Applications of the FF-Module Theorem and Related Results

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Abstract

Let p be a prime, M a finite group with $O_p(M) = 1$ and V a faithful $\mathbb{F}_pM$-module. In this paper we investigate the structure of M and V under various assumptions related to best offenders and quadratic action.

Introduction

This paper is the last part of a series of papers ([MS1], [MS2], [MS3], [MS4]) which form together with [GM1], [GM2], [Ch], [MeSt1] and [MeSt2] the module theoretic background for the classification of the finite groups of local characteristic p outlined in [MSS1] and in particular, for the Structure Theorem [MSS2].

Most of the results we present should be of independent interest since they give further insight in the action of offenders or more generally quadratic subgroups on modules. The results we prove come in two categories:

1) Consequences of the General FF-Module Theorem, the FF-Module Theorem and the Offender Theorem (see [MS4, Theorems 1,2 and 3]): Like the Strong Dual FF-Module Theorem 3.1, the Strong FF-Module Theorem 3.2, the General Point-Stabilizer Theorem 3.6, and the Q!FF-Module Theorem 4.6.

2) Statements about modules with quadratically acting subgroups: Like the Quadratic L-Lemma 2.9, and the Minimal Asymmetric Module Theorem 5.5.

Let H be a finite group. A *p-parabolic subgroup* of H is a subgroup that contains a Sylow p-subgroup of H; and we write parabolic subgroup if it is clear which prime p is meant. In contrast, for a genuine group of Lie type the subgroups containing a Borel subgroup are called *Lie parabolic*. For the definition of a genuine group of Lie type see [MS4]. Also the definitions for the various types of offenders can be found in [MS4].

Let $T \in \mathrm{Syl}_p(H)$ and V be an $\mathbb{F}_pH$-module. Then $P_H(V,T)) := \mathrm{O}^{p'}(\mathrm{C}_H(\mathrm{C}_V(T)))$ is called the *point-stabilizer* of H on V with respect to T.

The quadratic action on modules will be in the center of the investigation. Therefore the reader should be familiar with the basic properties of quadratic action, for example that a faithfully and quadratically acting group on an $\mathbb{F}_p$-vector space is an elementary abelian p-group (see [KS, 9.1.1]).

Several of the proofs require some knowledge of genuine groups of Lie type. So the reader should also have some understanding of these groups; in particular of their Lie parabolic structure.

In the following **GFMT** stands for the General FF-Module Theorem and **FMT** for the FF-Module Theorem and the Offender Theorem in [MS4]. The last two theorems we regard as one reference, so a reference like **FMT**(7) refers to case (7) in the FF-Module Theorem, then the reader is supposed to look up the properties of the offenders for this given case in the Offender Theorem. The reader might feel the need to have the article [MS4] at hand while following the proofs.

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1 The Kieler Lemma

In this section we discuss a property of finite groups of local characteristic p that is relevant for our investigations in Section 3. Recall that for p a prime, a finite group G has characteristic p if $\mathrm{C}_G(\mathrm{O}_p(G)) \leq \mathrm{O}_p(G)$; and G has local characteristic p if every p-local subgroup has characteristic p. We start with some elementary properties of groups of (local) characteristic p.

Lemma 1.1. *Let G be a finite group. Then G has characteristic p if and only if $\mathrm{F}^*(G) = \mathrm{O}_p(G)$.*

Proof. Suppose first that G has characteristic p. Put $L = \mathrm{O}_p(G)$. Then $\mathrm{C}_G(L) \leq L$. By [KS, 6.5.7(c)], $\mathrm{E}(G) = \mathrm{E}(L)\mathrm{C}_{\mathrm{E}(G)}(L) \leq L$. Also $\mathrm{O}_{p'}(G) \leq \mathrm{C}_G(L) \leq L$, and since $\mathrm{F}(G)$ is nilpotent, $\mathrm{F}(G) = \mathrm{O}_{p'}(\mathrm{F}(G))\mathrm{O}_p(G) \leq L$. Thus $\mathrm{F}^*(G) = \mathrm{E}(G)\,\mathrm{F}(G) \leq L \leq \mathrm{F}(G) \leq \mathrm{F}^*(G)$ and $\mathrm{F}^*(G) = L$.

Suppose next that $\mathrm{F}^*(G) = \mathrm{O}_p(G)$. By [KS, 6.5.8] $\mathrm{C}_G(\mathrm{F}^*(G)) \leq \mathrm{F}^*(G)$ and so G has characteristic p. □

Lemma 1.2. *Let G be a finite group of characteristic p and $S \in \mathrm{Syl}_p(G)$. Then the following hold:*

(a) Every subnormal subgroup of G has characteristic p.

(b) Every subgroup containing S has characteristic p.

(c) G has local characteristic p.

Proof. (a): Let $N \trianglelefteq\trianglelefteq G$. Then $\mathrm{F}^*(N) \leq \mathrm{F}^*(G)$ by [KS, 6.5.7]. Now (a) follows from 1.1.

(b): Let $S \leq H \leq G$. Then $\mathrm{O}_p(G) \leq \mathrm{O}_p(H)$, and (b) follows.

(c): Let P be a p-subgroup of G with $\mathrm{C}_{\mathrm{O}_p(G)}(P) \leq P$, and let Q be a p'-subgroup of $\mathrm{C}_G(P)$. Then $\mathrm{C}_{\mathrm{O}_p(G)}(P) \leq \mathrm{C}_{\mathrm{O}_p(G)}(Q)$ and so by Thompson's $P\times Q$-Lemma [KS, 8.2.8], $Q \leq \mathrm{C}_G(\mathrm{O}_p(G)) \leq \mathrm{O}_p(G)$. Thus $Q = 1$ and $\mathrm{C}_G(P)$ is a p-group.

Now let L be a p-local subgroup of G, so $L = \mathrm{N}_G(R)$ for some non-trivial p-subgroup $R \leq G$. Set $P := \mathrm{O}_p(L)$. Since $R \leq P$,

$$C_{\mathrm{O}_p(G)}(P) \leq \mathrm{C}_{\mathrm{O}_p(G)}(R) \leq L \cap \mathrm{O}_p(G) \leq P.$$

It follows that $\mathrm{C}_G(P)$ is a p-group. Since $\mathrm{C}_L(P) \leq \mathrm{C}_G(P)$ and $\mathrm{C}_L(P) \trianglelefteq L$, we get that $\mathrm{C}_L(P) \leq P$. □

Lemma 1.3. *Let G be a finite group. Then G has characteristic p if and only if $\mathrm{O}^p(G)$ has characteristic p.*

Proof. If G has characteristic p, then $\mathrm{O}^p(G)$ has characteristic p by 1.2(a).

Suppose $\mathrm{O}^p(G)$ has characteristic p. Note that $\mathrm{O}^p(\mathrm{F}^*(G)) \leq \mathrm{O}^p(G) \cap \mathrm{F}^*(G)$ and so by [KS, 6.5.7] $\mathrm{O}^p(\mathrm{F}^*(G)) \leq \mathrm{F}^*(\mathrm{O}^p(G))$. Thus by 1.1, $\mathrm{O}^p(\mathrm{F}^*(G))$ is a p-group. Hence $\mathrm{F}^*(G)$ is a p-group and so G has characteristic p by 1.1. □

Lemma 1.4. *Let G be a finite group of local characteristic p.*

(a) Let $N \trianglelefteq\trianglelefteq G$ and L be a p-local subgroup of G. Then $N \cap L$ has characteristic p.

(b) Every subnormal subgroup of G has local characteristic p.

(c) Let $S \in \mathrm{Syl}_p(G)$ and $N \trianglelefteq G$. Then SN has local characteristic p.

Proof. (a): Let $1 \neq P \leq G$ be a p-subgroup. Then $\mathrm{N}_G(P)$ has characteristic p since G is of local characteristic p. As $\mathrm{N}_N(P)$ is subnormal in $\mathrm{N}_G(P)$ we get from 1.2(a) that $N_N(P)$ has characteristic p.

(b): This follows from (a).

(c): Let $1 \neq P \leq S$. By (a), $\mathrm{N}_N(P)$ has characteristic p. Since $O^p(NS) \leq N$, $\mathrm{O}^p(\mathrm{N}_{NS}(P)) \trianglelefteq \mathrm{N}_N(P)$. We conclude from 1.2(a) that $\mathrm{O}^p(\mathrm{N}_{NS}(P))$ has characteristic p and then by 1.3 that $\mathrm{N}_{NS}(P)$ has characteristic p. □

Lemma 1.5 (Kieler Lemma). *Let G be a finite group of local characteristic p and $S \in \mathrm{Syl}_p(G)$, and let E be a subnormal subgroup of G such that $p \big| |E|$. Then $\mathrm{C}_E(\Omega_1\mathrm{Z}(S)) = \mathrm{C}_E(\Omega_1\mathrm{Z}(S \cap E))$.*

Proof. If $E = G$, there is nothing to prove. Thus, we may assume that $E \neq G$. Put $Z := \Omega_1\mathrm{Z}(S)$ and $Z_0 := \Omega_1\mathrm{Z}(S \cap E)$. We proceed by induction on $|G/E||G|$. Let M be a maximal normal subgroup of G containing E. Then $M < G$, $S \cap M \leq \mathrm{Syl}_p(M)$, and by 1.4(b) M has local characteristic p, so by induction

$$\mathrm{C}_E(Z_0) = \mathrm{C}_E(\Omega_1\mathrm{Z}(S \cap M)).$$

If $E \neq M$, then also by induction

$$\mathrm{C}_M(\Omega_1\mathrm{Z}(S \cap M)) = \mathrm{C}_M(Z),$$

and the lemma follows. Thus, we may assume that $E = M$. By 1.4(c) ES has local characteristic p, and so by induction we may assume

1°. *E is a maximal normal subgroup of G and $G = ES$.*

Set $C := \mathrm{C}_G(Z \cap E)$. Since p divides $|E|$, $S \cap E \neq 1$, and by (1°) $S \cap E$ is normal in S, so $1 \neq Z \cap E \leq Z_0$ and $\mathrm{C}_G(Z_0) \leq C$. Note that $|C/C \cap E| \leq |G/E|$ and that $\mathrm{N}_G(Z \cap E)$ has characteristic p. Since $C \trianglelefteq \mathrm{N}_G(Z \cap E)$ also C has characteristic p by 1.2(a). Hence, if $|C| < |G|$, then by induction

$$\mathrm{C}_E(Z) = \mathrm{C}_{C\cap E}(Z) = \mathrm{C}_{C\cap E}(Z_0) = \mathrm{C}_E(Z_0),$$

and the lemma follows. Thus, we may assume $G = \mathrm{C}_G(Z \cap E)$ and so

2°. *$Z \cap E \leq \mathrm{Z}(G)$, and G has characteristic p.*

We now treat the cases $Z \leq E$ and $Z \not\leq E$ separately.

Case 1: $Z \not\leq E$.

Put $V := \Omega_1 \mathrm{Z}(\mathrm{O}_p(G))$. By (2°) $Z \leq \mathrm{C}_G(\mathrm{O}_p(G)) \leq \mathrm{O}_p(G)$ and so $Z \leq V$. By (1°) $G = EZ$ and so $V = (V \cap E)Z$. In particular, $V = (V \cap E) \times Z_1$ for some $Z_1 \leq Z$. Since Z_1 is S-invariant, Gaschütz' Theorem [KS, 3.3.2] also gives a G-invariant complement Z_2 for $V \cap E$ in V. But then $Z_2 \leq \Omega_1 \mathrm{Z}(G)$, and $G = EZ$ yields

$$Z = Z_2 \times (Z \cap E) \text{ and } S = Z_2 \times (S \cap E).$$

It follows that $Z_0 = Z \cap E$ and $\mathrm{C}_E(Z) = \mathrm{C}_E(Z_0)$, and the lemma is proved in this case.

Case 2: $Z \leq E$.

Then $Z \leq Z_0$ and by (2°)

3°. $Z \leq \mathrm{Z}(G)$.

Hence $\mathrm{C}_E(Z) = E$ and it remains to show that E centralizes Z_0. Put $V := \Omega_1 \mathrm{Z}(\mathrm{O}_p(E))$. By 1.2(a), E has characteristic p, and so $Z_0 \leq C_E(\mathrm{O}_p(E)) \leq \mathrm{O}_p(E)$. Thus $Z \leq Z_0 \leq V$ and so also $\langle Z_0^G \rangle \leq V$. We now investigate the homomorphism

$$\pi : Z_0 \longrightarrow V \text{ with } x \mapsto \prod_{U \in E/S \cap E} x^U,$$

where the product runs over all cosets $U = (S \cap E)t, t \in E$ and $x^U := x^t$. Note that x^U is well defined since $S \cap E$ centralizes x.

The following elementary properties of π are easy to check:

– $(x\pi)^s = (x^s)\pi$ for all $x \in Z_0$ and $s \in S$.

– $Z_0\pi \leq \mathrm{C}_V(E) \leq Z_0$.

– $\pi|_{\mathrm{C}_V(E)}$ is the multiplication by $|E/S \cap E|$ and thus an automorphism.

– $\mathrm{C}_{\ker \pi}(S) \leq Z \leq \mathrm{C}_V(E)$ by (3°).

Let $x \in Z_0$. By the second property, $x\pi \in \mathrm{C}_V(E)$ and so by the third $x\pi = y\pi$ for some $y \in \mathrm{C}_V(E)$. Then $y^{-1}x \in \ker \pi$ and so $Z_0 \leq \mathrm{C}_V(E) \ker \pi$. From the third property we get that $\ker \pi \cap \mathrm{C}_V(E) = 1$. and so $Z_0 = \mathrm{C}_V(E) \times \ker \pi$. Moreover, the first property implies that $\ker \pi$ is S-invariant. Now the last property shows that $\ker \pi = 1$ and $Z_0 = \mathrm{C}_V(E)$. Hence $\mathrm{C}_E(Z) = E = \mathrm{C}_E(Z_0)$ and the lemma also holds in Case 2. □

Corollary 1.6. *Let H be a finite group, $E \trianglelefteq\trianglelefteq H$, and V be a finite dimensional $\mathbb{F}_pH$-module. Then*

$$\mathrm{C}_E(\mathrm{C}_V(T)) = \mathrm{C}_E(\mathrm{C}_V(T \cap E)) = \mathrm{C}_E(\mathrm{C}_{[V,E]}(T \cap E)) \textit{ for } T \in \mathrm{Syl}_p(H).$$

Proof. Observe that the corollary holds for H if and only if it holds for $H/\mathrm{C}_H(V)$. Thus, we may assume that V is a faithful H-module.

Let $G := V \rtimes H$ be the semidirect product of V and H, $\widetilde{E} := V \rtimes E$ and $S := V \rtimes T$. Then G is of characteristic p since V is faithful, so G is also of local characteristic p by 1.2(c). Moreover,

$$\mathrm{C}_V(T) = \Omega_1\mathrm{Z}(S) \text{ and } \mathrm{C}_V(T \cap E) = \Omega_1\mathrm{Z}(S \cap \widetilde{E}).$$

Hence 1.5 implies $\mathrm{C}_E(\mathrm{C}_V(T)) = \mathrm{C}_E(\mathrm{C}_V(T \cap E))$. This is the first equality of 1.6.

Pick $R \leq \mathrm{C}_V(T \cap E)$ such that $[V, E] + \mathrm{C}_V(T \cap E) = [V, E] \oplus R$. Then Gaschütz' Theorem shows that $[V, E] + R = [V, E] + \mathrm{C}_V(E)$ and thus

$$\mathrm{C}_V(T \cap E) = \mathrm{C}_{[V,E]+\mathrm{C}_V(E)}(T \cap E) = \mathrm{C}_{[V,E]}(T \cap E) + \mathrm{C}_V(E).$$

Now also the second equality of 1.6 follows. □

Lemma 1.7. *Let H be a finite group, $E \trianglelefteq\trianglelefteq H$, and V be a finite dimensional $\mathbb{F}_pH$-module. Let $A \leq T \in \mathrm{Syl}_p(H)$ such that A normalizes E and $V_0 \leq \mathrm{C}_V(\mathrm{O}^p(E))$. Put $\overline{V} := V/V_0$. Then the following hold:*

(a) $P_E(V, T \cap E) = P_E([V, E], T \cap E) = P_E([V, \mathrm{O}^p(E)], T \cap E) = \mathrm{O}^{p'}(E \cap P_H(V, T))$.

(b) $P_{EA}(V, T \cap EA) = \mathrm{O}^{p'}(EA \cap P_H(V, T))$.

(c) $P_E(\overline{V}, T \cap E) \leq P_E(V, T \cap E) \leq P_H(V, T)$ *and* $\mathrm{O}_p(P_H(V, T)) \cap E \leq \mathrm{O}_p(P_E(\overline{V}, T \cap E))$.

Proof. Note that for any two subgroups X and Y of a finite group, $\mathrm{O}^{p'}(X \cap Y) = \mathrm{O}^{p'}(X \cap \mathrm{O}^{p'}(Y))$. In particular,

$$\mathrm{O}^{p'}(\mathrm{C}_E(\mathrm{C}_V(T)) = \mathrm{O}^{p'}(E \cap P_H(V, T)).$$

(a): By 1.6 $\mathrm{O}^{p'}(\mathrm{C}_E(\mathrm{C}_V(T)) = P_E(V, T \cap E) = P_E([V, E], T \cap E)$ and so

$$P_E(V, T \cap E) = P_E([V, E], T \cap E) = \mathrm{O}^{p'}(E \cap P_H(V, T)).$$

From $P_E(V, T \cap E) = P_E([V, E], T \cap E)$ and induction on i we have $P_E(V, T \cap E) = P_E([V, E, i], T \cap E)$ for all $i \in \mathbb{Z}^+$, where $[V, E, 0] := V$ and $[V, E, i] := [[V, E, i-1], E]$ for $i > 0$. Since E acts nilpotently on $V/[V, \mathrm{O}^p(E)]$ and since $E/\mathrm{C}_E(V/[V, E, i])$ is a p-group we have $[V, \mathrm{O}^p(E)] = [V, E, k]$ for some $k \in \mathbb{Z}^+$. Thus

$$P_E(V, T \cap E) = P_E([V, E, k], T \cap E) = P_E([V, \mathrm{O}^p(E)], T \cap E).$$

and so (a) holds.

(b): We again apply 1.6 to $E \trianglelefteq EA$ and $E \trianglelefteq\trianglelefteq H$. Then

$$\mathrm{C}_{EA}(\mathrm{C}_V((T \cap E)A)) = A\mathrm{C}_E(\mathrm{C}_V((T \cap E)A)) = A\mathrm{C}_E(\mathrm{C}_V(T \cap E)) = A\mathrm{C}_E(\mathrm{C}_V(T)) = \mathrm{C}_{EA}(\mathrm{C}_V(T)),$$

and so

$$\begin{aligned} P_{EA}(V, (T \cap E)A) &= \mathrm{O}^{p'}(\mathrm{C}_{EA}(\mathrm{C}_V((T \cap E)A))) = \mathrm{O}^{p'}(\mathrm{C}_{EA}(\mathrm{C}_V(T))) \\ &= \mathrm{O}^{p'}(EA \cap P_H(V, T)). \end{aligned}$$

(c): By (a) $P_E(V, T \cap E) \leq P_H(V, T)$ and so $\mathrm{O}_p(P_H(V, T)) \cap E$ is normal in every subgroup of $P_E(V, T \cap E)$ containing $T \cap E$. Thus, it suffices to show $P := P_E(\overline{V}, T \cap E) \leq P_E(V, T \cap E)$.

Clearly $\mathrm{O}^p(P)$ centralizes $\mathrm{C}_V(T \cap E) + V_0/V_0$ and V_0. Hence $\mathrm{O}^p(P)$ centralizes $\mathrm{C}_V(T \cap E)$. From $P = \mathrm{O}^p(P)(T \cap E)$ we conclude that $P \leq \mathrm{C}_E(\mathrm{C}_V(T \cap E))$ and $P = \mathrm{O}^{p'}(P) \leq P_E(V, T \cap E)$. □

Lemma 1.8. *Let G be a finite group and N an abelian normal subgroup of G with $N \cap \Phi(G) = 1$. Then there exists a complement to N in G.*

Proof. If $|N| = 1$ lemma clearly holds. So suppose that $N \neq 1$. Then there exists a maximal subgroup M of G with $N \not\leq M$. Note that $N \cap M$ is normal in $G = NM$. Moreover, $(N \cap M) \cap \Phi(G) \leq N \cap \Phi(G) = 1$ and so by induction there exists a complement K to $N \cap M$ in G. Then $G = (N \cap M)K$ and so $M = (N \cap M)(M \cap K)$. Thus

$$G = NM = N(N \cap M)(M \cap K) = N(M \cap K)$$

and $N \cap (M \cap K) = (N \cap M) \cap K = 1$. Hence $M \cap K$ is a complement to N in G. $\square$

Lemma 1.9. *Let G be a finite group and π be a set of primes. Suppose that $\mathrm{O}_\pi(G) = 1$. Then $\Phi(G) = \Phi(\mathrm{O}^\pi(G))$.*

Proof. Put $L := \mathrm{O}^\pi(G)$. Since $L \trianglelefteq G$, $\Phi(L) \leq \Phi(G)$. Since $\Phi(G)$ is nilpotent and $\mathrm{O}_\pi(G) = 1$, $\Phi(G)$ is a π'-group and so $\Phi(G) \leq L$. Put $\overline{G} := G/\Phi(L)$ and $N := \Phi(G)$. Then $\Phi(\overline{L}) = 1$ and $\Phi(\overline{G}) = \overline{N}$. Note that $\Phi(\overline{N}) \leq \Phi(\overline{L}) = 1$ and since $\overline{N}$ is nilpotent, we conclude that $\overline{N}$ is abelian. Also $\overline{N} \cap \Phi(\overline{L}) = 1$, and so by 1.8 applied to $\overline{N}$ and $\overline{L}$ in place of N and G, there exists a complement to $\overline{N}$ in $\overline{L}$. Since $\overline{N}$ and $\overline{G}/\overline{L}$ are coprime and $\overline{N}$ is abelian, Gaschütz' theorem implies that there exist a complement to $\overline{N}$ in $\overline{G}$. Since $\overline{N} = \Phi(\overline{G})$ this gives $\overline{N} = 1$ and so $\Phi(G) = N = \Phi(L)$. $\square$

2 Quadratic Modules

Lemma 2.1. *Let H be a group and A, B, C be subgroups of H with $[A, B, C] = 1$.*

(a) If C normalizes $[B, C, A]$, then $[C, A, B] \leq [B, C, A]$.

(b) If C normalizes $[B, C, A]$ and $[A, C, B]$, then $[C, A, B] = [B, C, A]$.

Proof. Recall first that $[A, B] = [B, A]$ and $[A, B] \trianglelefteq \langle A, B\rangle$.

(a) Without loss $H = \langle A, B, C\rangle$. Since $[A, B] \trianglelefteq \langle A, B\rangle$ and C centralizes $[A, B]$ we conclude that $[A, B] \trianglelefteq H$ and that $\langle C^H\rangle$ centralizes $[A, B]$. In particular, $[[A, B], [B, C]] = 1$. Since B normalizes $[B, C]$ we get

$$\langle [B, C, A]^B\rangle = [B, C, \langle A^B\rangle] = [[B, C], A[A, B]]] = [B, C, A].$$

Thus B normalizes $[B, C, A]$. Note that also A and C normalize $[B, C, A]$ and so $[B, C, A] \trianglelefteq H$. Put $\overline{H} = H/[B, C, A]$. Then $[\overline{A}, \overline{B}, \overline{C}] = 1$ and $[\overline{B}, \overline{C}, \overline{A}] = 1$. Thus $[\overline{C}, \overline{A}, \overline{B}] = 1$ by the Three Subgroups Lemma. So (a) holds.

(b) By (a) $[C, A, B] \leq [B, C, A]$, and by (a) with the roles of A and B interchanged, $[C, B, A] \leq [A, C, B]$. Since $[B, C, A] = [C, B, A]$ and $[C, B, A] = [B, C, A]$ this gives $[A, C, B] = [C, A, B]$. $\square$

Suppose that $[A, B, C] = 1$ for some subgroups A, B, C of H. Then the assumption in (a) holds if A is a normal subgroup of H or if C is an abelian normal subgroups of H. Similarly the assumption in (b) hold if A and B are normal subgroups of H or if C is an abelian normal subgroup of H.

We remark that in (a) $[C, A, B]$ might be a proper subgroup of $[B, C, A]$ (and so the conclusion in (b) no longer has to hold if C does not normalize $[C, A, B]$.) Indeed let $H = \mathrm{Sym}(4)$, $A = \langle (12)(34)\rangle$, $B = \langle (12)\rangle$ and $C = \langle (23)\rangle$. Then $[A, B] = 1$ and so also $[A, B, C] = 1$. We have $[C, A] = \langle (23)(14)\rangle$, $[C, A, B] = \langle (12)(34)\rangle$, $[B, C] = \langle (123)\rangle$ and $[B, C, A] = \langle (12)(34), (13)(23)\rangle = \mathrm{O}_2(H)$. So $[B, C, A]$ is normal in H and $[C, A, B]$ is a proper subgroup of $[B, C, A]$.

Lemma 2.2. *Let M be a group, $\mathbb{K}$ a field, V an $\mathbb{K}M$-module and $A \leq M$. Suppose that M is generated by n conjugates of A in M.*

(a) Suppose that $\dim_{\mathbb{K}} V/\mathrm{C}_V(A) \leq r$. Then $\dim_{\mathbb{K}} V/\mathrm{C}_V(M) \leq nr$. If, in addition V is a non-trivial simple $\mathbb{K}M$-module, then $\dim_{\mathbb{K}} V \leq nr$.

(b) Suppose that $\dim_{\mathbb{K}}[V, A] \leq r$. Then $\dim_{\mathbb{K}}[V, M] \leq nr$. If, in addition V is a non-trivial simple $\mathbb{K}M$-module, then $\dim_{\mathbb{K}} V \leq nr$.

Proof. (a) Let $A_1, \dots A_n$ be conjugates of A in M with $M = \langle A_i \mid 1 \leq i \leq n\rangle$. Then

$$\bigcap_{i=1}^{n} \mathrm{C}_V(A_i) \leq \mathrm{C}_V(M)$$

and $\dim_{\mathbb{K}} V/\mathrm{C}_V(A_i) \leq r$. Thus $\dim_{\mathbb{K}} V/\mathrm{C}_V(M) \leq rn$. If V is a non-trivial simple $\mathbb{K}M$-module, then $\mathrm{C}_V(M) = 0$ and so (a) is proved.

(b) follows from (a) applied the $\mathbb{K}$-dual of V. □

Lemma 2.3. *Let M be a genuine group of Lie-type in characteristic p, V a simple $\mathbb{F}_pM$-module, $\mathbb{K} := \mathrm{End}_M(V)$, $S \in \mathrm{Syl}_p(M)$ and $1 \neq z \in \Omega_1\mathrm{Z}(S)$. Suppose that one of the following holds:*

(i) There exists $P \leq M$ with $S \leq P$ and $[V, z, \mathrm{O}_p(P)\mathrm{O}^p(P)] = 0$.

(ii) $[V, z, S] = 0$.

(iii) $[V, z]$ is 1-dimensional over $\mathbb{K}$.

Then $M \cong \mathrm{SL}_n(q), \mathrm{Sp}_n(q)$ or $\mathrm{SU}_n(q)$, and V is a corresponding natural module.

Proof. Suppose that (i) holds. Then $[V, z, \mathrm{O}_p(P)] = 0$ and so $[V, z] \leq \mathrm{C}_V(\mathrm{O}_p(P))$. By Smith's Lemma [MS4, 4.2] $\mathrm{C}_V(\mathrm{O}_p(P))$ is a simple $\mathbb{K}P$-module and thus $[V, z]$ is a simple $\mathbb{K}P$-module. Hence $[V, z, \mathrm{O}^p(P)] = 0$ implies that $[V, z, P] = 0$ and so $\dim_{\mathbb{K}}[V, z] = 1$.

Suppose (ii) holds. Then $[V, z] \leq \mathrm{C}_V(S)$. By Steinberg 's Lemma [MS4, 4.1] $\dim_{\mathbb{K}} \mathrm{C}_V(S)$ is 1-dimensional. Thus $\dim_{\mathbb{K}}[V, z] = 1$. and (ii) implies (iii). We have proved that (i) and (ii) imply (iii). So we may assume from now on that (iii) holds.

Without loss M is universal. Let $\overline{\mathbb{K}}$ be the algebraic closure of $\mathbb{K}$ and $\overline{V} = \overline{\mathbb{K}} \otimes_{\mathbb{K}} V$. Let $(\overline{M}, \sigma)$ be a σ-setup for M, see [GLS3, Definition 2.2.1]. So $M = \mathrm{O}^{p'}(\mathrm{C}_{\overline{M}}(\sigma))$ and by [GLS3, 2.8.2], $\overline{V}$ can be extended to rational $\overline{\mathbb{K}}\overline{M}$-module. Then S lies in maximal unipotent subgroup $\overline{S}$ of $\overline{M}$ and $1 \neq z \in \Omega_1\mathrm{Z}(\overline{S})$.

Put $\overline{B} = \mathrm{N}_{\overline{M}}(\overline{S})$. Then $\overline{B}$ is a Borel subgroup of $\overline{M}$ and since $\dim[\overline{V}, z]$ is 1-dimensional (over $\overline{\mathbb{K}}$), $[\overline{V}, z, \overline{S}] = 0$. Since by Steinberg 's Lemma [MS4, 4.1] $\mathrm{C}_{\overline{V}}(\overline{S})$ is 1-dimensional we conclude that $[\overline{V}, z] = \mathrm{C}_{\overline{V}}(\overline{S})$. Thus also $[\overline{V}, \langle z^{\overline{B}}\rangle] = \mathrm{C}_{\overline{V}}(\overline{S})$. Note that $\langle z^{\overline{B}}\rangle$ contains a root subgroup $\overline{Z}$ of $\overline{M}$. Then $\overline{Z} \leq \Omega_1\mathrm{Z}(\overline{S})$ and $[\overline{V}, \overline{Z}] = \mathrm{C}_V(\overline{S})$ is 1-dimensional. Observe that one of the following holds:

(a) $\mathrm{Z}(\overline{S})$ is a long root subgroup.

(b) $p = 3$ and $\overline{M} \cong G_2(\overline{\mathbb{K}})$, or $p = 2$, $\overline{M} \cong F_4(\overline{\mathbb{K}})$ and $\mathrm{Z}(\overline{S})$ is the product of a long and a short root subgroup.

(c) $p = 2$, $\overline{M} \cong B_n(\overline{\mathbb{K}})$ or $\overline{M} \cong C_n(\overline{\mathbb{K}})$ and $\mathrm{Z}(\overline{S})$ is the product of a long and a short root subgroup.

In case (b) we may apply the graph automorphism of $\overline{M}$, if necessary, and assume that $\overline{Z}$ is a long root subgroup. In case (c), since there exists an automorphism from $B_n(\overline{\mathbb{K}})$ to $C_n(\overline{\mathbb{K}})$ sending long root groups to short root groups and vice versa, we also may assume that $\overline{Z}$ is a long root subgroup. Thus in any case $\overline{Z}$ is a long root subgroup.

Suppose $\overline{M} = A_{n-1}(\overline{\mathbb{K}}) = \mathrm{SL}_n(\overline{\mathbb{K}})$. Then $\overline{M}$ is generated by n-conjugates of $\overline{Z}$ (see for example [MS4, 5.3]) and so by 2.2 $m := \dim_{\overline{\mathbb{K}}} \overline{V} \leq n$. By dimension reason we conclude that $m = n$ and the image of image of $\overline{M}$ in $\mathrm{GL}_{\overline{\mathbb{K}}}(\overline{V})$ is equal to $\mathrm{SL}_{\overline{\mathbb{K}}}(\overline{V})$. So $\overline{V}$ is a natural module for $\overline{M}$. We conclude that for $M = \mathrm{SL}_n(q)$ or $M = \mathrm{SU}_n(q)$, V is a corresponding natural module.

Suppose that $\overline{M} \cong C_n(\overline{\mathbb{K}}) = \mathrm{Sp}_{2n}(\overline{\mathbb{K}})$, $n \geq 2$. Let $\overline{P}^*$ the minimal Lie-parabolic subgroup of $\overline{M}$ with $\overline{B} \leq \overline{P}^*$ and $\overline{Z} \not\trianglelefteq \overline{P}^*$. Put $\overline{P} := \mathrm{O}^{p'}(\overline{P}^*)$. Then $\overline{P}/\mathrm{O}_p(\overline{P}) \cong \mathrm{SL}_2(\overline{\mathbb{K}})$. Let $\overline{K}$ be a Levi complement to $\mathrm{O}_p(\overline{P})$ in $\overline{P}$ and $\overline{R} = \overline{S} \cap \overline{K}$.

Note that $\overline{R}$ is a short root subgroup of $\overline{M}$ and so not conjugate to $\overline{Z}$. Nevertheless we will show that $\overline{R}$ acts quadratically on $\overline{V}$. For this let $\overline{N}$ be the natural $\mathrm{Sp}_{2n}(\overline{\mathbb{K}})$-module for $\overline{M}$ and $\overline{U} := [\overline{N}, \overline{R}]$. Then $[\overline{N}, \overline{Z}]$ is 1-dimensional, $\overline{U}$ is a 2-dimensional singular subspace of $\overline{N}$ and $[\overline{N}, \overline{Z}] \leq \overline{U}$. Put $\overline{D} := \mathrm{C}_{\overline{M}}(\overline{U}^{\perp})$. Then $\overline{D}$ is abelian and $\overline{Z}\,\overline{R} \leq \overline{D} \leq \overline{S}$. Thus $[\overline{V}, \overline{Z}, \overline{D}] = 0$ since $[\overline{V}, \overline{Z}] = \mathrm{C}_{\overline{V}}(\overline{S})$. It follows from [MS4, 3.4] that $\overline{D} = \langle \overline{Z}^{\overline{P}} \rangle$ and so $[\overline{V}, \overline{D}, \overline{D}] = \langle [\overline{V}, \overline{Z}, \overline{D}]^P \rangle = 0$. Thus $\overline{D}$ and so also $\overline{R}$ acts quadratically on $\overline{V}$.

By Smith's Lemma [MS4, 4.2], $\overline{W} := \mathrm{C}_{\overline{V}}(\mathrm{O}_p(\overline{P}))$ is a simple $\overline{\mathbb{K}P}$-module. Also $[\overline{W}, \overline{R}] \leq \mathrm{C}_{\overline{W}}(\overline{R}\mathrm{O}_p(\overline{P})) = C_{\overline{W}}(S)$ and so $[\overline{W}, R]$ is 1-dimensional. Hence $\overline{W}$ is a natural $\mathrm{SL}_2(\overline{\mathbb{K}})$-module for $\overline{P}$. Put $\overline{L} = \mathrm{O}^{p'}(\mathrm{N}_{\overline{M}}(\overline{Z}))$. Then $\overline{L}$ centralizes $[\overline{V}, \overline{Z}]$, and Smiths' Lemma implies that $C_{\overline{V}}(\mathrm{O}_p(\overline{L})) = \mathrm{C}_{\overline{V}}(\overline{L})$. Together with the Ronan-Smith's Lemma [MS4, 4.3] we conclude that $\overline{V}$ is natural $\mathrm{Sp}_{2n}(\overline{\mathbb{K}})$-module for $\overline{M}$. If $M \cong \mathrm{Sp}_{2n}(q)$ we conclude that V is a natural $\mathrm{Sp}_{2n}(q)$-module. Suppose $M \cong {}^2B_2(q)$. Then z is not a root element in $\overline{M}$ and so $[\dim V, z] = 2$, a contradiction.

Suppose now that $\overline{M} \cong G_2(\overline{\mathbb{K}})$ or $B_3(\overline{\mathbb{K}})$. Let $r = 2$ and 3, respectively. Let Φ be a root system for $\overline{M}$, N/H the corresponding Weyl-group and Φ_l the set of long root in Φ. Then Φ_l is a root system of type A_r. Also there exists $t \in N \setminus H$ with $tH \in Z(N/H)$ and t induces a graph automorphism on Φ_l and so also on the subgroup $\overline{L}$ of $\overline{M}$ generated by long root subgroups corresponding to Φ_l. Let U be a non-trivial composition factor for $\overline{\mathbb{K}}\,\overline{L}\langle t \rangle$ on $\overline{V}$. Since $[\overline{V}, \overline{Z}]$ is 1-dimensional, U is a simple $\overline{\mathbb{K}}\,\overline{L}$-module. Hence by the $A_r(\overline{\mathbb{K}})$-case treated above, U is natural $\mathrm{SL}_{r+1}(\overline{\mathbb{K}})$ module for $\overline{L}$. But this contradicts the action of t on $\overline{L}$.

Suppose $\overline{M} \cong B_n(\overline{\mathbb{K}})$ or $D_n(\overline{\mathbb{K}})$ for $n \geq 4$. Then $\overline{M}$ has subgroup isomorphic to $\mathrm{B}_3(\overline{\mathbb{K}})$ generated by long root subgroups, a contradiction to the $B_3(\overline{\mathbb{K}})$-case.

Suppose that $\overline{M} \cong E_r(\overline{\mathbb{K}})$, $6 \leq r \leq 8$. Then $\overline{M}$ has subgroups isomorphic to $D_{r-1}(\overline{\mathbb{K}})$ generated by long root subgroups, a contradiction to the $D_n(\overline{\mathbb{K}})$-case.

Suppose that $\overline{M} \cong F_4(\overline{\mathbb{K}})$. Then the long roots form a root system of type $D_4(\overline{\mathbb{K}})$, a contradiction to the $D_n(\overline{\mathbb{K}})$-case. $\square$

Definition 2.4. *Let L be a group and $A \leq L$. Then L is called A-minimal if $L = \langle A^L \rangle$ and A is contained in a unique maximal subgroup of L.*

This definition has been motivated by the L-Lemma in [PPS], which says that in a p-minimal group H every p-subgroup $A \leq H$ with $A \not\leq \mathrm{O}_p(H)$ is contained in a subgroup L which is $A\mathrm{O}_p(H)$-minimal.

Lemma 2.5. *Let L be a finite group, $A \leq L$ and suppose that L is A-minimal. Let M be the maximal subgroup of L containing A.*

(a) Let $K \trianglelefteq\trianglelefteq L$ with $A \leq K$. Then $K = L$.

(b) M is not subnormal in L and $\mathrm{N}_L(M) = M$.

(c) $L = \langle A, A^x \rangle$ for all $x \in L \setminus M$.

(d) A is not subnormal in L. In particular, $A \nleq \mathrm{F}(L)$, $A \neq 1$, $A \neq L$, and L is not nilpotent.

(e) If $B \lneq L$ with $A \leq B$, then L is B-minimal and M is the unique maximal subgroup containing B .

(f) If $K \trianglelefteq L$ with $L \neq KA$, then $K \leq M$ and L/K is AK/K-minimal with M/K being the maximal subgroup containing AK/K.

(g) If $K \trianglelefteq L$ such that L/K is nilpotent, then $L = KA$ and so $K \nleq M$.

(h) If A is a p-group, then L is p-minimal, M contains the normalizer of a Sylow p-subgroup and $\mathrm{O}_p(L) \leq \bigcap M^L$.

Proof. (a): Suppose that $K \neq L$. Then there exists a proper normal subgroup H of L with $K \leq H$. But then $L = \langle A^L \rangle \leq H$, a contradiction.

(b): By (a) M is not subnormal in L. Thus $M \leq \mathrm{N}_L(M) \lneq L$ and since M is a maximal subgroup of L, $\mathrm{N}_L(M) = M$.

(c): By (b), $M \neq M^x$. Since M (M^x) is the unique maximal subgroups of L containing A (A^x), we conclude that $\langle A, A^x \rangle$ is not contained in any maximal subgroup of L. Thus $L = \langle A, A^x \rangle$.

(d): If $A \trianglelefteq\trianglelefteq L$, then by (a) $L = A$ and so A is not contained in any maximal subgroup of L, which contradicts the definition of A-minimal. Since subgroups of nilpotent groups are subnormal, $A \nleq \mathrm{F}(L)$ and L is not nilpotent.

(e): Since $A \leq B \lneq L$ we have $B \leq M$ and M is the unique maximal subgroup of L containing B. Also $L = \langle A^L \rangle \leq \langle B^L \rangle$ and so $L = \langle B^L \rangle$. Thus L is B-minimal.

(f): Put $\overline{L} = L/K$. Since $KA \neq L$, $KA \leq M$ and so (e) shows that L is KA-minimal with $KA \leq M$. Hence $\overline{M}$ is the unique maximal subgroup of $\overline{L}$ containing $\overline{A}$. Since $L = \langle A^L \rangle$ implies $\overline{L} = \langle \overline{A}^{\overline{L}} \rangle$, $\overline{L}$ is $\overline{A}$-minimal.

(g): By (d) L/K is not AK/K-minimal. Hence (f) implies $L = KA$.

(h) Let S be a Sylow p-subgroups of L containing A. Then $A \trianglelefteq\trianglelefteq S$ and so by (d), $S \ntrianglelefteq L$. Thus $\mathrm{N}_L(S) \leq M$ and $S \neq L$. By (e), L is S-minimal and so L is p-minimal. Also $\mathrm{O}_p(L) \leq S \leq M$ and hence $\mathrm{O}_p(L) \leq \bigcap M^L$. □

Lemma 2.6. *Let H be a finite group. Suppose H has a quasisimple normal subgroup K such that $K/\mathrm{Z}(K)$ is a sporadic simple group or $\mathrm{U}_4(3)$. Then H is not 2-minimal and so H is not A-minimal for any 2-subgroup A of H.*

Proof. Replacing H by $H/\mathrm{C}_H(K)$ we may assume that $\mathrm{C}_H(K) = 1$. Then K is simple and H is isomorphic to a subgroup of $\mathrm{Aut}(K)$ containing K. Let $S \in \mathrm{Syl}_2(H)$. Since in all cases $\mathrm{Out}(K)$ is a 2-group, $H = KS$.

Suppose that K is a sporadic simple group. Then the list of maximal subgroups of K (for example in the ATLAS) shows that there exist distinct S-invariant maximal parabolic subgroups M_1, M_2 of K. Then M_1S, M_2S are distinct maximal subgroups of H and H is not 2-minimal.

Suppose $K \cong \mathrm{U}_4(3)$. Then $K \cap S$ is contained in exactly three maximal subgroup P_1, P_2, P_3 of K. Moreover, we can choose notation such that $P_1/Q_1 \cong \mathrm{Sym}(3) \times \mathrm{Sym}(3)$, $P_2/Q_2 \cong P_3/Q_3 \cong \mathrm{Alt}(6)$ and $P_2 \cap P_3 \nleq P_1$, where $Q_i = \mathrm{O}_p(P_i)$. If S normalizes P_2 and P_3, then P_2S and P_3S are distinct maximal subgroups of H. If S interchanges P_2 and P_3, then P_1S and $(P_2 \cap P_3)S$ are distinct maximal subgroups of H. So again H is not 2-minimal.

We have proved that H is not 2-minimal and thus by 2.5(h), H is also not A-minimal for any 2-subgroup A of H. $\square$

The next lemma establishes some basic properties of the group $L/\mathrm{O}_p(L)$ if L is A-minimal for a p-subgroup A. For $\mathcal{CK}$-groups L we will give the precise structure of $L/\mathrm{O}_p(L)$ in 2.9 under the hypothesis that L acts faithfully and A quadratically on an $\mathbb{F}_pL$-module V. Moreover, in 2.10 we will give the structure of such a module V under the additional hypothesis that V is simple and one of the cases 2.9(1) and (2) holds.

Lemma 2.7. *Let L be a finite group with $\mathrm{O}_p(L) = 1$, and let $1 \neq A \leq L$ be a p-subgroup. Suppose that L is A-minimal with M being the maximal subgroup of L containing A. Then the following hold, where $D := \bigcap M^L$ and $H := \mathrm{O}^p(L)$:*

(a) $H \nleq M$ and so $L = HA$ and $H \nleq D$.

(b) If $|A| = 2$, then $L \cong \mathrm{D}_{r^n}$, where r is an odd prime and $n \in \mathbb{Z}^+$.

(c) Let $N \trianglelefteq L$. Then either $N \leq D$, or $L = NA$, $H \leq N$ and $N \nleq M$.

(d) D is a p'-group, and $D = \Phi(L) = \Phi(H)$.

(e) $\mathrm{C}_L(a) \leq M$ for $1 \neq a \in \mathrm{Z}(A)$.

(f) H/D is the unique minimal normal subgroup of L/D.

(g) Suppose A is elementary abelian. Then one of the following holds:

1. *H/D is simple and $p \big| |H/D|$.*
2. *$|A| = p$, H is q-group, and H/D is an elementary abelian q-group for some prime $q \neq p$.*

Proof. (a): L/H is a p-group and so nilpotent. Thus (a) follows from 2.5(g).

(b): By 2.5(c) L is generated by two conjugates of A and so is a dihedral group of order $2m$, and m is the power of an odd prime since L is A-minimal.

(c): If $N \leq M$, then $N \leq D$ since $N \trianglelefteq L$. So suppose $N \nleq M$. Then $L = NA$, L/N is a p-group and $H = \mathrm{O}^p(L) \leq N$.

(d): Let $Q \in \mathrm{Syl}_p(D)$ with $A \leq \mathrm{N}_L(Q)$. Then by the Frattini argument $L = D\mathrm{N}_L(Q)$. Since $D \leq M$ this implies $\mathrm{N}_L(Q) \nleq M$ and so $L = \mathrm{N}_L(Q)$. Hence $Q \leq \mathrm{O}_p(L) = 1$ and D is a p'-group.

Suppose that there exists a maximal subgroup $\widetilde{M}$ of L not containing D. Then $L = \widetilde{M}D$, so $\widetilde{M}$ contains a Sylow p-subgroup of L. Thus we may assume that $A \leq \widetilde{M}$ and so $M = \widetilde{M}$ and $D \leq \widetilde{M}$, a contradiction. This shows that $D \leq \Phi(L)$. Conversely $\Phi(L) \leq M$ and so $D = \Phi(L)$. Since $\mathrm{O}_p(L) = 1$, 1.9 gives $\Phi(L) = \Phi(\mathrm{O}^p(L)) = \Phi(H)$, and (d) is proved.

(e): Observe that $A \leq \mathrm{C}_L(a)$ and $\mathrm{C}_L(a) \neq L$.

(f): By (d) $D \leq H$ and by (a), $H/D \neq 1$. Hence (f) is a direct consequence of (c).

(g): Assume first that $D = 1$. By (f) H is a minimal normal subgroup of L. So either H is an elementary abelian q-group (q a prime, $q \neq p$) or H is a direct product of non-abelian simple groups. In the first case $|A| = p$ since otherwise (e) and coprime action imply that $H = \langle \mathrm{C}_H(a) \mid 1 \neq a \in A\rangle \leq M$, a contradiction to (a). Thus (g:2) holds in this case.

In the second case let K be a component of H. Note that any A-invariant Sylow subgroup of H is contained in M and so by coprime action H is not a p'-group. By (a) we have $L = HA$. Thus

$H = \langle K^L \rangle = \langle K^A \rangle$ and K is not a p'-group. In particular, if A normalizes K we are done since then (g:1) holds. Thus we may assume that there exists $a \in A$ with $K^a \neq K$. Put

$$E := \langle \prod_{t \in \langle a \rangle} k^t \mid k \in K \rangle.$$

Then $E \leq \mathrm{C}_L(a)$ and by (e) $E \leq M$. On the other hand, by 2.5(h) M contains a Sylow p-subgroup of L and since K is not a p'-group, $M \cap K \neq 1$. It follows that

$$[M \cap K, E] = [M \cap K, K] = K \leq M.$$

Hence $H = \langle K^A \rangle \leq M$, which contradicts (a). So (g) holds if $D = 1$.

Assume next that H/D is a q-group for some prime q (not necessarily distinct from p). Then $H = DQ$ for $Q \in \mathrm{Syl}_q(H)$ and since $D = \Phi(H)$, $H = Q$. So H is a q-group and since $\mathrm{O}_p(L) = 1$, $q \neq p$.

In the general case we conclude that H/D is not a p-group, and since by (f) H/D is the unique minimal normal subgroup of L/D, $\mathrm{O}_p(L/D) = 1$. If $|AD/D| = p$, then also $|A| = p$ since by (d), $A \cap D = 1$. By 2.5(f), $\overline{L} := L/D$ is $\overline{A}$-minimal with $\overline{M}$ being the maximal subgroup containing $\overline{A}$. In particular, $\bigcap \overline{M}^{\overline{L}} = 1$ and we can apply (g) to $\overline{L}$. We conclude that (g) also holds for L. □

Notation 2.8. *A finite group L is called a $\mathcal{CK}$-group provided that each composition factor of L is isomorphic to one of the known finite simple groups.*[1]

Lemma 2.9 (**Quadratic L-Lemma).** *Let L be a $\mathcal{CK}$-group and V be a faithful $\mathbb{F}_pL$-module. Suppose that there exists $A \leq L$ such that A acts quadratically on V and that L is $A\mathrm{O}_p(L)$-minimal. Then one of the following holds for $\overline{L} = L/\mathrm{O}_p(L)$:*

1. *$\overline{L} \cong \mathrm{SL}_2(p^k)$.*

2. *$p = 2$ and $\overline{L} \cong \mathrm{Sz}(2^k)$, $k > 1$.*

3. *$p = 2$ and $\overline{L} \cong \mathrm{D}_{r^k}$, r an odd prime.*

Proof. Observe that the faithful quadratic action of A implies that A is an elementary abelian p-subgroup of L. Since by coprime action $\mathrm{O}_p(L)$ is the intersection of the centralizers in L of the non-trivial factors of an L-composition series of V, we also may replace V by the direct sum of the non-trivial L-composition factors on V and L by $\overline{L}$, so

1°. $\mathrm{O}_p(L) = 1$, *and V is a direct sum of non-trivial simple $\mathbb{F}_pL$-modules. In particular, $\mathrm{C}_V(L) = 0$ and $V = [V, L]$.*

We use the following notation: M is the unique maximal subgroup of L containing A, $D := \bigcap M^L$, $H := \mathrm{O}^p(L)$, and $A \leq T \in \mathrm{Syl}_p(L)$. Recall from 2.5(h) that $\mathrm{N}_L(T) \leq M$ and from 2.7(d) that

2°. $D = \Phi(L) = \Phi(H)$ *is a p'-group.*

Suppose that $|A| = 2$. Then 2.7(b) implies (3). Thus we may assume:

3°. $|A| > 2$.

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[1] This notion is weaker than that of a $\mathcal{K}$-group often used in the literature since it does not assume anything about sections.

By 2.5(e) we may assume without loss that A is a maximal quadratic subgroup of L, so

4°. $A = \mathrm{C}_L([V,A]) \cap \mathrm{C}_L(V/[V,A])$.

Next we prove:

5°. *$[V,a] = [V,A] = \mathrm{C}_V(A) = \mathrm{C}_V(a)$ for all $1 \neq a \in A$ and $|V| = |[V,b]|^2$ for all quadratic elements $b \in L^{\#}$.*

Let $1 \neq b$ be a quadratic element in L. Since D is a p'-group there exists $g \in L$ with $b^g \notin M$. Then $L = \langle A, b^g \rangle$ and so by (1°) $V = [V,L] = [V,A] + [V,b^g]$ and $\mathrm{C}_V(A) \cap \mathrm{C}_V(b^g) = \mathrm{C}_V(L) = 0$. Since $[V,A] \leq \mathrm{C}_V(A)$ and $[V,b] \leq \mathrm{C}_V(b)$ we conclude that $V = [V,A] \oplus [V,b^g]$, $\mathrm{C}_V(A) = [V,A]$ and $\mathrm{C}_V(b) = [V,b]$. In particular,

$$|[V,b]| = |V/C_V(b)| = |V/[V,b]| \text{ and } |V| = |[V,b]|^2.$$

For $a = b \in A$ we also get

$$[V,a] \leq [V,A] \leq \mathrm{C}_V(A) \leq \mathrm{C}_V(a) = [V,a].$$

Now (5°) follows.

6°. *$A \cap A^h = 1$ for all $h \in L \setminus \mathrm{N}_L(A)$.*

Let $h \in L$ and $1 \neq b \in A \cap A^h$. By (5°) $[V,A] = [V,b] = [V,A^h]$, and so by (4°) $A = A^h$.

7°. *$\mathrm{C}_L(a) \leq \mathrm{N}_L(A)$ for all $1 \neq a \in A$*

This follows from (6°).

8°. *$A \cap M^h = 1$ for every $h \in L \setminus M$.*

Let $h \in L$ with $B := A \cap M^h \neq 1$, and let $g \in L$ such that $B \leq T^g \leq M^h$. Then $\mathrm{C}_V(T^g) \leq \mathrm{C}_V(B)$ and so by (5°) $\mathrm{C}_V(T^g) \leq \mathrm{C}_V(A)$. Since $\mathrm{C}_V(L) = 0$ by (1°) we get $\langle A, T^g \rangle \neq L$. Thus $T^g \leq M$ and so $A^g \leq T^g \leq M \cap M^h$. Since A^g lies in a unique maximal subgroup of L this gives $M = M^h$ and so by 2.5(b), $h \in M$, a contradiction.

9°. *If V is is not a simple $\mathbb{F}_pL$-module, then (1) or (2) hold.*

Suppose that V is not simple. Let W be any simple $\mathbb{F}_pL$-submodule of V and put $E = \mathrm{C}_L(W)$. Since $\mathrm{C}_V(L) = 0$, also $\mathrm{C}_V(H) = 0$ and so $[W,H] \neq 0$. Hence, $H \not\leq E$ and by 2.7(c)

$$E \leq D \leq \Phi(L) \leq \mathrm{O}_{p'}(L).$$

In particular, E is a p'-group and A acts faithfully on W. By induction on $\dim V$ and by (3°) we get that $L/E \cong \mathrm{SL}_2(q)$ or $\mathrm{Sz}(q)$, $q = p^k > 2$. It follows that $\Phi(L/E) \leq \mathrm{Z}(L/E)$ and so $[\Phi(L), L] \leq E = \mathrm{C}_L(W)$. Since this holds for all simple $\mathbb{F}_pL$-submodules of V, (1°) and the faithful action of L on V yield $[L, \Phi(L)] = 1$. Thus $E \leq \Phi(L) \leq \mathrm{Z}(L)$. By 2.5, $L = L'A$ and since E is a p'-group, so $E \leq L'$. By [Gr] the p'-part of the Schur multipliers of $\mathrm{SL}_2(q)$ and $\mathrm{Sz}(q)$ is trivial. Hence $E = 1$ and (1) or (2) holds.

10°. *Suppose that p is odd. Then (1) holds.*

By (5°)

$$\dim[V,A] = \min\{\dim[V,b] \mid 1 \neq b \in L, [V,b,b] = 0\}$$

and by 2.5(c) L is generated by two conjugates of A. Observe that A acts quadratically on V satisfying the above minimality condition for $\dim[V,A]$. By (9°) we may assume that V is a simple $\mathbb{F}_pL$-module. Hence [Ho1, 4.3] (for $|A| = 3$), [Ho2, 4.2] (for $p = 3$ and $|A| > 3$) and [Ho3, 2.6] (for $p > 3$) show that $L \cong \mathrm{SL}_2(p^k)$.

We may assume from now on that $p = 2$.

11°. *$D = Z(L)$.*

Since by (2°) D is a $2'$-group, (3°) and coprime action show that

$$D = \langle \mathrm{C}_D(a) \mid 1 \neq a \in A \rangle \leq \mathrm{N}_L(A).$$

Then $[D,A] \leq D \cap A = 1$ and thus $D \leq \mathrm{Z}(L)$ since $L = \langle A^L \rangle$. Conversely, $\mathrm{Z}(L) \leq D$, and (11°) follows.

12°. *H is quasisimple.*

By (11°) and (2°) $\mathrm{Z}(L) = D = \Phi(H) \leq H$. Put $K := H'$. Since $H/\mathrm{Z}(L)$ is simple by 2.7(g) and (3°), we conclude that $H = \mathrm{Z}(L)K$ and K is quasisimple. Since $L/\mathrm{Z}(L)K$ is a 2-group, L/K is nilpotent and so by 2.5(e) $L = KA$ and $H \leq K$; in particular, $H = K$ and H is quasisimple.

13°. *Suppose that $p = 2$. Then (1) or (2) holds.*

Put $\tilde{L} := L/\mathrm{Z}(L)$. Recall that H is quasisimple by (12°). We discuss the possibilities for H using that $\tilde{H}$ is a known simple group.

Suppose that $\tilde{H}$ is a group of Lie-type over a field in characteristic 2. Put $S := T \cap H$ and let Δ be the set of minimal Lie-parabolic subgroups of $\tilde{H}$ containing $\tilde{S}$. Since $H \not\leq M$, we can choose $\tilde{P} \in \Delta$ with $\tilde{P} \not\leq \tilde{M}$. Then $\tilde{L} = \langle \tilde{P}, \tilde{A} \rangle$ and so $\tilde{H} = \langle \tilde{P}^{\tilde{A}} \rangle$. It follows that A acts transitively on Δ and $|\Delta| \leq 2$. In particular, $\mathrm{N}_L(S)$ is a maximal subgroup of L containing A, and so $M = \mathrm{N}_L(S)$.

Suppose that $\tilde{H} \cong \mathrm{SL}_2(2^k)$. If $A \leq H$, then (1) holds. So suppose that $A \not\leq H$ and let $a \in A \setminus S$. Since $\mathrm{Out}(\tilde{H})$ is cyclic and A is elementary abelian, $|AH/H| = 2$, $k = 2l$ for some $l \in \mathbb{Z}^+$, and S is elementary abelian with $|\mathrm{C}_S(a)| = |\mathrm{C}_S(A)| = 2^l$. Thus all involutions in $T \setminus S$ are conjugate under S. Since S acts regularly on $S^L \setminus \{S\}$, a Frattini argument gives $M = S(M \cap M^x)$ and $S \cap M^x = 1$, where $x \in L \setminus M$. Thus $T = S(T \cap M^x)$ and $T \cap M^x$ has order 2. Since all involutions in $T \setminus S$ are conjugate under S, we can choose x such that $A \cap M^x$ has order 2. But this contradicts (8°).

Suppose next that $\tilde{H} \cong \mathrm{Sz}(2^k)$. Then H has no outer automorphism of order 2, so $A \leq H$ and (2) holds.

Finally suppose that $\tilde{H} \cong \mathrm{U}_3(2^k), \mathrm{L}_3(2^k)$ or $\mathrm{Sp}_4(2^k)$. Assume that $A \cap \mathrm{Z}(S) \neq 1$. Then $S \leq \mathrm{C}_L(\mathrm{Z}(S) \cap A) \leq \mathrm{N}_L(A)$ by (7°), so $[S,A]$ is elementary abelian. The action of T on S shows that $A \leq S \leq H$. Hence $H = L$, and since A acts transitively on Δ, $|\Delta| = 1$ and $L/\mathrm{Z}(L) \cong \mathrm{U}_3(2^k)$. It follows that $A \leq \Omega_1(S) = \Omega_1\mathrm{Z}(S)$, and there exists $U \leq L$ with $A \leq \Omega_1\mathrm{Z}(S) \leq U$ and $U \cong \mathrm{SL}_2(2^k)$. But then $U \not\leq M = \mathrm{N}_L(S)$ and $U = L$, a contradiction.

We have shown that $A \cap \mathrm{Z}(S) = 1$. Suppose that $A \cap H \neq 1$ and let $1 \neq a \in A \cap H$. Since all involutions of H are 2-central, $\mathrm{C}_H(a)$ contains a Sylow 2-subgroup S_0 of H. Now (7°) implies $S_0 = S$ since $M = N_L(S)$, and so $a \in A \cap \mathrm{Z}(S) = 1$, a contradiction. Hence $A \cap H = 1$. On the other hand $|A| \geq 4$ by (3°), and so, since the Sylow 2-subgroups of $\mathrm{Out}(\mathrm{U}_3(2^k))$ are cyclic, $\tilde{H} \cong \mathrm{L}_3(2^k)$ or $\mathrm{Sp}_4(2^k)$ and $A_0 := \mathrm{C}_A(\Delta) \neq 1$.

Let R be a root subgroup of H contained in $\mathrm{Z}(S)$. Then $X := \mathrm{C}_R(A_0) \neq 1$ and again by (7°), $[X, A] \leq H \cap A = 1$. Thus $A \leq \mathrm{C}_L(X) \leq \mathrm{N}_L(R)$. This rules out the $\mathrm{Sp}_4(q)$-case, so $H/\mathrm{Z}(H) \cong \mathrm{L}_3(q)$. It follows that $R = \mathrm{Z}(S)$ and $|A/\mathrm{C}_A(R)| = 2$. But then again by (7°), $1 \neq [R, A] \leq [\mathrm{C}_H(\mathrm{C}_A(R)), A] \leq A \cap H = 1$ a contradiction.

Suppose now that $\tilde{H}$ is not a group of Lie-Type over a field in characteristic 2. Since L is A-minimal, 2.6 shows that $\tilde{H}$ is not a sporadic group. If $\tilde{H}$ is a group of Lie Type over a field of odd characteristic, then by [MeSt1] $\tilde{H} \cong \mathrm{U}_4(3)$. But then L is not 2-minimal by 2.6, again a contradiction.

It remains to consider the case $\tilde{H} \cong \mathrm{Alt}(n)$, $n \geq 7$. Since L is 2-minimal, $n = 2^k + 1$, with $k \geq 3$ and $\widetilde{M \cap H} \cong \mathrm{Alt}(2^k)$. Let X be a non-trivial orbit for A on $\{1, \ldots, n\}$. If $|X| \neq 2^k$, then $A \leq \mathrm{N}_L(X) \not\leq M$, a contradiction. So $|X| = 2^k$. Since A is elementary abelian we conclude that A acts regularly on X, so $|A| = 2^k$ and $\tilde{A} \leq \tilde{H}$. Moreover, $\tilde{A}$ is uniquely determined in $\tilde{H}$ up to conjugation under $\mathrm{Sym}(n)$. It follows that $\tilde{A}$ is contained in a subgroup $\tilde{H}_0$ of $\tilde{H}$ isomorphic to $\mathrm{L}_2(2^k)$. But $\tilde{H}_0 \not\leq \tilde{M}$. This final contradiction completes the proof of 2.9. □

Lemma 2.10. *Let $L \cong \mathrm{SL}_2(q)$ or $\mathrm{Sz}(q)$, $q = p^k$, where $p = 2$ in the latter case, and let V be a non-trivial simple $\mathbb{F}_pL$-module. Suppose that L is A-minimal for some $A \leq L$ with $[V, A, A] = 0$. Then V is a corresponding natural module.*

Proof. Put $\mathbb{K} := \mathrm{End}_L(V)$. Then $\mathbb{K}$ is (isomorphic to) a subfield of $\mathbb{F}_q$. Put $W := \mathbb{F}_q \otimes_{\mathbb{K}} V$. Then W is a simple $\mathbb{F}_qL$-module and $[W, A, A] = 0$. Let N be the natural $\mathbb{F}_qL$-module. Then N is the only non-trivial basic module for L and so by Steinberg's Tensor Product Theorem [St, 1.31], $W \cong \bigotimes_{\sigma \in \Sigma} N^\sigma$, where Σ is a subset of $\mathrm{Aut}(\mathbb{F}_q)$. Since V is non-trivial, $\Sigma \neq \emptyset$.

Suppose for a contradiction that $|\Sigma| \geq 2$. Then $q > 2$ and since L is A-minimal, $|A| \geq 3$. Then by [MS3, 6.5] $|\Sigma| = 2$. Let $\Sigma = \{\mu, \sigma\}$. Replacing W by $W^{\mu^{-1}}$ we may assume that $\mu = \mathrm{id}_{\mathbb{F}_q}$. By [MS3, 4.9] A acts λ-dependent on $N^\mu = N$ and on N^σ, where λ is a homomorphism from A to $(\mathbb{F}_q, +)$. Since A acts λ-dependent on N, A acts $\lambda \circ \sigma$-dependent on N^σ. Thus $\lambda = \lambda \circ \sigma$. Let $\mathbb{F}_{q_0}$ be the fixed-field of σ. It follows that A is contained in a subgroup of L isomorphic to $\mathrm{SL}_2(q_0)$ and $\mathrm{Sz}(q_0)$, respectively, which contradicts the assumption that L is A-minimal.

Thus $\Sigma = \{\sigma\}$ for some σ in $\mathrm{Aut}(\mathbb{F})$. Since N is a simple $\mathbb{F}_pL$-module we conclude that W, N^σ, N and V all are isomorphic as $\mathbb{F}_pL$-modules. □

Lemma 2.11. *Let H be a finite group, $L \trianglelefteq H$ and M a parabolic subgroup of H with $H = LM$. Let V be a finite $\mathbb{F}_pH$-module and W an $\mathbb{F}_pM$-submodule of V. Suppose that $[X/Y, L] = 0$ whenever X/Y is a simple $\mathbb{F}_pH$-section of V with $X \cap W \not\leq Y$. Then $[W, \mathrm{O}^p(L)] = 0$, and W is an H-submodule of V.*

Proof. Note that $L = (M \cap L)\mathrm{O}^p(L)$ since $H = LM$ and M is parabolic. Hence $[W, \mathrm{O}^p(L)] = 0$ implies that W is an H-module. Thus it suffices to show that $[W, \mathrm{O}^p(L)] = 0$. We do this using induction on $\dim V$.

We may assume $V \neq 0$ and so V has a minimal (non-zero) H-submodule U. Then $W + U/U$ and V/U in place of W and V satisfy the hypothesis. Thus by induction $W + U$ is a H-submodule of V and $[W, \mathrm{O}^p(L)] \leq U$.

Suppose first that $W \cap U \neq 0$. As U is a simple $\mathbb{F}_pH$-section of V, we conclude that $[U, \mathrm{O}^p(L)] = 0$. Thus $[W, \mathrm{O}^p(L)] = [W, \mathrm{O}^p(L), \mathrm{O}^p(L)] \leq [U, \mathrm{O}^p(L)] = 0$.

Suppose next that $W \cap U = 0$. Again since M is parabolic, Gaschütz' Theorem shows that there exists an H-submodule D in $W + U$ with $W + U = D \oplus U$. Since $[W + U, \mathrm{O}^p(L)] \leq U$ we get that

$[D, \mathrm{O}^p(L)] \le D \cap U = 0$. On the other hand $W + U/D$ is a simple $\mathbb{F}_pH$-section with $W \not\le D$, so by our hypothesis $[W, \mathrm{O}^p(L)] \le D$. Now similarly as above

$$[W, \mathrm{O}^p(L)] = [W, \mathrm{O}^p(L), \mathrm{O}^p(L)] \le [D, \mathrm{O}^p(L)] = 0.$$

□

The next two results describes situations in which dual offenders arise in a natural way. Compare this, for example, with the case $c = 1$ in the qrc-Lemma [MS5, 4.6].

Lemma 2.12. *Let H be a finite group, $T \in \mathrm{Syl}_p(H)$ and $R := [\mathrm{O}_p(H), \mathrm{O}^p(H)]$, and let V be an $\mathbb{F}_pH$-module and Y be a T-submodule of V with $V = \langle Y^H \rangle \ne Y$. Then one of the following holds:*

1. *$[V, R] = 0$ and $\mathrm{C}_{\mathrm{O}_p(H)}(Y) \trianglelefteq H$.*
2. *R is a non-trivial strong dual offender on Y.*
3. *There exist $\mathrm{O}_p(H)\mathrm{O}^p(H)$-submodules $Z_1 \le X_1 \le Z_2 \le X_2$ such that for $i = 1, 2$, X_i/Z_i is a non-trivial simple $\mathrm{O}^p(H)$-module and $X_i \cap Y \not\le Z_i$.*

Proof. Suppose first that $[Y, R] = 0$. Since R is normal in H and $V = \langle Y^H \rangle$ we get $[V, R] = 0$. Moreover, $[\mathrm{C}_{\mathrm{O}_p(H)}(Y), \mathrm{O}^p(H)] \le R \le \mathrm{C}_{\mathrm{O}_p(H)}(Y)$ and so $H = T\mathrm{O}^p(H)$ normalizes $\mathrm{C}_{\mathrm{O}_p(H)}(Y)$. Thus (1) holds in this case. Hence, we may assume now:

1°. $[Y, R] \ne 0$.

Let W be an $\mathrm{O}_p(H)\mathrm{O}^p(H)$-submodule of V. We want to apply 2.11 with $H_0 := \mathrm{O}_p(H)\mathrm{O}^p(H)$, $O^p(H)$ and $T \cap \mathrm{O}_p(H)\mathrm{O}^p(H)$ in place of H, L and M. If $[V, \mathrm{O}^p(H)] \not\le W$, then also $[Y, \mathrm{O}^p(H)] \not\le W$ since $V = \langle Y^H \rangle$. Hence 2.11, applied to V/W and $Y + W/W$ in the roles of V and Y, gives an H_0-chieffactor X_2/Z_2 with $W \le Z_2$ such that $[X_2, O^p(H)] \not\le Z_2$ and $Y \cap X_2 \not\le Z_2$. If also $[Y \cap W, \mathrm{O}^p(H)] \ne 0$, then 2.11, applied to W and $Y \cap W$ in the roles of Y, gives an H_0-chieffactor X_1/Z_1 of W such that $[X_1, O^p(H)] \not\le Z_1$ and $Y \cap X_2 \not\le Z_1$. Hence (3) holds. Thus we may assume:

2°. *$[V, \mathrm{O}^p(H)] \le W$ or $[Y \cap W, \mathrm{O}^p(H)] = 0$ for every $\mathrm{O}_p(H)\mathrm{O}^p(H)$-submodule W.*

For $W_0 := [V, \mathrm{O}_p(H)]$ this either gives $[V, \mathrm{O}^p(H)] \le W_0$ or $[W_0 \cap Y, \mathrm{O}^p(H)] = 0$. In the first case $Y + W_0 = V$ since $V = \langle Y^H \rangle = \langle Y^{\mathrm{O}^p(H)} \rangle$, and so $V/Y = [V/Y, \mathrm{O}_p(H)]$. The nilpotent action of $\mathrm{O}_p(H)$ on V/Y gives $V = Y$, a contradiction. Hence

3°. $[W_0 \cap Y, \mathrm{O}^p(H)] = 0$.

In particular, $[Y, \mathrm{O}_p(H)] \le \mathrm{C}_Y(\mathrm{O}^p(H)) \le \mathrm{C}_Y(R)$. By (1°) we can pick $y \in Y \setminus \mathrm{C}_Y(R)$. It follows that

4°. $[y, \mathrm{O}_p(H)] \le \mathrm{C}_Y(\mathrm{O}^p(H)) \le \mathrm{C}_Y(R)$.

Put $U := \langle y^{H_0} \rangle$. Then $[Y \cap U, \mathrm{O}^p(H)] \ne 0$ and thus by (2°):

5°. *$[V, \mathrm{O}^p(H)] \le U$. In particular, $V = Y + U$ and $[V, R] \le Y + [U, R]$.*

By (4°) $[y, R] = [\langle y^{\mathrm{O}_p(H)}\rangle, R] = \langle [y, R]^{\mathrm{O}_p(H)}\rangle$ and $[y, R] \leq \mathrm{C}_Y(\mathrm{O}^p(H))$. Thus

$$[y, R] = \langle [y, R]^{H_0}\rangle = [\langle y^{H_0}\rangle, R] = [U, R].$$

Hence by (5°), $[[V, \mathrm{O}^p(H)], R] \leq [U, R] = [y, R] \leq Y$, and again by (5°), $[V, R] \leq Y$ and so $[V, R] \leq W_0 \cap Y$. Now (3°) implies $[V, R, \mathrm{O}^p(H)] = 0$, and 2.1 gives $[\mathrm{O}^p(H), R, V] = [V, \mathrm{O}^p(H), R]$. Since by coprime action $R = [R, \mathrm{O}^p(H)]$,

$$[R, V] = [\mathrm{O}^p(H), R, V] = [V, \mathrm{O}^p(H), R] \leq [y, R].$$

But then

$$[y, R] \leq [V, R] \leq [y, R].$$

We have shown that $[Y, R] = [y, R]$ for all $y \in Y \setminus \mathrm{C}_Y(R)$, so R is a strong dual offender on Y, and (2) holds. □

Corollary 2.13. *Let H be a finite group, V a $\mathbb{F}_pH$-module, M a maximal parabolic subgroup of H, Y an M-submodule of V, and $R := [\mathrm{O}_p(H), \mathrm{O}^p(H)]$. Suppose that there exists a p-subgroup $A \leq M$ such that*

(i) A acts quadratically on V,

(ii) H is $A\mathrm{O}_p(H)$-minimal, and

(iii) $V = \langle Y^H\rangle$ and $Y \neq V$.

Then one of the following holds:

1. *$[V, R] = 0$ and $\mathrm{C}_{\mathrm{O}_p(H)}(Y) \trianglelefteq H$.*
2. *R is a non-trivial dual offender on Y.*
3. *Let $g \in H \setminus M$ and A_0 be an elementary abelian p-subgroup of $H/\mathrm{O}_p(H)$ of maximal order. Then $|Y^g/\mathrm{C}_{Y^g}(A)| \geq |A_0|^2 \geq |A\mathrm{O}_p(H)/\mathrm{O}_p(H)|^2$.*

Proof. Since M is parabolic, M contains a Sylow p-subgroup T of H. In particular, Y is a T-submodule and $\mathrm{O}_p(H) \leq M$. Hence M is the unique maximal subgroup of H containing $A\mathrm{O}_p(H)$, and we can apply 2.12. If 2.12(1) or (2) holds, then (1) or (2) holds. So we may assume that 2.12(3) holds. Choose $X_i, Z_i, 1 \leq i \leq 2$, with the properties given in 2.12(3), and pick $g \in H \setminus M$.

Case 1. $|A_0| = p$.

Then $|A/A\cap\mathrm{O}_p(H)| \leq |A_0| = p$ by the maximality of A_0, so $|A/A\cap\mathrm{O}_p(H)| = p$ since $A \not\leq \mathrm{O}_p(H)$ by (ii). Put $B := A^{g^{-1}}$ and let H_0 be minimal in H with $B \leq H_0$ and $H = \mathrm{O}_p(H)H_0$. Replacing A by a conjugate under $\mathrm{O}_p(H)$ we may assume that $A \in B^{H_0}$. Pick $a \in A \setminus \mathrm{O}_p(H)$. Then

$$\langle a, B\rangle\mathrm{O}_p(H) = \langle A, A^{g^{-1}}\rangle\mathrm{O}_p(H) = H,$$

so $H_0 = \langle a, B\rangle$ by the minimality of H_0.

If $|Y/\mathrm{C}_Y(B)| \geq p^2$, then (3) holds. So we may assume that $|Y/\mathrm{C}_Y(B)| \leq p$. Put $D := \mathrm{C}_Y(H_0)$, and let E be maximal in Y with $[E, a] \leq D$. Then $H_0 = \langle a, B\rangle$ normalizes $\mathrm{C}_E(B)$. Since A and B are conjugate in H_0 we conclude that A centralizes $\mathrm{C}_E(B)$ and so $\mathrm{C}_E(B) = D$. From $|E/\mathrm{C}_E(B)| \leq$

$|Y/\mathrm{C}_Y(B)| = p$ we conclude $|E/D| \le p$. Since $E/D = \mathrm{C}_{Y/D}(a)$ and a acts quadratically on Y/D, we get $[Y, a] \le E$ and

$$|Y/D| = |\mathrm{C}_{Y/D}(a)||[Y/D, a]| \le |E/D|^2 \le p^2.$$

Put $L := \mathrm{O}_p(H)\mathrm{O}^p(H)$ and note that $L = \mathrm{O}_p(L)(L \cap H_0)$. Thus X_i/Z_i is a non-trivial simple $L \cap H_0$-module. It follows that $X_i \cap D \le Z_i$ for $i = 1, 2$. Since $X_1 \cap Y \not\le Z_1$, this gives

1°. $X_1 \cap Y \not\le D$.

Moreover, if $X_2 \cap Y \le (Z_2 \cap Y) + D$ then $X_2 \cap Y = (Z_2 \cap Y) + (X_2 \cap D) \le Z_2$, a contradiction. Thus

2°. $Y \ne (Z_2 \cap Y) + D$,

and $D < D + (X_1 \cap Y) \le D + (Z_2 \cap Y) < Y$. Hence $|Y/D| = p^2$, $|E/D| = p$, and $E = [Y, A] + D = \mathrm{C}_Y(A)$. In particular, there are $p+1$ hyperplanes of Y containing D, one of which is $\mathrm{C}_Y(A)$. Hence A is transitive on the other p hyperplanes, one of which is $\mathrm{C}_Y(B)$. Therefore,

3°. $Y = \mathrm{C}_Y(A) \cup \bigcup \mathrm{C}_Y(B)^A = \bigcup_{F \in A^{H_0}} \mathrm{C}_Y(F)$.

Set $U := \bigcap Z_2^H$. By (2°) and (3°) there exists $F \in A^{H_0}$ with $[(Z_2 \cap Y) + D, F] = 0$. Hence

$$Z_2 \cap Y \le \bigcap Z_2^F = \bigcap Z_2^{LF} = \bigcap Z_2^H = U \le Z_2$$

and so

4°. *$Z_2 \cap Y = U \cap Y$; in particular $Z_2 \cap Y$ is M-invariant.*

Assume that $[Z_2 \cap Y, B] = 0$. Then by (4°), $Z_2 \cap Y$ is invariant under $H = \langle M, B\rangle$. So $Z_2 \cap Y \le D$, a contradiction to $X_1 \cap Y \le Z_2 \cap Y$ and (1°).

Thus $[Z_2 \cap Y, B] \ne 0$. Since $Z_2 \cap Y = U \cap Y$ and $|Y/\mathrm{C}_Y(B)| = p$ this gives $Y = (U \cap Y) + \mathrm{C}_Y(B)$. It follows that $Y + U = \mathrm{C}_Y(B) + U$ is invariant under $\langle M, B\rangle = H$, and $\langle B^H\rangle$ centralizes $Y + U/U$. Thus

$$[X_2 \cap Y, \mathrm{O}^p(H)] \le [Y, \mathrm{O}^p(H)] \le U \le Z_2,$$

a contradiction since $X_2 \cap Y \not\le Z_2$ and X_2/Z_2 is a non-trivial simple $\mathrm{O}^p(H)$-module. This completes the analysis of (Case 1).

Case 2. $|A_0| > p$.

Put $q := |A_0|$. Then by 2.9 $H/\mathrm{O}_p(H) \cong \mathrm{SL}_2(q)$ or $\mathrm{Sz}(q)$, and M is the normalizer of a Sylow p-subgroup of H. In particular $H = \mathrm{O}_p(H)\mathrm{O}^p(H)$ and by 2.10 X_i/Z_i is a natural module. Set $D_i := \mathrm{C}_{(Y \cap X_i)+Z_i/Z_i}(\mathrm{O}_p(M))$. Then D_i has order q. Since $H = \langle \mathrm{O}_p(M), B\rangle$ we have $\mathrm{C}_{D_i}(B) = 0$. Thus $Y \cap X_i/(Y \cap Z_i) + \mathrm{C}_{Y \cap X_i}(B)$ has order at least q and $|Y/\mathrm{C}_Y(B)| \ge q^2$, so (3) holds. This treats (Case 2) and the lemma is proved. □

3 Applications of the FF-Module Theorem

The purpose of this section is to derive some useful corollaries from **GFMT** and **FMT**. One idea is to eliminate some or most of the cases of **FMT** assuming that the offenders or the modules have particular properties. The properties we come up with play an important role in the classification program of the finite simple groups of characteristic p, but are also of independent interest. A typical example is the Strong Dual Offender Theorem below which can be applied in the situation described in 2.12.

As already mentioned in the introduction, we assume the reader to familiar with the notation used in **GFMT** and **FMT**.

Theorem 3.1 (**Strong Dual FF-Module Theorem).** *Let M be a finite $\mathcal{CK}$-group such that $K := \mathrm{F}^*(M)$ is quasisimple, and let V be a faithful simple $\mathbb{F}_pM$-module. Suppose that $A \leq M$ is a strong dual offender on V and $M = \langle A^M \rangle$. Then one of the following holds, where q is a power of p:*

1. *$M \cong \mathrm{SL}_n(q)$ or $\mathrm{Sp}_{2n}(q)$, and V is a corresponding natural module.*
2. *$p = 2$, $M \cong \mathrm{Alt}(6)$ or $\mathrm{Alt}(7)$, V is a spin-module of order 2^4 and $A \cong \langle (12)(34), (13)(24) \rangle$. (Note that in the $\mathrm{Alt}(6)$ case, V might also be viewed as an natural $\mathrm{Alt}(6)$-module with $A \cong \langle (12)(34), (34)(56) \rangle$).*
3. *$p = 2$, $M \cong \mathrm{O}^{\epsilon}_{2n}(2)$ or $\mathrm{Sym}(n)$, V is the corresponding natural module, and $|A| = 2$.*

Proof. By [MS4, 1.5(c)] A is a quadratic best offender on V. Hence $M = \mathrm{J}_M(V)$ and K is a J-component of M, so M satisfies the hypothesis of **FMT**. Let V^* be the dual of V. By [MS4, 1.5 (c)], A is also an offender on V^*. By [MS4, 1.8(a) -(c)] $|V^*/\mathrm{C}_{V^*}(A)| = |[V, A]|$ and so

$$(*) \qquad |[V, A]| \leq |A|.$$

We now discuss the cases given in **FMT**.

Case FMT (1),(2): $M \cong \mathrm{SL}_n(q)$ or $\mathrm{Sp}_{2n}(q)$, and V is a natural module.

Then (1) holds.

Case FMT (3): $M \cong \mathrm{SU}_n(q)$, and V a is natural module.

Let f be the unitary form on V (with corresponding field automorphism α) left invariant by M. Pick $0 \neq u \in [V, A]$ and choose $v \in V$ such that $f(u, v) \neq 0$ and $f(u, v) + f(u, v)\alpha \neq 0$. Observe that by [MS4, 3.2(c)] $\mathrm{C}_V(A) = [V, A]^\perp$, so $v \notin \mathrm{C}_V(A)$. Since A is a strong dual offender, there exists $a \in A$ such that $[v, a] = u$. Hence

$$f(v, v) = f(va, va) = f(v + u, v + u) = f(v, v) + f(u, u) + f(v, u) + f(v, u)\alpha.$$

It follows that $f(u, u) \neq 0$. On the other hand, by [MS4, 3.2(e)] $[V, A]$ is isotropic and thus $f(u, u) = 0$, a contradiction.

Case FMT (4): $M \cong \Omega_n(q)^\epsilon$ or $p = 2$ and $M \cong \mathrm{O}^{\epsilon}_{2n}(q)$, and V is a natural module.

Let h be the quadratic form on V left invariant by M and s be the corresponding symmetric form. Suppose there exists a singular $0 \neq u \in [V, A]$. Choose $v \in V$ such that $s(u, v) \neq 0$. As in the previous case there exists $a \in A$ such that $[v, a] = u$. Hence

$$h(v) = h(va) = h(v + u) = h(v) + h(u) + s(v, u) = h(v) + s(v, u),$$

so $s(v, u) = 0$, a contradiction.

We have shown that $[V, A]$ is totally non-singular. On the other hand, by [MS4, 3.2(e)] $[V, A]$ is isotropic. Hence [MS4, 3.1] shows that $\dim_{\mathbb{K}}[V, A] = 1$, and by [MS4, 3.4(e)] $|A| = 2$. As A is an offender, also $q = 2$, and (3) holds.

Case FMT (5): $p = 2$, $M \cong \mathrm{G}_2(q)$ and V is a natural module.

Let $A \leq T \in \mathrm{Syl}_2(M)$. We have $|A| = |V/\mathrm{C}_V(A)| = |\mathrm{C}_V(A)| = q^3$, $\mathrm{C}_V(A) = [V, A]$, and $\mathrm{Z}(T) \leq \mathrm{C}_G(A) \leq A$. Note that $\mathrm{N}_M(A)$ and $\mathrm{N}_M(\mathrm{Z}(T))$ are distinct maximal Lie parabolic subgroups of M containing T. Thus $\mathrm{N}_M(\mathrm{Z}(T))$ does not normalize $\mathrm{C}_V(A)$ and so $\mathrm{C}_V(\mathrm{Z}(T)) \neq \mathrm{C}_V(A)$. Hence A is not a strong offender. On the other hand, A is a strong dual offender with $|[V, A]| = |\mathrm{C}_V(A)| = q^3 = |A|$ and so by [MS4, 1.5(d)], A is a strong offender, a contradiction.

Case FMT (6): $M \cong \mathrm{SL}_n(q)$, $n \geq 5$, and V is the exterior square of a natural module.

In this case $[V, A] = \mathrm{C}_V(A)$ and $|V/\mathrm{C}_V(A)| = |A| = q^{n-1}$. Thus $|[V, A]| = \frac{q^{\binom{n}{2}}}{q^{n-1}} = q^{\binom{n-1}{2}}$ and so by $(*)$, $q^{\binom{n-1}{2}} \leq q^{n-1}$, a contradiction to $n \geq 5$.

Case FMT (7): $M \cong \mathrm{Spin}_7(q)$, and V is a spin module.

Then $|A| \geq q^4$ and there exists an M-invariant quadratic form on V. By the orthogonal case treated above $|A| = 2$, a contradiction.

Case FMT (8): $M \cong \mathrm{Spin}^+_{10}(q)$, and V is a half-spin module.

A similar argument as in the $\mathrm{G}_2(q)$-case gives a contradiction.

Case FMT (9): $M \cong 3{}^{\cdot}\mathrm{Alt}(6)$ and $|V| = 2^6$.

We have $|V/\mathrm{C}_V(A)| = |A| = 4$ and $|[V, A]| = 16$, a contradiction to $(*)$.

Case FMT (10): $M \cong \mathrm{Alt}(7)$ and $|V| = 2^4$.

In this case (2) holds.

Case FMT (11),(12): $M \cong \mathrm{Alt}(n)$ or $\mathrm{Sym}(n)$, and V is a natural module.

Let W be the permutation module with basis $(v_1, v_2, \ldots, v_n)$, $\tilde{W} := W/\mathrm{C}_W(M)$ and $W_0 := [W, M]$. Then $V \cong \tilde{W_0}$. So A is a strong dual offender on $\tilde{W_0}$.

Suppose first that $n \neq 2k$. Then $[v_1 + v_n, A] = \langle v_1 + v_2 \rangle$ and so $[\tilde{W_0}, A] = [\tilde{v_1} + \tilde{v_n}, A] = \langle \tilde{v_1} + \tilde{v_2} \rangle$. Thus $k = 1$ and (3) holds.

Suppose next that $n = 2k$. Then we are in one of the cases (12) (1) – (4) of **FMT**. In case (12) (3) A does not act quadratically on V, so this case is not possible.

Assume case (12)(4). Then $n = 8 = |A|$. So $M \cong \Omega_6^+(2)$ or $\mathrm{O}_6^+(2)$ and V is the corresponding natural module, so the Case **FMT** (4) treated above gives a contradiction.

Thus, we are in case (12)(1) or (12)(2). In both cases all orbits of A on $\{1, 2, \ldots, n\}$ have length 1 or 2. Say the non-trivial orbits of A are $\{1, 2\}, \{3, 4\}, \ldots, \{2k-1, 2k\}$. Then $[\tilde{W}_0, A] = [\tilde{v_1} + \tilde{v_3}, A] \leq \langle \tilde{v_1} + \tilde{v_2}, \tilde{v_3} + \tilde{v_4} \rangle$. It follows that $k = 3$ and $n = 6$. If $A \leq M'$ then $M \cong \mathrm{Alt}(6)$ and (2) holds; and if $A \not\leq M'$ then $M \cong \mathrm{Sym}(6) \cong \mathrm{Sp}_4(2)$ and (1) holds. $\square$

Theorem 3.2 (Strong FF-Module Theorem). *Let M be a finite $\mathcal{CK}$-group such that $K := \mathrm{F}^*(M)$ is quasisimple, and let V be a faithful simple $\mathbb{F}_pK$-module. Suppose that $A \leq M$ is a strong offender on V and $M = \langle A^M \rangle$. Then one of the following holds, where q is a power of p:*

1. *$M \cong \mathrm{SL}_n(q)$ or $\mathrm{Sp}_{2n}(q)$ and V is a corresponding natural module.*

2. *$p = 2$, $M \cong \mathrm{Alt}(6)$, $3{\cdot}\mathrm{Alt}(6)$ or $\mathrm{Alt}(7)$, $|V| = 2^4$, 2^6 or 2^4, respectively, and $|A| = 4$.*

3. *$p = 2$, $M \cong \mathrm{O}_{2n}^{\epsilon}(2)$ or $\mathrm{Sym}(n)$, V is a corresponding natural module, and $|A| = 2$.*

Proof. By [MS4, 1.6] A is a best offender on V and so we can apply **FMT** again.

Assume first that V and V^* are isomorphic as $\mathbb{F}_pM$-modules. By [MS4, 1.7] A is a strong dual offender on V and we are done by 3.1.

So we may assume that V is not a selfdual $\mathbb{F}_pM$-module. In particular, (see for example [MS4, 3.2(a)]) there does not exists a non-degenerated M-invariant symplectic, symmetric or unitary form on V. This excludes all cases but **FMT** (1),(6),(8),(9) and (10). The first and the last two of these remaining cases appear in the theorem.

Suppose V is the exterior square of a natural $\mathrm{SL}_n(q)$-module W for $n \geq 5$. Then $A = \mathrm{C}_M(U)$ for some hyperplane U of W. Let $1 \neq t, s \in A$, $x \in W \setminus U$, $y := [x, t]$ and $z := [x, s]$. Since $n > 2$ we can choose s such that $\mathbb{F}_q y \neq \mathbb{F}_q z$. Note that $y \in U \leq C_W(A)$ and so

$$[x \wedge y, t] = y \wedge y = 0 \text{ and } [x \wedge y, s] = z \wedge y \neq 0.$$

Thus $\mathrm{C}_V(t) \neq \mathrm{C}_V(A)$, a contradiction.

Suppose that $M \cong \mathrm{Spin}_{10}^+(q)$ and V is the half-spin module. Let $T \leq A$ be a long root subgroup. Then $N_M(T)$ and $\mathrm{N}_M(A)$ are distinct maximal Lie-parabolic subgroups of M and so $\mathrm{C}_V(T) \neq \mathrm{C}_V(A)$, a contradiction. $\square$

Definition 3.3. *Let M be a finite group and V faithful $\mathbb{F}_pM$-module. Then $\mathcal{AP}_M(V)$ is the set of non-trivial best offenders A of M on V such that $A \leq \mathrm{O}_p(P)$ for some point-stabilizer P of M on V.*

The next lemma will be used in Theorem 3.5 to show that there are no over-offenders in the radical of a point-stabilizer. This fact can be used nicely. For example, in pushing up problems for the radical of a point stabilizer.

Lemma 3.4. *Let V be an $\mathbb{F}A$-module and suppose that $[V, A]$ is 1-dimensional over $\mathbb{F}$. If A is an offender on V, then $|V/\mathrm{C}_V(A)| = |A/\mathrm{C}_A(V)|$ and the canonical commutator map $A/\mathrm{C}_A(V) \to \mathrm{Hom}_{\mathbb{F}}(V/\mathrm{C}_V(A), [V, A])$ is an isomorphism.*

Proof. Clearly the commutator map is an injective homomorphism. Since $[V, A]$ is 1-dimensional

$$|\mathrm{Hom}_{\mathbb{F}}(V/\mathrm{C}_V(A), [V, A])| = |V/\mathrm{C}_V(A)|.$$

Hence if A is an offender on V, then $|V/\mathrm{C}_V(A)| \leq |A/\mathrm{C}_A(V)|$ and the lemma holds. $\square$

Theorem 3.5 (Point-Stabilizer Theorem). *Let M be a finite $\mathcal{CK}$-group with $\mathrm{O}_p(M)=1$ and let V be a faithful $\mathbb{F}_pM$-module. Suppose that $M=\langle \mathcal{AP}_M(V)\rangle$ and that there exists a $\mathrm{J}_M(V)$-component K with $V=[V,K]$ and $\mathrm{C}_V(K)=0$. Let $A\in\mathcal{AP}_M(V)$ and let P be a point-stabilizer for M on V with $A\le \mathrm{O}_p(P)$. Then the following hold:*

(a) $M\cong \mathrm{SL}_n(q)$, $\mathrm{Sp}_{2n}(q)$, $\mathrm{G}_2(q)$ or $\mathrm{Sym}(n)$, q a power of p, where $p=2$ in the last two cases, and $n\equiv 2,3 \pmod 4$ in the last case.

(b) V is a corresponding natural module.

(c) Put $\mathbb{F}:=\mathrm{End}_M(V)$, $q:=|\mathbb{F}|$ and $Z:=\mathrm{C}_V(P)$. Then Z is 1-dimensional over $\mathbb{F}$, and one of the following holds:

1. *$M\cong\mathrm{SL}_n(q)$, $[V,A]=Z$, and $A=\mathrm{C}_M(\mathrm{C}_V(A))\cap \mathrm{C}_M(V/Z)$.*
2. *$M\cong\mathrm{Sp}_{2n}(q)$, $Z\le [V,A]\le Z^\perp$, and $A=\mathrm{C}_M(\mathrm{C}_V(A))\cap\mathrm{C}_M(Z^\perp/Z)$.*
3. *$M\cong \mathrm{G}_2(q)$, $[V,A]=\mathrm{C}_V(A)$, $|V/\mathrm{C}_V(A)|=|A|=q^3$, and $A\trianglelefteq P$.*
4. *$M\cong\mathrm{Sym}(n)$, $n\equiv 2,3 \pmod 4$, $n>6$, $|A|=2$, and $A\trianglelefteq P$.*

(d) $|V/\mathrm{C}_V(A)|=|A|$, and V is a simple $\mathbb{F}_pK$-module.

Proof. By [MS4, 2.2] either $K\cong \mathrm{C}_3$ (and $p=2$) or Q_8 (and $p=3$) and $|V|=p^2$, or K is quasisimple. In the first two cases M is isomorphic to $\mathrm{SL}_2(2)$ and $\mathrm{SL}_2(3)$, respectively, and the theorem follows easily. Thus, we may assume that K is quasisimple. Since $V=[V,K]$, **GFMT** implies that K is the unique J-component of V and V is a semisimple $\mathbb{F}_pM$-module. Let $T\in\mathrm{Syl}_p(P)$ and note that $A\le\mathrm{O}_p(P)\le T\in\mathrm{Syl}_p(M)$.

Let V_1 be simple M-submodule of V. Then by **GFMT**(h) V is either a direct sum of copies of V_1, or $M\cong\mathrm{SL}_n(q)$, $n\ge 4$, and V is a direct sum of copies of V_1 and copies of V_1^*.

Suppose for a contradiction that V is not a direct sum of copies of V_1. So $M\cong \mathrm{SL}_n(q)$, $n\ge 4$, V_1 is a natural $\mathrm{SL}_n(q)$-module, and there exists a simple $\mathbb{F}_pM$ submodule V_2 of V with $V_2\cong V_1^*$. Let $\{1,2\}=\{i,j\}$, $U_i=\mathrm{C}_{V_i}(T)$ and $H_i=[V_i,T]$. Note that H_i is the hyperplane of V_i corresponding the 1-space U_j in $V_j\cong V_i^*$. Then

$$P=P_M(V_1\oplus V_2,T)=\mathrm{O}^{p'}(\mathrm{C}_M(U_1)\cap\mathrm{C}_M(U_2))=\mathrm{C}_M(U_1)\cap\mathrm{C}_M(V_1/H_1)=\mathrm{C}_M(U_2)\cap\mathrm{C}_M(V_2/H_2).$$

It follows that H_i/U_i is a simple P-module, and so $[H_i,\mathrm{O}_p(P)]=U_i$, $i=1,2$. Recall that $A\le\mathrm{O}_p(P)$.

Let $D\le A$ be a non-trivial minimal offender on V. Assume that $[H_i,D]\ne 0$. Then $[H_i,D]=U_i$, and U_i is 1-dimensional over $\mathrm{End}_K(V_i)$. By 3.4 D is not an over-offender on H_i, so by [MS4, 1.3] $\mathrm{C}_D(H_i)=1$ and $V=H_i+\mathrm{C}_V(D)$. But this is absurd since $V_j\not\le V_i+\mathrm{C}_V(D)$.

Thus $[H_1+H_2,D]=0$, so D centralizes a hyperplane in V_1 and V_1^*. Hence $|D|\le q$ and $|V/\mathrm{C}_V(D)|\ge q^2$, and D is not an offender, a contradiction.

We have shown that V is the direct direct sum of copies of V_1. It follows that $P_M(V_1,T)=P$. So (M,V_1,A) satisfies the hypothesis of Theorem 3.5. If the theorem holds for (M,V_1,A), then $|V_1/\mathrm{C}_{V_1}(A)|=|A|$ and $V=V_1\mathrm{C}_V(A)$. Hence $[V,K]\le[V,\langle A^M\rangle]\le V_1$ and $V=V_1$, and we are done.

Thus, it suffices to prove 3.5 for (M,V_1,A), in other words we are allowed to assume that $V=V_1$, so V is a simple M-module.

We now discuss the different possibilities for V and M as listed in **FMT**. Put $U:=\mathrm{C}_V(T)$ and $\mathbb{F}:=\mathrm{End}_M(V)$.

FMT (1): Suppose that V is a natural $\mathrm{SL}_n(q)$-module. Then $\dim_{\mathbb{F}}U=1$ and $U=[V,\mathrm{O}_p(P)]$. Hence 3.4 gives (c:1) and (d).

FMT (2)-(4): Suppose that V is a natural $\mathrm{Sp}_{2n}(q)$-, $\Omega_n^\epsilon(q)$-,$\mathrm{O}_n^\epsilon(q)$- or $\mathrm{SU}_n(q)$-module for M (with $p=2$ in the $\mathrm{O}_n^\epsilon(q)$-case). Then $|\mathbb{F}|=q$ in the first two cases and $|\mathbb{F}|=q^2$ in the last. Moreover, from the structure of these modules we get the following elementary facts: U is a 1-dimensional singular $\mathbb{F}$-subspace, $[U^\perp, \mathrm{O}_p(P)] \le U$, and $\mathrm{C}_{\mathrm{O}_p(P)}(U^\perp)$ has order q, 1, 1, and q, respectively. Moreover, $\mathrm{C}_V(a) \le U^\perp$ for all $a \in \mathrm{O}_p(P)^\sharp$. Thus

$$|\mathrm{C}_A(U^\perp)| \le q \le |\mathbb{F}|,$$

and

$$|A| \ge |V/\mathrm{C}_V(A)| = |V/U^\perp||U^\perp/\mathrm{C}_{U^\perp}(A)| = |\mathbb{F}||U^\perp/\mathrm{C}_{U^\perp}(A)|.$$

Now by 3.4 with $U^\perp$ in place of V,

$$|U^\perp/\mathrm{C}_{U^\perp}(A)| \ge |A/\mathrm{C}_A(U^\perp)| \ge q^{-1}|A|.$$

We conclude that equality holds in the three preceding equations. Hence $|\mathrm{C}_A(U^\perp)| = |\mathbb{F}|$, which excludes the natural $\Omega_n^\epsilon(q)$-, $\mathrm{O}_n^\epsilon(q)$-, and $\mathrm{SU}_n(q)$-module. Thus (c:2) and (d) hold.

FMT (5): Suppose that V is a natural $\mathrm{G}_2(q)$-module for M. Then $q^3 = |V/\mathrm{C}_V(A)| = |A|$, $\mathrm{C}_M(A) = A$ and A is quadratic on V. Put $D = \mathrm{Z}(\mathrm{O}_p(P))$, then $D \le \mathrm{C}_M(A) \le A$. Since $|\mathrm{C}_V(D)| = q^3$ we conclude that $\mathrm{C}_V(A) = \mathrm{C}_V(D)$ and then $A = \mathrm{C}_M(\mathrm{C}_V(D)) \trianglelefteq P$. So (c:3) holds.

FMT (6): Suppose V is the exterior square of a natural $\mathrm{SL}_n(q)$-module W with $n \ge 5$. Then $[W, A]$ is a hyperplane of W. On the other hand, $P/\mathrm{O}_p(P) \cong \mathrm{SL}_2(q) \times \mathrm{SL}_{n-2}(q)$ and $[W, \mathrm{O}_p(P)]$ is 2-dimensional. Since $A \le \mathrm{O}_p(P)$ and $n \ge 5$ we have a contradiction.

FMT (7),(8): Suppose that V is a spin module for $\mathrm{Spin}_7(q)$ or a half-spin module for $\mathrm{Spin}_{10}^+(q)$. Let W be the natural $\Omega_7(q)$- and $\Omega_{10}^+(q)$-module, respectively, so $|W^\perp| = q$ if q is even in the $\mathrm{Spin}_7(q)$-case, and $W^\perp = 0$ in all other cases.

We will show that $|\mathrm{C}_W(A)/W^\perp| \le q^2$. For this let $A^* \le M$ be a maximal best offender with $A \le A^*$. Then by **FMT** $|A^*/A| \le q$ and $\mathrm{O}^{p'}(\mathrm{N}_M(A))/A^* \cong \mathrm{Sp}_4(q)$ and $\mathrm{Spin}_8^+(q)$, respectively. Thus $\mathrm{N}_M(A) = \mathrm{N}_M(U)$ for some 1-dimensional singular subspace U of W, and $A \le A^* \le \mathrm{O}_p(\mathrm{N}_M(A))$. Hence $[U^\perp, A] \le U$ and $\mathrm{C}_W(A) \le U^\perp$. By 3.4 applied with $V = U^\perp$,

$$|U^\perp/\mathrm{C}_{U^\perp}(A)| \ge |A/\mathrm{C}_A(U^\perp)|.$$

Suppose that $M \cong \mathrm{Spin}_7(q)$ and p is odd. Then $|\mathrm{C}_A(U^\perp)| = 1$, $|U^\perp| = q^6$, $|A| \ge q^4$ and $W^\perp = 0$. Thus

$$q^4 \le |A| \le |U^\perp/C_{U^\perp}(A)| \text{ and } q^4|C_{U^\perp}(A)| \le |U^\perp| = q^6,$$

so $|\mathrm{C}_W(A)/W^\perp| = |\mathrm{C}_W(A)| = |\mathrm{C}_{U^\perp}(A)| \le q^2$.

Suppose that $M \cong \mathrm{Spin}_7(q)$ and $p = 2$. Then $|\mathrm{C}_A(U^\perp)| \le q$, $|U^\perp| = q^6$ and $|A| \ge q^4$ and $|W^\perp| = q$. Thus

$$q^3 \le |A/C_A(U^\perp)| \le |U^\perp/C_{U^\perp}(A)| \text{ and } q^3|C_{U^\perp}(A)| \le |U^\perp| = q^6,$$

so $|\mathrm{C}_W(A)/W^\perp| = |\mathrm{C}_{U^\perp}(A)/W^\perp| \le q^2$.

Suppose that $M \cong \mathrm{Spin}_{10}^+(q)$. Then $|\mathrm{C}_A(U^\perp)| = 1$, $|U^\perp| = q^9$, $|A| = |A^*| = q^8$ and $W^\perp = 0$. Thus

$$q^8|C_{U^\perp}(A)| = |A||C_{U^\perp}(A)| \le |U^\perp| = q^9,$$

and $|\mathrm{C}_W(A)/W^\perp| = |\mathrm{C}_W(A)| = |\mathrm{C}_{U^\perp}(A)| \le q \le q^2$.

We prove that $|\mathrm{C}_W(A)/W^\perp| \le q^2$ in all cases. On the other hand $\mathrm{C}_W(\mathrm{O}_p(P))$ contains maximal singular subspace of W and $A \le \mathrm{O}_p(P)$, so $|\mathrm{C}_W(A)/W^\perp| \ge q^3$ a contradiction.

FMT (9): Suppose that $p = 2$ and $M \cong 3{\cdot}\mathrm{Alt}(6)$. Then $|[V, \mathrm{O}_p(P)]| = 4$ and $|[V, A]| = 16$, which contradicts $A \le \mathrm{O}_p(P)$.

FMT (10): Suppose that $M \cong \mathrm{Alt}(7), p = 2$ and $|V| = 2^4$. Then $P \cong \mathrm{SL}_3(2)$ and $\mathrm{O}_p(P) = 1$, which contradicts $A \le \mathrm{O}_p(P)$.

FMT (11),(12): Suppose that $p = 2$ and V is a natural $\mathrm{Alt}(n)$-module for K. Assume first that $n = 5$ or 8. Then $K \cong \Omega_4^-(2)$ and $\Omega_8^+(2)$, respectively, and $M \cong \mathrm{O}_4^-(2)$ or $\mathrm{O}_8^+(2)$ if $K \ne M$. Thus, these cases have been treated earlier in case **FMT**(2) – (4).

Hence, we may assume that $n \ge 6$ and $n \ne 8$. Let $(\Omega_1, \dots, \Omega_l)$ be the orbits of T on $\Omega = \{1, \dots, n\}$. Note that $|\Omega_i| \ne |\Omega_j|$ for $i \ne j$, so we may assume that $|\Omega_1| < \dots < |\Omega_l|$.

Suppose first that $l = 1$. Then $P \cong \big(\mathrm{Sym}(\frac{n}{2}) \wr \mathrm{C}_2\big) \cap M$, and since $n \ne 8$ and $n \ge 6$, $\mathrm{O}_p(P) = 1$, which contradicts $A \le \mathrm{O}_p(P)$.

Suppose now that $l \ne 1$. Then $P \cong \big(\bigtimes_{i=1}^{l} \mathrm{Sym}(\Omega_i)\big) \cap M$. Observe that $\mathrm{O}_2(\mathrm{Sym}(\Omega_i)) = 1$ unless $|\Omega_i| = 2$ or 4. Thus $|A| \le 8$.

If A is generated by transpositions, then $A = \mathrm{Sym}(\Omega_j)$ for some $j \in \{1, 2\}$ and $|A| = 2$. Thus either $j = 1$ and $n \equiv 2 \mod 4$, or $|\Omega_1| = 1$, $j = 2$, and $n \equiv 3 \mod 4$. Hence (c:4) holds in this case, and since $|A| = 2$ obviously also (d) holds.

If A is not generated by transpositions, then we are in case (12)(2) or (12)(3) of **FMT**. In both cases $|V/\mathrm{C}_V(A)| = |A|$, so (d) holds. In the first case $|A| = 2^{\frac{n}{2}-1} \le 8$, and so $n = 6$, since n is even and $n \ne 8$. Hence $|A| = 4$, and A is conjugate to $\langle (12), (34)(56)\rangle$. Then $M \cong \mathrm{Sym}(6) \cong \mathrm{Sp}_4(2)$ and (c:2) holds.

In the second case we get $|A| = 2^{\frac{n}{2}} \le 8$ and so $n = 6$ and $|A| = 8$. But then A is conjugate to $\langle (12), (34)(56), (35)(46)\rangle$, and again (c:2) holds. □

Theorem 3.6 (General Point-Stabilizer Theorem). *Let M be a finite $\mathcal{CK}$-group with $\mathrm{O}_p(M) = 1$ and let V be a faithful $\mathbb{F}_pM$-module. Put $\mathcal{AP} := \mathcal{AP}_M(V)$ and suppose that $\mathcal{AP} \ne \emptyset$. Then there exists an M-invariant set $\mathcal{N}$ of subnormal subgroups of M such that the following hold:*

(a) $\langle \mathcal{AP} \rangle = \bigtimes \mathcal{N}$, and $N = \langle A \in \mathcal{AP} \mid A \le N\rangle$ for all $N \in \mathcal{N}$.

(b) For all $N_1 \ne N_2 \in \mathcal{N}$, $[V, N_1, N_2] = 0$.

(c) Put $\overline{V} = V/\mathrm{C}_V(\mathcal{N}))$. Then $[\overline{V}, \mathcal{N}] = \bigoplus_{N \in \mathcal{N}} [\overline{V}, N]$.

(d) Let $N \in \mathcal{N}$. Then $(N, [\overline{V}, N])$ satisfies the hypothesis of 3.5 in place of (M, V).

(e) For all $N \in \mathcal{N}$, $\mathrm{C}_V(N) = \mathrm{C}_V(\mathrm{O}^p(N))$ and $[V, \mathrm{O}^p(N)] = [V, N]$.

(f) Let $A \in \mathcal{AP}$. Then

(a) $|V/\mathrm{C}_V(A)| = |A|$,

(b) $A = \bigtimes_{N \in \mathcal{N}} A \cap N$,

(c) $A \cap N \in \mathcal{AP}$ for all $N \in \mathcal{N}$ with $A \cap N \ne 1$.

Proof. Let $A \in \mathcal{AP}$ and let P be a point stabilizer of M on V with $A \leq \mathrm{O}_p(P)$. Choose $T \in \mathrm{Syl}_p(P)$ with $A \leq T$. Then $P = P_M(V,T)$. Put $M_0 := \langle \mathcal{AP} \rangle$ and $T_0 := T \cap M_0$. Then by 1.7(a) $P_0 := P_{M_0}(V, T_0) \leq P$ and so $A \leq \mathrm{O}_p(P_0)$. Thus replacing M by M_0 we may assume that $M = \langle \mathcal{AP} \rangle$. In particular $M = \mathrm{J}_M(V)$.

Put

$$\mathcal{J} := \mathcal{J}_M(V),\ \overline{V} := V/\mathrm{C}_V(\mathcal{J}) \text{ and } W := [V,\mathcal{J}] + \mathrm{C}_V(\mathcal{J})/\mathrm{C}_V(\mathcal{J}),$$

and for $K \in \mathcal{J}$,

$$W_K := [W,K] \text{ and } \widetilde{M} := M/\mathrm{C}_M(W_K).$$

Note that A normalizes K and thus also W_K. An application of **GFMT** gives:

1°. $\mathrm{O}^p(M) = \mathrm{F}^*(M) = \langle \mathcal{J} \rangle$, *and* W *is a faithful semisimple* $\mathbb{F}_pM$*-module.*

2°. $W = \bigoplus_{K \in \mathcal{J}} W_K$ *and so* $\mathrm{C}_W(A) = \bigoplus_{K\in\mathcal{J}} \mathrm{C}_{W_K}(A)$.

3°. $\mathrm{F}^*(\widetilde{M}) = \widetilde{K} = \mathrm{O}^p(\widetilde{M})$.

4°. A *is a best offender on* W.

By (4°) and [MS4, 1.2] A is also a best offender on W_K. Put $P_1 := P_M(\overline{V},T)$. By 1.7(c) (with $H = E := M$) we have $P_1 \leq P$ and $A \leq \mathrm{O}_p(P_1)$. Put $P_2 := P_{KT}(\overline{V},T)$. Then clearly $P_2 \leq P_1$ and so $A \leq \mathrm{O}_p(P_2)$. Since $K = \mathrm{O}^p(KT)$, 1.7(a) (with $E := KT$) gives $P_2 = P_{KT}(W_K,T)$. Since by (3°) $\widetilde{K} = \mathrm{O}^p(\widetilde{M})$ we have $\widetilde{M} = \widetilde{K}\widetilde{T}$ and so $\widetilde{P}_2 = P_{\widetilde{M}}(W_K,\widetilde{T})$. So either $\widetilde{A} = 1$ or $\widetilde{A} \in \mathcal{A}P_{\widetilde{M}}(W_K)$. Moreover, the semisimplicity of W given in (1°) yields $W_K = [W_K,K]$, $\mathrm{C}_{W_K}(\widetilde{K}) = 0$ and $\mathrm{O}_p(\tilde{M}) = 1$. Thus:

5°. *If* $\widetilde{A} \neq 1$, *then* $(\widetilde{M}, W_K, \widetilde{A})$ *satisfies the hypothesis of 3.5.*

In particular, by 3.5(d):

6°. $|W_K/\mathrm{C}_{W_K}(A)| = |\widetilde{A}| = |A/C_A(W_K)|$.

Note that (6°) also holds if $\widetilde{A} = 1$.We now use the following additional notation for $K \in \mathcal{J}$:

$$K^\perp := \prod_{E \in \mathcal{J}\setminus\{K\}} E,\ A_K := \mathrm{C}_A([W,K^\perp]).$$

From (6°) and [MS4, 1.1] applied with $\mathcal{W} = \{W_K \mid K \in \mathcal{J}\}$ we conclude

7°. $|W/\mathrm{C}_A(W)| = |A/\mathrm{C}_A(W)|$ *and* $A = \bigtimes_{K\in\mathcal{J}} A_K = A_K \times \mathrm{C}_A(W_K)$.

Recall that A is a best offender on W_K. Since $A = A_K \times \mathrm{C}_A(W_K)$ and $W = W_K + \mathrm{C}_A(A_K)$ we have

8°. A_K *is a best offender on* W_K *and on* W.

Since by (7°) $|W/\mathrm{C}_W(A)| = |A|$ and also $|V/\mathrm{C}_V(A)| \leq |A|$, we conclude that $\mathrm{C}_V(\mathcal{J}) \leq \mathrm{C}_V(A)$ and $\overline{\mathrm{C}_{[V,\mathcal{J}]}(A)} = \mathrm{C}_W(A)$, and

9°. $\mathrm{C}_V(\mathcal{J}) \leq \mathrm{C}_V(A)$ *and* $\overline{\mathrm{C}_{[V,\mathcal{J}]}(A)} = \mathrm{C}_W(A)$, *and* $|V/\mathrm{C}_V(A)| = |A|$.

Let D be an offender in M on V such that $D \leq \mathrm{O}_p(P)$. Let $B \leq D$ with $|B||\mathrm{C}_V(B)|$ maximal. Then B is a best offender on V. We claim that $|B||\mathrm{C}_V(B)| = |V|$. If $B = 1$, this is obvious and if $B \neq 1$ this follows from (9°) applied to B instead of A. Since D is an offender, $|V| \leq |D||\mathrm{C}_V(D)|$ and so $|B||\mathrm{C}_V(B)| = |V| \leq |D||\mathrm{C}_V(D)|$. Thus

10°. *All offenders in $O_p(P)$ are best offenders on V*

From (9°) and $M = \langle \mathcal{A}\mathcal{P} \rangle$, we have $C_V(\mathcal{J}) = C_V(M)$ and $|[V,K]+C_V(A)/C_V(A)| = |W/C_W(A)| = |A|$, so $V = [V,K] + C_V(A)$ since A is an offender on V.

Put

$$M_K := C_M([W, K^\perp]), \text{ and } \mathcal{N} := \{M_K \mid K \in \mathcal{J}\},$$

so $A_K \le M_K$ and by (2°) also $K \le M_K$. We will now show the statements of the conclusion for $\mathcal{N}$.

Since $K^\perp$ acts faithfully on $[W, K^\perp]$, $[M_K, K^\perp] = 1$. Thus the Three Subgroups Lemma implies $[V, K^\perp, M_K] = 1$ and so $V = [V,K] + [V,K^\perp] + C_V(A) = [V,K] + C_V(A_K)$. It follows that $[V,K] = [V, M_K]$ and $C_V(K) = C_V(M_K)$. This gives (b), (c) and (e). Moreover, (f:a) and (f:b) follow from (7°) and (9°), respectively. Note that $C_{W_K}(A_K) = C_{W_K}(A)$. So (9°) implies that $|[V,K]/C_{[V,K]}(A_K)| = |W_K/C_{W_K}(A_K)|$. Since $V = [V,K] + C_V(A_K)$ and A_K is an offender on W_K we conclude that A_K an offender on V. Note that $A_K \le A \le O_p(P)$ and so by (10°) A_K is a best offender on V. Thus (f:c) holds.

In particular, M_K is generated by best offenders, so the second part of (a) holds. The first part of (a) follows from (b) and the fact that M acts faithfully on W.

Note that $M_K C_M(W_K) = M$. Hence (d) follows from (5°), and all parts of the conclusion are proved. □

Corollary 3.7. *Let M be a finite $\mathcal{CK}$-group with $O_p(M) = 1$ and V be a faithful $\mathbb{F}_pM$-module. Let A be an offender in M on V such that $A \le O_p(P)$ for some point stabilizer P for M on V. Then A is a best offender.*

Proof. This is Step 10° in 3.6. □

Corollary 3.8. *Let $L \cong SL_n(q)$, $q = p^k$, X be a natural $\mathbb{F}_pSL_n(q)$-module for L, and P be a point stabilizer of L on X. Put $X^* = \mathrm{Hom}_{\mathbb{F}_p}(X, \mathbb{F}_p)$, the dual of X. Suppose that V is a $\mathbb{F}_pL$-module with $0 \ne V = [V,L]$ and that there exists $1 \ne A \le O_p(P)$ with $|V/C_V(A)| \le |A|$. Then one of the following holds:*

1. *$V = V_1 \oplus \ldots \oplus V_m$ where $V_1, \ldots, V_m$ are isomorphic to X^* and $1 \le m < n$.*
2. *V is isomorphic X.*
3. *$A = O_p(P)$, $n = 3$, $q = 2$, $|V| = 2^4$, and $V/C_V(L) \cong X$.*
4. *$A = O_p(P)$, $n = 2$, $q = 2^k$, $V/C_V(L) \cong X$, and V is a quotient of the natural $\Omega_3(q)$-module.*
5. *$A = O_p(P)$, $n \ge 4$, and V is isomorphic to the exterior square of X^* over $\mathbb{F}_q$. Moreover, $C_V(A) = [V,A]$, and as an $\mathbb{F}_pP$-module A is a natural $\mathbb{F}_pSL_{n-1}(q)$-module dual to $X/C_X(A)$, $V/C_V(A))$ is isomorphic to A, and $C_V(A)$ is isomorphic to the exterior square of A over $\mathbb{F}_q$.*

Proof. Observe that $O_p(P)$ is elementary abelian, so A is an offender on V. Put $\overline{V} := V/C_V(O^p(L))$. Let $1 \ne B \le A$ such that $|B||C_V(B)|$ is maximal. Then B is a best offender on A. Note that either L is quasisimple or $n = 2$ and $q = p \le 3$. In any case we conclude that $L = \langle B^L \rangle$ and so $L = J_L(V)$. Thus we can apply **GFMT** to L and V. In particular, since $V = [V,L]$, $\overline{V}$ is a semisimple $\mathbb{F}_pL$-module. Let Y be a simple $\mathbb{F}_pL$-submodule of $\overline{V}$. By **GFMT**(d) B is a best offender on $\overline{V}$ and so by [MS4, 1.2] B is a best offender on Y. Thus we can apply **FMT** also to (L,Y). Only the cases **FMT**(1), (4) (with $n = 3$ and $\epsilon = +$) and (6) apply to our situation, since $\Omega_6^+(q) \cong L_4(q)$.

Suppose first that $n \ge 5$ and Y is the exterior square of a natural $SL_n(q)$-module Z. Then by **GFMT**(g), $Y = \overline{V}$, and B is the centralizer of an $\mathbb{F}_q$-hyperplane in Z. Since $B \le A \le O_p(P)$ this shows that Z is dual to X and $A = B = O_p(P)$. By [MS4, 6.1], $C_V(L) = 0$ and so (5) holds.

Suppose next that $n = 4$ and Y is a natural $\Omega_6^+(q)$-module for L. Note that $\mathrm{C}_Y(\mathrm{O}_p(P)) = [Y, \mathrm{O}_p(P)]$ is a 3-dimensional singular subspace of Y. Thus also $[Y, B]$ is singular. Since Y is an offender we conclude from [MS4, 3.4] that $[Y, B]$ is 3-dimensional, $A = B = \mathrm{O}_p(P)$, and $|Y/\mathrm{C}_Y(B)| = |B| = q^3$. Thus $Y = \overline{V}$. Suppose that $\mathrm{C}_Y(L) \neq 0$. Then by [MS4, 6.1], $q = 2$, and V is isomorphic to a submodule of the permutation module for $\mathrm{SL}_4(2) \cong \mathrm{Alt}(8)$. But B acts transitively on the eight points permuted by $\mathrm{Alt}(8)$, and so $\mathrm{C}_Y(B) = \mathrm{C}_Y(L)$ and $|Y/\mathrm{C}_Y(B)| = 2^6 > |B|$, a contradiction. Thus $\mathrm{C}_V(L) = 0$ and (5) holds.

Suppose that Y is isomorphic to X. Since $A \leq \mathrm{O}_p(P)$ the Point Stabilizer Theorem 3.5 shows that $|Y/\mathrm{C}_Y(B)| = |B|$. Thus $\overline{V} = Y$. If $\mathrm{C}_V(L) = 0$, (2) holds. So suppose $\mathrm{C}_V(L) \neq 0$. Then by [MS4, 8.4] $p = 2$, $|\mathrm{C}_Y(L)| \leq q$, and either $n = 2$, or $n = 3$ and $q = 2$. In the first case (4) holds. So suppose $n = 3$ and $q = 2$. Then by [MS4, 8.4] $|B| = 4$ and $A = B = \mathrm{O}_2(L)$. Hence (3) holds.

It remains to treat the case where Y (and every other simple L-submodule of $\overline{V}$) is isomorphic to the dual of X, so $\overline{V}$ is the direct sum of m natural modules dual to X, and by **GFMT**(g) $n > 2$. Then $q^m = |\overline{V}/\mathrm{C}_{\overline{V}}(B)| \leq |B| \leq |\mathrm{O}_p(P)| = q^{n-1}$ and so $m \leq n-1$. If $\mathrm{C}_V(L) = 0$, then (1) holds. So suppose $\mathrm{C}_V(L) \neq 0$. Since $V = [V, L]$ there exists an $\mathbb{F}_pL$-submodule U of V with $U = [U, L]$, $\mathrm{C}_U(L) \neq 0$, and $\overline{U}$ simple. Then [MS4, 6.1] shows $n = 3$ and $q = 2$. Since B is an offender on U, [MS4, 8.4] shows that $|B| = 4$ and $|[\overline{U}, B]| = 2$. But this is impossible, since $B \leq \mathrm{O}_2(P)$ and $\overline{U}$ is dual to X. □

Theorem 3.9 (Rank 1 Theorem). *Let M be a finite $\mathcal{CK}$ group with $\mathrm{O}_p(M) = 1$ and V be a faithful finite dimensional $\mathbb{F}_pM$-module, and let K be $\mathrm{J}_M(V)$-component. Suppose that the point stabilizers of K on V are p-groups. Then $K \cong \mathrm{SL}_2(q)'$, q a power of p, and $[V,K]/\mathrm{C}_{[V,K]}(K)$ is a natural $\mathrm{SL}_2(q)'$-module.*

Proof. Let $T \in \mathrm{Syl}_p(K)$ and $P = P_K(V, T)$. By 1.7(a), P is also a point stabilizer of K on $[V, K]$ and so by 1.7(c) (applied with $V := [V, K]$ and $V_0 := \mathrm{C}_{[V,K]}(K)$), P contains a point stabilizer P^* of K on $X := [V, K]/\mathrm{C}_{[V,K]}(K)$.

Suppose first that X is a homogeneous K-module, that is, $X \cong Y^m$ for some simple $\mathbb{F}_pK$-module Y. Then P^* is the point stabilizer of K on Y. Since P^* is a p-group we conclude from **GFMT** that $K \cong \mathrm{SL}_2(q)$, Y is a natural $\mathrm{SL}_2(q)$ module and $X \cong Y$. So the theorem holds in this case.

Suppose next that X is not an homogeneous K-module. Then by **GFMT** $K \cong \mathrm{SL}_n(q)$, $n \geq 4$, and $X \cong Y^r \oplus Y^{*s}$, where Y is a natural $\mathrm{SL}_n(q)$ and $r, s \in \mathbb{Z}^+$. Then $P^* = \mathrm{O}^{p'}(C_K(\mathrm{C}_Y(T)) \cap \mathrm{C}_K(Y/[Y,T])$ and so $P^*/\mathrm{O}_p(P^*) \cong \mathrm{SL}_{n-2}(q)$, a contradiction since P^* is p-group and $n \geq 4$. □

4 $Q!$-Modules

The results of this section are important for the investigation of finite groups G which possess a large subgroup. Here a p-subgroup $Q \leq G$ is called large if

$$(*) \qquad \mathrm{C}_G(Q) \leq Q \text{ and } \mathrm{N}_G(A) \leq \mathrm{N}_G(Q) \text{ for all } 1 \neq A \leq \mathrm{Z}(Q).$$

Observe that most of the simple groups of Lie type in characteristic p possess a large subgroup, namely $\mathrm{O}_p(\mathrm{N}_G(Z))$, where Z is a long root subgroup. The exceptions are the groups $\mathrm{F}_4(2^n)$, $\mathrm{Sp}_4(2^n)$ and ${}^2\mathrm{G}_2(3^n)$.

For the investigation of p-local subgroups containing a large subgroup by means of their action on elementary abelian p-subgroups the above property $(*)$ can be easily transformed into a property of modules, so-called $Q!$-modules. We recall the definition here from [MS4, 6.2]:

Definition 4.1. *Let H be a finite group, V an $\mathbb{F}_pH$ module, and Q a p-subgroup of H. Then V is called a $Q!$-module for H if Q is not normal in H and*

$$\text{(Q!)} \qquad Q \trianglelefteq \mathrm{N}_H(A) \textit{ for all } 1 \neq A \leq \mathrm{C}_V(Q).$$

In this section H is a finite group with $\mathrm{O}_p(H) = 1$, Q is a p-subgroup of H, and V a faithful $Q!$-module for $\mathbb{F}_pH$. Put $H^\circ := \langle Q^H \rangle$.

Recall that a p-subgroup $A \leq H$ is a weakly closed subgroup of H if A is the only conjugate of A in T for some $T \in \mathrm{Syl}_p(H)$.

Lemma 4.2. *(a) Q is a weakly closed subgroup of H.*

(b) $H^\circ = \langle Q^h \mid h \in H^\circ \rangle$.

(c) $\mathrm{C}_H(H^\circ/\mathrm{Z}(H^\circ)) = \mathrm{C}_H(H^\circ)$.

(d) Let $H^\circ \leq L \leq H$ and W be a non-zero L-submodule of V. Then $\mathrm{C}_L(W) \leq \mathrm{C}_L(H^\circ)$. In particular, $\mathrm{C}_{H^\circ}(W)$ is a p'-group.

(e) $\mathrm{C}_V(H^\circ) = 0$.

(f) Let $Q \leq L \leq H$ with $Q \ntrianglelefteq L$. Then V is $Q!$-module for $\mathbb{F}_pL$.

(g) Let $L \trianglelefteq\trianglelefteq H$ with $[L, Q] \neq 1$. Then $\mathrm{C}_V(\langle L^Q \rangle) = 0$.

Proof. (a): Let $Q \leq T \in \mathrm{Syl}_p(H)$ and $A := \mathrm{C}_V(T)$. By $Q!$, $Q \trianglelefteq \mathrm{N}_H(A)$, in particular $Q \trianglelefteq \mathrm{N}_H(T)$. Thus, if $Q^h \leq T$ then also $Q^h \trianglelefteq \mathrm{N}_H(T)$. By Burnside's Theorem [Go, 7.2.1] any two normal subgroups of T are conjugate in H if and only if they are conjugate in $\mathrm{N}_H(T)$. Hence $Q = Q^h$.

(b): Let $H_0 := \langle Q^h \mid h \in H^\circ \rangle$. Then (a) shows that $Q^H = Q^{H_0}$ and so $H_0 = H^\circ$.

(c): Set $D := \mathrm{C}_H(H^\circ/\mathrm{Z}(H^\circ))$. Note that $\mathrm{Z}(H^\circ)$ is a p'-group since $\mathrm{O}_p(H) = 1$ and that $\mathrm{Z}(H^\circ)$ centralizes Q. Hence $Q = \mathrm{O}_p(Q\mathrm{Z}(H^\circ))$. Since $[D, Q\mathrm{Z}(H^\circ)] \leq \mathrm{Z}(H^\circ) \leq Q\mathrm{Z}(H^\circ)$ we conclude that D normalizes $Q\mathrm{Z}(H^\circ)$ and Q. So $[D, Q] \leq Q \cap \mathrm{Z}(H^\circ) = 1$ and thus $[D, H^\circ] = 1$.

(d): Since $\mathrm{C}_W(Q) \neq 0$, $Q!$ implies that $\mathrm{C}_{H^\circ}(W) \leq \mathrm{N}_H(\mathrm{C}_W(Q)) \leq \mathrm{N}_H(Q)$. Thus

$$[\mathrm{C}_{H^\circ}(W), Q] \leq Q \cap \mathrm{C}_{H^\circ}(W) \leq \mathrm{O}_p(\mathrm{C}_{H^\circ}(W)) \leq \mathrm{O}_p(H^\circ) \leq \mathrm{O}_p(H) = 1.$$

Now (b) implies $\mathrm{C}_{H^\circ}(W) \leq \mathrm{Z}(H^\circ)$. This shows the additional claim since $\mathrm{Z}(H^\circ)$ is a p'-group. Moreover, $[\mathrm{C}_L(W), H^\circ] \leq \mathrm{C}_{H^\circ}(W) \leq \mathrm{Z}(H^\circ)$ and so by (c), $\mathrm{C}_L(W) \leq \mathrm{C}_L(H^\circ)$.

(e): Since $Q \ntrianglelefteq H$, $Q \neq 1$ and so $\mathrm{C}_{H^\circ}(\mathrm{C}_V(H^\circ)) = H^\circ$ is not a p'-group. Thus (d) implies $\mathrm{C}_V(H^\circ) = 0$.

(f): This follows immediately from the definition of a $Q!$-module.

(g): Put $L^* = \langle L^Q \rangle$. Since L is subnormal in H, so is L^*. Thus $\mathrm{O}_p(L^*) \leq \mathrm{O}_p(H) = 1$. If $Q \trianglelefteq QL^*$ we get $1 \neq [Q, L] \leq [Q, L^*] \leq Q \cap L^* \leq \mathrm{O}_p(L^*) = 1$, a contradiction. Thus $Q \ntrianglelefteq L^*Q$ and by (f), V is a $Q!$-module for L^*Q. By (e) applied to L^*Q, $\mathrm{C}_V(\langle Q^{L^*Q} \rangle) = 0$. In particular, $\mathrm{C}_V(L^*Q) = 0$ and so also $\mathrm{C}_V(L^*) = 0$. □

Recall from [MS2] that an $\mathbb{F}_pH$-module U is called quasisimple if $U = [U, H]$, $U/\mathrm{C}_U(H)$ is a simple H-module, and $\mathrm{O}_p(H/\mathrm{C}_H(U)) = 1$.

Lemma 4.3. *Let K be a $\mathrm{J}_H(V)$-component such that $[V, K]$ is a quasisimple K-module. Then $K \leq [K, H^\circ] \leq H^\circ$.*

Proof. Otherwise [MS4, 2.10] implies $[K, H^\circ] = 1$. Put $W = ([V, K] + \mathrm{C}_V(K))/\mathrm{C}_V(K)$. Then W is a simple K-module. Since $[K, Q] = 1$, $\mathrm{C}_W(Q)$ is a non-trivial K-submodule and so $[W, Q] = 0$. By 4.2(b) $H^\circ = \langle Q^{H^\circ}\rangle$ and thus $[W, H^\circ] = 0$ since H° normalizes W. Thus $[V, K, H^\circ, K] = 0$. Since $[H^\circ, K] = 1$ the Three Subgroups Lemma gives $[V, K, K, H^\circ] = 1$. By 4.2, $\mathrm{C}_V(H^\circ) = 0$, thus $[V, K, K] = 0$, a contradiction since $[V, K]$ is quasisimple. □

Lemma 4.4. *Let Y be a p-subgroup of H with $\mathrm{C}_Y([V, Y]) \neq 1$ and $[H^\circ, Y] \neq 1$. Then $\mathrm{C}_Y(H^\circ) = 1$.*

Proof. Put $X := \mathrm{C}_Y(H^\circ)$ and $Z := \mathrm{C}_Y([V, Y]) \cap \mathrm{Z}(Y)$. Note that $Z \neq 1$. Suppose that $X \neq 1$. Then $[V, X]$ is a non-zero $H^\circ Z$-submodule of V. Since Z centralizes $[V, X]$ we conclude from 4.2(d), that Z centralizes H°. Since $[V, Y, Z] = 0$ and $[Z, Y] = 0$ the Three Subgroups Lemma gives $[V, Z, Y] = 0$. Thus $[V, Z]$ is a non-zero $H^\circ Y$ submodule centralized by Y and another application of 4.2(d) gives $[H^\circ, Y] = 1$, a contradiction. □

Lemma 4.5. *Suppose that one the following holds.*

(i) $\mathrm{F}^*(H) \cong \mathrm{Alt}(n)$, $n \geq 5$, *and* $[V, H]$ *is a natural* $\mathbb{F}_p\mathrm{Alt}(n)$*-module for* $\mathrm{F}^*(H)$*, or*

(ii) $H \cong \mathrm{Alt}(7)$ *and* $|[V, H]| = 2^4$.

Then (i) holds, and either $n = p$ or (n, p) is one of $(5, 2), (6, 2), (8, 2), (6, 3)$.

Proof. Clearly every $\mathbb{F}_pH$-submodule of V is also a $Q!$-module, so we may assume that $V = [V, H]$. Since Q is not normal in H, $Q \neq 1$. Moreover, replacing H by H° we may assume that $H = H^\circ$ and so $H = \mathrm{O}^{p'}(H)$.

If (ii) holds, then $\mathrm{C}_H(v) \cong L_3(2)$ and so $\mathrm{O}_2(\mathrm{C}_H(v)) = 1$ for every $0 \neq v \in V$. For $0 \neq v \in \mathrm{C}_V(Q)$ this gives $Q \leq \mathrm{O}_2(\mathrm{C}_H(v)) = 1$, a contradiction. Thus (i) holds.

Put $K := \mathrm{F}^*(H) \cong \mathrm{Alt}(n)$. Note that either $\mathrm{Aut}(K) \cong \mathrm{Sym}(n)$, or $n = 6$, $|\mathrm{Aut}(K)/\mathrm{Inn}(K)| = 4$. For $p \neq 2$ this implies $H = \mathrm{O}^{p'}(H) \cong \mathrm{Alt}(n)$. For $p = 2$ and $n = 6$, $\mathrm{Sym}(6)$ is the largest subgroup of $\mathrm{Aut}(K)$ acting on the natural $\mathbb{F}_2\mathrm{Alt}(6)$-module. So for $p = 2$, $H \cong \mathrm{Alt}(6)$ or $H \cong \mathrm{Sym}(n)$. In any case we may assume that $H \leq \mathrm{Sym}(n)$. Let $\Omega := \{1, 2, \dots, n\}$ and W be the permutation module with basis $(w_i \mid i \in \Omega)$; and for $\Psi \subseteq \Omega$ let $w_\Psi := \sum_{i\in\Psi} w_i$. Set $W_0 := [W, H]$ and $\overline{W} := W/\mathrm{C}_W(H)$. Then $\overline{W_0} \cong V$.

Assume first that Q does not act transitively on Ω. Let Ψ be any Q-invariant subset of Ω such that $\Psi \neq \Omega$, Q acts non-trivially on Ψ, and $p \,\big|\, |\Psi|$. Then $w_\Psi \in W_0$ and $\overline{w}_\Psi \neq 0$. Note that $\mathrm{Alt}(\Psi)$ centralizes $\overline{w}_\Psi$, so $Q!$ implies that $\mathrm{Alt}(\Psi)$ normalizes the image of Q in $\mathrm{Sym}(\Psi)$. It follows that either $p = 2$ and $|\Psi| = 2$ or 4, or $p = 3$ and $|\Psi| = 3$. In all cases Ψ is a non-trivial Q-orbit. Since this holds for all possible choices of Ψ and $n \geq 5$, Ω is the union of two non-trivial Q-orbits or of one non-trivial Q-orbit and at most $p - 1$ fixed points of Q. This gives one of the following possibilities:

$$n = 2 + 4,\ n = 4 + 4,\ n = 4 + 1 \text{ for } p = 2,$$

$$n = 3 + 3,\ n = 3 + 2 \text{ for } p = 3.$$

Suppose $p = 3$ and $n = 5$. Say $Q = \langle(1, 2, 3)\rangle$. Then Q centralizes $w := w_{1234} - w_5$. Observe that $w \in W_0$, $\overline{w} \neq 0$ and $\mathrm{C}_H(\overline{w}) = \mathrm{Alt}(4)$, a contradiction to $Q!$. So $(n, p) \neq (5, 3)$ and the lemma holds in the intransitive case.

Assume next that Q acts transitively on Ω. Let $(X_i)_{i\in\mathbb{F}_p}$ be a Q-invariant partition of Ω into p sets of size $\frac{n}{p}$. Pick $g \in Q$ with $X_0^g \neq X_0$ and choose notation such that $X_i^g = X_{i+1}$ for all $i \in \mathbb{F}_p$. Define $w_0 = \sum_{i\in\mathbb{F}_p} i w_{X_i}$. Then

$$w_0^g = \sum_{i\in\mathbb{F}_p} i w_{i+1} = \sum_{j\in\mathbb{F}_p}(j-1)w_j = \sum_{j\in\mathbb{F}_p} j w_j - \sum_{j\in\mathbb{F}_p} w_j = w_0 - w_\Omega$$

Thus Q centralizes $\overline{w_0}$ and $\overline{w}_0 \neq 0$. Note that $\mathrm{Alt}(X_0)$ centralizes $\overline{w}_0$ and so by $Q!$, $\mathrm{Alt}(X_0)$ normalizes Q. Thus $[\mathrm{Alt}(X_0), Q]$ is a p-group. Since $\mathrm{Alt}(X_0)^g = \mathrm{Alt}(X_1)$ this implies that $\mathrm{Alt}(X_0)$ is p-group. Hence one of the following holds: $|X_0| = 1$ and $n = p$; $|X_0| = 2 = p$ and $n = 4$; or $|X_0| = 3 = p$ and $n = 9$. In the first case the lemma holds. Since $n \geq 5$, the second case is impossible. So suppose $|X_0| = 3 = p$. Put $D = (\mathrm{Sym}(X_0) \times \mathrm{Sym}(X_1) \times \mathrm{Sym}(X_2)) \cap \mathrm{Alt}(n)$ and $E = \mathrm{Alt}(X_0) \times \mathrm{Alt}(X_1) \times \mathrm{Alt}(X_2)$. Then $D/E \cong \mathrm{C}_2 \times \mathrm{C}_2$ and Q acts non-trivially on D/E. Thus $[D, Q]$ is a not 3-group, a contradiction to $Q!$ and $D \leq \mathrm{C}_H(\overline{w}_0)$. □

Theorem 4.6 (Q!FF-Module Theorem). *Let H be a finite group with $\mathrm{O}_p(H) = 1$ and Q be a p-subgroup of H, and let V be a faithful $Q!$-module for H. Put $H^\circ := \langle Q^H \rangle$ and $J := \mathrm{J}_H(V)$. Suppose that there exists an offender Y in H such that $[H^\circ, Y] \neq 1$ and that one of the following holds:*

(i) Y is quadratic on V.

(ii) Y is a best offender on V.

(iii) $\mathrm{C}_Y([V, Y]) \neq 1$.

(iv) $\mathrm{C}_Y(H^\circ) = 1$.

Then one of the following holds, where q is a power of p:

1. There exists an H-invariant set $\mathcal{K}$ of subgroups of H such that:

- *(a) For all $K \in \mathcal{K}$, $K \cong \mathrm{SL}_2(q)$ and $[V, K]$ is a natural module for K,*
- *(b) $J = \bigtimes_{K\in\mathcal{K}} K$ and $V = \bigoplus_{K\in\mathcal{K}} [V, K]$,*
- *(c) Q acts transitively on $\mathcal{K}$,*
- *(d) $H^\circ = \mathrm{O}^p(J)Q$.*

2. Put $R := \mathrm{F}^*(J)$. Then

- *(a) R is quasisimple, $R \leq H^\circ$, and either $J = R$ or $p = 2$ and $J \cong \mathrm{O}^\pm_{2n}(q)$, $\mathrm{Sp}_4(2)$ or $\mathrm{G}_2(2)$.*
- *(b) $\mathrm{C}_V(R) = 0$, $[V, R]$ is a semisimple J-module, and H acts faithfully on $[V, R]$.*
- *(c) Put $J^0 := J \cap H^\circ$. Then one of the following holds:*
  1. *(a) $R = J^0 \cong \mathrm{SL}_n(q)$, $n \geq 3$, $\mathrm{Sp}_{2n}(q)$, $n \geq 3$, $\mathrm{SU}_n(q)$, $n \geq 8$, or $\Omega^\pm_n(q)$, $n \geq 10$.*
     (b) $[V, R]$ is the direct sum of at least two isomorphic natural modules for R.
     (c) $H^\circ = R\mathrm{C}_{H^\circ}(R)$.
     (d) If $V \neq [V, R]$ then $R \cong \mathrm{Sp}_{2n}(q)$, $p = 2$, and $n \geq 4$.
  2. *(a) $[V, R]$ is a simple R-module.*
     (b) Either $H^\circ = R = J^0$ or $H^\circ \cong \mathrm{Sp}_4(2)$, $3{}^{\cdot}\mathrm{Sym}(6)$, $\mathrm{SU}_4(q).2$ ($\cong \mathrm{O}^-_6(q)$ and $[V, R]$ the natural $\mathrm{SU}_4(q)$-module), or $\mathrm{G}_2(2)$.
     (c) One of the cases **FMT** *(1) - (9), (12) applies to $(J, [V, R])$, where $n = 6$ in case (12).*
  3. *$p = 2$, $J = R \cong \mathrm{SL}_4(q)$, H°/R has order two and induces a graph automorphism on R, and V is the direct sum of two non-isomorphic natural modules.*

Proof. Note that (i) implies (iii), and by Timmesfeld's Replacement Theorem [KS, 9.2.3], also (ii) implies (iii). By 4.4, (iii) implies (iv). So in any case:

1°. $\quad \mathrm{C}_Y(H^\circ) = 1.$

Let $\mathcal{J}$ be the set of J-components of H, and let X be a minimal offender in Y. By [MS4, 1.3] X is a quadratic best offender and so $X \leq J$. Now (1°) implies

2°. $\quad [H^\circ, X] \neq 1$. *In particular,* $[H^\circ, J] \neq 1$.

According to (2°) and [MS4, 2.2(b)] there exists $R \in \mathcal{J}$ with $R \leq H^\circ$. Next we prove:

3°. $\quad [R, H^\circ] \neq 1.$

Suppose that $[R, H^\circ] = 1$. Since $R \leq H^\circ$ it follows that R is abelian. So [MS4, 2.2(d)] implies that $p = 2$, $R \cong \mathrm{C}_3$ and $|[V, R]| = 4$. Thus $[V, R]$ is a simple R-module. Hence 4.3 gives $R \leq [R, H^\circ] = 1$, a contradiction.

According to (3°) we may choose R such that $[R, Q] \neq 1$. Put $R^* := \langle R^Q \rangle$.

4°. $\quad \mathrm{C}_V(R^*) = 0.$

As a J-component, R is subnormal in H. Thus (4°) follows from 4.2(g).

5°. $\quad \mathcal{J} = R^Q$, *so* $R^* = \langle \mathcal{J} \rangle$.

Otherwise there exists $\widetilde{R} \in \mathcal{J} \setminus R^Q$. Put $U := [V, \langle \widetilde{R}^Q \rangle]$. Since $R^Q \cap \widetilde{R}^Q = \emptyset$, [MS4, 2.2(f)] gives $[U, \langle R^Q \rangle] = 0$, which contradicts (4°).

6°. $\quad W := [V, R^*]$ *is a semisimple, faithful J-module, and* $R^* = \bigtimes_{K \in \mathcal{J}} K = \mathrm{F}^*(J) \leq H^\circ$.

By (4°) $\mathrm{C}_V(R^*) = 0$. Hence (6°) follows from (5°) and [MS4, 8.3].

7°. $\quad$ *H acts faithfully on W.*

Set $D := \mathrm{C}_H([V, R^*])$. Since W is a faithful J-module, $R^* \cap D \leq J \cap D = 1$. Thus $[R^*, D] = 1$, and the Three Subgroups Lemma shows that $[V, D, R^*] = 0$. Now (7°) follows from (4°).

Let S be a Sylow p-subgroup of H with $Q \leq S$. Since R and R^* are subnormal in H, $R \cap S$ and $R^* \cap S$ are Sylow p-subgroups of R and R^*, respectively. Since $Q \leq S$, also $R^*Q \cap S$ is a Sylow p-subgroup of R^*Q.

8°. $\quad$ *Let W_0 be an R^*Q-submodule of W and $Z_0 := \mathrm{C}_{[W_0,R]}(S \cap R)$. Then $\mathrm{C}_R(Z_0) \leq \mathrm{N}_H(Q)$.*

From 1.6, applied to $E = R$, $H = R^*Q$, $T = S \cap R^*$ and W_0, we get that $\mathrm{C}_R(Z_0) = \mathrm{C}_R(\mathrm{C}_{W_0}(S \cap R^*))$. Since $\mathrm{C}_{W_0}(S \cap R^*) \leq \mathrm{C}_V(Q)$, the Q!-property gives $\mathrm{C}_H(\mathrm{C}_{W_0}(S \cap R^*)) \leq \mathrm{N}_H(Q)$. So (8°) holds.

Put $Z := \mathrm{C}_{[V,R]}(S \cap R)$, $P := \mathrm{C}_R(Z)$ and $\Omega := \{v \mid 0 \neq v \in \mathrm{C}_V(Q^h), h \in H\}$. Note that $S \cap R \leq P$ and by (8°) $P \leq \mathrm{N}_H(Q)$; in particular $[P, Q] \leq \mathrm{O}_p(R^*)$.

9°. $\quad$ *Suppose that R^* acts transitively on Ω. Then $H^\circ = R^*Q$.*

Let $v \in \Omega$ such that $[v, Q] = 0$. By the Frattini argument $H = R^*\mathrm{C}_H(v)$ and by Q!, $\mathrm{C}_H(v) \leq \mathrm{N}_H(Q)$. Thus $H = R^*\mathrm{N}_H(Q)$, R^*Q is normal in H, and $R^*Q \leq H^\circ \leq R^*Q$.

Case 1. *Suppose that $\mathcal{J} \neq \{R\}$. Then (1) holds.*

By (5°) R^* is the direct product of at least two Q-conjugates of R. On the other hand by (8°) $[\mathrm{C}_R(Z), Q] \leq Q$. We conclude from the direct product that P is a p-group. By (6°), $\mathrm{C}_{[V,R]}(R) = 0$ and so the Rank 1 Theorem 3.9 shows that $R \cong \mathrm{SL}_2(q)'$ and $[V, R]$ is a natural module for R.

Then (5°) implies (1:a) and (1:c). Moreover, **GFMT** gives the first part of (1:b). To show the second part of (1:b) it suffices to show that $V = [V, R] + \mathrm{C}_V(R)$. Let A be a minimal offender on V. Since Q acts transitively on $\mathcal{J}$ we may assume $[R, A] \neq 1$. Since A is not an over-offender on $[V, R]$ and A is minimal we conclude from [MS4, 1.2] that $\mathrm{C}_A([V, R]) = 1$, $|A| = q = |V/\mathrm{C}_V(A)|$, $[\mathrm{C}_V(R), A] = 0$ and $RA \cong \mathrm{SL}_2(q)$. Thus RA is generated by two conjugates of A and so $|V/\mathrm{C}_V(RA)| \leq q^2$. Hence $V = [V, R] + \mathrm{C}_V(R)$ and (1:b) holds.

To prove (1:d) let Ω^* be the set of elements in V such that $[v, \widetilde{R}] \neq 0$ for all $\tilde{R} \in \mathcal{J}$. Since R acts transitively on $[V, R]$ and $\mathrm{C}_V(R^*) = 0$, and since we have already proved that $V = [V, R^*]$, R^* acts transitively on Ω^*. Since Q acts transitively on $\mathcal{J}$ we conclude that $\Omega = \Omega^*$. Thus by (9°), $H^\circ = R^*Q$ and (1:d) holds.

According to (Case 1) we may assume from now on that $\mathcal{J} = \{R\}$. In particular, $W = [V, R]$ and $\mathrm{C}_V(R) = 0$.

Case 2. *Suppose that R is solvable. Then (1) holds.*

By [MS4, 2.2(d)]) (or **FMT**) $p = 2$ or 3, $R \cong \mathrm{SL}_2(p)'$ and $[V, R]$ is a natural $\mathrm{SL}_2(p)'$-module. Since $\mathrm{C}_V(R) = 0$, [MS4, 8.4] (or coprime action) gives $V = [V, R]$ and so (1) holds.

According to (Case 1) and (Case 2) we may assume from now on that R is quasisimple. We will show that (2) holds.

Case 3. *Suppose that W is not a homogeneous R-module. Then (2) and (2:c:3) hold.*

By **GFMT** we have $R \cong \mathrm{SL}_n(q)$ and $[V, R] \cong N^r \oplus N^{*s}$, where N and N^* are simple natural R-submodule in W dual to each other. Moreover, $r, s \geq 1$ and $n \geq 4$.

Set $U := \mathrm{C}_N(R \cap S)$ and $U^* = \mathrm{C}_{N^*}(R \cap S)$. Then U and U^* are 1-dimensional over $\mathbb{F}_q$ and $P = \mathrm{C}_R(Z) = \mathrm{C}_R(U) \cap \mathrm{C}_R(U^*)$. From (8°) we get $[P, Q] \leq P \cap Q \leq \mathrm{O}_p(P)$.

Assume first that Q acts trivially on the Dynkin diagram of R. Put $W_0 := \langle N^Q \rangle$. Then $W_0 \cong N^{t'}$ for some $t' \leq r$, and by (8°) $\mathrm{C}_R(U) \leq \mathrm{N}_R(Q)$. By symmetry also $\mathrm{C}_R(U^*) \leq \mathrm{N}_R(Q)$. Thus $R = \langle \mathrm{C}_R(U), \mathrm{C}_R(U^*) \rangle \leq \mathrm{N}_M(Q)$. But then $[R, Q] \leq Q \cap R \leq \mathrm{O}_p(R) = 1$, which contradicts (3°).

Assume now that Q acts non-trivially on the Dynkin diagram of R. Then clearly $p = 2$. Also $[P, Q] \leq \mathrm{O}_2(P)$ implies that $n = 4$ and that Q does not induce a non-trivial field or field graph automorphism on R. By **GFMT** $\sqrt{r} + \sqrt{s} \leq \sqrt{4} = 2$ and so $r = s = 1$. Thus $W \cong N \oplus N^*$. An application of [MS4, 8.4] also gives $W = V$.

Let $f : N \times N^* \to \mathbb{F}_q$ be a surjective R-invariant $\mathbb{F}_q$-bilinear map. Then R acts transitively on $\Omega^* = \{n + n^* \mid f(n, n^*) = 0\}$ and $\Omega = \Omega^*$. Thus by (9°), $H^\circ = RQ$ and so (Case 3) is proved.

Case 4. *Suppose that W is a homogeneous but not simple R-module. Then (2) and (2:c:1) hold.*

GFMT shows that R is a genuine group of Lie-type of type A_n $(n \geq 2)$, B_n $(n \geq 5)$, C_n $(n \geq 3)$, D_n $(n \geq 5)$, ${}^2\mathrm{A}_n$ $(n \geq 4)$, or ${}^2\mathrm{D}_n$ $(n \geq 5)$, and $W \cong N^r$ for some natural module N and some $r \geq 2$. Put $U = \mathrm{C}_N(S \cap R)$. Then $P = \mathrm{C}_R(U)$, $\mathrm{N}_R(U)$ is a maximal parabolic subgroup of R, and $P/\mathrm{O}_p(P)$ is a group of Lie-type A_{n-1}, B_{n-1}, C_{n-1}, D_{n-1}, ${}^2\mathrm{A}_{n-2}$ or ${}^2\mathrm{D}_{n-2}$, respectively. Note that $[P, Q] \leq \mathrm{O}_p(P)$. If R is of type A_n, P is not invariant under any graph automorphism of R, and if

R is type D_n, any graph automorphism also induces a graph automorphism on P/Q. Thus Q does not induce graph automorphisms on R. Also if R has roots of more than one length, so does P/Q. Thus Q does not induces any field automorphism on R. Since Q is a p-group, Q does not induces any diagonal automorphism on R and so $Q \leq R\mathrm{C}_H(R)$. Thus $H^\circ = R\mathrm{C}_{H^\circ}(R)$.

Suppose that $V \neq [V,R]$. We apply [MS4, 8.4]. Since $\mathrm{C}_V(R) = 0$ and $[V,K]$ is not simple, we are in case (f) of [MS4, 8.4], and $R \cong \mathrm{Sp}_{2n}(q)$, $p = 2$, and $n \geq 4$.

Case 5. *Suppose that W is a simple R-module.*

Clearly we can apply **FMT** to (J,W). In the cases (10) – (12) of **FMT** 4.5 shows that one of the following holds:

$$\begin{array}{lll} J \cong \mathrm{Sym}(5) \cong \mathrm{O}_4^-(2), & J \cong \mathrm{Alt}(8) \cong \Omega_6^+(2), & J \cong \mathrm{Sym}(8) \cong \mathrm{O}_6^+(2), \\ J \cong \mathrm{Alt}(6) \cong \mathrm{Sp}_4(2)', & J \cong \mathrm{Sym}(6) \cong \mathrm{Sp}_4(2), & \end{array}$$

and W is the corresponding natural orthogonal or symplectic module. Thus, one of the cases (1) – (9) of **FMT** holds or $J \cong \mathrm{Alt}(6)$. In particular, (2:a) holds. It remains to determine H°. Observe first that in all cases $\mathrm{N}_R(P)$ acts transitively on $\mathrm{C}_W(P)$. Also $H = R\mathrm{N}_H(P)$ and so the Frattini argument gives $H = R\mathrm{C}_H(v) \leq R\mathrm{N}_H(Q)$, where $0 \neq v \in \mathrm{C}_V(PQ)$. Thus $H^\circ = RQ$ and so $\mathrm{O}_p(RQ) \leq \mathrm{O}_p(H) = 1$.

If $Q \leq R$, (2:c:2) holds. So suppose $Q \not\leq R$. Then some element of Q induces an outer automorphism on R. If $R \cong \mathrm{G}_2(2)'$, then $H^\circ \cong \mathrm{G}_2(2)$. If $R \cong \mathrm{Sp}_4(2)'$ then $R \cong \mathrm{Sp}_4(2)$ and if $R \cong 3^{\cdot}\mathrm{Alt}(6)$, then $H \cong 3^{\cdot}\mathrm{Sym}(6)$ since H° acts on W. Thus again (2:c:2) holds.

In all the remaining case R is genuine group of Lie-type in characteristic p. If R has Lie-rank 1, $R \cong \mathrm{SL}_2(q)$. Hence either W is a natural $\mathrm{SL}_2(q)$-module and (1) holds, or W is a natural $\Omega_4^-(2)$-module. In the latter case $P \cong \mathrm{Alt}(4)$, so $[P,Q] \leq \mathrm{O}_2(P)$ implies $Q \leq R$, which is not the case.

Suppose now that R has rank at least two. Note that $\mathrm{N}_R(P)$ is a maximal parabolic subgroup of R and $P = \mathrm{O}^{p'}(\mathrm{N}_R(P))$. Since $[P,Q] \leq \mathrm{O}_p(P)$ and $Q \not\leq R$ we can argue as in (Case 4) and conclude that Q induces non-trivial field automorphisms on R. Moreover, R must have root groups defined over two different fields (and so **FMT** implies $R \cong \mathrm{SU}_n(q), n \geq 4$, or $R \cong \Omega_{2n}^\epsilon(q)$, $2n \geq 6$), and Q can only centralize root groups defined over the smaller field. Since Q centralizes $P/O_p(P)$, this shows that all roots groups in $P/O_p(P)$ must have order q .

The last condition rules out the natural $\Omega_{2n}^\epsilon(q)$-module and shows that $n = 4$ if W is a natural $\mathrm{SU}_n(q)$-module. Also Q induces a field automorphism of order 2 and so $p = 2$. Thus (2:c:2) follows. □

The next result is inspired by a situation that arises in applications of the quadratic L-lemma 2.9. Let L be a finite group, V be an elementary abelian normal p-subgroup with $[V, \mathrm{O}^p(L)] = V$, and A be an elementary abelian p-subgroup with $A \trianglelefteq A\mathrm{O}_p(L)$. Then Y and V normalize each other, so V acts quadratically on Y, and vice versa.

Suppose that $H := L/\mathrm{C}_L(V)$ and $A := Y\mathrm{C}_L(V)/\mathrm{C}_L(V)$ satisfy the hypothesis of 2.9 and that $V/\mathrm{C}_V(O^p(L))$ is a simple L-module. Then $H \cong \mathrm{SL}_2(q)$, $\mathrm{Sz}(q)$ or D_{2r}, and by 2.10 $V/\mathrm{C}_V(\mathrm{O}^p(L))$ is a natural module for H. The structure of these modules shows that V is a strong offender on Y and

$$(*) \qquad [Y,V] = [Y,X] \text{ for all } X \leq V \text{ with } |XC_V(Y)/C_V(Y)| \geq 4.$$

In other words, the module Y possesses a non-trivial strong offender V with the additional property $(*)$. Such modules are investigated in the next result, where as usual the module is called V rather than Y.

Theorem 4.7. *Let H be a finite group with $\mathrm{O}_p(H) = 1$, and let V be a faithful $Q!$-module. Suppose that there exists $1 \neq W \leq H$ such that*

(i) W is a strong offender on V; and

(ii) $[X, W] = [V, W]$ for all $X \leq V$ with $|X/\mathrm{C}_X(W)| > 2$.

Put $H^\circ := \langle Q^H \rangle$, $K^* := \langle W^H \rangle$, $K := \langle W^{K^*} \rangle$ and $\mathcal{K} := K^H$. Then

$$K^* = \bigtimes \mathcal{K} \text{ and } [V, K^*] = \bigoplus_{R \in \mathcal{K}} [V, R].$$

Moreover, one of the following holds:

1. (a) $K \trianglelefteq H$, $H^\circ = \mathrm{O}^p(K)Q$, and $\mathrm{C}_V(K) = 0$.

(b) $K \cong \mathrm{SL}_n(q)$, $n \geq 3$, $\mathrm{Sp}_{2n}(q)$, $\mathrm{Sp}_4(2)'$, $\mathrm{O}^\epsilon_{2n}(2)$ or $3{}^{\cdot}\mathrm{Alt}(6)$; q a power of p , $p = 2$ in the last three cases; and $[V, K]$ is a corresponding natural module.

(c) Either $H^\circ \leq K$ or $K \cong \mathrm{Sp}_4(2)'$ and $H^\circ \cong \mathrm{Sp}_4(2)$ or $K \cong 3{}^{\cdot}\mathrm{Alt}(6)$ and $H^\circ \cong 3{}^{\cdot}\mathrm{Sym}(6)$.

(d) If $K \cong \mathrm{O}^\epsilon_{2n}(2)$, then $|W| = 2$.

2. (a) Q acts transitively on $\mathcal{K}$, $H^\circ = \mathrm{O}^p(K^*)Q$, and $V = [V, K^*]$.

(b) $K \cong \mathrm{SL}_2(q)$, and $[V, K]$ is a corresponding natural module.

3. (a) $p = 2$, $K \cong \mathrm{SL}_n(2)$, $n \geq 3$, $V = [V, K]$ is the direct sum of two isomorphic natural modules for K, and $|V/\mathrm{C}_V(W)| = 4$.

(b) $K \trianglelefteq H$, $K \leq H^\circ$, and $H^\circ \cong \mathrm{SL}_n(2)$ or $\mathrm{SL}_n(2) \times \mathrm{SL}_2(2)$.

4. (a) $p = 2$, $K \cong \mathrm{SL}_n(2)$, $n \geq 3$, $V = \mathrm{C}_V(K^*) \oplus [V, K^*]$, $[V, K]$ is the direct sum of two isomorphic natural modules for K, and $|V/\mathrm{C}_V(W)| = 4$.

(b) $K^* \trianglelefteq H$, $[K^*, H^\circ] = 1$ and $H^\circ \cong \mathrm{SL}_2(2)$.

Proof. Set $\overline{V} := V/\mathrm{C}_V(K)$. Since W is a strong offender, [MS4, 1.6] implies that W is a quadratic best offender on V and so $K^* = \mathrm{J}_{K^*}(V)$. Set $\mathcal{J} = \mathcal{J}_{K^*}(V)$ and $\mathcal{J}_W = \{J \in \mathcal{J} \mid [J, W] \neq 1\}$. **GFMT** implies that $\langle \mathcal{J} \rangle = \bigtimes \mathcal{J}$, $J = [J, W]$ for all $J \in \mathcal{J}_W$, and $K^*/\langle \mathcal{J} \rangle$ is an elementary abelian p-group. The latter fact shows that $K = \langle \mathcal{J}_W \rangle W$ and $\langle \mathcal{J}_W \rangle = \mathrm{O}^p(K) = [\mathrm{O}^p(K), W]$. Pick $J \in \mathcal{J}_W$, and let Y be a minimal non-trivial J-submodule of V. By [MS4, 2.11] Y is a quasisimple and so perfect as an K-module. Then $[Y, W] \neq 0$, and by [MS4, 1.9] Y is K^*-invariant.

Case 1. *Suppose $[V, \mathrm{O}^p(K)] \nleq Y$.*

Since $\mathrm{O}^p(K) = [\mathrm{O}^p(K), W]$ we conclude that $[V, W] \nleq Y$ and $[Y, W] \neq [V, W]$. Now (ii) implies that $p = 2$ and $|Y/\mathrm{C}_Y(W)| = 2$. Moreover $|V/\mathrm{C}_V(W)| \geq 4$ and since W is an offender, $|W| \geq 4$. Since W is a strong offender and $[Y, W] \neq 0$, $\mathrm{C}_W(Y) = \mathrm{C}_W(V) = 1$. Thus $1 = \mathrm{C}_W(Y) = \mathrm{C}_W(J) = \mathrm{C}_W(\overline{Y})$ by [MS4, 2.4], and W acts faithfully on $\overline{Y}$. Since $|\overline{Y}/\mathrm{C}_{\overline{Y}}(W)| = 2$, this gives $|[\overline{Y}, W]| = |W| \geq 4$. In particular, $\overline{Y}$ is not a selfdual JW-module since (for example by [MS4, 1.8]), $|\overline{Y}/\mathrm{C}_{\overline{Y}}(W)| = |[\overline{Y}^*, W]$. We now apply **FMT**. Then the properties $|\overline{Y}/\mathrm{C}_{\overline{Y}}(W)| = 2 < |[\overline{Y}, W]|$ and $\overline{Y}$ being not self-dual eliminates all cases apart from the case where $K/\mathrm{C}_K(Y) \cong \mathrm{SL}_n(2)$, $n \geq 3$, $|Y/\mathrm{C}_Y(W)| = 2$. Now an application of [MS4, 8.4] yields $\mathrm{C}_Y(K) = 0$ and so:

1°. *$K/\mathrm{C}_K(Y) \cong \mathrm{SL}_n(2)$, $n \geq 3$, $|Y/\mathrm{C}_Y(W)| = 2$, and Y is a natural $\mathrm{SL}_n(2)$-module.*

Assume that $\mathrm{O}^2(K) \neq J$. Then there exist $J_1 \in \mathcal{J}_W$ with $J_1 \neq J$ and a simple J_1-submodule Y_1 in V satisfying (1°) in place of J and Y. Then $|Y + Y_1/\mathrm{C}_{Y+Y_1}(W)| \geq 4$, and (ii) implies that $[V,W] \leq Y + Y_1$. Hence $[V, K^*] = Y + Y_1$, $Y = [V,J]$ and $Y_1 = [V,J_1]$. By 4.3 $JJ_1 \leq H^\circ$ and so $[H^\circ, W] \neq 1$. Thus we can apply 4.6. But the only case in 4.6 with more than one $\mathrm{J}_H(V)$-component is 4.6(1), where there $\mathrm{J}_H(V)$-components are isomorphic to $\mathrm{SL}_2(q)$. This contradicts (1°).

We have shown that $\mathrm{O}^2(K) = J$, so $K = JW$. Moreover, by (1°) W induces inner automorphisms in J. Since $\mathrm{O}_2(K) \leq \mathrm{O}_2(H) = 1$, we conclude that

$$K = J,\ [V,J] \not\leq Y \text{ and } \mathrm{C}_J([V,J]) = 1,$$

in particular by (1°) $J \cong \mathrm{SL}_n(2)$, $n \geq 3$. Moreover, [MS4, 8.4] shows that $V = \mathrm{C}_V(K) \oplus [V,K]$. Now

$$K^* = \bigtimes \mathcal{K}, \quad [V,K^*] = \bigoplus_{J \in \mathcal{K}} [V,J] \text{ and } V = \mathrm{C}_V(K^*) \oplus [V,K^*].$$

follow from [MS4, 2.2].

By **GFMT**, $[\overline{V}, J]$ is semisimple J-module and so there exists a minimal non-trivial J-submodule Y_1 with $Y \neq Y_1$. Put $Y_2 := Y$. Then (1°) applies to (Y_i, J) for $i = 1,2$. In particular, $|Y_i/\mathrm{C}_{Y_i}(W)| = 2$ and $|[Y_i, W]| = |W| > 2$ for $i = 1,2$. Hence, Y_1 is not isomorphic to the dual of Y_2 and so Y_1 and Y_2 are isomorphic natural modules. As above, (ii) implies $[V,W] \leq Y_1 + Y_2$ and $[V,J] = Y_1 \oplus Y_2$.

If $[H^\circ, W] \neq 1$, we can apply 4.6. By (1°), Case 1 of 4.6 does not hold. Hence by 4.6(2:b) $\mathrm{F}^*(\langle \mathcal{K} \rangle)$ is quasisimple and so $\mathcal{K} = \{J\}$ and $J \trianglelefteq H$. If $[H^\circ, W] = 1$, we have $[J, H^\circ] = 1$. In any case H° normalizes J and so also $[V,J]$. Since $[V,J] = Y_1 \oplus Y_2$, the normalizer of the image of J in $\mathrm{Aut}([V,J])$ is isomorphic to $\mathrm{SL}_n(2) \times \mathrm{SL}_2(2)$. Since $\mathrm{O}_2(H^\circ) = 1$ and H° is generated by 2-elements we conclude that $H^\circ/\mathrm{C}_{H^\circ}([V,J])$ is isomorphic to $\mathrm{SL}_n(2)$, $\mathrm{SL}_2(2)$ or $\mathrm{SL}_n(2) \times \mathrm{SL}_2(2)$. By 4.2(d), $\mathrm{C}_{H^\circ}([V,J]) \leq \mathrm{Z}(H^\circ)$, and by [Gr] the Schur multipliers of the above groups are 2-groups, so $\mathrm{C}_{H^\circ}([V,J]) \leq \mathrm{O}_2(H^\circ) = 1$.

Suppose $J \leq H^\circ$. Then $K = J \trianglelefteq H^\circ$, $[J, H^\circ] \neq 1$ and by 4.2(g) $\mathrm{C}_V(K) = \mathrm{C}_V(J) = 0$. Hence $V = [V,K]$ and (3) holds.

Suppose that $J \not\leq H^\circ$. Then $[J, H^\circ] = 1$, $J \cong \mathrm{SL}_n(2)$ and $H^\circ \cong \mathrm{SL}_2(2)$. It follows that also $[K^*, H^\circ] = 1$, and (4) holds.

Case 2. $\quad [V, \mathrm{O}^p(K)] = Y$.

In this case clearly $\mathrm{O}^p(K) = J$ and so $K = JW$. By 4.3, $J \leq H^\circ$. Thus $[H^\circ, W] \neq 1$, and we are allowed to apply 4.6. If $J \ntrianglelefteq H$, 4.6 shows that (2) holds. So suppose $J \trianglelefteq H$ and so also $K \trianglelefteq H$. Comparing 4.6 with 3.2 we see that (1) holds if $K \ncong \mathrm{SL}_2(q)$. In the latter case we are in case (2) with $K = K^*$. $\square$

Lemma 4.8. *Let $1 \neq A \leq H$ be a strong dual offender on V. Put $K^* := \langle A^H \rangle$, $K := \langle A^{K^*} \rangle$ and $\mathcal{K} := K^H$. Then one of the following holds.*

1. *(a) $K \trianglelefteq H$, $H^\circ = \langle Q^K \rangle$ and $\mathrm{C}_V(K) = 0$.*
   (b) $K \cong \mathrm{SL}_n(q)$, $n \geq 3$, $\mathrm{Sp}_{2n}(q)$, $\mathrm{Alt}(6)$, or $\mathrm{O}^\epsilon_{2n}(2)$, q a power of p , $p = 2$ in the last two cases; and $[V,K]$ is a corresponding natural module.
   (c) Either $H^\circ \leq K$ or $K \cong \mathrm{Sp}_4(2)'$ and $H^\circ \cong \mathrm{Sp}_4(2)$.
   (d) If $K \cong \mathrm{O}^\epsilon_{2n}(2)$, then $|W| = 2$.
2. *(a) Q acts transitively on $\mathcal{K}$ and $H^\circ \leq \langle \mathcal{K} \rangle Q$*
   (b) $V = \bigoplus_{R \in \mathcal{K}} [V,R]$, $K \cong \mathrm{SL}_2(q)$, and $[V,K]$ is a natural $\mathrm{SL}_2(q)$-module for K.

Proof. By [MS4, 1.5] A is a quadratic best offender on V. Let J be a $\mathrm{J}_H(V)$-component with $[J, A] \neq 1$ and W a quasisimple J-submodule of V. Then $1 \neq [W, A] \leq W$ by [MS4, 2.6], and since A is a strong dual offender, $[V, A] = [W, A]$. Thus J is the unique $\mathrm{J}(V)$-component of H not centralized by A and $K = JA$. Moreover, $W = [V, J] = [W, K]$ and so by 4.3, $J \leq H^\circ$. Hence $[H^\circ, A] \neq 1$, and the lemma follows from 4.6 and 3.1. □

5 Minimal asymmetric modules

In this section H is a finite group and V is an $\mathbb{F}_pH$-module.

Definition 5.1. *Let A and B be p-subgroups of H with $A \leq B$. Then V is a minimal asymmetric $\mathbb{F}_pH$-module (with respect to $A \leq B$) provided that*

(i) $A \trianglelefteq \mathrm{N}_H(B)$*, and B is a weakly closed subgroup of H,*

(ii) $[V, A, B] = [V, B, A] = 0$*,*

(iii) $\langle A^H \rangle$ *does not act nilpotently on V,*

(iv) $\langle A^F \rangle$ *acts quadratically on V for every proper subgroup F of H with $B \leq F$.*

Note that conditions (i) and (iii) imply that A is non-trivial normal subgroup of B.

As many definitions in the previous section, the above definition is motivated be the investigation of the p-local structure of finite groups of local characteristic p. Consider the following situation:

G is a finite group of characteristic p, $S \in \mathrm{Syl}_p(G)$, and A is an abelian normal subgroup of S with $A \not\leq O_p(G)$. Put $B := C_S(A)$. Then $A \leq B$ since A is abelian. Suppose that B is weakly closed in G and that $[A, A^g] = 1$ for all $g \in G$ such that $\langle A, A^g \rangle$ is p-group. Then for every $A \leq L \leq G$ with $A \leq \mathrm{O}_p(L)$, $X := \langle A^{\mathrm{O}_p(G)L} \rangle$ is p-group and so abelian. In particular, X acts quadratically on $\mathrm{O}_p(G)$ and $\langle A^{\mathrm{N}_G(B)} \rangle$ is abelian. So replacing A by $\langle A^{N_G(B)} \rangle$ we may assume that $A \trianglelefteq N_G(B)$. Now let H be a subgroup of G minimal with $B \leq H$ and $A \not\leq \mathrm{O}_p(H)$, and let V be elementary abelian H-invariant section of $\mathrm{O}_p(G)$ not centralized by $\langle \mathrm{O}^p(\langle A^H \rangle)$. Then V is a minimal asymmetric module for H.

Lemma 5.2. *Let V be a minimal asymmetric $\mathbb{F}_pH$-module with respect to the subgroups $A \leq B$. Suppose that $\mathrm{C}_V(H) = 0$. Then the following hold:*

(a) AA^g acts quadratically on V for every $g \in H$ with $[A, A^g] \leq A \cap A^g$.

(b) $\langle A^P \rangle$ acts quadratically on V for every $P \leq H$ with $A \leq \mathrm{O}_p(P) \leq \mathrm{N}_H(A)$.

Proof. (a): By 5.1(ii) A acts quadratically on V. Put $F := \langle B, B^g \rangle$. Suppose first that $F = H$. Then $[V, A, A^g] \leq [V, A] \cap [V, A^g] \leq \mathrm{C}_V(\langle B, B^g \rangle) = \mathrm{C}_V(H) = 0$. By symmetry, $[V, A^g, A] = 0$ and so AA^g is quadratic.

Suppose next that $F \neq H$. By 5.1(iv), $\langle A^F \rangle$ acts quadratically on V. Since B is a weakly closed subgroup of H there exists $f \in F$ with $B^{gf} = B$. Since $A \trianglelefteq \mathrm{N}_G(B)$, this gives $A^{gf} = A$. Thus $A^g \in A^F$, and AA^g is quadratic.

(b): Note that $A^g \trianglelefteq \mathrm{O}_p(P)$ for every $g \in P$. Hence (b) follows from (a). □

Lemma 5.3. *Let G be a group, $T \leq G$, and $g \in G$ with $[T, T^g] = 1$. Then $T' \leq [T, \langle g \rangle]$.*

Proof. Note that $T \le T^g[T, \langle g \rangle]$ and so

$$T' = [T, T] \le [T, T^g[T, \langle g \rangle]] = [T, [T, \langle g \rangle]] \le [T, \langle g \rangle].$$

$\square$

Lemma 5.4. *Let V be a faithful simple minimal asymmetric $\mathbb{F}_pH$-module with respect to $A \le B$. Put $L := \langle A^H \rangle$ and $K := \mathrm{F}^*(H)$. Then $H = KB$, $K = [K, A] \le L$, $L = KA$, and one of the following holds:*

1. *$|B| = 2$ and $H = L \cong \mathrm{D}_{2r}$, r an odd prime.*
2. *$|A| = 2$, $L \cong \mathrm{SU}_3(2)'$, $B \cong \mathrm{C}_4$ or Q_8, and V is a natural $\mathrm{SU}_3(2)'$-module for L.*
3. *$|B| = 3$, $H = L \cong \mathrm{SL}_2(3)$, and V is a natural $\mathrm{SL}_2(3)$-module for L.*
4. *K is quasisimple and not a p'-group, $H = KB$, V is a simple $\mathbb{F}_pK$-module, and H acts $\mathbb{K}$-linearly on V, where $\mathbb{K} = \mathrm{End}_K(V)$.*

Proof. Let $B \le T \in \mathrm{Syl}_p(\mathrm{N}_H(B))$. From 5.1(ii) and the Three Subgroups Lemma we get that $[B, A, V] = 0$. Hence the faithful action of H and the quadratic action of A give $A \le \Omega_1\mathrm{Z}(B)$. Moreover, $T \in \mathrm{Syl}_p(H)$ since B is weakly closed in H. Thus we have:

1°. *$A \le \Omega_1\mathrm{Z}(B)$, $A \trianglelefteq T$, and $T \in \mathrm{Syl}_p(H)$.*

Next we show:

2°. *Let R be a B-invariant subgroup of H with $[R, A] \not\le \mathrm{O}_p(R)$. Then $H = RB$ and $K \le R$. In particular, $H = KB$, $K = [K, A]$, and $L = KA$.*

Note that $A \not\le \mathrm{O}_p(RB)$; in particular $\langle A^{RB} \rangle$ does not act quadratically on V. Thus by 5.1(iv), $RB = H$ and therefore $\mathrm{O}^p(H) \le R$. Since V is a faithful simple H-module, $\mathrm{O}_p(H) = 1$ and so $K = \mathrm{O}^p(K) \le \mathrm{O}^p(H) \le R$.

Again since $\mathrm{O}_p(H) = \mathrm{O}_p(K) = 1$, $[K, A, A] \ne 1 = \mathrm{O}_p([K, A])$ and $A \not\le \mathrm{O}_p([K, A]B)$. Hence for $R = [K, A]$ we get $H = [K, A]B$ and $K = [K, A]$. Since $KA \trianglelefteq KB = H$ also $L = KA$ follows.

According to (2°) the initial statements in the conclusion hold, so it remains to establish one of (1) – (4).

Suppose that $|B| = 2$. By Baer's Theorem there exists $g \in H$ such that $\langle B, B^g \rangle$ is not a 2-group. Choose $\langle B, B^g \rangle$ minimal with that property. Then $\langle B, B^g \rangle \cong \mathrm{D}_{2r}$, r a prime, and by (2°) $\langle B, B^g \rangle = H$, so (1) holds. Hence, we may assume from now on:

3°. $|B| > 2$.

Let W be a simple $\mathbb{F}_pL$-submodule of V with $[W, A] \ne 0$. Then $[W, A] \le \mathrm{C}_W(B)$. Since B normalizes L, B also normalizes W, and since $H = KB = LB$, we get $W = V$. So:

4°. *V is a simple $\mathbb{F}_pL$-module.*

Now let W be a Wedderburn component for K on V [2] and suppose that $V \neq W$. Since $L = KA$, (4°) implies $V = \langle W^A \rangle$. Hence A acts transitively on the set of Wedderburn components for K on V and therefore $W^A = W^B$. By [MS3, 2.11] $|W^A| = 2$ and since $|B| > 2$ we get $\mathrm{N}_B(W) \neq 1$. But $[W, \mathrm{N}_B(W)] \leq \mathrm{C}_W(A) = 0$, and by (1°) $\mathrm{N}_B(W)$ centralizes A. So $\mathrm{N}_B(W) \leq \mathrm{C}_B(\langle W^A \rangle) = \mathrm{C}_B(V) = 1$, a contradiction.

Thus $W = V$, and V a is homogeneous $\mathbb{F}_p K$-module. It follows that the number of simple $\mathbb{F}_p K$-submodules in V is not divisible by p, see for example [Go, 3.5.6]. Hence A normalizes a simple $\mathbb{F}_p K$-submodule. Since $L = KA$, (4°) implies that V is a simple $\mathbb{F}_p K$-module.

Put $\mathbb{K} = \mathrm{End}_K(V)$. Observe that by Schur's Lemma and Wedderburn's Theorem $\mathbb{K}$ is a finite field. Moreover, H acts $\mathbb{K}$-semilinearly on V. Suppose that A does not act $\mathbb{K}$-linearly on V. Then by [MS3, 2.14] [3], $|A| = 2$, and since the non-trivial element in A inverts an element in $\mathbb{K} \setminus \{0\}$, $\mathrm{C}_V(A) = [V, A]$. Thus $[V, B] \leq \mathrm{C}_V(A) = [V, A] \leq \mathrm{C}_V(B)$, and B acts quadratically on V. So using [MS3, 2.14] one more time, $|B| = 2$, a contradiction. Thus A acts $\mathbb{K}$-linearly on V and $[V, A]$ is a non-trivial $\mathbb{K}$-subspace of V centralized by B. So also B acts $\mathbb{K}$-linearly on V. Since $H = KB$, H acts $\mathbb{K}$-linearly on V. We have proved:

5°. *V is a simple $\mathbb{F}_p K$-module, $\mathbb{K}$ is a field, and H acts $\mathbb{K}$-linearly on V.*

Since the image of $\mathrm{C}_H(K)$ in $\mathrm{End}(V)$ is contained in $\mathbb{K}$, (5°) implies that $\mathrm{C}_H(K)$ is a cyclic p'-group and $\mathrm{C}_H(K) = \mathrm{Z}(H)$. Clearly $\mathrm{Z}(H) \leq \mathrm{F}^*(H) = K$ and so $\mathrm{Z}(H) \leq \mathrm{Z}(K)$. Thus

6°. $\mathrm{Z}(K) = \mathrm{C}_H(K) = \mathrm{Z}(H)$ *is a cyclic p'-group. In particular,* $\mathrm{C}_B(K) = 1$, *and K is not abelian.*

Case 1. *Suppose that K is a p'-group.*

By (2°), A centralizes every proper B-invariant subgroup of K, and by (6°) K is not abelian. Hence [Go, 5.3.7] shows that K is special and that $K/\Phi(K)$ is a simple $\mathbb{F}_r B$-module. By (6°) K is extraspecial. Moreover, by coprime action $\mathrm{C}_B(K/\mathrm{Z}(K)) \leq \mathrm{C}_B(K) = 1$, and so $K/\mathrm{Z}(K)$ is a faithful simple $\mathbb{F}_r B$-module. Hence $\mathrm{Z}(B)$ is cyclic and since $A \leq \Omega_1 \mathrm{Z}(B)$, $|A| = p$.

Suppose first that p is odd. Since A is quadratic and $K = [K, A]$ we can apply [MS4, 7.1]. Hence $K \cong \mathrm{Q}_8$, $KA \cong \mathrm{SL}_2(3)$ and $p = 3$. As V is a simple $\mathbb{F}_3 K$-module, it has dimension 2, and (3) holds.

Suppose next that $p = 2$. Then r is odd, and by [Go, 5.3.9(i) and 5.3.10] K has exponent r. Let Y be a maximal abelian subgroup of K and put $\mathcal{X} = \{X \leq Y \mid Y = X \times \mathrm{Z}(K)\}$. Then K acts transitively on $\mathcal{X}$. Moreover $\mathrm{C}_V(\mathrm{Z}(K)) = 0$, and coprime action shows that

$$(*) \qquad V = \bigoplus_{X \in \mathcal{X}} \mathrm{C}_V(X).$$

Note that $\mathrm{N}_K(X) = Y = X\mathrm{Z}(K)$ for $X \in \mathcal{X}$. Hence $\mathrm{N}_K(X)$ acts as $\mathbb{K}$-scalar multiplication on $\mathrm{C}_W(X)$. Then $(*)$ and the simplicity of W as an $\mathbb{F}_p K$-module yield $\dim_{\mathbb{K}} \mathrm{C}_W(X) = 1$ for each $X \in \mathcal{X}$.

Let $1 \neq t \in A$ and put $X_t := [Y, t]$ and $V_t := \mathrm{C}_V(X_t)$. Since t inverts $K/\mathrm{Z}(K)$ and centralizes $\mathrm{Z}(K)$, $X_t \in \mathcal{X}$, so $\dim_{\mathbb{K}} V_t = 1$ and $V_t \leq \mathrm{C}_V(t)$. Moreover, t does not fix any other element in $\mathcal{X}$, and $(*)$ shows that $\mathrm{C}_V(t) = V_t \oplus [V, t]$. In particular, $\dim_{\mathbb{K}} \mathrm{C}_V(t)/[V, t] = 1$ and $[\mathrm{C}_V(t), B] \leq [V, t]$. Since $[V, A, B] = 0$, B acts quadratically on $V/[V, t]$ and $\mathrm{C}_V(t)$; so $B/\mathrm{C}_B(V/[V, t])$ and $B/C_B(\mathrm{C}_V(t))$ are elementary abelian. Thus

$$[V, \Phi(B), B] \leq [V, t, B] = 0 \text{ and } [V, B, \Phi(B)] \leq [\mathrm{C}_V(t), \Phi(B)] = 0.$$

[2] i.e., a maximal homogeneous K-submodule of V

[3] Note the misprint in [MS3, 2.14]: Instead of '$|\mathbb{K}|$ is a cubic $\mathbb{E}A$-module' it should read 'V is a cubic $\mathbb{E}A$-module'

Since B acts faithfully on V, the Three Subgroups Lemma yields $\Phi(B) \leq \mathrm{Z}(B)$, and the quadratic action of $\Phi(B)$ on V shows that $\Phi(B)$ is elementary abelian. Since $\mathrm{Z}(B)$ is cyclic, this gives $\Phi(B) = A$. In particular, there exists $f \in B$ with $f^2 = t$.

On the other hand, f centralizes $[V,t]$, so $[V,t] \leq \mathrm{C}_V(f)$. Put $V_0 := \{v \in V \mid [v,f] \in [V,t]\}$. Then f acts quadratically on V_0 and so $V_0 \leq \mathrm{C}_V(t)$. Moreover, the map given by $v \mapsto [v,f] + [V,t]$ shows that $V/V_0 \cong [V,f]/[V,t]$. From $[V,f] \leq \mathrm{C}_V(t) = V_t \oplus [V,t]$ and $\dim_{\mathbb{K}} V_t = 1$ we conclude that

$$1 \leq \dim_{\mathbb{K}} V/\mathrm{C}_V(t) \leq \dim_{\mathbb{K}} V/V_0 = \dim_{\mathbb{K}}[V,f]/[V,t] \leq 1.$$

So $\dim V/\mathrm{C}_V(t) = 1 = \dim_{\mathbb{K}}[V,t]$ and $\dim_{\mathbb{K}} V = 3$. In particular, $|\mathcal{X}| = 3$ and $|K| = 3^3$. Thus (2) holds in this case.

Case 2. *K is not a p'-group.*

By (2°) $[\mathrm{O}_{p'}(K), A] = 1$ and $K = [K,A]$, so $\mathrm{O}_{p'}(K) \leq \mathrm{Z}(K)$ and by (6°) $\mathrm{O}_{p'}(K) \leq \mathrm{Z}(H)$. Since $K = \mathrm{F}^*(K)$ and $\mathrm{O}_p(H) = 1$ we conclude that there exists a component E of K with $p \big| |E/\mathrm{Z}(E)|$. Since $K = [K,A]$ we have $[E,A] \neq 1$ and by (2°) $K = \langle E^B \rangle$. Put $F := \mathrm{C}_H(A) \cap \mathrm{C}_H([V,A])$. Note that $B \leq F$ and by the Three Subgroups Lemma, $[V,F,A] = 0$. Since $\mathrm{Z}(E) \leq \mathrm{Z}(K)$, $\mathrm{Z}(E)$ acts as $\mathbb{K}$-scalar multiplication on V, and so $F \cap \mathrm{Z}(E) = 1$.

Suppose that $E \cap F \neq 1$. Then $E \cap F \not\leq \mathrm{Z}(E)$ and so $A \leq \mathrm{C}_H(E \cap F) \leq \mathrm{N}_H(E)$. Let U be a simple $\mathbb{F}_p E$-submodule of V. Since $E \trianglelefteq K$ and V is a simple K-module, $\mathrm{C}_V(E) = 0$ and thus $[U,E] \neq 0$. As E is quasisimple this gives $\mathrm{C}_U(E) \leq \mathrm{Z}(E)$. Then $0 \neq [U, F \cap E] \leq \mathrm{C}_U(A)$ and so A normalizes U. Hence A normalizes all simple $\mathbb{F}_p E$-submodules of V and since $L = \langle A^K \rangle$, the same is true for L. Thus U is L-invariant, and (4°) shows that $V = U$. It follows that $\mathrm{C}_H(E)$ is abelian, so $K = E$ and (4) holds.

Suppose next that $E \cap F = 1$. Then $E \cap B = 1$. If $B \leq \mathrm{N}_H(E)$, (4) holds. So we may assume that $B \not\leq \mathrm{N}_H(E)$. Pick $b \in B$ with $E \neq E^b$. Then by 5.3 $(E \cap T)' \leq [E \cap T, \langle b \rangle] \leq [T,B] \leq B$. Since $E \cap B = 1$ we conclude that $E \cap T$ is abelian. By Burnside's Transfer Theorem,

$$(**) \qquad E \cap T \not\leq \mathrm{Z}(\mathrm{N}_E(T \cap E)),$$

and so $\mathrm{N}_E(T \cap E)$ is not a p-group. Put $D = \langle A^{\mathrm{N}_H(T \cap K)} \rangle$. Since $B \leq N_H(T \cap K) < H$, 5.1(iv) shows that D acts quadratically on V. Hence D is an elementary abelian p-group and $D \leq F$. Since $\mathrm{N}_E(T \cap E) \leq \mathrm{N}_E(T \cap K)$ we conclude that $[\mathrm{N}_E(T \cap E), A]$ is p-group. Since $\mathrm{N}_E(T \cap E)$ is not a p-group this gives $A \leq \mathrm{N}_H(E)$ and so $[\mathrm{N}_E(T \cap E), A] \leq D \cap E \leq F \cap E = 1$. Thus $\mathrm{N}_E(T \cap E) \leq \mathrm{N}_H(A)$. Since $B \leq F$ and $\mathrm{N}_H(A)$ normalizes F, this implies $[\mathrm{N}_E(T \cap E), B] \leq F$. By 5.3 $(\mathrm{N}_E(T \cap E))' \leq [\mathrm{N}_E(T \cap E), \langle b \rangle] \leq F$, and so $E \cap F = 1$ implies that $\mathrm{N}_E(T \cap E)$ is abelian, a contradiction to $(**)$. □

Theorem 5.5 (**Minimal Asymmetric Module Theorem**). *Let H be a $\mathcal{CK}$-group, $A \leq B \leq H$ and V be a faithful simple $\mathbb{F}_p H$-module. Suppose that V is a minimal asymmetric $\mathbb{F}_p M$-module with respect to A and B and that $\mathrm{F}^*(H)$ is quasisimple with $p \big| |\mathrm{F}^*(H)|$. Then one of the following holds for $L := \langle A^H \rangle$:*

1. *$L \cong \mathrm{SL}_n(q), \mathrm{Sp}_{2n}(q), \mathrm{SU}_n(q), {}^3\mathrm{D}_4(q)$, $\mathrm{Spin}_7(q)$, $\mathrm{Spin}_8^-(q)$, $\mathrm{G}_2(q)'$ or $\mathrm{Sz}(q)$, where q is a power of p, V is the corresponding natural module for L, and A is a long root subgroup of L.*

2. *$L \cong \mathrm{Sym}(2^k + 2)$, $k \geq 3$, $|A| = 2$, A is generated by a transposition, and V is the corresponding natural module.*

3. *$L \cong 3{\cdot}\mathrm{Alt}(6)$, $|A| = 2$ and $|V| = 2^6$.*

Proof. Observe that $\mathrm{O}_p(H) = 1$ since V is simple and faithful. Put $K := \mathrm{F}^*(H)$ and let $S \in \mathrm{Syl}_p(H)$ with $B \leq S$.

1°. *$B \trianglelefteq S$ and $A \trianglelefteq S$.*

The first statement follows since B is a weakly closed subgroup of H. The second then follows from the fact that $A \trianglelefteq \mathrm{N}_H(B)$.

2°. *$A \leq \mathrm{O}_p(U)$ for every $B \leq U < H$.*

Observe that $\langle A^U \rangle$ is quadratic on V, so $\langle A^U \rangle$ is an elementary abelian normal p-subgroup of U.

3°. *$K = [K, A]$, $H = KB$ and $L = KA = \langle A^L \rangle$.*

See 5.4.

4°. *V is a simple $\mathbb{F}_p K$-module.*

See 5.4.

5°. *There exists $g \in H$ such that AA^g acts quadratically on V and $|AA^g| > 2$.*

If $|A| > 2$ we can choose $g = 1$. So suppose that $|A| = 2$. Then $p = 2$ and so by assumption $|K|$ has even order. Since $K \leq [H, A]$ we conclude from Glauberman's Z^*-theorem [Gl] that there exists $g \in H$ with $[A, A^g] = 1$ and $A \neq A^g$. Now 5.2(a) shows that AA^g is quadratic.

6°. *Suppose that A is a maximal quadratic subgroup of H. Then $|A| > 2$, and $\mathrm{N}_M(A)$ is the unique maximal subgroup of H containing B. In particular, H is B-minimal.*

Since A is a maximal quadratic subgroup, (5°) shows that $|A| > 2$.

Let $B \leq U < H$. Then $\langle A^U \rangle$ is quadratic and so by the maximality of A, $A = \langle A^U \rangle$ and $A \trianglelefteq U$. Hence $\mathrm{N}_H(A)$ is the unique maximal subgroup of H containing B. By (3°) $H = KB = \langle A^L \rangle B = \langle B^H \rangle$ and so H is B-minimal.

Put $\overline{H} := H/\mathrm{Z}(K)$.

Case 1. *Suppose that p is odd and $\overline{K}$ is not a group of Lie type defined over a field of characteristic p.*

By [Ch, Theorem A] $p = 3$ and the maximal quadratic subgroups of H have order 3. In particular $|A| = p = 3$, and A is a maximal quadratic subgroup of H. Moreover, $\overline{L} \cong \mathrm{PGU}_n(2)$, $\mathrm{Alt}(n), n \neq 6$, $\mathrm{D}_4(2)$, $\mathrm{G}_2(4)$, $\mathrm{Sp}_6(2)$, Co_1, Sz, or J_2. Observe that $\overline{L}$ has no outer automorphism of order 3, unless $\overline{L} \cong \mathrm{D}_4(2)$. In the $\mathrm{D}_4(2)$-case, $\mathrm{C}_{\overline{L}}(\overline{A}) \cong C_3 \times \mathrm{U}_4(2)$ and so the conjugacy class of $\overline{A}$ under $\overline{L}$ is not invariant under the outer automorphism of order three. Hence in any case $H = L$.

According to [ATLAS] we can choose a subgroup $\overline{U}$ of $\overline{H}$ as in the following chart:

$\overline{H}$	$\mathrm{PGU}_n(2)$	$\mathrm{Alt}(n), 3 \nmid n$	$\mathrm{Alt}(3m), 3m \geq 9$	$\mathrm{D}_4(2)$	$\mathrm{G}_2(4)$
$\overline{U}$	$\mathrm{C}_3 \wr \mathrm{Sym}(n)/C_3$	$\mathrm{Alt}(n-1)$	$\mathrm{O}^{3'}(\mathrm{Sym}(3) \wr \mathrm{Sym}(m))$	$\mathrm{O}^{3'}(\mathrm{Sym}(3) \wr \mathrm{Sym}(4))$	$\mathrm{U}_3(3)$
$\overline{H}$	$\mathrm{Sp}_6(2)$	Co_1	Sz	J_2	
$\overline{U}$	$\mathrm{Sym}(3) \wr \mathrm{Sym}(3)$	$3^6.2.\mathrm{Mat}_{12}$	$3^5\mathrm{Mat}_{11}$	$\mathrm{U}_3(3)$	

In each case $\overline{U}$ contains a Sylow 3-subgroup of $\overline{H}$ and $\mathrm{Z}(\mathrm{O}^{3'}(U)) = 1$. So we may assume that $\overline{B} \leq \overline{U}$. Then by (6°), $\overline{A} \trianglelefteq \overline{U}$ and so $\overline{A} \leq \mathrm{Z}(\mathrm{O}^{3'}(\overline{U})) = 1$, a contradiction.

Case 2. *Suppose that $p = 2$ and $\overline{K}$ is not a group of Lie type defined over a field of characteristic 2.*

By (5°), [MeSt1, Theorem] and [MeSt2, Theorem 1] we have $\overline{K} \cong \mathrm{Alt}(n), n = 7$ or $n > 8$, $\mathrm{U}_4(3)$, Mat_{12}, Mat_{22}, Mat_{24}, J_2, Sz, Co_2 or Co_1.

Suppose first that $\overline{K} \cong \mathrm{Alt}(n)$, $n = 7$ or $n > 8$. Then $\overline{H} \cong \mathrm{Alt}(n)$ or $\mathrm{Sym}(n)$. If B acts transitively on $\Omega = \{1, \dots, n\}$, then $n > 8$ and there exists $B \leq U \leq H$ with $\overline{U} \cong (\mathrm{Sym}(\frac{n}{2}) \wr \mathrm{C}_2) \cap \overline{H}$. But then $\mathrm{O}_2(\overline{U}) = 1$, which contradicts (2°).

Hence there exists a B-invariant proper subset Ψ of Ω. Put $U := \mathrm{N}_H(\Psi)$. Then $B \leq U < H$ and $\overline{U} \cong (\mathrm{Sym}(\Psi) \times \mathrm{Sym}(\Omega \setminus \Psi)) \cap \overline{H}$. Since $\overline{A} \leq \mathrm{O}_2(\overline{U})$ by (2°), we conclude that one of Ψ and $\Omega \setminus \Psi$, say Ψ, has size 2 or 4. Since n is none of $4, 6$ and 8, $\mathrm{O}_2(\mathrm{Sym}(\Omega \setminus \Psi)) = 1$ and so $A \leq \mathrm{C}_H(\Omega \setminus \Psi)$.

Assume $|\Psi| = 2$. Then $\overline{A} = \mathrm{O}_2(\overline{U})$, and $\overline{A}$ is generated by a transposition. In particular, $A \not\leq \mathrm{O}_2(\mathrm{N}_H(\Delta))$ for any $\Delta \subset \Omega \setminus \Psi$, so B acts transitively on $\Omega \setminus \Psi$. Hence (2) holds in this case.

So we may assume that B has no orbit of length less than 4. Then $|\Psi| = 4$, $n > 8$, and $\langle \overline{A}^{U} \rangle = \mathrm{O}_2(\overline{U})$. Since $n > 8$ and $\mathrm{Z}(K)$ is a $2'$-group, [Gr] shows that $\mathrm{Z}(K) = 1$. Hence $\mathrm{O}_2(U)$ acts quadratically on V. This shows that V is not the natural module and so by [MeSt2, Theorem 4] V is a spinmodule.

Note that there exists $g \in H$ with $\Psi \cap \Psi^g = \emptyset$ since $n > 8$. Hence $A \leq \mathrm{C}_H(\Psi^g)$ and $[A, A^g] = 1$. Now 5.2(a) shows that AA^g acts quadratically on V. But AA^g does not act quadratically on the spinmodule by [MS4, 7.5].

Suppose that $\overline{K} \cong \mathrm{U}_4(3)$. By [MS4, 7.7-7.9] there exists an elementary abelian subgroup Q of order 2^4 in K such that $\mathrm{N}_H(Q)$ contains a Sylow 2-subgroup of H, Q is not quadratic on V, $\mathrm{N}_H(Q)/QZ(K) \cong \mathrm{Alt}(6)$ or $\mathrm{Sym}(6)$ and Q is the corresponding natural module for $\mathrm{N}_H(Q)$. In particular, Q is the unique non-trivial normal 2-subgroup of $\mathrm{N}_H(Q)$. Since $\mathrm{N}_H(Q)$ is contains a Sylow 2-subgroup of H, we may assume that $B \leq \mathrm{N}_H(Q)$. Hence by (2°) $A \leq \mathrm{O}_2(\mathrm{N}_H(Q))$ and so $Q = \langle A^{\mathrm{N}_H(Q)} \rangle$. But then Q acts quadratically on V, a contradiction.

Suppose now that $\overline{K}$ is a sporadic group and so $\overline{K} \cong \mathrm{Mat}_{12}$, Mat_{22}, Mat_{24}, J_2, Sz, Co_2 or Co_1. Then by 2.6 H is not B-minimal, so by (6°) A is not a maximal quadratic subgroup of H.

Assume that $|A| \geq 4$. Then H possesses a quadratic subgroup of order at least eight. Thus, by [MeSt2, Theorem 2] $L = K \cong 3^{\cdot}\mathrm{Mat}_{22}$. Hence there exists $S \leq U \leq H$ with $U/\mathrm{O}_2(U) \cong \mathrm{Sym}(5)$. As $\mathrm{O}_2(U) \cap K$ is the unique minimal normal subgroup of U, we get that $\langle A^U \rangle = \mathrm{O}_2(U) \cap K$, so $\mathrm{O}_2(U) \cap K$ is quadratic on V by 5.2. But this contradicts [MeSt2, Theorem 3].

Thus $|A| = 2$ and so $A \leq \mathrm{Z}(S)$. In all the seven cases for $\overline{K}$ given above $\mathrm{C}_S(S \cap K) = \mathrm{Z}(S \cap K)$ and $|\mathrm{Z}(S \cap K)| = 2$. Hence $A = \mathrm{Z}(S \cap K)$ and $L = K$. Put $U := \mathrm{C}_K(A)$. Then in all cases there exists $g \in K$ with $A^g \leq U$ but $A^g \not\leq \mathrm{O}_2(U)$. Now 5.2 implies that AA^g is quadratic and so $\mathrm{C}_U([V, A]) \not\leq \mathrm{O}_2(U)$. But this contradicts the action of U on V, see [MeSt2, Theorem 3].

Case 3. *Suppose that $\overline{K}$ is a group of Lie-type defined over a field of characteristic p.*

Let Δ be the set of minimal Lie-parabolic subgroups of $\overline{K}$ containing $\overline{S \cap K}$ and $\mathbb{K} := \mathrm{End}_L(V)$. By (4°), Schur's Lemma and Wedderburn's Theorem, $\mathbb{K}$ is a finite field.

7°. *Either $L = K$ or $\overline{L} \cong \mathrm{Sp}_4(2)$, $\mathrm{G}_2(2)$ or ${}^2\mathrm{G}_2(3)$.*

Let $M_1, M_2, \dots, M_t$ be the maximal subgroups of H containing S. Then $A \leq \mathrm{O}_p(M_i)$. In particular, $[\mathrm{N}_K(S \cap K), A] \leq S \cap K$. Hence no element of A induces a non-trivial field automorphism on K.

Suppose for a contradiction that A acts non-trivially on Δ. Then $|\Delta| > 1$ and there exists a minimal Lie-parabolic P of K containing $S \cap K$ with $A \not\leq \mathrm{N}_H(P)$. Put $M_P = \langle P, B \rangle$. Suppose that

$A \le \mathrm{O}_p(M_P)$. Then $[P, A] \le \mathrm{O}_p(M_P) \cap K \le S \cap K \le P$, a contradiction to $A \not\le \mathrm{N}_H(P)$. Thus $A \not\le \mathrm{O}_p(M_P)$ and the definition of a minimal asymmetric module implies that $M_P = H$ and so B acts transitively on Δ. It follows that $p = 2$, $|\Delta| = 2$, and $\overline{K} \cong \mathrm{L}_3(q)$ or $\mathrm{Sp}_4(q)'$. Then $S \cap K$ has exactly two maximal elementary abelian normal subgroups Q_1 and Q_2. Moreover, since A acts non-trivially on Δ, A does not normalize Q_i, $i = 1, 2$. It follows that $[Q_1Q_2, A]$ is not elementary abelian, a contradiction to $[Q_1Q_2, A] \le [S, A] \le A$.

We have shown that A acts trivially on Δ and that no element of A induces a non-trivial field automorphism. Hence either $L = K$ or $\overline{L} \cong \mathrm{Sp}_4(2)$, $\mathrm{G}_2(2)$, ${}^2\mathrm{F}_4(2)$ or ${}^2\mathrm{G}_2(3)$. But in the ${}^2\mathrm{F}_4(2)$-case, all involutions of L are contained in K, a contradiction since $L = KA$.

8°. *Suppose that B acts non-trivially on Δ. Then $A \le \mathrm{Z}(S \cap L)$.*

Suppose first that $p = 2$ and $\overline{L} \cong \mathrm{L}_3(q)$ or $\mathrm{Sp}_4(q)'$. As above let Q_1 and Q_2 be the two maximal elementary abelian normal subgroups of $S \cap L$. Recall from the structure of $\overline{L}$ that

(i) all involutions of $S \cap L$ are contained in $Q_1 \cup Q_2$,

(ii) $Q_1 \cap Q_2 = \mathrm{Z}(S \cap L)$, and

(iii) B is transitive on $\{Q_1, Q_2\}$.

The first property shows that there exists i, say $i = 1$, such that $A \le Q_1$. Since B normalizes A, (iii) shows that $A \le Q_1 \cap Q_2$ Now (ii) gives $A \le \mathrm{Z}(S \cap L)$.

Suppose next that $p = 2$ and $\overline{L} \cong \mathrm{F}_4(q)$. Set $U := \mathrm{C}_L(\mathrm{Z}(S \cap L))$. Then $U/\mathrm{O}_2(U) \cong \mathrm{Sp}_4(q)$ and $\mathrm{O}_2(U) \le \mathrm{O}^2(U)$. Since B acts non-trivially on Δ, B acts non-trivially on $U/\mathrm{O}_2(U)$. Thus $\mathrm{O}_2(U) \le \mathrm{O}^2(U) = [\mathrm{O}^2(U), B] \le \langle B^U \rangle$. Since $[V, A \cap \mathrm{Z}(S \cap L), B] = 0$ we conclude that

$$0 \neq [V, A \cap \mathrm{Z}(S \cap L)] \le \mathrm{C}_V(\mathrm{O}^2(U)) \le \mathrm{C}_V(\mathrm{O}_2(U)).$$

Now 2.3 gives a contradiction.

Suppose finally that $p = 2$ and $\overline{L} \cong \mathrm{L}_n(q), n \ge 4$, $\mathrm{D}_n(q)$, $n \ge 4$ or $\mathrm{E}_6(q)$, or $p = 3$ and $\overline{L} \cong \mathrm{D}_4(q)$. Put $Z := \mathrm{Z}(S \cap L)$, $U := \mathrm{N}_H(Z)$ and $Q := \mathrm{O}_p(U \cap L)$. Then Z is a long root subgroup and $\Phi(Q) = Z$. Note also that $A_1 := \langle A^U \rangle$ is an elementary abelian normal subgroup of U.

Assume that $\overline{L} \not\cong \mathrm{L}_4(q)$. Then U acts simply on Q/Z. It follows that all abelian normal p-subgroups of U in Q are contained in Z and so $A \le A_1 \le Z$.

So assume that $\overline{L} \cong \mathrm{L}_4(q)$. Let P be the maximal subgroup of L with $S \le P$ and $P \neq U$. Then $A \le \mathrm{O}_2(P)$ and P acts simply on $\mathrm{O}_2(P)$. Thus $\mathrm{O}_2(P) = \langle A^P \rangle$ acts quadratically on V. Since $S \cap L \le \langle \mathrm{O}_2(P)^U \rangle$ we conclude that $[V, Z, S \cap L] = 0$. It follows from 2.3 that V is a natural $\mathrm{SL}_n(q)$-module for L, a contradiction, since B interchanges the two isomorphism classes of natural $\mathrm{SL}_n(q)$-modules for L.

9°. *Suppose that there exists a long root subgroup R in K such that $A \le R$, $[V, A] = [V, R]$ and $\mathrm{C}_V(A) = \mathrm{C}_V(R)$. Then $A = R$.*

Note that $[V, R, B] = [V, A, B] = 0$ and $[V, B] \le \mathrm{C}_V(A) = \mathrm{C}_V(R)$. Hence the Three Subgroup Lemma gives $[R, B, V] = 0$, so since V is faithful, $B \le \mathrm{C}_H(R)$. As B is a weakly closed subgroup of H, a Frattini argument yields

$$\mathrm{N}_H(R) = \mathrm{C}_H(R)(\mathrm{N}_H(R) \cap \mathrm{N}_H(B)) \le \mathrm{N}_H(A).$$

Since $\mathrm{N}_H(R)$ acts simply on R, we get $R = A$.

Case 4. *Suppose that $|\Delta| = 1$.*

Then $\overline{K} \cong \mathrm{L}_2(q)$, $\mathrm{U}_3(q)$, $\mathrm{Sz}(q)$ or ${}^2\mathrm{G}_2(q)'$. As ${}^2\mathrm{G}_2(q)'$ has abelian Sylow 2-subgroups, this group does not have any non-trivial quadratic module, see for example [Go, 3.8.4]. Hence this case is excluded. Since K is not solvable we have $q > 2$, so $L = K$ and $A \leq S \cap K$ by (7°).

Put $P := \mathrm{N}_H(L \cap S)$ and $Z := \mathrm{Z}(S \cap K)$. Then L has the following properties:

(i) P acts simply on Z.

(ii) Either $S \cap K = Z$, or $Z = \Phi(S \cap K)$ and P acts simply on $(S \cap K)/Z$.

(iii) If $\overline{L} \cong \mathrm{U}_3(q)$, then $\mathrm{C}_P(Z)$ acts simply on $(S \cap K)/Z$.

(iv) $S \cap K$ is a TI-subgroup and $|Z| = q$.

Since $\langle A^P \rangle$ is abelian we conclude from (i) and (ii) that $Z = \langle A^P \rangle$, so Z acts quadratically V.

Let $Z \leq U \leq K$ such that U is minimal with respect to $Z \not\leq \mathrm{O}_p(U)$. By (iv), $\mathrm{O}_p(U) = 1$ and $Z \trianglelefteq \mathrm{N}_U(T)$ for $Z \leq T \in \mathrm{Syl}_p(U)$. Thus 2.9 implies $U \cong \mathrm{SL}_2(q)$ or $\mathrm{Sz}(q)$ since $|Z| = q > 2$. If $L = U$ then by 2.10 V is a natural module, and (9°) implies case (1) of the theorem.

So suppose $L \neq U$. Since $3 \nmid |\mathrm{Sz}(q)|$, $\mathrm{SL}_2(q)$ is not involved in $\mathrm{Sz}(q)$ and so $\overline{L} \cong \mathrm{U}_3(q)$. Since $A \not\leq \mathrm{O}_p(U)$ and $U \neq H$, we have $B \not\leq U$. From (iii) we conclude that $S \cap K \leq \langle B^{\mathrm{C}_P(A)} \rangle$. Hence $[V, A, S \cap K] = 1$, and $Z = \langle A^P \rangle$ gives $[V, Z, S \cap K] = 1$. So by 2.3 V is the natural module. Now (9°) implies case (1) of the theorem.

We assume from now on that $|\Delta| > 1$. Let $L_i, 1 \leq i \leq |\Delta|$ be the maximal subgroups of L containing $L \cap S$. Put $A_i := \langle A^{L_i} \rangle$ and $E_i := \mathrm{O}^{p'}(L_i)$.

10°. *A_i acts quadratically on V; in particular A_i is an elementary abelian normal p-subgroup of L_i.*

Suppose first that B acts trivially on Δ. Then L_i is B-invariant. Since L_iB is a proper subgroup, the definition of a minimal asymmetric modules shows that A_i acts quadratically on V.

Suppose next that B acts non-trivially on Δ. Then by (8°) $A \leq \mathrm{Z}(S \cap L) \leq \mathrm{O}_p(L_i)$. Since $\mathrm{O}_p(L_i) \leq S \cap L \leq \mathrm{N}_H(B) \leq \mathrm{N}_H(A)$, 5.2(b) shows that A_i acts quadratically on V.

Case 5. *Suppose that $|\Delta| = 2$.*

Then $\overline{K}$ is isomorphic to one of the following groups:

$$\mathrm{L}_3(q),\ \mathrm{PSp}_4(q)',\ \mathrm{U}_4(q),\ \mathrm{U}_5(q),\ \mathrm{G}_2(q)',\ {}^2\mathrm{F}_4(q)',\ {}^3\mathrm{D}_4(q).$$

Suppose that $\overline{K} \cong \mathrm{Sp}_4(2)' \cong \mathrm{Alt}(6)$. If $\mathrm{Z}(K) \neq 1$, then [MS4, 7.4] shows that $|V| = 2^6$ and $A \leq K$. Since $A \leq \mathrm{O}_p(L_1) \cap \mathrm{O}_p(L_2)$ we have $|A| = 2$ and so case (3) holds. If $\mathrm{Z}(K) = 1$, then [MeSt2, Theorem 4] shows that V is a natural $\mathrm{Sp}_4(2)'$-module for K. Choose notation such that L_1 is a point stabilizer for L on V. Then $[V, A_1]$ is a singular subspace of V invariant under L_1 and so $|[V, A_1]| = 2$. Thus $|A_1| = 2$, $L \cong \mathrm{Sp}_4(2)$, $A = A_1$, and A is a long root subgroup of L. So case (1) of the theorem holds. Suppose that $\overline{K} \cong \mathrm{G}_2(2)'$. Then [MS4, 7.6] implies that V is natural $\mathrm{G}_2(q)'$-module for K. Choose notation such that $Z := \mathrm{Z}(L_1) \neq 1$. Then Z is a long root subgroup of K, $|Z| = 2$, $Z \leq K$, and Z is the unique non-trivial elementary abelian normal subgroup of L_1. Thus $Z = A_1 = A$, $L = \langle Z^M \rangle = K \cong \mathrm{G}_2(2)'$, and case (1) of the theorem holds.

Therefore, we may assume from now on that $q > 2$ in the $\mathrm{PSp}_4(q)'$- and $\mathrm{G}_2(q)'$-case. Hence by (7°) $L = K$.

If $\mathrm{Z}(E_i) \leq \mathrm{Z}(K)$ for some i, we choose our notation such that $\mathrm{Z}(E_2) \leq \mathrm{Z}(K)$. Otherwise we choose our notation such that $[A, E_2] \neq 1$. Then in any case $A \not\leq \mathrm{Z}(E_2)$. Put

$$Z_2 := [\mathrm{Z}(\mathrm{O}_p(E_2)), E_2],\ V_i := \mathrm{C}_V(\mathrm{O}_p(E_i)),$$

and let Z be the root subgroup with $Z \leq \mathrm{Z}(S \cap L)$ and $[Z, E_2] \neq 1$.

We use the following properties of the groups given above:

(i) Z_2 is the unique normal subgroup of E_2 minimal with respect to $[Z_2, E_2] \neq 1$.

(ii) $Z \leq Z_2$.

(iii) Either $\overline{L} \cong \mathrm{L}_3(q)$ or $[Z, E_1] = 1$.

(iv) If $\overline{L} \cong \mathrm{PSp}_4(q)$, $\mathrm{U}_4(q)$ or $\mathrm{U}_5(q)$, then $\mathrm{O}_p(E_1)\mathrm{O}^p(E_1) \leq \langle Z_2^{E_1} \rangle$. Indeed $E_1 \leq \langle Z_2^{E_1} \rangle$, except for $\overline{L} \cong \mathrm{U}_5(2)$.

All these properties can be found in [DS] by first going to the table on page 98 to get the value of the parameter b and then to look up the properties in those chapters where this value of b is treated.

By (i) and (ii) $Z \leq Z_2 \leq A_2$. Suppose that $\overline{L} \cong \mathrm{L}_3(q)$. Then $A_i = \mathrm{O}_p(E_i)$, $A \leq A_1 \cap A_2 \leq Z$, and $A_1 A_2 = S \cap L$. Thus $[V, A, S \cap L] = 0$. Hence by 2.3 V is a natural $\mathrm{SL}_3(q)$-module and by (9°) case (1) of the theorem holds.

Suppose now that $\overline{L} \not\cong \mathrm{L}_3(q)$. Then $Z \trianglelefteq E_1$ by (iii). Put $W_1 := \langle Z_2^{E_1} \rangle$. Since $[V, Z, Z_2] \leq [V, A_2, A_2]$ and by (10°) $[V, A_2, A_2] = 0$, we get

11°. $\quad [V, Z, W_1] = 0.$

Suppose that $\overline{L} \cong \mathrm{PSp}_4(q)$, $\mathrm{U}_4(q)$ or $\mathrm{U}_5(q)$. Then by (iv) $\mathrm{O}_p(E_1)\mathrm{O}^p(E_1) \leq W_1$ and thus $[V, Z, \mathrm{O}_p(E_1)\mathrm{O}^p(E_1)] = 0$. So by 2.3 V is a natural module. The action of E_1 on the natural module reveals that any quadratic normal subgroup of E_1 is contained in Z. Thus $A \leq A_1 \leq Z$ and by (9°) again case (1) of the theorem holds.

For the remaining cases

$$\overline{L} \cong \mathrm{G}_2(q),\ q > 2, {}^2\mathrm{F}_4(q)', {}^3\mathrm{D}_4(q)$$

let Γ be the coset graph of L with respect to L_1 and L_2. All the properties of the action of L on Γ we use here can be found in [DS], in Section 10 for $\mathrm{G}_2(q)$ and ${}^3\mathrm{D}_4(q)$, in Section 12 for ${}^2\mathrm{F}_4(q)'$. In particular the value of b in these cases is 2, 2 and 3, respectively.

Choose a path $(\alpha_1, \alpha_2, \ldots \alpha_d)$ of minimal length $d-1$ such that $\alpha_1 = L_1, \alpha_2 = L_2$ and $Z \not\leq \mathrm{O}_p(L_d)$, where L_j is the stabilizer of α_j in L. Then $d - 1 = b + 1$. Note that for $i = 1, 2$ this is compatible with our earlier notation since $L_1 = \alpha_1$ and $L_2 = \alpha_2$. Let $\Delta(\alpha_i)$ be the set of neighbors of α_i. Put

$$E_i := \mathrm{O}^{p'}(L_i),\ V_i := \mathrm{C}_V(\mathrm{O}_p(L_i)),\ J_i := \langle V_k^{E_i} \mid 1 \leq k \leq d,\ \alpha_k \in \Delta(\alpha_i) \rangle,$$

and, for i even, $Z_i := [\mathrm{Z}(\mathrm{O}_p(E_i)), E_i]$. Observe that $Z \leq \mathrm{O}_p(L_{d-1}) \leq L_d$ and by (11°) $[V_d, Z, W_1] = 0$, so $[V_d, Z]$ is centralized by $R := \langle \mathrm{O}_p(L_d), W_1 \rangle$.

Suppose that $\overline{L} \cong \mathrm{G}_2(q)$, $q > 2$, or ${}^3\mathrm{D}_4(q)$. Then $d = 4$, $Z_2 \leq \mathrm{O}_p(L_1)$, $Z_2 \not\leq \mathrm{Z}(\mathrm{O}_p(E_1))$ and $\mathrm{Z}(\mathrm{O}_p(E_1))/\Phi(\mathrm{O}_p(E_1))$ is the unique maximal L_1-submodule of $\mathrm{O}_p(E_1))/\Phi(\mathrm{O}_p(E_1))$; in fact $\Phi(\mathrm{O}_p(E_1)) = \mathrm{Z}(\mathrm{O}_p(E_1))$ and $\mathrm{O}_p(E_1)/\Phi(\mathrm{O}_p(E_1))$ is a simple E_1-module unless $\overline{L} \cong \mathrm{G}_2(q)$ and $p = 3$ or $q = 4$. Since $L_1 = (L_1 \cap L_2)E_2$, L_1 normalizes W_1. It follows that $W_1 = \mathrm{O}_p(E_1)$ and $R = \langle E_2, E_3 \rangle = L$. So $\mathrm{C}_V(R) = 0$ and $[V_4, Z] = 0$. Thus $[V_4, E_4] = 0$ and so also $[V_2, E_2] = 0$.

Now Steinberg's Lemma shows that V_2 is 1-dimensional over $\mathbb{K}$ and $V_2 = \mathrm{C}_V(S \cap K)$. Moreover, $W_1 = \mathrm{O}_p(E_1)$ and (11°) imply that $[V, Z] \leq V_1$.

Since $Z_2 \leq \mathrm{O}_p(E_1)$, Z_2 centralizes V_1 and so $[J_2, Z_2] = 0$. Since by definition of Γ, α_4 and α_2 are conjugate under L we conclude that $[J_4, Z_4] = 0$. Then also $[Z, Z_4, J_4] = 0$ since $Z \leq L_4$ and $Z_4 \trianglelefteq L_4$. Hence the Three Subgroup Lemma gives $[J_4, Z, Z_4] = 0$. Since also $\mathrm{O}_p(E_1)$ centralizes $[Z, J_4]$, we conclude that $E_2 = \langle \mathrm{O}_p(E_1), Z_4 \rangle$ centralizes $[J_4, Z]$. Thus $[J_4, Z] \leq V_2$, so also $[J_4, Z]$ is 1-dimensional.

Let $1 \neq z \in Z$ and α_5 be a neighbor of α_4 in Γ different from α_3. Then $0 \neq [V_5, z] \leq [J_4, Z] \leq V_2$, so $[V_5, z]$ and thus also $V_5/\mathrm{C}_{V_5}(z)$ are 1-dimensional over $\mathbb{K}$. Observe that

$$\mathrm{C}_{V_5}(z) \leq V_5 \cap V_5^z \leq \mathrm{C}_V(\langle \mathrm{O}_p(E_5), \mathrm{O}_p(E_5)^z \rangle) = \mathrm{C}_V(E_4) = V_4.$$

Thus V_5/V_4 is 1-dimensional. So also V_1/V_2 is 1-dimensional. Since V_2 is 1-dimensional, we get that V_1 is 2-dimensional over $\mathbb{K}$. Hence V_1 is a natural $\mathrm{SL}_2(q^\epsilon)$-module for E_1, where $\epsilon = 1$ in the $\mathrm{G}_2(q)$-case and $\epsilon = 3$ in ${}^3D_4(q)$-case. Note also that E_2 centralizes V_2. By Ronan-Smith's Lemma [MS4, 4.3] this determines V up to isomorphism and it follows that V is the natural module for L. According to (9°), in order to establish that (1) holds, it remains to show in these cases that $A \leq Z$.

If $\overline{L} \cong \mathrm{G}_2(q)$, q not a power of 3 or $\overline{L} \cong {}^3\mathrm{D}_4(q)$, then Z is the unique maximal elementary abelian normal subgroup of E_1 and so $A \leq A_1 \leq Z$. Suppose $\overline{K} \cong \mathrm{G}_2(3^k)$ and $A \not\leq Z$. Then also $[A, E_1] \neq 1$ and the set-up is symmetric in 1 and 2. As we have seen above, $V_2 = \mathrm{C}_V(S \cap K) = \mathrm{C}_V(E_2)$, so by symmetry also $V_1 = \mathrm{C}_V(S \cap K) = \mathrm{C}_V(E_1)$ and $V_1 = V_2$, a contradiction.

Suppose finally that $\overline{L} \cong {}^2F_4(q)'$. Then W_1 is abelian, $W_1 \leq \mathrm{O}_p(E_2)$ and $d = 5$. Let $X_1 = \bigcap_{\alpha \in \Delta(\alpha_1)} \mathrm{O}_p(L_\alpha)$, so $W_1 \leq X_1$. Observe that X_1 centralizes J_1, so X_7 centralizes J_7 since α_1 and α_7 are conjugate. Thus by (11°), $E_5 = \langle W_1, X_7 \rangle$ centralizes $[J_7, Z]$. It follows that $[J_7, Z] = V_5$, $[V_5, E_5] = 1$ and $[J_7, Z]$ is 1-dimensional over $\mathbb{K}$. Observe that $\mathrm{O}^p(L_7)$ does not centralize J_7 and so there exists a composition factor W for $\mathbb{K}L_7$ on J_7 not centralized by $\mathrm{O}^p(E_7)$. But $E_7/\mathrm{O}_p(E_7) \cong \mathrm{Sz}(q)$. Thus by 2.3 $\dim_{\mathbb{K}}[W, Z] > 1$, a contradiction.

Case 6. *Suppose that* $|\Delta| = t > 2$.

According to (7°) $L = K$. We divide the groups under consideration into two classes:

(I) $\overline{K} \cong \mathrm{PSp}_{2n}(q)$, q odd and $n \geq 3$, $\mathrm{PSU}_n(q), n \geq 6$, $\mathrm{F}_4(q)$, q odd, ${}^2\mathrm{E}_6(q)$, $\mathrm{E}_6(q)$, $\mathrm{E}_7(q)$, $\mathrm{E}_8(q)$.

(II) $\overline{K} \cong \mathrm{L}_n(q)$, $n \geq 4$, $\mathrm{P}\,\Omega_n(q)$, $n \geq 7$, $\mathrm{F}_4(q)$, q even.

We first discuss the groups in (I). They all have the following properties in common:

(i) $Z := \mathrm{Z}(S \cap K)$ is a long root subgroup of L.

(ii) $\mathrm{N}_L(Z)$ is a maximal subgroup of L (and we choose $L_t = \mathrm{N}_L(Z)$).

(iii) $\Phi(\mathrm{O}_p(L_t)) = Z$.

(iv) $\mathrm{O}_p(L_t)/Z$ is a simple E_t-module.

(v) $E_t/\mathrm{O}_p(E_t)$ is quasisimple.

Since $\mathrm{N}_L(S \cap K)$ acts simply on Z we have $Z \leq A_i$ for $i = 1, \ldots, t$. Observe that by (iii) and (iv), $A_t = Z$, and so $A \leq Z$. Put $\widetilde{L}_t := L_t/\mathrm{O}_p(L_t)\mathrm{Z}(K)$ and $R := \langle A_1^{L_t} \rangle$. If $\overline{K} \cong \mathrm{PSp}_{2n}(q)$ or $\mathrm{PSU}_n(q)$ choose L_1 to be the normalizer of a maximal singular subspace (of the natural module). Then $A_1 \not\leq \mathrm{O}_p(E_t)$ and by (v) $\widetilde{R} = \widetilde{E}_t$. In the other cases of (I) except $\overline{K} \cong \mathrm{E}_7(q)$, choose L_1 such that $E_1/\mathrm{O}_p(E_1) \cong \Omega_m^\epsilon(q)$, $(E_1 \cap E_t)/\mathrm{O}_p(E_1 \cap E_t) \cong \Omega_{m-2}^\epsilon(q)$, and $Z_1 := \langle Z^{E_1} \rangle$ is a natural $\Omega_m^\epsilon(q)$-module. For $\overline{K} \cong E_7(q)$ choose L_1 such that $E_1/\mathrm{O}_p(E_1) \cong \mathrm{E}_6(q)$ and so $(E_1 \cap E_t)/\mathrm{O}_p(E_1 \cap E_t) \cong \mathrm{Spin}_{10}^+(q)$, and $Z_1 := \langle Z^{E_1} \rangle$ is a simple E_1-module of order q^{27}. Since $Z \leq A_1$, we get $Z_1 \leq A_1$ and thus $[A_1, \mathrm{O}_p(E_t)] \not\leq Z$ since $[Z_1, \mathrm{O}_p(E_t)] \not\leq Z$. Hence (iii) shows that $A_1 \not\leq \mathrm{O}_p(E_t)$ and so by (v) $\widetilde{R} = \widetilde{E}_t$.

We have shown $\widetilde{E}_t = \widetilde{R}$. Hence (iii) and (iv) imply

$$(*) \qquad E_t = R.$$

As $Z \leq A_i$ and A_i is quadratic on V, we get that $[V, Z, R] = 0$. Hence $(*)$ implies that $[V, Z, E_t] = 0$. So by 2.3 $K \cong \mathrm{Sp}_{2n}(q)$ or $\mathrm{SU}_n(q)$, and V is the natural module. Hence (9°) implies case (1) of the theorem.

We now discuss the groups in (II). Suppose $\overline{K} \cong L_n(q)$, $n > 3$. Let P_1 and P_2 be the p-minimal subgroups of H with $S \cap K \leq P_i$ and $Z := \mathrm{Z}(S \cap K)$ not normal in P_i. As $n > 3$, P_1 and P_2 commute. Put $P = P_1P_2$ and $D = \langle A^P \rangle$. Then $Z \leq D$ and $[V, Z, D] = 0$, so $D \leq \mathrm{N}_L(Z)$ but $D \not\leq \mathrm{O}_p(N_L(Z))$, and $\langle D^{\mathrm{N}_L(Z)} \rangle = \mathrm{O}^{p'}(\mathrm{N}_L(Z))$. Hence $[V, Z, \mathrm{O}^{p'}(\mathrm{N}_L(Z))] = 0$ and by 2.3 V is a natural module. Let L_1 and L_2 be the normalizers of a 1-dimensional subspace and hyperplane, respectively. Then $A \leq \mathrm{O}_p(L_1) \cap \mathrm{O}_p(L_2) = Z$, and (9°) implies case (1) of the theorem holds.

Suppose $\overline{K} \cong \mathrm{P}\,\Omega_n^\epsilon(q)$, $n \geq 7$. Let d be the dimension of a maximal singular subspace (of the natural module) and for $i = 1, 2, d$ choose notation such that L_i normalizes a singular i-subspace. Set $Z_2 := \mathrm{Z}(E_2)$, $B_i := \langle Z_2^{L_i} \rangle$ and, if $p = 2$ and n is odd, $Z_1 := \mathrm{Z}(E_1)$. We will use the following properties:

(i) If p is odd or n is even, then $\mathrm{Z}(S \cap K) = Z_2$, and if $p = 2$ and n is odd, then Z_1 and Z_2 are non-conjugate root subgroups with $\mathrm{Z}(S \cap K) = Z_1Z_2$. Moreover, $Z_2 = B_d \cap \mathrm{Z}(S \cap K) = B_d \cap \mathrm{Z}(\mathrm{O}_p(L_2))$.

(ii) B_d is the unique minimal normal subgroup of L_d in $\mathrm{O}_p(L_d)$.

(iii) $B_d = \mathrm{O}_p(L_d)$ or $\mathrm{O}_p(L_d)/B_d$ is a simple L_d-module.

(iv) If $B_d \neq \mathrm{O}_p(L_d)$ and $\mathrm{O}_p(L_d)$ is abelian, then $p = 2$ and n is odd.

(v) $B_1 = \mathrm{O}_p(L_1)$

(vi) If $d \geq 4$ and $\overline{K} \not\cong \Omega_8^+(q)$ then $L_2 = \langle (B_1B_d)^{L_2} \rangle$.

(vii) If $\overline{K} \cong \Omega_8^+(q)$, then L_3 normalizes a singular 4-space and $L_2 = \langle (B_1B_3B_4)^{L_2} \rangle$.

(viii) L_2 acts simply on $\mathrm{O}_p(L_2)/\mathrm{Z}(\mathrm{O}_p(L_2))$.

Suppose first that $A_d \neq B_d$. Then $B_d \neq \mathrm{O}_p(L_d)$, $A_d = \mathrm{O}_p(L_d)$, and $\mathrm{O}_p(L_d)$ is elementary abelian. Thus $p = 2$ and n is odd. Then Z_1 is a root subgroup and $Z_1 \leq A_d \not\leq \mathrm{O}_2(E_1)$, $E_1 = \langle A_d^{E_1} \rangle$, and $[V, Z_1, E_1] = 0$. Hence by 2.3 V is a natural $\mathrm{Sp}_{n-1}(q)$-module. In particular, Z_1 is the unique maximal quadratic normal subgroup of L_1 and so $A \leq A_1 \leq Z_1$. Now (9°) implies that case (1) of the theorem holds.

Suppose that $A_d = B_d$. Then $A \le B_d$ and so $1 \ne A \cap \mathrm{Z}(S \cap K) \le B_d \cap \mathrm{Z}(S \cap K) = Z_2$. Since $\mathrm{N}_K(S \cap K)$ acts simply on Z_2, $Z_2 = \langle (A \cap \mathrm{Z}(S \cap K))^{\mathrm{N}_K(S \cap K)} \rangle$ and therefore $Z_2 \le A_1 \cap A_d$. Since $\mathrm{O}_p(L_1) = B_1 = \langle Z_2^{L_1} \rangle$ we get $A_1 = \mathrm{O}_p(L_1)$. Thus [MS4, 7.11] implies that V is a (half-)spin-module. If $d \ge 4$ and $\overline{K} \not\cong \Omega_8^+(q)$ then $L_2 = \langle (B_1 B_d)^{L_2} \rangle = \langle (A_1 A_d)^{L_2} \rangle$ and so $[V, Z_2, L_2] = 0$, a contradiction to 2.3. If $\overline{K} \cong \Omega_8^+(q)$, then $A_3 = B_3$, $L_2 = \langle (B_1 B_3 B_4)^{L_d} \rangle = \langle (A_1 A_3 A_4)^{L_2} \rangle$, and we obtain the same contradiction. Thus $d = 3$. Suppose that $A \not\le Z_2$. Since $A_3 \cap \mathrm{Z}(\mathrm{O}_p(L_2)) = Z_2$ we conclude that $A \not\le \mathrm{Z}(\mathrm{O}_p(L_2))$. Since L_2 acts simply on $\mathrm{O}_p(L_2)/\mathrm{Z}(\mathrm{O}_p(L_2))$ we get $\mathrm{O}_p(L_2) = A_2 \mathrm{Z}(\mathrm{O}_p(L_2))$, a contradiction since A_2 is abelian. Hence $A \le Z_2$ and (9°) yields case (1) of the theorem.

Suppose finally that $\overline{K} \cong \mathrm{F}_4(q)$ and $p = 2$. Let Z_1 and Z_2 be the two root subgroups with $\mathrm{Z}(S \cap K) = Z_1 Z_2$. For $i = 1, 2$ let $L_i := \mathrm{N}_L(Z_i)$. Since $Z_1 \cap Z_2 = 1$ we can choose notation such that $A \cap \mathrm{Z}(S \cap K) \not\le Z_1$. Then $Y_1 := \Omega_1 \mathrm{Z}(\mathrm{O}_2(E_1)) \le \langle (A \cap \mathrm{Z}(S \cap K))^{E_1} \rangle \le A_1$, and so Y_1 is quadratic on V. Note that $Z_2 \le Y_1$, $Y_1 \not\le \mathrm{O}_2(E_2)$ and $E_2/\mathrm{O}_2(E_2)$ is quasisimple. We conclude that $E_2 \le \langle Y_1^{E_2} \rangle$ and $[V, Z_2, E_2] = 0$, which contradicts 2.3. $\square$

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