_E(Y_{\widetilde{C}}) = E$$

Thus $E \leq C_G(x)$ and $E!$ implies $C_G(x) \leq \widetilde{C}$. □

The reader might want to verify that $L_n(q)$ is an example of a group which fulfills $Q!$ but not $E!$.

In this section we assume $Q!$ and that G has rank at least two. For $L \in \mathcal{L}$ define $L^\circ = \langle Q^g \mid g \in G, Q^g \leq L\rangle$.

Lemma 2.4.2 *Suppose $Q!$.*

(a) *$\widetilde{C}^\circ = Q$, in particular, any p-subgroup of G contains at most one conjugate of Q.*

(b) *If $L \in \mathcal{L}$ with $Q \leq O_p(L)$, then $L \leq \widetilde{C}$. In particular, if $1 \neq X \leq Z(Q)$ then $N_G(X) \leq \tilde{C}$.*

(c) *If $Q_1, Q_2 \in Q^G$ with $Z(Q_1) \cap Z(Q_2) \neq 1$, then $Q_1 = Q_2$.*

(d) *Let $L \in \mathcal{L}$ with $Q \leq L$. Then*

  (da) *$L^\circ = \langle Q^{L^\circ}\rangle$*

  (db) *$L = L^\circ(L \cap \widetilde{C})$.*

  (dc) *$[C_L(Y_L), L^\circ] \leq O_p(L)$.*

  (dd) *If L acts transitively on $Y_L^\sharp$, then $L^\circ = N_G(Y_L)^\circ$.*

  (de) *If $L^\circ \neq Q$, then $C_{Y_L}(L^\circ) = 1$.*

Proof: (a) Let $g \in G$ with $Q^g \le \widetilde{C}$. We may assume that $Q^g \le S$. Then $Z(S) \le C_G(Q^g)$ and thus $S \le C_G(x) \le \widetilde{C}^g$ for $1 \ne x \in Z(S)$. Since $N_G(S) \le N_G(\Omega_1 Z(S)) \le \widetilde{C}$ we conclude that S is in a unique conjugate of $\widetilde{C}$, so $\widetilde{C} = \widetilde{C}^g$ and $Q = Q^g$.

(b) By (a) $Q = O_p(L)^\circ \trianglelefteq L$ and so $L \le N_G(Q) = \widetilde{C}$. By $Q!$ we have $C_G(X) \le \widetilde{C}$, so $Q \le O_p(C_G(X)) \le O_p(N_G(X))$ and we are done.

(c) As $\langle Q_1, Q_2\rangle \le C_G(Z(Q_1) \cap Z(Q_2))$, we get from $Q!$ and (a) that $Q_1 = Q_2$.

(d) By (a) each Sylow p-subgroup of L° contains a unique G-conjugate of Q. Thus Sylow's Theorem gives

$$\{Q^g \mid Q^g \le L\} = Q^{L^\circ} = Q^L,$$

in particular (da) holds and by the Frattini argument $L = L^\circ N_L(Q)$. Then also (db) holds since $N_L(Q) \le \widetilde{C}$. Note that $C_{Y_L}(Q) \ne 1$, so $C_L(Y_L) \le L \cap \widetilde{C}$ by $Q!$. Thus

$$[C_L(Y_L), Q] \le C_L(Y_L) \cap Q \le O_p(C_L(Y_L)) \le O_p(L),$$

and (dc) follows from (da).

Let $Q^g \le N_G(Y_L)$. Then there exists $1 \ne x \in C_{Y_L}(Q^g)$. If L is transitive on $Y_L^\#$, then x is also centralized by an L-conjugate of Q. On the other hand, by $Q!$ and (a) $C_G(x)$ contains a unique conjugate of Q. Hence $Q^g \le L$ and $N_G(Y_L)^\circ = L^\circ$.

(de) follows immediately from $Q!$ and (a). □

2.4.1 The Structure Theorem

In this section we assume $Q!$ and that G has rank at least two. Our goal is to determine the action of L on Y_L for all $L \in \mathcal{L}(S)$ with $L \not\le \widetilde{C}$.

For this let $\mathcal{M}^\ddagger(S)$ be the set of all $M \in \mathcal{M}(S)$ such that

$$\mathcal{M}(L) = \{M\} \text{ for all } L \in \mathcal{L}_M(S) \text{ with } M = LC_M(Y_M)$$

To explain the relevance of this set we define a partial ordering on a certain subset of $\mathcal{L}(S)$. For $L \in \mathcal{L}$ define $L^\dagger = LC_G(Y_L)$ and so $L = L^\dagger$ iff $C_G(Y_L) \le L$. Then clearly Y_L is a p-reduced normal subgroup of $L^\dagger$ and so $Y_L \le Y_{L^\dagger}$. Thus $C_G(Y_{L^\dagger}) \le C_G(Y_L) \le L^\dagger$. We conclude that every $L \in \mathcal{L}$ is contained in a member of

$$\mathcal{L}^\dagger = \{L \in \mathcal{L} \mid C_G(Y_L) \le L\}$$

For $L_1, L_2 \in \mathcal{L}^\dagger(S)$ we define

$$L_1 \ll L_2 \Leftrightarrow L_1 = (L_1 \cap L_2)C_G(Y_{L_1})$$

The following lemma has an elementary proof:

Lemma 2.4.3 (a) $\ll$ *is a partial ordering on* $\mathcal{L}^\dagger(S)$.

(b) *$\mathcal{M}^\ddagger(S)$ is precisely the set of maximal elements in $\mathcal{L}^\dagger(S)$ with respect to $\ll$.*

(c) *If $L, H \in \mathcal{L}(S)$ with $L^\dagger \ll H^\dagger$, then $Y_L \le Y_{H^\dagger}$ and $L^\circ \le H^\circ$.* □

Let $L \in \mathcal{L}(S)$ with $L \not\le \widetilde{C}$. As we have said earlier, we want to determine the action of L on Y_L. This will be done using a particular point of view based on the following elementary observations.

By the preceding lemma $L^\dagger \ll M$ for some $M \in \mathcal{M}^\ddagger(S)$, so

$$L = (L \cap M)C_L(Y_L) = (L \cap M)(L \cap \widetilde{C})$$

since $C_L(Y_L) \le \widetilde{C}$; in particular, also $M \not\le \widetilde{C}$.

It is easy to see that

$$L = \langle \mathcal{P}_L(S)\rangle N_L(S) = \langle \mathcal{P}_L(S)\rangle (L \cap \widetilde{C}),$$

so there exists $P \in \mathcal{P}_L(S)$ with $P \not\le \widetilde{C}$.

According to these observations it suffices to study the action of M on Y_M, where $M \in \mathcal{M}^\ddagger(P)$ for a given $P \in \mathcal{P}(S)$ with $P \not\le \widetilde{C}$. This point of view allows a case subdivision that requires another definition:

For $L \in \mathcal{L}(S)$ we write $gb(L) = 1$ if $Y_M \not\le Q$ for some $M \in \mathcal{L}(L)$, and $gb(L) > 1$ otherwise. In the above discussion we now distinguish the cases $gb(P) > 1$ and $gb(P) = 1$. These two cases are treated in the next two sections. We remark that of the actual groups have $gb(P) = 1$. Indeed among the groups of Lie Type in characteristic p, only ${}^2F_4(2^k)$, ${}^3D_4(q)$ and (for $p \ne 3$) $G_2(q)$ fulfill $E!$, $\operatorname{rank} G > 1$ and $gb(P) > 1$.

We further set

$$\mathcal{L}^\circ = \{L \in \mathcal{L} \mid O^p(L) \le L^\circ\} \text{ and } \mathcal{P}^\circ = \mathcal{P} \cap \mathcal{L}^\circ.$$

Note that for $P \in \mathcal{P}(S)$, $P \in \mathcal{P}^\circ$ iff $P \not\le \widetilde{C}$.

2.4.1.1 The Structure Theorem for $Y_M \le Q$

In this section we discuss a proof of the following theorem:

Theorem 2.4.4 (M-Structure Theorem for $Y_M \le Q$) *Suppose $Q!$ and that $P \in \mathcal{P}^\circ(S)$ with $gb(P) > 1$. Let $M \in \mathcal{M}^\ddagger(P)$. Then one of the following two cases holds for $\overline{M} := M/C_M(Y_M)$ and $M_0 := M^\circ C_S(Y_M)$:*

(a) (aa) *$\overline{M_0} \cong SL_n(p^k)$ or $Sp_{2n}(p^k)$ and $C_{\overline{M}}(\overline{M_0}) \cong C_q$, $q|p^k - 1$, or $\overline{M} \cong Sp_4(2)$ and $\overline{M_0} \cong Sp_4(2)'$ (and $p = 2$),*

(ab) *$[Y_M, M^\circ]$ is the corresponding natural module for $\overline{M_0}$,*

(ac) *$C_{M_0}(Y_M) = O_p(M_0)$, or $p = 2$ and $M_0/O_2(M_0) \cong 3Sp_4(2)'$.*

(b) (ba) $P = M_0S$, $Y_M = Y_P$, *and there exists a unique normal subgroup* P^* *of* P *containing* $O_p(P)$ *such that*

(bb) $\overline{P^*} = K_1 \times \cdots \times K_r$, $K_i \cong SL_2(p^k)$, $Y_M = V_1 \times \cdots \times V_r$, *where* $V_i := [Y_M, K_i]$ *is a natural* K_i*-module,*

(bc) Q *permutes the subgroups* K_i *of (bb) transitively,*

(bd) $O^p(P) = O^p(P^*) = O^p(M_0)$, *and* $P^*C_M(Y_P)$ *is normal in* M*,*

(be) *either* $C_{M^0}(Y_P) = O_p(M_0)$, *or* $p = 2$, $r > 1$, $K_i \cong SL_2(2)$, *and* $C_{M_0}(Y_P)/O_2(M_0) = Z(M_0/O_2(M_0))$ *is a 3-group.*

A second look at the situation discussed in section 1.2 (with M_1 corresponding to M) might help the reader to appreciate the conclusion of the Structure Theorem. In section 1.2 we have assumed that $F^*(M_1/O_p(M_1))$ is quasisimple. Here we get a similar statement as a conclusion in part (a), and part (b) shows that only for "small groups" it is not true (in fact, this case later will be ruled out in the P!-Theorem).

In section 1.2 we found that Y_{M_1} is an FF-module or a $2F$-module for M_1, where the second case is basically ruled out here by the hypothesis $Y_M \leq Q$. But in the FF-module case a glance at the FF-Module Theorem 1.1.2 shows that by far not all possible groups actually occur in the conclusion of the Structure Theorem. In the following we want to demonstrate, using the groups $Sym(I)$ and $G_2(2^k)$ as examples, how these additional groups are ruled out.

Suppose that $\overline{M} \cong Sym(I)$, $|I| \geq 9$, $p = 2$, and $Y := [Y_M, M]$ is the non-central irreducible constituent of the natural permutation module for $Sym(I)$. To describe the action of M on Y let V be a $GF(2)$-vector space with basis v_k, $k \in I$, and set $v_J = \sum_{k\in J} v_k$ for every $J \subseteq I$. Then $Sym(I)$ acts on V via $v_k \mapsto v_{kx}$, $x \in Sym(I)$. Let $V_e := \{v_J \mid J \subseteq I,\ |J| \text{ even}\}$ and $\overline{V}_e = V_e + \langle v_I\rangle/\langle v_I\rangle$. Then $\overline{V}_e$ is the irreducible constituent meant above, so $Y = \overline{V}_e$.

Assume first that Q does not act transitively on I. Then there exists a proper Q-invariant subset J of I with $|J| \leq |I \setminus J|$ and $\overline{v}_J \in C_Y(Q)$. Hence Q! gives $C_M(\overline{v}_J) \leq \widetilde{C}$ and $Q \leq O_2(C_M(\overline{v}_J))$. Note that $C_{\overline{M}}(a_J) \cong Sym(J) \times Sym(I \setminus J)$ (respectively $Sym(I) \wr C_2$, if $|J| = |I \setminus J|$). By 2.4.2(b) $\overline{Q} \neq 1$, we conclude that $O_2(Sym(J)) \neq 1$ or $O_2(Sym(I \setminus J)) \neq 1$. Since $|I| \geq 9$, $|J| \leq |I \setminus J|$, and $O_2(Sym(n)) = 1$ for all $n \geq 5$, we get that $|J| \in \{2, 4\}$ and $O_2(Sym(I \setminus J)) = 1$. Thus $\overline{Q} \leq Sym(J)$, and Q centralizes every $\overline{v}_{J^*}$ for $J^* \subseteq I \setminus J$. Choose such an J^* with $|J^*| = 2$. Then $C_{\overline{M}}(\overline{v}_{J^*}) = Sym(J^*) \times Sym(I \setminus J^*)$ and $\overline{Q} \leq O_2(C_{\overline{M}}(\overline{v}_{J^*})) \cap O_2(Sym(J))$. We conclude that $\overline{Q} = 1$ since $O_2(Sym(I \setminus J^*)) = 1$. But this is impossible by 2.4.2(b).

Assume now that Q is transitive on I. Let J be an orbit of a maximal subgroup of Q that contains the stabilizer of a point. Then $|J| = \frac{1}{2}|I|$ and $\overline{Q}$ centralizes $\overline{v}_J$ since $\overline{v}_J = \overline{v}_{I\setminus J}$. Now a similar argument as above leads to a contradiction.

As a second example let $p = 2$, $\overline{M} \cong G_2(q)$, $q = 2^k$, and $Y := [Y_M, M]$ be the module of order q^6. In addition, suppose that there exists $g \in G$ with $YY^g \leq M \cap M^g$ and $[Y, Y^g] \neq 1$. Then it is easy to see that $|\overline{Y^g}| = q^3$ and

$|[Y, Y^g]| = q^3$. Let $1 \neq x \in [Y, Y^g]$. Since M act transitively on Y, there exist $h \in M$ such that $[x, Q^h] = 1$ From $Q!$ applied to Q^h, $C_G(x) \leq \widetilde{C}^h$. From the hypothesis $Y_M \leq Q$ we get $Y \leq Q^h$ and so $Y \leq O_2(C_G(x))$; in particular

$$Y \leq O_2(C_{M^g}(x)) \text{ for all } 1 \neq x \in [Y, Y^g].$$

This contradicts the action of M^g on Y^g.

Outline of a proof for 2.4.4: Let H be minimal in M with $S \leq H$ and $M = HC_M(Y_M)$. Then by definition of $\mathcal{M}^{\ddagger}(S)$, M is the unique maximal p-local subgroup containing H. Let $Y = Y_H (= Y_M)$.

We consider the following cases:

(a) *[The Orthogonal Case]* $p = 2$, $\overline{H} \cong O^{\epsilon}_{2n}(2)$, $[Y, H]$ is the natural module and $C_H(y) \not\leq M$ for every non-singular element $y \in [Y, H]$.

(b) *[The Symmetric Case]* (a) does not hold and there exists $g \in G$ with $YY^g \leq H \cap H^g$ and $[Y, Y^g] \neq 1$.

(c) *[The Non Abelian Asymmetric Case]* Neither (a) nor (b) holds and there exists $L \in \mathcal{L}$ with $O_p(H) \leq L$ and $Y \not\leq O_p(L)$.

(d) *[The Abelian Asymmetric Case]* None of (a),(b) or (c) holds.

In the following we show how these cases arise from the amalgam method and how they are dealt with.

Choose $P_1 \in \mathcal{P}_{\widetilde{C}}(S)$ with $P_1 \not\leq M$ and P_1 minimal. Since M is the unique maximal p-local containing H, $O_p(\langle H, P_1 \rangle) = 1$ and we can apply the amalgam method to the pair (H, P_1). For notation see [DS].

Assume that b is even. Let (α, α') be a critical pair. Then $Q!$ shows that $G_\alpha \sim H$, and we obtain $g \in G$ with $YY^g \leq H \cap H^g$ and $[Y, Y^g] \neq 1$. Hence either the Symmetric Case or the Orthogonal Case holds. Note that the symmetry in H and H^g allows to assume that Y^g is an offender on Y.

Suppose that there exists $1 \neq x \in [Y, Y^g]$ that is p-central in both, H and H^g. Then our hypothesis $Y \leq Q$ and $Q!$ imply that $Y^g \leq O_p(C_H(x))$, and (after a technical reduction to one component of $H/C_H(Y)$) the Point Stabilizer Theorem 1.1.3 applies. This gives the desired conclusion since the preceding discussion already ruled out $Sym(n)$ and $G_2(2^k)$.

Suppose now that $[Y, Y^g]$ does not contain such an element. Then (again omitting a reduction to components) the FF-Module Theorem 1.1.2 shows that Case (b) of the Structure Theorem holds, or that $|[Y, Y^g]| = 2$. The latter possibility leads to the Orthogonal Case.

Assume now that b is odd and (α, α') is a critical pair. Then again $G_\alpha \sim H$. If there exists $1 \neq x \in Y_\alpha$ with $[x, O^p(G_{\alpha'})] = 1$, then $Y_\alpha \not\leq O_p(C_{G_{\alpha'}}(x))$ and the Non-Abelian Asymmetric Case (or (a) or (b)) hold.

Suppose that $[x, O^p(G_{\alpha'})] \neq 1$ for all $x \in Y_\alpha^{\#}$ (in the actual proof we do not use $O^p(G_{\alpha'})$ but a possibly smaller subgroup of $G_{\alpha'}$). Using the action of Y_α on $V_{\alpha'}$ one can show the existence of a strong offender on Y_α. Here an offender A

on a module V is called strong, if $C_V(a) = C_V(A)$ for all $a \in A \setminus C_A(V)$. This rules out most of the cases of the FF-module Theorem 1.1.2, and we get what we want, (except that it does not rule out $SL_n(q)$ on a direct sum of natural modules, a case which we will not discuss here).

This leaves us with the Orthogonal Case or the Non-Abelian Asymmetric Case. In the Orthogonal Case we choose L minimal with $C_H(x) \leq (L \cap H) C_H(Y)$ and $L \not\leq M$, where x is a non-singular vector (i.e. a non-p-central element) in $[Y, H]$. Let z be a non-zero singular vector in $[Y, H]$ perpendicular to x, so z is p-central in H. Then $[z, Q^h] = 1$ for some $h \in H$. Let $Q_z := Q^h$. We now show that $O_p(\langle Q_z, L\rangle) = 1$, $[Q_z, C_L(z)] \leq Q_z \cap L$, and that z and y are not conjugate in G. Then 2.1.5 gives the shape of L, and one obtains a contradiction.

It remains to discuss the Non-Abelian Asymmetric Case. Let $U \in \mathcal{L}(O_p(H))$ with $Y \not\leq O_p(U)$ such that first $|U \cap H|_p$ is maximal and then U is minimal. Let $T \in Syl_p(U \cap H)$. If $N_G(T) \not\leq M$, then considering the amalgam $(H, N_G(T))$ we obtain $g \in G$ with $YY^g \in H \cap H^g$ and $[Y, Y^g] \neq 1$. But this contradicts the assumptions of the Non-Abelian Asymmetric Case. Hence $N_G(T) \leq M$, in particular T is a Sylow p-subgroup of U. If $Q \not\leq U$ we can apply 2.1.5 and get a contradiction. So $Q \leq U$. Since $Y \leq Q$ but $Y \not\leq O_p(U)$, we have $U \not\leq \widetilde{C}$, and 2.4.2(de) implies $C_{Y_U}(U) = 1$.

Let $T \leq X < U$. Then by minimality of U, $Y \leq O_p(X)$. Since $O_p(X) \leq T \leq H$ we get $\langle Y^X \rangle \leq H$. Hence $\langle Y^X \rangle$ is abelian since we are not in the symmetric case. So 1.1.4 gives the structure of $U/O_p(U)$ and Y_U. Moreover, in most cases we can conclude that Y_U is a strong dual offender on Y and in all cases we get some strong dual offender on Y. Here a group A is called a strong dual offender on a module V if A acts quadratically on V and $[v, A] = [V, A]$ for all $v \in V \setminus C_V(A)$. The existence of a strong dual offender on Y together with the FF-Module Theorem 1.1.2 gives the desired conclusion. □

2.4.1.2 The Structure Theorem for $Y_M \not\leq Q$

In this section we outline a proof of the following theorem. (It might be worthwhile to mention that given $E!$ we do not need to assume in this section that G is of local characteristic p but only that G is of parabolic characteristic p.)

Theorem 2.4.5 (M-Structure Theorem for $Y_M \not\leq Q$) *Let $M \in \mathcal{M}(S)$ with M° maximal and put $\overline{K} = F^*(M^\circ / C_{M^\circ}(Y_M))$. Suppose $E!$, $Y_M \not\leq Q$ and that $M^\circ S$ is not p-minimal, then one of the following holds*

1. *$\overline{K}$ is quasisimple and isomorphic to $SL_n(q)$, $\Omega_n^{\pm}(q)$, or $E_6(q)$. In case of $\overline{K} \cong SL_n(q)$, or $E_6(q)$ no element in M induces a diagram automorphism.*
2. *$\overline{K} \cong SL_n(q)' \circ SL_m(q)'$.*
3. *$p = 2$ and $\overline{K} \cong Alt(6)$, $3Alt(6)$, $Sp_8(2)$, M_{22}, or M_{24}*

4. $p = 3$ *and* $\overline{K} \cong M_{11}$ *or* M_{12}

Moreover, the module Y_M *is a 2F–module with quadratic or cubic offender and contains a module* V *as in the table below.*

K	prime	module	example
$SL_n(q)$	p	ext. square	$\Omega_{2n}(q)$
$SL_n(q)$	p odd	sym. square	$Sp_{2n}(q)$
$SL_n(q^2)$	p	$V(\lambda_1) \otimes V(\lambda_1^\sigma)$	$SU_{2n}(q)$
$SL_3(2)$	2	natural	$G_2(3).2$
$Alt(6)$	2	natural	Suz
$3Alt(6)$	2	6-dim	M_{24}
$Sp_8(2)$	2	8-dim	F_2
$\Omega_n^{\pm}(q)$	p	natural	$\Omega_{n+2}^{\pm}(q)$
$\Omega_{10}^{\pm}(q)$	p	half spin	$E_6(q)$
$E_6(q)$	p	$V(\lambda_1)$	$E_7(q)$
M_{11}	3	5-dim	Co_3
$2M_{12}$	3	6-dim	Co_1
M_{22}	2	10-dim	$M(22)$
M_{24}	2	11-dim	$M(24)$

The proof of the above theorem corresponds to the discussion of the Cases (I) and (II) in section 1.2. Let $L \in \mathcal{L}_{\widetilde{C}}$ be minimal with $Y_M \leq S \cap L \in Syl_p(L)$ and $Y_M \not\leq O_p(L)$. Note that such a choice is possible since $Y_M \not\leq Q$. Let $Y_M \leq P < L$ and $S \cap P \in Syl_p(P)$. Then by the minimal choice of L, $Y_M \leq O_p(P)$ and so $\langle Y_M^P \rangle \leq O_p(P) \leq S \leq M$. We now consider the following two cases separately:

(1F) There exists $g \in G$ such that $1 \neq [Y_M, Y_M^g] \leq Y_M \cap Y_{M^g}$ and $Y_M Y_M^g \leq M \cap M^g$.

(2F) $\langle Y_M^P \rangle$ is abelian for all $Y_M \leq P < L$ with $S \cap P \in Syl_p(P)$.

In the 1F-Case, possibly after replacing g be g^{-1}, we may assume that $A := Y_M^g$ is a quadratic offender on Y_M.

In the 2F-Case 1.1.5 can be used to get a cubic $2F$-offender on Y_M as in case (III) of 1.2.

The FF-module Theorem1.1.2 and the 2F-module Theorem 1.1.6 now allow us to identify the components (or solvable variants of components) of $M^\circ/C_{M^\circ}(Y_M)$ which are not centralizes by A. In the iF-case one can show that A centralizes all but i of the components. Let K be the product of the components not centralized by A. By 2.4.2(de), $M^\circ/C_{M^\circ}(Y_M)$ acts essentially faithful on $[Y_M, K]$. This allows also to obtain information about all of $M^\circ/C_{M^\circ}(Y_M)$.

2.4.2 The $P!$-Theorem

In this section we assume $Q!$ and that G has rank at least 2. Note that this implies that $\mathcal{P}^\circ(S) \neq \emptyset$. We investigate the members of $\mathcal{P}^\circ(S)$, and distinguish the two cases $\langle \mathcal{P}^\circ(S)\rangle \notin \mathcal{L}$ and $\langle \mathcal{P}^\circ(S)\rangle \in \mathcal{L}$. Detailed proofs for the following two theorems can be found in [PPS].

Theorem 2.4.6 (The P! Theorem,I) *Suppose $Q!$ hold and $\langle \mathcal{P}^\circ(S)\rangle \notin \mathcal{L}$. Then*

(a) *p is odd.*

(b) *$Q = \mathrm{B}(S)$, $\widetilde{C} = N_G(\mathrm{B}(S))$ and $|Q|$ has order q^3, q a power of p.*

(c) *$P^\circ \sim q^2 SL_2(q)$ for all $P \in \mathcal{P}^\circ(S)$*

Outline of a Proof: Let $L = N_G(B(S))$. By our assumption not every element of $\mathcal{P}^\circ(S)$ is in L. We first investigate an element $P \in \mathcal{P}^\circ(S)$ with $P \not\leq L$. Observe that $Q!$ implies that $\Omega_1 \mathrm{Z}(X) = 1$ for every $X \in \mathcal{P}^\circ(S)$, so by 2.1.3 $P \in \mathcal{P}U_4(B(S))$; i.e.

($*$) $P = \langle N_P(B(S)), P_0 \mid P_0 \leq P, P_0 \in \mathcal{P}U_3(B(S))\rangle$.

An application of 2.1.4 and a short argument show that for the groups P_0 in ($*$):

(1) $Y_P = O_p(P_0) = O_p(P)$,

(2) P_0/Y_P, Y_P is a natural $SL_2(q)$-module for P_0/Y_P ($q = p^m$), and $|\mathrm{B}(S)| = q^3$.

(3) P_0 is normal in P, and $P = SP_0$.

Suppose that $p = 2$. Then $|\mathcal{A}(S)| = 2$, so $L = SO^2(L) \leq N_G(A)$ for all $A \in \mathcal{A}(S)$. It is now easy to see that there exist exactly two maximal 2-local subgroups containing S. One of them is $\widetilde{C}$ and so $\langle \mathcal{P}^\circ(S)\rangle$ is contained in the other. But this contradicts our hypothesis.

So p is odd. Suppose that $Q \not\leq \mathrm{B}(S)$. Then (ii) shows that $q = p^{pk}$ for some integer $k \geq 1$. Moreover, $[Y_P, Q]$ has order at least $p^{(2p-1)k}$. Let $V = \langle [Y_P, Q]^{\widetilde{C}}\rangle$. Note that an elementary abelian p-subgroup of S not contained in $\mathrm{B}(S)$ has order at most p^{2k+1}. Since $p > 2$, $(2p-1)k > 2k+1$ and so $V \leq \mathrm{B}(S)$.

In particular, $Z(V) \leq C_{\mathrm{B}(S)}([Y_P, Q]) \leq Y_P$. It follows that either $V \leq Y_P$ or $[V, Y_P] = Z(V) = Z(\mathrm{B}(S))$. In both cases $\langle Y_P^{\widetilde{C}}\rangle$ acts trivially on the series $1 \leq Z(V) \leq V \leq Q$, so $Y_P \leq Q$ since $\widetilde{C}$ has characteristic p. As Y_P is not normal in $\widetilde{C}$ we get $\mathrm{B}(S) = \langle Y_P^{\widetilde{C}}\rangle \leq Q$. In particular $\mathrm{B}(S) = \mathrm{B}(Q)$, so $\widetilde{C} = L$ and $Q \leq O_p(N_P(\mathrm{B}(S))) = \mathrm{B}(S)$.

We have proved that $Q \leq \mathrm{B}(S)$, so by 2.4.2(b) $L \leq \widetilde{C}$. In particular $P \not\leq L$ for every $P \in \mathcal{P}^\circ(S)$, and (1) – (3) hold for every $P \in \mathcal{P}^\circ(S)$. It remains to prove that $Q = \mathrm{B}(S)$.

Suppose that $Q \neq \mathrm{B}(S)$. Again as $\widetilde{C}$ is of characteristic p, we get that $Z(\mathrm{B}(S)) < Q$. Note that $N_P(\mathrm{B}(S))$ acts irreducibly on $Y_P/Z(\mathrm{B}(S))$ and $\mathrm{B}(S)/Y_P$.

It follows that either $Y_P \leq Q$ or $Y_P \cap Q = Z(\mathrm{B}(S))$. The first case gives $Q = Y_P$ contrary to our assumption. The second case shows, with an argument as above using the series $1 \leq \Omega_1 Z(Q) \leq Q$, that $\Omega_1 Z(Q) \neq Z(\mathrm{B}(S))$, so Q is elementary abelian of order q^2.

For every $P \in \mathcal{P}^\circ(S)$ let t_P be an involution in P that maps onto the central involution of P_0/Y_P. Then t_P normalizes $\mathrm{B}(S)$ and so also Q. We conclude that t_P inverts $Z(\mathrm{B}(S))$ and $\mathrm{B}(S)/Q$ and centralize $Q/Z(\mathrm{B}(S))$. There exists $X \in \mathcal{P}^\circ(S)$ with $Y_X \neq Y_P$. Let $u = t_P t_X$. Then u centralizes $Z(\mathrm{B}(S))$, $\mathrm{B}(S)/Q$ and $Q/Z(\mathrm{B}(S))$. So u induces a p-element on $\mathrm{B}(S)$ and since $N_G(\mathrm{B}(S))$ has characteristic p, $\mathrm{B}(S)\langle u\rangle$ is a p-group. By (1)-(3) we conclude that $u \in B(S)$ and $t_P \mathrm{B}(S) = t_X \mathrm{B}(S)$. But then $Y_P = [\mathrm{B}(S), t_P] = [\mathrm{B}(S), t_X] = Y_X$, a contradiction.

We have shown that $Q = \mathrm{B}(S)$, and the lemma is proved. □

We say that $P!$ holds in G provided that:

(P!-1) There exists a unique $P \in \mathcal{P}^\circ(S)$.

(P!-2) $P^\circ/O_p(P^\circ) \cong SL_2(q)$, q a power of p.

(P!-3) Y_P is a natural module for P°.

(P!-4) $C_{Y_P}(S \cap P^\circ)$ is normal in $\widetilde{C}$.

Theorem 2.4.7 (The P! Theorem,II) *Suppose that*

(i) *$Q!$ holds and G has rank at least 2.*

(ii) *P is a maximal element of $\mathcal{P}^\circ(S)$ and $gb(P) > 1$.*

(iii) $M := \langle \mathcal{P}^\circ(S) \rangle \in \mathcal{L}$

Then $P!$ holds in G.

Outline of a Proof: Applying the Structure Theorem 2.4.4 to some $\widetilde{M} \in \mathcal{M}^\ddagger(M)$ it is fairly easy to see that $P = M$. In case (a) of the Structure Theorem 2.4.4 $P^\circ/O_p(P^\circ) \cong SL_2(q)$ and Y_p is the natural $P^\circ/O_p(P^\circ)$-module. In this case we define $Z_0 := C_{Y_P}(P^\circ \cap S)$ $(= \Omega_1 Z(S \cap P^\circ))$. In case (b) of the Structure Theorem we define $Z_0 := C_{Y_P}(S \cap P^*)$, where P^* is as given there.

The main step in the proof of the P!-Theorem is to show that Z_0 is normal in $\widetilde{C}$. Suppose not and let $\widetilde{P} \in \mathcal{P}_{\widetilde{C}}(S)$ be minimal with $Z_0 \ntrianglelefteq \widetilde{P}$. Another application of the Structure Theorem shows that $O_p(\langle P, \widetilde{P}\rangle) = 1$. So we can apply the amalgam method to the pair $(P, \widetilde{P})$.

For $\gamma = \widetilde{P}g$ put $\widetilde{C}_\gamma = \widetilde{C}^g$. Let (α, α') be a critical pair. Suppose that $\alpha \sim \widetilde{P} \sim \alpha'$. Then both Q_α and $Q_{\alpha'}$ contain a conjugate of Q. Since $1 \neq [Z_\alpha, Z_{\alpha'}] \leq Z(Q_\alpha) \cap Z(Q_{\alpha'})$ we conclude from 2.4.2(c) that $\widetilde{C}_\alpha = \widetilde{C}_{\alpha'}$ and so $Z_\alpha Z_{\alpha'} \leq Z_{\widetilde{C}_\alpha}$. Thus $[Z_\alpha, Z_{\alpha'}] = 1$, a contradiction.

So we may assume that $\alpha = P$. Since $Y_P \leq Q \leq O_p(\widetilde{P})$ we have $b > 1$. Suppose that $b = 2$. By the Structure Theorem Q (and so also Q_β) acts transitively on the "components" of G_α/Q_α. Hence $Z_0 = [Z_\alpha, Z_{\alpha'}]$. This is used to show that $Z_0 \trianglelefteq G_\beta$, a contradiction.

Thus $b \geq 3$. A lengthy amalgam argument now leads to contradiction.

We have established that Z_0 is normal in $\widetilde{C}$. In Case (a) of the Structure Theorem we are done. So suppose that Case (b) of the Structure Theorem holds. Since $N_P(Z_0) \leq \widetilde{C}$, $Q \leq O_p(N_P(Z_0))$. Since Q acts transitively on the components we conclude that $q = p = 2$.

Note that $\widetilde{M}$ is the unique maximal 2-local subgroup of G containing P. Suppose that $N_G(\mathrm{B}(S)) \not\leq \widetilde{M}$. Then $O_2(\langle P, N_G(\mathrm{B}(S))\rangle = 1$ and 2.1.4 gives a contradiction. Hence $N_G(\mathrm{B}(S)) \leq \widetilde{M}$. Since $\mathrm{B}(S) \leq C_G(Z_0)$ and Z_0 is normal in $\widetilde{C}$, the Frattini argument implies $\widetilde{C} = (\widetilde{C} \cap \widetilde{M})C_G(Z_0)$.

Let K be the one of the $Sym(3)$-components of $P/O_p(P)$, T a subgroup of index 2 in S with $N_S(K) \leq T$, $X = \langle([Y_P, K] \cap Z_0)^T\rangle$ and $L = N_G(X)$. Then $\langle K, T, C_G(Z_0)\rangle \leq L$. Since Q acts transitively on the components of $P/O_p(P)$, $Q \not\leq T$ and $P = \langle L \cap P, Q\rangle$. Thus $O_2(\langle Q, L\rangle) = 1$. Suppose that T is not a Sylow 2-subgroup of L. Since T is of index 2 in a Sylow 2-subgroup of G, $N_L(T)$ contains a Sylow 2-subgroup of L and G. But $N_L(T) \leq N_G(\mathrm{B}(S)) \leq \widetilde{M}$ and so $N_L(T)$ contains a Sylow 2-subgroup of $\widetilde{M}$. One concludes that $P \leq L$, a contradiction.

Thus T is a Sylow 2-subgroup of L. Since $C_L(\Omega_1\mathrm{Z}(T)) \leq C_G(\Omega_1\mathrm{Z}(S)) \leq \widetilde{C}$ and $|Q/Q \cap T| = 2$ we get $C_L(\Omega_1\mathrm{Z}(T)) \trianglelefteq QC_L(\Omega_1\mathrm{Z}(T))$. So we can apply 2.1.5 to $\Sigma = L^Q$ and $R = O_p(C_L(\Omega_1\mathrm{Z}(T)))$. A little bit of more work gives a contradiction. □

2.4.3 The $\widetilde{P}$! Theorem

Suppose that G fulfills $Q!$ and $P!$. We say that $\widetilde{P}!$ holds in G provided that

($\widetilde{P}$!-1) There exists at most one $\widetilde{P} \in \mathcal{P}(S)$ such that $\widetilde{P}$ does not normalize P° and $M := \langle P, \widetilde{P}\rangle \in \mathcal{L}$.

($\widetilde{P}$!-2) If such a $\widetilde{P}$ exists then,

- (a) $M \in \mathcal{L}^\circ$.
- (b) $M^\circ/C_{M^\circ}(Y_M) \cong SL_3(q), Sp_4(q)$ or $Sp_4(2)'$
- (c) Y_M is a corresponding natural module.

In this section we outline a proof of the following theorem from [MMPS]:

Theorem 2.4.8 (The $\widetilde{P}$! Theorem) *Suppose $Q!$ and $gb(P) > 1$ for some $P \in \mathcal{P}^\circ(S)$. Then one of the following is true:*

1. *G fulfills $\widetilde{P}!$.*

2. *Let $\widetilde{P} \in \mathcal{P}(S)$ with $\widetilde{P} \not\leq N_G(P^\circ)$ and $M := \langle P, \widetilde{P}\rangle \notin \mathcal{L}$. Then*

   (a) $p = 3$ *or* 5.

   (b) $M/O_p(M) \cong SL_3(p)$

   (c) $O_p(M)/Z(O_p(M))$ *and* $Z(O_p(M))$ *are natural* $SL_3(p)$*-modules for* $M/O_p(M)$ *dual to each other.*

Outline of a Proof: We may assume that $\widetilde{P}$! does not hold. Then there exists $P_1 \in \mathcal{P}(S)$ such that $M_1 = \langle P, P_1\rangle \in \mathcal{L}$ and $P_1 \not\leq N_G(P^\circ)$. The Structure Theorem 2.4.4 shows that $M_1/O_p(M_1) \cong SL_3(q)$ or $Sp_4(q)$ (or some variant of $Sp_4(2)$) and that Y_{M_1} is a corresponding natural module. In particular, if P_1 were unique $\widetilde{P}$! would hold. Hence we can choose P_2 having the same properties as P_1 and $P_1 \neq P_2$. Define $M_2 = \langle P, P_2\rangle$. The Structure Theorem also implies that $\langle M_1, M_2\rangle \notin \mathcal{L}$ and so we can apply the amalgam method to (M_1, M_2). Fairly short and elementary arguments show that $b \leq 2$. In the $b = 1$ case one easily gets $M_i' \cong 2^4 Sp_4(2)'$ and then obtains a contradiction to $Y_{M_i} \leq Q$. Fairly routine arguments in the $b = 2$ case show that $M_i \sim q^{3+3}SL_3(q)$ or $q^{3+3+3}SL_3(q)$. A little extra effort rules out the second of this possibilities. But the proof that $q = 3$ or 5 in the remaining case currently is a rather tedious commutator calculation. □

The next lemma collects some information about $\widetilde{C}/O_p(\widetilde{C})$ which can be easily obtained using Q!, P! and $\widetilde{P}$!:

Lemma 2.4.9 *Suppose Q!, P!, $\widetilde{P}$! and that G has rank at least three. Let $L = N_G(P^\circ)$. Then*

(a) $N_G(T) \leq L \cap \widetilde{C}$ *for all* $O_p(\widetilde{C} \cap L) \leq T \trianglelefteq S$.

(b) *There exists a unique* $\widetilde{P} \in \mathcal{P}_{\widetilde{C}}(S)$ *with* $\widetilde{P} \not\leq L$.

(c) $\widetilde{P}/O_p(\widetilde{P}) \sim SL_2(q).p^k$.

(d) $\widetilde{C}/Q$ *has a unique component* K/Q. *Moreover,* $\widetilde{P} \leq KS$.

(e) $\widetilde{C} = K(L \cap \widetilde{C})$, $L \cap \widetilde{C}$ *is a maximal subgroup of* $\widetilde{C}$ *and* $O_p(\widetilde{C} \cap L) \neq Q$.

(f) *Let* $Z_0 = C_{Y_P}(S \cap P^\circ)$ *and* $V = \langle Y_P^{\widetilde{C}}\rangle$. *Then* $Z_0 \trianglelefteq V$ *and* $V \leq Q$.

(g) *Let* $D = C_{\widetilde{C}}(K/O_p(K))$. *Then D is the largest normal subgroup of $\widetilde{C}$ contained in L and D/Q is isomorphic to a section of the Borel subgroup of* $\mathrm{Aut}(SL_2(q))$.

(h) *Let* $\overline{V} = V/Z_0$. *Then*

   (ha) $[\overline{V}, Q] = 1$

   (hb) $C_{\widetilde{C}}(\overline{V}) \leq D$ *and* $C_{\widetilde{C}}(\overline{V}) \cap C_{\widetilde{C}}(Z_0) = Q$.

(hc) *Let* $1 \neq X \leq Y_P/Z_0$*. Then* $N_{\widetilde{C}}(X) \leq \widetilde{C} \cap L$*.*

(hd) $\widetilde{C} \cap L$ *contains a point-stabilizer for* $\widetilde{C}$ *on* $\overline{V}$*.*

(g) $\langle \widetilde{C}, L \rangle \notin \mathcal{L}$*.*

2.4.4 The Small World Theorem

Given $Q!$ and $P \in \mathcal{P}^\circ(S)$. We say that $gb(P) = 2$ if $gb(P) > 1$ and $\langle Y_P^E \rangle$ is not abelian. If neither $gb(P) = 1$ nor $gb(P) = 2$ for P we say that $gb(P)$ is at least three.

Theorem 2.4.10 (The Small World Theorem) *Suppose* $E!$ *and let* $P \in \mathcal{P}^\circ(S)$*. Then one of the following holds:*

1. *G has rank 1 or 2.*

2. $gb(P) = 1$ *or* $gb(P) = 2$*.*

Outline of a Proof: Assume that G has rank at least three and that $gb(P)$ is at least three . In the exceptional cases of the $P!$-theorems (2.4.7, 2.4.6) one easily sees that $gb(P) = 1$. Thus $P!$ holds. Also in the exceptional case of the $\widetilde{P}!$ -Theorem 2.4.8 one gets $gb(P) = 1$ or $gb(P) = 2$. Thus $\widetilde{P}!$ holds. We proved

Step 1 *$P!$ and $\widetilde{P}!$ hold.*

2.4.9 gives us a good amount of information about E. We use the notation introduced in 2.4.9.

Since $\langle \widetilde{C}, L \rangle \notin \mathcal{L}$, we can apply the amalgam method to the pair $(\widetilde{C}, L)$. A non-trivial argument shows

Step 2 *One of the following holds:*

1. $O_p(\widetilde{C} \cap L)/Q$ *contains a non-trivial quadratic offender on* $\overline{V}$*.*

2. *There exists a non-trivial normal subgroup* A *of* $\widetilde{C} \cap L/Q$ *and normal subgroups* $Y_P \leq Z_2 \leq Z_3 \leq V$ *of* $\widetilde{C} \cap L$ *such that:*

   (a) A *and* V/Z_3 *are isomorphic as* $\mathbb{F}_p C_{\widetilde{C} \cap L}(Y_P)$*-modules.*

   (b) $|Z_3/Z_2| \leq |A|$*.*

   (c) $[\overline{V}, A] \leq \overline{Z_2} \leq C_{\overline{V}}(A)$*. In particular, A is a quadratic* $2F$*-offender.*

   (d) $[\overline{x}, A] = \overline{Y_P}$ *for all* $x \in Z_3 \setminus Z_2$*.*

   (e) *Let* $Z_1 = \langle Y_P^{\widetilde{P}} \rangle$*. Then* $Y_1 \leq Z_2$ *and* $\overline{Z_1}$ *is a natural* $SL_2(q)$*-module for* $\widetilde{P} \cap C_{\widetilde{C}}(Z_0)$*.*

We remark that 1. and 2. of Step 2 correspond to the $b > 3$- and $b = 3$-Case for the amalgam $(\widetilde{C}, L)$.

Let $X = C_V(O^p(K))$. Using 1.1.1, 1.1.3 and 2.4.9 (and the Z^*-theorem [Gl]) to deal with the case $|A| = 2$) it is not too difficult to derive

Step 3 *$K/O_p(K) \cong SL_n(q)$, $(n \geq 3)$, $Sp_{2n}(q)'$,$(n \geq 2)$ or $G_2(q)'$,$(p = 2)$. Moreover, V/X is the natural module for $K/O_p(K)$ and $\widetilde{C} \cap L$ contains a point-stabilizer for $\widetilde{C}$ on V/W.*

An amalgam argument now shows that $X = Z_0$. In particular, K acts transitively on $\overline{V}$. Hence all elements in V are conjugate under K to an element of Y_P. From this it is not to difficult to show that $b = 3$ in the amalgam $(\widetilde{C}, L)$. Finally also the $b = 3$ case leads to a contradiction.

□

We finish this section with

2.4.4.1 The open "$gb = 2$"-Problem

Suppose $P \in \mathcal{P}^\circ(S)$, $gb(P) = 2$ and that G has rank at least three. Determine the shape of $\widetilde{C}$ and P

Note that by the definition of $gb(P) = 2$, $Y_P \leq Q$ and $\langle Y_P^{\widetilde{C}} \rangle$ is not abelian. So it should be possible to treat the $gb = 2$ problem with the methods of Parker/Rowley from [PR].

2.4.5 Rank 2

In this section we consider the case where $Q!$ holds and G has rank 2. The general idea is to show that $\langle P, \widetilde{P} \rangle$ is a weak BN-pair and then apply the Delgado-Stellmacher weak BN-pair Theorem [DS]. More precisely we try to characterize the situations where no weak BN-pair can be found. The following theorem has been proved in [Ch1] and [Ch2]

Theorem 2.4.11 (The Rank 2 Theorem) *Suppose $Q!$, $P!$, and $\widetilde{P}!$ and that G has rank 2. Choose $\widetilde{P} \in \mathcal{P}_{\widetilde{C}}(S)$ such that*

(i) *$\langle P, \widetilde{P} \rangle \notin \mathcal{L}$.*

(ii) *$H := \langle P \cap \widetilde{C}, \widetilde{P} \rangle$ is minimal with respect to (i).*

(iii) *$\widetilde{P}$ is minimal with respect to (i) and (ii).*

Then one of the following holds:

1. $Y_P \not\leq O_p(\widetilde{P})$

2. *$(N_H(P^\circ)P^\circ, H)$ is a weak BN-pair.*

3. *(H, P) has the same shape as a suitable pair of parabolic subgroups in one of the following groups.*

    1. *For $p = 2$, $U_4(3).2^e$, $G_2(3).2^e$, $D_4(3).2^e$, $HS.2^e$, F_3, $F_5.2^e$ or Ru.*
    2. *For $p = 3$, $D_4(3^n).3^e$, Fi_{23} or F_2.*
    3. *For $p = 5$, F_2.*
    4. *For $p = 7$, F_1.*

We will not go into the details of this proof. It is a rather technical application of the amalgam method applied to the pair $(N_H(P^\circ)P^\circ, H)$.

The Rank 2 Theorem leaves as in the rank 2 case with the following open problem.

2.4.5.1 The open "Rank 2, gb=1"-Problem

Suppose $Q!$ holds and there exists $P, \widetilde{P} \in \mathcal{P}(S)$ such that $\langle P, \widetilde{P} \rangle \notin \mathcal{L}$, $P \in \mathcal{P}^\circ(S)$ and $gb(P) = 1$. Determine the structure of P.

2.4.6 $gb = 1$

In this section we assume $E!$ and that G has rank at least 3. We investigate the case where $Y_M \not\leq Q$ for some $M \in \mathcal{M}(S)$ with M° maximal. Put $M_0 = M^\circ S$. The Structure Theorem 2.4.5 tells us the action of $M^\circ/O_p(M^\circ)$ on Y_M.

But we can get a lot more information. Let us consider one example. Suppose $\overline{M_0} = F^*(M_0/O_p(M_0)) \cong SL_n(q)$ and Y_M is the natural module. Then $\overline{M_0}$ has the following Dynkin diagram

We have that $C_{\overline{M_0}}(\Omega_1(Z(S)) \cap Y_M)$ is a maximal parabolic, which then by $E!$ is in $\widetilde{C}$. Hence there is a unique minimal parabolic P in M_0 which is not in $\widetilde{C}$. Notice that most of our groups we aim at are groups of Lie type in which $\widetilde{C}$ is a maximal parabolic. So there is a unique $P \in \mathcal{P}^\circ(S)$. Hence we are going to approach this situation.

But this unique P does not exist in general, as one can see in Case 2 of the structure theorem 2.4.5, in Case 3 with $M^\circ/O_2(M^\circ) \cong 3A_6$ and Y_M a 6-dimensional module, and in Case 4 with $M^\circ/O_3(M^\circ) \cong M_{11}$ and Y_M a 5-dimensional module. Hence one cannot expect a similar theorem as 2.4.7 for "$gb = 1$".

To be able to state the theorems in this section we need to introduce some notation:

Let H^* be a finite group. We say the G is of *identification type* H^* provided that:

(I1) There exist $T^* \in \mathrm{Syl}_P(H^*)$ and $I^* \subseteq \mathcal{P}_{H^*}(T^*)$ with $H^* = \langle I^* \rangle$.

(I2) There exists $H \leq G$ with $C_G(H) = 1 = O_p(H)$ and $M_0 \leq N_G(H)$.

(I3) Let $T = S \cap H$. Then there exists $I \subseteq \mathcal{P}_H(T)$ with $H = \langle I \rangle$.

(I4) There exists a bijection $I \to I^*, L \mapsto L^*$ such that for all $J \subset I$,

$$\langle J \rangle / O_p(\langle J \rangle) \cong \langle J^* \rangle / O_p(\langle J^* \rangle).$$

(I5) There exist $M^*, C^* \in \mathcal{L}_{H^*}(T^*)$ such that $M_0 \cap H$ has the same structure as M^* and $\widetilde{C} \cap H$ has the same structure as $\widetilde{C}^*$.

Theorem 2.4.12 *Suppose $E!$, $gb(P) = 1$, $\operatorname{rank} G > 2$ and $\mathcal{P}^\circ(S) \neq \{P\}$. If $p = 2$ then G is of identification type M_{24}, He or $L_n(q)$.*

So suppose from now on that $\mathcal{P}^\circ(S) = \{P\}$.

Here is another observation. Let $P \in \mathcal{P}^\circ_{M_0}(S)$. Then in our example P corresponds to an end node of the Dynkin diagram of M_0. Hence (in most cases) there is a unique $\widetilde{P}$ in $\mathcal{P}_{M_0}(S)$ with $\widetilde{P} \not\leq N_G(P^\circ)$. Let us consider the group G we aim at, a group of Lie type. Then again in most cases P corresponds to an end node of the Dynkin diagram of G and there exists a unique minimal parabolic in $\mathcal{L}(\mathcal{S})$ not normalizing P°.

Unfortunately this is not true in general, for example if Y_M is the exterior square of the natural $SL_n(q)$–module. To analyze this situation, we consider $P_1 \neq P_2$ in $\mathcal{P}(S)$ such that P_i does not normalize P° for $i = 1, 2$. Let $L = \langle P_1, P, P_2 \rangle$. The case $O_p(L) = 1$ should be approachable with the amalgam method, (see the open problem at the end of the section).

So suppose that $L \in \mathcal{L}$. From the structure theorem we conclude that $L^\circ / C_{L^\circ}(Y_L) \cong SL_n(q), n \geq 4$ (on the exterior square), M_{24} (on a 11-dimensional module) or M_{22} (on a 10-dimensional module.) These cases lead to the different groups in our next theorem.

Theorem 2.4.13 *Suppose $E!$, $\operatorname{rank} G > 2$, $\mathcal{P}(S) = \{P\}$ and $gb(P) = 1$. Furthermore, assume that there exist $P_1 \neq P_2 \in \mathcal{P}(S)$ with $P_i \not\leq N_G(P^\circ)$ and $\langle P_1, P, P_2 \rangle \in \mathcal{L}$. Then G is of identification type $\Omega_n^{\pm}(q)$ or (for $p = 2$) Co_2, $M(22)$, Co_1, J_4, or $M(24)'$*

From now on we can assume that there is a unique P in $\mathcal{P}^\circ(S)$ and a unique $\widetilde{P} \in \mathcal{P}_{M_0}(S)$ which does normalize P°.

Theorem 2.4.14 *Suppose $E!$, $\operatorname{rank} G > 2$, $\mathcal{P}(S) = \{P\}$, $gb(P) = 1$ and that there exists a unique $\widetilde{P} \in \mathcal{P}(S)$ with $\widetilde{P} \not\leq N_G(P^\circ)$. If $p = 2$, then G is of identification type $U_n(q)$, ${}^2E_6(q)$, $E_6(q)$, $E_7(q)$, Sz,F_2 or F_1.*

In the remainder of this section, we will illustrate in some examples the basic ideas of the proof of the theorems. All the examples will be for $p = 2$.

Example 2.4.15 *Let $K = F^*(M_0/O_2(M_0)) \cong M_{24}$ and assume that Y_M contains an 11-dimensional submodule V with $|Y_M : V| \leq 2$. Assume further that V is the module in which $L = C_K(C_V(S)) \cong 2^6 3Sym(6)$. Then G is of identification type J_4 or $M(24)'$.*

For L we have the following series in V

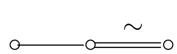
$$1 < V_1 < V_2 < V,$$

where $|V_1| = 2$, V_2/V_1 is the 6–dimensional $3Sym(6)$–module and V/V_1 is the 4–dimensional $Sym(6)$–module. As $\widetilde{C} \cap M/O_2(M_0)$ contains L, we see that $QO_2(M_0)/O_2(M_0)$ is the elementary abelian subgroup of order 2^6 in L.

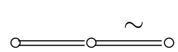
Suppose $V \leq Q$. By 1.1.2 V is not an FF–module and we conclude that $W = \langle (Y_M \cap Q)^{\widetilde{C}} \rangle$ is elementary abelian. Hence $W \leq O_2(M_0)$, i.e. $[Y_M, W] = 1$. But as $Y_M \not\leq Q$ and $[Q, Y_M] \leq W$ this contradicts $C_{\widetilde{C}}(Q) \leq Q$.

So we have $V \not\leq Q$. This now gives $V_2 = [V, Q]$ and $V_1 = [V_2, Q]$. Define $W := \langle V_2^{\widetilde{C}} \rangle$. Then V acts quadratically and nontrivially on W. Further from M_0 we see that for any $x \in V$ we have $|[W/V_1, x]| \leq 2^4$. Let L_1 be the preimage of L in M_0. Then $L_1/O_2(L_1) \cong 3Sym(6)$. Let $U = \langle V^{\widetilde{C}} \rangle Q$. Then $C_U(W)Q/Q \leq Z(U/Q)$. Hence as $|[W/V_1, x]| \leq 2^4$ for $x \in V$, we see that $[F(U/Q), V] = 1$. So there is some component U_1 of U/Q containing L_1'/Q. If U_1 is a group of Lie type defined over a field of characteristic 2 we see that it has to be $F_4(2)$ or $Sp_{2n}(2)$, for some n. But in both cases the $Sp_4(2)$–parabolic has no elementary abelian normal subgroup of order 16. So U_1 is not a group of Lie type defined over a field of characteristic 2. As VQ/Q acts quadratically an application of 1.1.1 yields that $U_1 \cong 3U_4(3)$ or $3M_{22}$. This now tells us that V_1 is normal in U and that W/V_1 involves exactly one nontrivial irreducible module, which is 12–dimensional. As $[V, Q] \leq W$, we see that $[U, Q] \leq W$. This shows that $L_1/O_2(L_1)$ possesses exactly three nontrivial chief-factors in $O_2(L_1)$, two of them 6–dimensional and one 4-dimensional. Since L_1 has a a 4–dimensional and a 6-dimensional factor in V and a 6-dimensional factor in $QO_2(M_0)/O_2(M_0)$, we get that $[F^*(M_0/O_2(M_0)), O_2(M_0)] \leq V$. This shows that $O_2(M_0) = Y_M$ and so $O_2(M_0) = V$ or $|O_2(M_0) : V| = 2$.

In both cases we get that Q is extraspecial of order 2^{13} and that $\widetilde{C}/Q$ is an automorphism group of $3M_{22}$ or $3U_4(3)$. In the former case we have $\widetilde{C}/Q \cong 3Aut(M_{22})$ and so G is of identificationtype J_4. So assume the latter case. Then we have that $\widetilde{C}/Q \cong 3U_4(3).2$, or $3U_4(3).4$. Now M_0 has a geometry with diagram



and $\widetilde{C}$ has one with diagram

.

The intersection is the geometry for L_1. Let $P \in \mathcal{P}^{\circ}_{M_0}(S)$. Then P centralizes the foursgroup on which P_0 acts nontrivially. Hence $\langle P_0, P\rangle = P_0P$. This shows that we have a geometry with diagram

and that G is of identification type $M(24)'$.

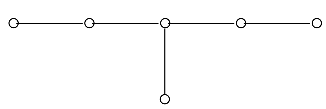
Example 2.4.16 *Let $K = F^*(M_0/O_2(M_0)) \cong \Omega^+_{10}(q)$, q a power of 2, and assume that Y_M contains V the half spin module. Then G is of identification type $E_6(q)$.*

Let $L = C_K(C_V(S))$. Then $L \sim q^{\binom{n}{2}} L_n(q)$ and $QO_2(M_0)/O_2(M_0) = O_2(L)$. Note that V has the following L–series

$$1 < V_1 < V_2 < V,$$

where $|V_1| = q$, $|V_2/V_1| = q^{10}$, $|V/V_2| = q^5$. As in 2.4.15 $V \not\leq Q$, and so $V_2 = [V, Q] = V \cap Q$. Now $|V/V \cap Q| = q^5$ and $V/V \cap Q$ is a natural module for $L_1/O_2(L_1)$, where L_1 is the pre-image of L. We can now proceed as in 2.4.15. Let U be as before, then we again see that $[F(U/Q), V] = 1$. Let U_1 be a component of U/Q containing $O^2(L_1/Q)$. Because of quadratic action and the fact that $|VQ/Q| \geq 32$, we get with 1.1.1 that U_1 is a group of Lie type in characteristic two and the list of possible groups U_1 and the corresponding modules W in Q. As L_1 induces in some W on $C_W(O_2(O^2(L_1)))$ the 10 - dimensional module, we see that W is not a $V(\lambda)$ where λ belongs to an end node of the Dynkin diagram of U_1. Hence the possible groups U_1 are $SL_n(q)$, $Sp_{2n}(q)$ or $U_n(q)$. Further for $t \in V$ we have $[W, t] \leq C_W(O_2(O^2(L_1)))$ and $C_{L_1}(t)$ has to act on this group. This shows that $|[W, t]| = q^6$ or q^4. This in the first place shows that $U_1 \cong SL_n(q)$ and then that $W = V(\lambda_2)$ or $V(\lambda_3)$. In both cases we have $U_1 \cong SL_6(q)$, as $|[W, t]| \leq q^6$. Moreover, $[W, t]$ is not the natural $C_L(t)$–module in the case $|[W, t]| = q^6$. If we have $W = V(\lambda_2)$, then $W/C_W(V)$ is a 5-dimensional L_1–module, but there is no such module in $O_2(L)$. So we have that $W = V(\lambda_3)$. Now we see that $L/O_2(L)$ induces in $QO_2(M_0)$ exactly two 10-dimensional and one 5-dimensional module, as $[V, Q] \leq \langle V_2^{\widetilde{C}}\rangle$. But in V this group induces one 10-dimensional module and one 5-dimensional one. Further in $QO_2(M_0)/O_2(M_0)$ we see another 10-dimensional module. This shows $[K, O_2(M_0)] = V$. Again $Y_M = O_2(M_0)$ and so $Y_M = V$. Now we see that $M^\circ \cong q^{16}\Omega^+_{10}(q)$ and $U \cong q^{1+20}SL_6(q)$. The intersection is the $SL_5(q)$-parabolic. Now in this case we are in the situation of 2.4.14, so any minimal parabolic not in M_0 normalizes P°.

We try to show that $H = \langle M^\circ, U\rangle$ has a parabolic system with an E_6 – diagram.



We have that $M_0 = M^\circ S$ and so there might be some field automorphism involved. But these field automorphisms are also field automorphisms on L, so they induce field automorphisms on $U/O_2(U)$. This shows that U and M^0 have a common Sylow 2–subgroup, and so G is of identificationtype $E_6(q)$.

2.4.6.1 The open "$\widetilde{P}$!,gb=1"-Problem 2

Suppose $E!$, $\operatorname{rank} G > 2$, $\mathcal{P}^\circ(S) = \{P\}$, $gb(P) = 1$ and that there exist $P_1, P_2 \in \mathcal{P}(S)$ such that for $i = 1$ and 2:

(i) $P_i \not\leq N_G(P^\circ)$.

(ii) $M_i := \langle P, P_i \rangle \in \mathcal{L}$

(iii) $\langle M_1, M_2 \rangle \notin \mathcal{L}$.

Determine the shape of M_1 and M_2.

As a starting point towards a solution of the preceding problem we observe

Lemma 2.4.17 *Suppose $E!$ and $\mathcal{P}^\circ(S) = \{P\}$. Let $\widetilde{P} \in \mathcal{P}(S)$ with $\widetilde{P} \neq P$, $L := \langle P, \widetilde{P} \rangle \in \mathcal{L}$ and $\widetilde{P} \not\leq N_G(P^\circ)$. Then $L \in \mathcal{L}^\circ$ and $L^\circ/C_{L^\circ}(Y_L) \cong SL_3(q)$, $Sp_4(q)$, $\Omega_5(q)$ (and p odd), $Alt(6)$ (and $p = 2$), or $2.M_{12}$ (and $p = 3$).*

Note that in all cases of the preceding lemma $L/O_p(L)$ has a weak BN-pair of rank 2. Hence [StTi] provides a solution to the above open problem. But we believe that our stronger assumptions allow for a shorter solution.

3 The Global Analysis

We have not yet devoted much time to this part of the project, but here are some thoughts.

The main tool to identify the group G is via a diagram geometry for a non-local parabolic subgroup H of G. Usually we will not only know the diagram but also the group induced on each of the residues and so the isomorphism type of each of the residues. This allows to identify the geometry and then the group H.

For example if the diagram is the diagram of the spherical building of rank at least four, then the isomorphism type of the residues uniquely determines the building. This follows from the classification of spherical buildings, but can actually be proved using only a small part of the theory of buildings.

For many of diagrams which we encounter, classification results are available in the literature. At this time we have not decided which of these results we will quote and which ones we will revise as part of our program.

The situation when $\mathcal{M}(S) = \{M_1, M_2\}$ is different. If (M_1, M_2) is a weak BN-pair associated to a BN-pair of rank 2 defined over a not to small field, one tries to recover the Weyl-group. For p odd, this probably requires a $\mathcal{K}$-group assumptions not only for the p-local subgroups but also for some 2-locals. Once the Weyl-group has been identified, H can be recognized as a group of Lie-type, see [BS].

Suppose (M_1, M_2) is not associated to a BN-pair of rank 2. If $p = 2$, the knowledge of the parabolic subgroups often allows to determine the order of G by counting involutions. The actual identification will be done by some ad hoc methods depending on the group. If p is odd, the group is probably better left unidentified.

After the group H is identified, one still needs to deal with situations where $H \neq G$. Usually our choice of the group H will allow us to show that $H =$ is the p-core with respect to S, but some exceptions will have to be dealt with. The strongly p-embedded situation has been discussed before. If G has rank 1 the CGT-theorem 2.2.3 will limit the structure of H. For $p = 2$ this y hopefully will lead to a contradcition, while for $p \neq 2$ we might not be able to identify G.

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