The General FF-module Theorem

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Abstract

Let p be a prime, M a finite group with $\mathrm{O}_p(M) = 1$, V a faithful $\mathbb{F}_pM$-module and J the subgroup of M generated by the best offenders on V. In this paper we determine structure of J and the action of J on V.

Introduction

Let p be a prime, M a finite group and V a finite dimensional $\mathbb{F}_pM$-module, where $\mathbb{F}_p$ is the prime field in characteristic p. A subgroup $A \le M$ is an *offender* on V if

1. $A/\mathrm{C}_A(V)$ is an elementary abelian p-group, and
2. $|V/\mathrm{C}_V(A)| \le |A/\mathrm{C}_A(V)|$;

and A is a *non-trivial* offender on V, if in addition $[V, A] \ne 0$. Moreover, V is called an FF-module for M if some subgroup of M is a non-trivial offender on V. Faithful simple FF-modules for groups of Lie type in equicharacteristic have been classified by Cooperstein [Co] (the case $p = 2$) and Meixner [M] (the case $p \ne 2$) and for arbitrary nearly simple groups by Guralnick, R. Lawther and G. Malle [GM1], [GM2], [GLM].

These results have been of great importance for the local theory of finite groups since such FF-modules are closely related to the failure of the Thompson-factorization in groups of characteristic p. In fact, for a finite group G and a normal elementary abelian p-subgroup X the elementary abelian p-subgroups of maximal order in G provide examples for offenders on X; and so G possesses non-trivial offenders on X if $[X, \mathrm{J}(S)] \ne 1$, where $S \in Syl_p(G)$. The action of such elementary abelian subgroups have an additional property that is reflected in the following definition.

A subgroup $A \le M$ is a *best offender* on V if

(i) $A/\mathrm{C}_A(V)$ is an elementary abelian p-group, and

(ii) $|B||\mathrm{C}_V(B)| \le |A||\mathrm{C}_V(A)|$ for every subgroup $B \le A$.

It is easy to see (using $B := \mathrm{C}_A(V)$) that every best offender is an offender. Indeed, a best offender A on V is an offender on every A-submodule of V; and this property characterizes best offenders (see 1.2).

In this paper we use this slightly stronger definition to derive a result about FF-modules that is free from the restriction to simple modules. It includes the above mentioned FF-module theorems, but also in these cases it gives more information about the size and action of offenders on V.

Most of the time we will treat groups like $\mathrm{Alt}(6) \cong \mathrm{Sp}_4(2)'$, $\mathrm{SU}_3(3) \cong \mathrm{G}_2(2)'$ and ${}^2\mathrm{F}_4(2)'$ together with the groups of Lie-Type. We therefore use the following definition.

Definition. A *genuine group of Lie-type in characteristic* p is a group isomorphic to $\mathrm{O}^{p'}(\mathrm{C}_{\overline{K}}(\sigma))$, where $\overline{K}$ is a semisimple $\overline{\mathbb{F}_p}$-algebraic group, $\overline{\mathbb{F}_p}$ is the algebraic closure of $\mathbb{F}_p$, and σ is Steinberg endomorphism of $\overline{K}$, see [GLS3, Definition 2.2.2] for details. A simple group of Lie-type in characteristic p is a non-abelian composition factor of a genuine group of Lie-type in characteristic p.

Before stating our main result we give some further definitions.

Definition. The normal subgroup of M generated by the best offenders of M on V is denoted by $\mathrm{J}_M(V)$. A non-trivial subgroup K of $\mathrm{J}_M(V)$ is a $\mathrm{J}_M(V)$*-component* if K is minimal with respect to $K = [K, \mathrm{J}_M(V)]$. The set of these components we denote by $\mathcal{J}_M(V)$.

A finite group H is a called a $\mathcal{CK}$*-group* provided that each composition factor of H is one of the known finite simple groups.

Let $\mathcal{S}$ be a set of subgroups of M. We often write $[V, \mathcal{S}]$ and $C_V(\mathcal{S})$ rather than $[V, \langle\mathcal{S}\rangle]$ and $C_V(\langle\mathcal{S}\rangle)$. Similarly, we write $\bigtimes \mathcal{S}$ rather than $\bigtimes_{A\in\mathcal{S}} A$.

The $\mathbb{F}_pM$-module V is *perfect* if $V = [V, M]$, *simple* if $V \neq 0$ and 0 is the only proper $\mathbb{F}_pM$-submodule of V, and *quasisimple* if V is perfect, $O_p(M/C_M(V)) = 1$ and $V/C_V(M)$ is simple. Moreover, M acts *simply* on V if V is a simple M-module; and M acts *nilpotently* on V if there exists a finite series $0 = V_0 \leq V_1 \leq V_{k-1} \leq V_k = V$ of $\mathbb{F}_pM$-submodules of V with $[V_i, M] \leq V_{i-1}$ for all $1 \leq i \leq k$.

Let A be a subgroup of M. Then

- A is a *strong dual offender* on V if A acts nilpotently on V and $[V, A] = [v, A]$ for every $v \in V \setminus \mathrm{C}_V(A)$;
- A is a *strong offender* on V if A is an offender on V and $\mathrm{C}_V(A) = \mathrm{C}_V(a)$ for every $a \in A\setminus\mathrm{C}_A(V)$ (note that the last condition is equivalent to $\mathrm{C}_A(V) = \mathrm{C}_A(v)$ for all $v \in V \setminus \mathrm{C}_V(A)$);
- A is an *over-offender* on V if A is an offender and $|A/\mathrm{C}_A(V)| > |V/\mathrm{C}_V(A)|$.

Finally we call V a *natural* $\mathbb{F}_pK$-module for M if $M/C_M(V) \cong K$, and there exists a quadratic, bilinear or sesquilinear form f on V left invariant by M such that for K, $\mathbb{K} := \mathrm{End}_M(V)$, $\dim_{\mathbb{K}} V$ and f one of the following cases holds:

K	$\dim_{\mathbb{K}} V$	$\mathbb{K}$	f
$\mathrm{SL}_n(p^k)$	n	$\mathbb{F}_{p^k}$	zero-form
$\mathrm{Sp}_{2n}(p^k)$	$2n$	$\mathbb{F}_{p^k}$	non-deg. symplectic
$\mathrm{O}_n^\epsilon(p^k)$	n	$\mathbb{F}_{p^k}$	non-deg. quadratic
$\Omega_n^\epsilon(p^k)$	n	$\mathbb{F}_{p^k}$	non-deg. quadratic
$\mathrm{SU}_n(p^k)$	n	$\mathbb{F}_{p^{2k}}$	non-deg. unitary
$\mathrm{G}_2(2^k)$	6	$\mathbb{F}_{2^k}$	non-deg. symplectic
$\mathrm{Sym}(2n)$	$2n-2$	$\mathbb{F}_2$	zero-form
$\mathrm{Alt}(2n)$	$2n-2$	$\mathbb{F}_2$	– ॥ –
$\mathrm{Sym}(2n+1)$	$2n$	$\mathbb{F}_2$	– ॥ –
$\mathrm{Alt}(2n+1)$	$2n$	$\mathbb{F}_2$	– ॥ –

In the last four cases V is meant to be the simple composition factor of the $\mathbb{F}_2$-permutation module for $\mathrm{Sym}(2n)$ and $\mathrm{Sym}(2n+1)$, respectively.

Note that in the above definition a *non-degenerate quadratic form* is a quadratic form that is non-zero on every non-zero element in the radical of the associated symmetric form. Also observe that $\mathrm{O}_{2n+1}(2^k) \cong \mathrm{Sp}_{2n}(2^k)$ and V is a central extension of a natural $\mathrm{Sp}_{2n}(2^k)$-module. This extension does not split if $n > 1$ or $k > 1$.

In general, M can have more than one natural module. For example, for $n = 5$, $\mathrm{Alt}(5) \cong \mathrm{SL}_2(4) \cong \Omega_4^-(2)$, so M has three natural modules, the natural $\mathrm{SL}_2(4)$-module, the natural $\Omega_4^-(2)$-module, and the natural $\mathrm{Alt}(5)$-module, the latter two being isomorphic.

In addition, $M \cong \mathrm{SL}_n(q)$, $n > 2$, has two natural $\mathrm{SL}_n(q)$-modules that are not isomorphic due to the graph automorphism of $\mathrm{SL}_n(q)$. Similarly, $M \cong \mathrm{Spin}_8^+(q)$ has three natural $\Omega_8^+(q)$-modules. In the literature two of these are called half-spin modules depending which epimorphism from M to $\Omega_8^+(q)$ one chooses.

Theorem 1 (**General FF-Module Theorem).** *Let M be a finite $\mathcal{CK}$-group with $\mathrm{O}_p(M) = 1$ and V be a faithful finite dimensional $\mathbb{F}_pM$-module. Suppose that $J := \mathrm{J}_M(V) \neq 1$. Then for $\mathcal{J} := \mathcal{J}_M(V)$, $W := [V, \mathcal{J}] + \mathrm{C}_V(\mathcal{J})/\mathrm{C}_V(\mathcal{J})$, $K \in \mathcal{J}$ and $\overline{J} := J/\mathrm{C}_J([W, K])$ the following hold:*

(a) K is either quasisimple, or $p = 2$ or 3 and $K \cong \mathrm{SL}_2(p)'$.

(b) $[V, K, L] = 0$ for all $K \neq L \in \mathcal{J}$, and $W = \bigoplus_{K\in\mathcal{J}}[W, K]$.

(c) $J^pJ' = \mathrm{O}^p(J) = \mathrm{F}^*(J) = \bigtimes \mathcal{J}$.

(d) W is a faithful semisimple $\mathbb{F}_pJ$-module.

(e) If $A \leq M$ is a best offender on V, then A is a best offender on W.

(f) $\overline{K} = \overline{\mathrm{F}^*(J)} = \mathrm{O}^p(\overline{J})$ and $\mathrm{C}_J([W, K]) = \mathrm{C}_J([V, K])$.

(g) Either $[W, K]$ is a simple $\mathbb{F}_pK$-module, or one of the following holds, where q is a power of p:

1. *$\overline{J} \cong \mathrm{SL}_n(q)$, $n \geq 3$, and $[W, K] \cong N^r \oplus N^{*s}$, where N is a natural $\mathrm{SL}_n(q)$-module, N^* its dual, and r, s are integers with $0 \leq r, s < n$ and $\sqrt{r} + \sqrt{s} \leq \sqrt{n}$.*
2. *$J \cong \mathrm{Sp}_{2m}(q)$, $m \geq 3$, and $[W, K] \cong N^r$, where N is a natural $\mathrm{Sp}_{2m}(q)$-module and r is a positive integer with $2r \leq m + 1$.*
3. *$\overline{J} \cong \mathrm{SU}_n(q)$, $n \geq 8$, and $[W, K] \cong N^r$, where N is a natural $\mathrm{SU}_n(q)$-module and r is a positive integer with $4r \leq n$.*
4. *$\overline{J} \cong \Omega_n^\epsilon(q)$ with p odd if n is odd, or $\overline{J} \cong \mathrm{O}_n^\epsilon(q)$ with $p = 2$ and n even.[1] Moreover, $n \geq 10$ and $[W, K] \cong N^r$, where N is a natural $\Omega_n^\epsilon(q)$-module and r is a positive integer with $4r \leq n - 2$.*

(h) If $[W, K]$ is not a homogeneous $\mathbb{F}_pK$ module, then (g:1) holds with $r \neq 0 \neq s$ and $n \geq 4$.

Theorem 2 (**FF-Module Theorem).** *Let $M \neq 1$ be a finite $\mathcal{CK}$-group and V be a faithful $\mathbb{F}_pM$-module. Put*

$\mathcal{D} := \{A \leq M \mid$ *there exists $1 \neq B \leq A$ such that $[V, B, A] = 0$ and A and B are offenders on V*$\}$.[2]

Suppose that V is a simple $\mathbb{F}_p\,\mathrm{J}_M(V)$-module and $M = \langle\mathcal{D}\rangle$. Then one of the following holds, where q is a power of p:

[1] The odd-dimensional orthogonal groups in characteristic 2 are covered in case (g:2).

[2] Note here that $\mathcal{D}$ contains all quadratic offenders and by the Timmesfeld Replacement Theorem [KS, 9.2.3], also all best offenders in M on V.

1. $M \cong \mathrm{SL}_n(q)$, $n \geq 2$, and V is a natural $\mathrm{SL}_n(q)$-module.

2. $M \cong \mathrm{Sp}_{2n}(q)$, $n \geq 1$, and V is a natural $\mathrm{Sp}_{2n}(q)$-module.

3. $M \cong \mathrm{SU}_n(q)$, $n \geq 4$, and V is a natural $\mathrm{SU}_n(q)$-module.

4. $M \cong \Omega^+_{2n}(q)$ for $2n \geq 6$, $M \cong \Omega^-_{2n}(q)$ for $p = 2$ and $2n \geq 6$, $M \cong \Omega^-_{2n}(q)$ for p odd and $2n \geq 8$, $M \cong \Omega_{2n+1}(q)$ for p odd and $2n+1 \geq 7$, $M \cong O^-_4(2)$, or $M \cong \mathrm{O}^\epsilon_{2n}(q)$ for $p = 2$ and $2n \geq 6$, and V is a corresponding natural module.

5. $M \cong \mathrm{G}_2(q)$, $p = 2$, and V is a natural $\mathrm{G}_2(q)$-module (of order q^6).

6. $M \cong \mathrm{SL}_n(q)/\langle -\mathrm{id}^{n-1}\rangle$, $n \geq 5$, and V is the exterior square of a natural $\mathrm{SL}_n(q)$-module.

7. $M \cong \mathrm{Spin}_7(q)$, and V is a spin module of order q^8.

8. $M \cong \mathrm{Spin}^+_{10}(q)$, and V is a half-spin module of order q^{16}.

9. $M \cong 3.\mathrm{Alt}(6)$, $p = 2$ and $|V| = 2^6$.

10. $M \cong \mathrm{Alt}(7)$, $p = 2$, and $|V| = 2^4$.

11. $M \cong \mathrm{Sym}(n)$, $p = 2$, n odd, $n \geq 3$, and V is a natural $\mathrm{Sym}(n)$-module.

12. $M \cong \mathrm{Alt}(n)$ or $\mathrm{Sym}(n)$, $p = 2$, n is even, $n \geq 6$, and V is a corresponding natural module.

Theorem 3 (Best Offender Theorem). *Let $M \neq 1$ be a finite group, $T \in \mathrm{Syl}_p(M)$, and V be a faithful $\mathbb{F}_p M$-module, and let $A \leq T$ be an non-trivial offender on V.*

(a) *Suppose that $M \cong \mathrm{G}_2(q)$, $p = 2$, and V is a natural $\mathrm{G}_2(q)$-module. Then $\mathrm{N}_M(A)$ is a maximal Lie-parabolic subgroup, $|A| = |V/\mathrm{C}_V(A)| = q^3$, $[V, A] = \mathrm{C}_V(A)$, and $\mathrm{C}_T(A) = A$.*

(b) *Suppose that $M \cong \mathrm{SL}_n(q)/\langle -\mathrm{id}^{n-1}\rangle$, $n \geq 5$, and V is the exterior square of the natural $\mathrm{SL}_n(q)$-module W. Let U be the (unique) T-invariant $\mathbb{F}_q$-hyperplane of W. Then $A = C_M(U)$. In particular, A is uniquely determined in T, $C_T(A) = A$, $[V, A] = \mathrm{C}_V(A)$ and $|V/\mathrm{C}_V(A)| = |A| = q^{n-1}$.*

(c) *Suppose that $M \cong \mathrm{Spin}_7(q)$, and V is a spin module of order q^8. Then $\mathrm{C}_V(A) = [V, A]$, $|V/\mathrm{C}_V(A)| = q^4 \leq |A| \leq q^5$, and if A is maximal, then $|A| = q^5$, $\mathrm{C}_T(A) = A$, $\mathrm{O}^{p'}(\mathrm{N}_M(A))/A \cong \mathrm{Sp}_4(q)$, and A is uniquely determined in T.*

(d) *Suppose that $M \cong \mathrm{Spin}^+_{10}(q)$, and V is a half-spin module of order q^{16}. Then $[V, A] = \mathrm{C}_V(A)$, $q^8 = |A| = |V/\mathrm{C}_V(A)|$, $\mathrm{O}^{p'}(\mathrm{N}_M(A)/A) \cong \mathrm{Spin}^+_8(q)$, and A is uniquely determined in T.*

(e) *Suppose that $M \cong 3.\mathrm{Alt}(6)$, $p = 2$ and $|V| = 2^6$. Then $[V, A] = \mathrm{C}_V(A)$, $|[V, A]| = |\mathrm{C}_V(A)| = 16$, $|V/\mathrm{C}_V(A)| = |A| = 4$, and A is uniquely determined in T.*

(f) *Suppose that $M \cong \mathrm{Alt}(7)$, $p = 2$ and $|V| = 2^4$. Then $[V, A] = \mathrm{C}_V(A)$, $|[V, A]| = |\mathrm{C}_V(A)| = 4$, $|V/\mathrm{C}_V(A)| = |A| = 4$, and A is uniquely determined in T.*

(g) *Suppose that $M \cong \mathrm{Sym}(n)$, $p = 2$, n odd, and V is a natural $\mathrm{Sym}(n)$-module. Then every offender on V is a quadratic best offender, A is generated by commuting transpositions and $|V/\mathrm{C}_V(A)| = |[V, A]| = |A|$.*

(h) *Suppose that* $M \cong \mathrm{Alt}(n)$ *or* $\mathrm{Sym}(n)$*,* $p = 2$*,* n *is even,* $n \geq 6$*, and* V *is a corresponding natural module. Then every offender on* V *is a best offender, and there exists a set of pairwise commuting transpositions* $t_1, \dots, t_k$ *such that one of the following holds:*

1. $A = \langle t_1, \dots, t_k \rangle$*, and either* $n \neq 2k$*,* $[V, A] \leq \mathrm{C}_V(A)$ *and* $|[V, A]| = |V/\mathrm{C}_V(A)| = |A|$ *or* $n = 2k$*,* $[V, A] = \mathrm{C}_V(A)$ *and* $2|V/\mathrm{C}_V(A)| = |A|$*.*
2. $n = 2k$ *and* $A = \langle t_1t_2, t_2t_3 \dots, t_{l-1}t_l, t_{l+1}, t_{l+2}, \dots, t_k \rangle$ *for some* $2 \leq l \leq k$*,* $[V, A] = \mathrm{C}_V(A)$ *and* $|V/\mathrm{C}_V(A)| = |A|$*.*
3. $n = 2k$ *and* $A = \langle t_1t_2, s_1s_2, t_3, t_4 \dots, t_k \rangle$*, where* s_1, s_2 *are transpositions distinct from* t_1 *and* t_2 *and* s_1s_2 *moves the same four symbols as* t_1t_2*,* A *is not quadratic and* $|[V, A]| = |V/\mathrm{C}_V(A)| = |A|$*.*
4. $n = 8 = |A|$*,* A *acts regularly on* $\{1, 2, \dots, 8\}$*,* $[V, A] = C_V(A)$ *and* $|V/\mathrm{C}_V(A)| = |A|$*.*

In particular, if $A \leq \mathrm{Alt}(n)$ *and* $n \neq 8$*, then* $n = 2k$ *and* $A = \langle t_1t_2, t_2t_3, \dots, t_{k-1}t_k \rangle$*.*

Note that in all cases of the FF-Module Theorem M is generated by quadratic best offenders.

In the following list we give the module structure of A, $V/C_V(A)$ and $[V, A]$ considered as a $N_M(A)$-modules in the cases (a) – (d) of the Offender Theorem, as it can be deduced from the action of M on V. Put $P := O^{p'}(N_M(A))$.

Case	$P/\mathrm{O}_p(P)$	A	$[V, A]$	$V/C_V(A)$	Remarks
(a)	$\mathrm{SL}_2(q)$	U	U^*	U	$[U, P]$ a nat. $\mathrm{SL}_2(q)$-module
(b)	$\mathrm{SL}_{n-1}(q)$	U	$\bigwedge^2(U)$	U	U a nat. $\mathrm{SL}_{n-1}(q)$-module
(c)	$\mathrm{Sp}_4(q)$	nat. $\Omega_5(q)$	nat. $\mathrm{Sp}_4(q)$	nat. $\mathrm{Sp}_4(q)$	$V/C_V(A) \cong [V, A]$
					$A/C_A(P) \not\cong V/\mathrm{C}_V(A)$
(d)	$\mathrm{Spin}_8^+(q)$	nat. $\Omega_8^+(q)$	nat. $\Omega_8^+(q)$	nat. $\Omega_8^+(q)$	pairwise non-isom.

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1 Linear Algebra and Offenders

In this section p is a prime, M a finite group and V a finite dimensional $\mathbb{F}_pM$-module.

Lemma 1.1. *Let* $A \leq M$ *and* $\mathcal{W}$ *a set of* A*-submodules of* V *with* $V = \bigoplus \mathcal{W}$*. Suppose that* A *is a faithful offender on* V *but not an over-offender on* W *for any* $W \in \mathcal{W}$*. Let* $W \in \mathcal{W}$ *and put* $A_W = \bigcap_{W \neq U \in \mathcal{W}} \mathrm{C}_A(U)$*. Then*

(a) $|A| = |V/\mathrm{C}_V(A)|$*.*

(b) $A = \bigtimes_{W \in \mathcal{W}} A_W = A_W \times \mathrm{C}_A(W)$*.*

(c) $|A/\mathrm{C}_A(W)| = |W/C_W(A)| = |W/C_W(A_W)| = |A_W|$*.*

Proof. Since A is not an over-offender on W, $|A/C_A(W)| \leq |W/C_W(A)|$, and since $V = \bigoplus \mathcal{W}$, $|V/\mathrm{C}_V(A)| = \prod_{W\in\mathcal{W}} |W/C_W(A)|$. Since A is an offender on V this gives

$$(*) \qquad |A| \geq |V/\mathrm{C}_V(A)| = \prod_{W\in\mathcal{W}} |W/C_W(A)| \geq \prod_{W\in\mathcal{W}} |A/\mathrm{C}_A(W)|.$$

Put $B = \bigtimes_{W\in\mathcal{W}} A/\mathrm{C}_A(W)$ and let $B_W = A/\mathrm{C}_A(W)$ be viewed as a subgroup of B. So B is the internal direct product of the B_W, $W \in \mathcal{W}$. Consider the homomorphism

$$\phi : A \to B, a \to (a\mathrm{C}_A(W))_{W\in\mathcal{W}}.$$

Since V is a faithful A-module and $V = \bigoplus \mathcal{W}$, $\ker\phi = \bigcap_{W\in\mathcal{W}} \mathrm{C}_A(W) = \mathrm{C}_A(V) = 1$ and ϕ is injective. By $(*)$ $|A| \geq |B|$. Thus ϕ is surjective and so an isomorphism. Moreover, equality holds everywhere in $(*)$. In particular, (a) and the first equality in (c) hold.

Let $a \in A$. Then $a\phi \in B_W$ if and only if $a \in \mathrm{C}_A(U)$ for all $W \neq U \in \mathcal{W}$ and so if and only if $a \in A_W$. Thus $A_W\phi = B_W$. Also $a \in \mathrm{C}_A(W)$ if and only if the W-coordinate of $a\phi$ is 1 and so if and only if $a\phi \in \bigtimes_{W\neq U\in\mathcal{W}} B_W$. Thus $\mathrm{C}_A(W)\phi = \bigtimes_{W\neq U\in\mathcal{W}} B_W$. Since $B = \bigtimes_{W\in\mathcal{W}} B_W$ and ϕ is an isomorphism, (b) holds.

From (b) we get that $C_W(A) = C_W(A_W)$ and $|A_W| = |A/C_A(W)|$. Hence the (already proved) first equality in (c) gives also the second and third equality in (c). □

Lemma 1.2. *Let $A \leq M$. Then A is a best offender on V if and only if A is an offender on every A-submodule of V.*

Proof. If A is a best offender, then by [MS1, 2.5] A is an offender on every A-submodule of V.

Conversely, suppose A is an offender on every A-submodule of V. Then A is an offender on V and so elementary abelian. Let $B \leq A$ and put $W := \mathrm{C}_V(B)$. Clearly

$$(*) \qquad B \leq \mathrm{C}_A(W) \text{ and } \mathrm{C}_W(A) = \mathrm{C}_V(A).$$

As A is an offender on W, $|W/\mathrm{C}_W(A)| \leq |A/\mathrm{C}_A(W)|$, and $(*)$ implies that

$$|B||W| \leq |B||A/\mathrm{C}_A(W)||\mathrm{C}_W(A)| \leq |A||\mathrm{C}_V(A)|.$$

This shows that A is a best offender on V. □

Lemma 1.3. *Suppose that B is a minimal offender on V and W is a B-submodule of V. Then B is a quadratic best offender on W, and one of the following holds:*

1. *B is an over-offender on W.*

2. *$[W, B] = 0$.*

3. *$\mathrm{C}_B(W) = \mathrm{C}_B(V)$ and $V = W + \mathrm{C}_V(B)$.*

Proof. Let $D \leq B$. Since B is a minimal offender,

$$|D||\mathrm{C}_V(D)| \leq |V||C_D(V)| \leq |V||\mathrm{C}_B(V)| \leq |B||\mathrm{C}_V(B)|$$

and so B is a best offender. By the Timmesfeld Replacement Theorem [KS, 9.2.3], $\mathrm{C}_B([V, B])$ is a non-trivial offender on V and so by minimality $B = \mathrm{C}_B([V, B])$. Thus B is quadratic.

Assume that B is not an over-offender on W. Then $|B/\mathrm{C}_B(W)| = |W/\mathrm{C}_W(B)|$ and

$$|V/\mathrm{C}_V(B) + W| = |V/\mathrm{C}_V(B)||W/\mathrm{C}_W(B)|^{-1} \leq |B||B/\mathrm{C}_B(W)|^{-1} = |\mathrm{C}_B(W)|.$$

Hence $\mathrm{C}_B(W)$ is an offender on V, and the minimality of B gives either $B = \mathrm{C}_B(W)$ or $\mathrm{C}_B(W) = \mathrm{C}_B(V)$. In the first case (2) holds. In the second case

$$V = \mathrm{C}_V(B) + W$$

and (3) follows.

□

Lemma 1.4. *Suppose that $A \leq M$ acts nilpotently on V. Then the following are equivalent:*

(a) A is a strong dual offender on V.

(b) Let $0 \leq U \leq Y \leq V$ be any chain of A-submodules with $[Y/U, A] = 0$. Then $[V, A] \leq U$ or $Y \leq \mathrm{C}_V(A)$.

(c) A is a strong dual offender on V^*.

Proof. Suppose (a) holds. Let U and Y be as in (b) and suppose that $Y \not\leq \mathrm{C}_V(A)$. Pick $v \in Y \setminus \mathrm{C}_V(A)$. Then

$$[V, A] = [v, A] \leq [Y, A] \leq U.$$

Thus (a) implies (b).

Suppose next that (b) holds. To show that (a) holds, let $v \in V \setminus \mathrm{C}_V(A)$ and put $Y := \langle v^A \rangle$ and $U := [v, A]$. Since $[v^k, a] = [v, a]^k$ for all $k \in \mathbb{Z}, a \in A$, $U = [\langle v \rangle, A]$. So Y and U are A-submodules, $U \leq Y$ and A centralizes Y/U. Since $v \in Y$, $Y \not\leq \mathrm{C}_V(A)$ and so (b) implies that $[V, A] \leq U$. Hence $[v, A] = U = [V, A]$ and (a) holds.

By 1.8(c), (b) holds for V if and only if it holds for V^* in place of V. Thus the above argument with V^* in place of V shows that (b) and (c) are equivalent. □

Lemma 1.5. *Let A be a strong dual offender on V. Then the following hold:*

(a) A is quadratic on V.

(b) A is a strong dual offender on every A-submodule of V and V^*.

(c) A is best offender on V and on V^*.

(d) If $|[V, A]| = |A|$, then A is a strong offender on V.

Proof. Since by 1.4 A is also a strong dual offender on V^* it suffices to prove the statements for V.

(a): Since A acts nilpotently on V there exists $v \in V \setminus \mathrm{C}_V(A)$ with $[v, A] \leq \mathrm{C}_V(A)$. By definition of a strong dual offender we conclude that $[V, A] = [v, A] \leq \mathrm{C}_V(A)$ and so A is quadratic.

(b): This follows immediately from the definition of a strong dual offender.

(c): Let $v \in V \setminus \mathrm{C}_V(A)$. Since A is quadratic on V, $[v, A] = \{[v, a] \mid a \in A\}$ and so

$$(*) \qquad |[V, A]| = |[v, A]| = |A/\mathrm{C}_A(v)| \leq |A|.$$

Thus by 1.8 $|V^*/\mathrm{C}_{V^*}(A)| \leq |A|$. So A is an offender on V^*. By (b) this is also true for any A-submodule of V^*. Thus by 1.2 A is a best offender on V^*. By symmetry, A is also a best offender on V.

(d): Suppose $|[V,A]| = |A|$. Then by $(*)$

$$|A| \leq |A/\mathrm{C}_A(v)| \leq |A| \text{ for every } v \in V \setminus \mathrm{C}_V(A).$$

Hence $\mathrm{C}_A(v) = 1$ and so $\mathrm{C}_V(a) = \mathrm{C}_V(A)$ for all $a \in A^\sharp$. □

Lemma 1.6. *Let A be a strong offender on V. Then A is a quadratic best offender on V.*

Proof. Let W be an A-submodule of V with $[W,A] \neq 0$. Then $\mathrm{C}_A(W) = 1$ and so

$$|W/\mathrm{C}_W(A)| \leq |V/\mathrm{C}_V(A)| \leq |A| = |A/\mathrm{C}_A(W)|.$$

Hence A is an offender on W and so by 1.2, A is a best offender on V.

To show that A is quadratic we may assume that $[V,A] \neq 0$. Put $B = \mathrm{C}_A([V,A])$. By the Timmesfeld Replacement Theorem [KS, 9.2.3], $[V,B] \neq 0$ and since A is a strong offender, $\mathrm{C}_V(B) = \mathrm{C}_V(A)$. Since $[V,A,B] = 0$ we conclude that $[V,A,A] = 0$ and so A is quadratic. □

Lemma 1.7. *Let A be a subgroup of M. Suppose V is self-dual as an $\mathbb{F}_pA$-module. Then A is a strong offender iff $|V/\mathrm{C}_V(A)| = |A|$ and A is a strong dual offender.*

Proof. Suppose first that A is strong offender and let $1 \neq a \in A$. Then $\mathrm{C}_V(a) = \mathrm{C}_V(A)$ and since V is self-dual, $[V,a] = [V,A]$ by 1.8(c). Let $v \in V \setminus \mathrm{C}_V(A)$. Then $\mathrm{C}_A(v) = 1$ and so $|[v,A]| \geq |A|$. Hence

$$|A| \leq |[v,A]| \leq |[V,A]| = |[V,a]| = |V/\mathrm{C}_V(a)| = |V/\mathrm{C}_V(A)| \leq |A|,$$

and equality holds everywhere. Thus $[v,A] = [V,A]$ and so A is a strong dual offender.

Suppose now that $|V/\mathrm{C}_V(A)| = |A|$ and A is a strong dual offender. Since V is self-dual we get $|[V,A]| = |A|$. Thus by 1.5(d), A is a strong offender. □

Lemma 1.8. *Suppose that $\mathbb{K}$ is a field and V is a $\mathbb{K}$-space. The following hold for $A \leq \mathrm{GL}_{\mathbb{K}}(V)$ and U a $\mathbb{K}$-subspace of V:*

(a) $\dim_{\mathbb{K}} V = \dim_{\mathbb{K}} V^*$.

(b) $\dim_{\mathbb{K}} U + \dim_{\mathbb{K}} U^\perp = \dim_{\mathbb{K}} V$.

(c) $[V,A]^\perp = \mathrm{C}_{V^*}(A)$ *and* $\mathrm{C}_V(A)^\perp = [V^*,A]$.

(d) $[V,A,A] = 0 \iff [V^*,A,A] = 0$.

(e) $\mathrm{C}_M(\mathrm{C}_V(A)) \cap \mathrm{C}_M(\mathrm{C}_{V^*}(A))$ *is the largest subgroup* $Y \leq M$ *with* $\mathrm{C}_V(Y) = \mathrm{C}_V(A)$ *and* $[V,Y] = [V,A]$.

(f) If A is quadratic on V, then $\dim_{\mathbb{K}}[V,A] + \dim_{\mathbb{K}} V/\mathrm{C}_V(A) \leq \dim_{\mathbb{K}} V$.

Proof. (a), (b) and (c) are well-known and easy to prove statements from linear algebra; and (e) follows from (c).

(d): $[V,A,A] = 0$ iff $[V,A] \leq \mathrm{C}_V(A)$ iff $\mathrm{C}_V(A)^\perp \leq [V,A]^\perp$ iff $[V^*,A] \leq \mathrm{C}_{V^*}(A)$ iff $[V^*,A,A] = 0$.

(f): Since A is quadratic, $[V,A] \leq \mathrm{C}_V(A)$. Thus

$$\dim_{\mathbb{K}} V = \dim_{\mathbb{K}}[V,A] + \dim_{\mathbb{K}} \mathrm{C}_V(A)/[V,A] + \dim_{\mathbb{K}} V/\mathrm{C}_V(A).$$

□

Lemma 1.9. *Let $\mathbb{F}$ be a finite field of characteristic p, V a finite dimensional $\mathbb{F}H$-module, and $N \trianglelefteq H$. Put $\mathbb{K} := \mathrm{End}_{\mathbb{F}N}(V)$ and suppose that V is a self-dual simple $\mathbb{F}N$-module. Then the following hold:*

(a) There exists an N-invariant non-degenerate symmetric, symplectic or unitary $\mathbb{K}$-form s on V.

(b) There exists a homomorphism $\rho : H \to \mathrm{Aut}_{\mathbb{F}}(\mathbb{K})$ with $h \mapsto \rho_h$ such that $h \in H$ acts ρ_h-semi-linearly on the right $\mathbb{K}$-vector space V; i.e., $(v + w)h = vh + wh$ and $(vk)h = (vh)(k\rho_h)$ for $v, w \in V$ and $k \in \mathbb{K}$.

(c) There exists a map $\lambda : H \to \mathbb{K}^\sharp$ with $h \mapsto \lambda_h$ such that the map $H \to \mathbb{K}^\sharp \rtimes \mathrm{Aut}_{\mathbb{F}}(K), h \to \lambda_h\rho_h$ is a homomorphism and

$$(vh, wh)s = (v, w)s\lambda_h\rho_h$$

for all $v, w \in V$, $h \in H$.

(d) Let U be a $\mathbb{K}$-subspace of V and put $U^\perp = \{v \in V \mid (u, v)s = 0 \text{ for all } u \in U\}$. Then $U^\perp$ is $N_H(U)$-invariant.

(e) Let U be a non-zero $\mathbb{K}$-subspace of V such that $C_H(U)$ acts simply on $V/U^\perp$. Then U is 1-dimensional over $\mathbb{K}$.

(f) Put $H_0 = \ker\rho$. Then s is $O^{p'}(H_0)N$-invariant.

Proof. Recall that $\mathbb{K}$ is a finite field of characteristic p since V is finite and simple. It is convenient to write V in the following as a right $\mathbb{K}$-vector space since we write the action of $\mathbb{K}$ on V from the right.

Put $V^* := \mathrm{Hom}_{\mathbb{K}}(V, \mathbb{K})$ and $W := \mathrm{Hom}_{\mathbb{F}}(V, \mathbb{F})$. Let $\mu : \mathbb{K} \to \mathbb{F}$ be any non-zero $\mathbb{F}$-linear map and define

$$\tau : V^* \to W \text{ by } u \to u \circ \mu.$$

(Recall that our mappings act from the right, so $v(u \circ \mu) = (vu)\mu$.)

Let $0 \neq u \in V^*$. Then $Vu = \mathbb{K}$ and so there exists $v \in V$ with $vu \notin \ker\mu$. Thus $v.u\tau = vu\mu \neq 0$. In particular $u\tau \neq 0$ and $\ker\tau = 0$. Since τ is $\mathbb{F}$-linear and

$$\dim_{\mathbb{F}} V^* = \dim_{\mathbb{F}} \mathbb{K} \dim_{\mathbb{K}} V^* = \dim_{\mathbb{F}} \mathbb{K} \dim_{\mathbb{K}} V = \dim_{\mathbb{F}} V = \dim_{\mathbb{F}} W$$

we conclude that τ is an $\mathbb{F}$-isomorphism. For $n \in N$, $v \in V$ and $u \in V^*$ we have

$$v.un\tau = v.un.\mu = vn^{-1}u\mu = vn^{-1}.u\tau = v.u\tau n$$

and so $un\tau = u\tau n$. Thus τ is an $\mathbb{F}N$-isomorphism. Since V is self-dual as an $\mathbb{F}N$-module, this shows that V and V^* are isomorphic $\mathbb{F}N$-modules. Hence the set $\mathcal{H}$ of $\mathbb{F}N$-isomorphisms from V to V^* is non-empty.

For $k \in \mathbb{K}$ let

$$\overline{k} : V^* \to V^* \text{ defined by } x\overline{k} : v \mapsto vk.x \quad (x \in V^*,\ v \in V).$$

Then $\overline{k} \in End_{\mathbb{F}N}(V^*) =: \overline{\mathbb{K}}$ and $k \mapsto \overline{k}$ induces an isomorphism of fields from $\mathbb{K}$ to $\overline{\mathbb{K}}$.

Let $\beta \in \mathcal{H}$. Then $\beta \circ \overline{k} \circ \beta^{-1}$ is $\mathbb{F}$-linear and so

$$\sigma_\beta : \mathbb{K} \to \mathbb{K} \text{ with } k \mapsto \beta \circ \overline{k} \circ \beta^{-1}$$

is an $\mathbb{F}$-linear automorphism of $\mathbb{K}$. Since $\beta \circ \overline{k} = k\sigma_\beta \circ \beta$ we get

1°. *β is σ_β^{-1}-semi-linear.*

Let $\delta \in \mathcal{H}$ and put $l = \delta \circ \beta^{-1}$. Then l is $\mathbb{F}N$-linear and so $l \in \mathbb{K}$. Thus:

2°. *For all $\beta, \delta \in \mathcal{H}$ there exists $l \in \mathbb{K}$ with $\delta = l \circ \beta$.*

It follows that

$$k\sigma_\delta = \delta \circ \overline{k} \circ \delta^{-1} = l \circ \beta \circ \overline{k} \circ \beta^{-1} \circ l^{-1} = l \circ k\sigma_\beta \circ l^{-1}.$$

Since $\mathbb{K}$ is commutative, this implies $k\sigma_\delta = k\sigma_\beta$. Thus $\sigma_\delta = \sigma_\beta$ is independent from $\beta \in \mathcal{H}$. So we just write σ for σ_β.

Let $\mathcal{F}$ be the set of all N-invariant non-zero functions $s : V \times V \to \mathbb{K}$ which are $\mathbb{K}$-linear in the first coordinate and $\mathbb{F}$-linear in the second, where N-invariant means that $(vn, wn)s = (v, w)s$ for all $v, w \in V$ and $n \in N$. Clearly, all these forms are non-degenerate since V is a simple $\mathbb{F}N$-module.

For $\beta \in \mathcal{H}$ define $s_\beta : V \times V \to \mathbb{K}, (v, w) \to v.w\beta$. Then $s_\beta \in \mathcal{F}$ and so also $\mathcal{F} \neq \emptyset$. Conversely, for $s \in \mathcal{F}$ define $\beta_s : V \to V^*$ by $v.w\beta_s = (v, w)s$. Then $\beta_s \in \mathcal{H}$, and (1°) applied to β_s gives:

3°. *Each $s \in \mathcal{F}$ is a σ^{-1}-sesquilinear $\mathbb{K}$-form.*

Define $s^* : V \times V \to \mathbb{K}, (v, w) \to (w, v)s\sigma$. Then s^* is N-invariant, $\mathbb{K}$-linear in the first coordinate and σ-semi-linear in the second coordinate. In particular, $s^* \in \mathcal{F}$ and so (3°) implies. Hence

4°. *$\sigma = \sigma^{-1}$, and either $\sigma = \mathrm{id}_\mathbb{K}$ or σ has order 2.*

We now will prove (a) – (f).

(a): Put $t = s + s^*$. Then $t = t^*$. Suppose first that $t \neq 0$. If $\sigma = \mathrm{id}_\mathbb{K}$, then t is an N-invariant symmetric $\mathbb{K}$-form; and if $|\sigma| = 2$, then t is an N-invariant unitary $\mathbb{K}$-form. So (a) holds in this case.

Suppose next that $t = 0$. Then $s = -s^*$. Assume $\operatorname{char} \mathbb{K} = 2$, then $s = s^*$ and so s is a symmetric or unitary $\mathbb{K}$-form. Assume $\operatorname{char} \mathbb{K} \neq 2$. If $\sigma = \mathrm{id}_\mathbb{K}$ then s is a symplectic $\mathbb{K}$-form. If $|\sigma| = 2$ pick $x \in \mathbb{K}$ with $x \neq x\sigma$ and put $y := x - x\sigma$. Then $y\sigma = -y$. Hence $(sy)^* = s^*.y\sigma = sy$ and so sy is a N-invariant unitary $\mathbb{K}$-form on V. Again (a) hold.

(b): Since $N \trianglelefteq H$, it is readily verified that for $k \in \mathbb{K}$ and $h \in H$ the map $V \to V, v \mapsto vh^{-1}kh$ is in $\mathbb{K}$. Thus $\rho_h \in \mathrm{Aut}_\mathbb{F}(\mathbb{K})$ where

$$v.k\rho_h = vh^{-1}kh \text{ for all } k \in \mathbb{K},\ h \in H.$$

A simple calculation shows that $\rho : H \to \mathrm{Aut}_\mathbb{F}(\mathbb{K})$ with $h \mapsto \rho_h$ is a homomorphism and h acts ρ_h-semi-linearly on V.

(c): Fix $h \in H$ and define

$$s_h : V \times V \to \mathbb{K}, (v, w) \mapsto (vh, wh)s\rho_h^{-1}.$$

Using that $\mathrm{Aut}(\mathbb{K})$ is abelian, it is straight forward to verify that $s_h \in \mathcal{F}$. By (2°), $\beta_{s_h} = k_h \circ \beta_s$ for some $k_h \in \mathbb{K}$. Thus for all $v, w \in V$

$$(vh, wh)s\rho_h^{-1} = (v, w)s_h = v.w\beta_{s_h} = v.wk_h\beta_s = (v, wk_h)s = (v, w)s.k_h\sigma$$

Define $\lambda_h = k_h\sigma$, then

$$(vh, wh)s = (v, w)s\lambda_h\rho_h.$$

It is readily verified that the map $H \to \mathbb{K}^\sharp \rtimes \mathrm{Aut}_{\mathbb{F}}(\mathbb{K}), h \to \lambda_h\rho_h$ is a homomorphism.

(d): Let $v \in U^\perp, h \in N_H(U)$ and $u \in U$. Then

$$(u, vh)s = (uh^{-1}, v)s\lambda_h\rho_h = 0.$$

(e): Let D be a 1-dimensional $\mathbb{K}$-subspace of U. Then by (d), $D^\perp$ is $C_H(U)$-invariant. Since $U^\perp \leq D^\perp$ and $C_H(U)$ is simple on $V/U^\perp$ we get $U^\perp = D^\perp$ and $U = D$.

(f) For $a, b \in H_0$ the homomorphism given in (c) yields

$$\lambda_{ab}\rho_{ab} = \lambda_{ab} = \lambda_a\rho_a\lambda_b\rho_b = \lambda_a\lambda_b.$$

Hence $\lambda\mid_{H_0}$ is a homomorphism from H_0 in $\mathbb{K}^\sharp$. Since $\mathbb{K}^\sharp$ is a p'-group, (f) follows. □

2 *J*-Components

In this section p is a prime, M is a finite group with $\mathrm{O}_p(M) = 1$, and V is a finite dimensional faithful $\mathbb{F}_pM$-module such that $\mathrm{J}_M(V) \neq 1$.

Notation 2.1. *Put $J := \mathrm{J}_M(V)$ and $\mathcal{J} := \mathcal{J}_M(V)$. Let $\mathcal{I}$ be the set of solvable J-components, $\mathcal{K}$ be the set of perfect J-components, $E := \langle\mathcal{K}\rangle$, and $I := \langle\mathcal{I}\rangle$.*

Lemma 2.2. *The following hold:*

(a) $\mathrm{C}_M(J/\mathrm{Z}(J)) = \mathrm{C}_M(J)$.

(b) Let N be a J-invariant subgroup of M with $[N, J] \neq 1$. Then there exists $K \in \mathcal{J}$ with $K \leq N$.

(c) $\mathcal{J} \neq \emptyset$, $\mathcal{J} = \mathcal{I} \cup \mathcal{K}$, and $\mathcal{K}$ is the set of components of J.

(d) Let $K \in \mathcal{I}$. Then either $p = 2$, $K \cong \mathrm{C}_3 \cong \mathrm{SL}_2(2)'$, and $[V, K] \cong \mathbb{F}_2^2$, or $p = 3$, $K \cong Q_8 \cong \mathrm{SL}_2(3)'$, and $[V, K] \cong \mathbb{F}_3^2$.

(e) $[W, K] = [W, K, K]$ for every $K \in \mathcal{J}$ and every K-submodule W of V.

(f) $[K, F] = 1$ and $[V, K, F] = 0$ for every $K, F \in \mathcal{J}$ with $K \neq F$.

(g) $\mathrm{C}_J(IE) = \mathrm{Z}(J)$, or $p = 2$ and $\mathrm{C}_J(IE) = \mathrm{Z}(J)I$. So in both cases $\mathrm{C}_J(IE)$ is an abelian p'-group.

(h) Let $U \leq M$ and $K \in \mathcal{J}$. Then either $[K, U] = 1$ or $[W, K] \leq [W, [K, U]]$ for every K-submodule $W \leq V$.

Proof. (a) Put $R = \mathrm{C}_M(J/\mathrm{Z}(J))$ and let T be a p-subgroup of J. Since $\mathrm{O}_p(M) = 1$, $\mathrm{O}_p(\mathrm{Z}(J)) = 1$ and so $\mathrm{Z}(J)$ is a p'-group, Since $[\mathrm{Z}(J), T] = 1$, we conclude that $T = \mathrm{O}_p(\mathrm{Z}(J)T)$. So, as $[R, T] \leq \mathrm{Z}(J)$, R normalizes T and $[R, T] \leq T \cap \mathrm{Z}(J) = 1$. Since J is generated by p-groups, this means $[R, J] = 1$ and so $R = \mathrm{C}_M(J)$.

(b): By (a), $[N, J] \not\leq \mathrm{Z}(J)$. So by [MS1, 3.1] there exists $K \in \mathcal{J}$ with $K \leq [N, J]$.

(c) and (d) follow from [MS1, 3.2], and [MS1, 3.4], and (f) is The Other $P(G, V)$-Theorem in [MS1].

(e): By (c) and (d) K is generated by p'-elements. Hence (e) follows from elementary properties of coprime action.

(g): Put $C := \mathrm{C}_J(IE)$. Clearly $\mathrm{Z}(J) \leq C$. Hence, by (b) either $C = \mathrm{Z}(J)$, or there exists a J-component in C. Assume the latter case. Then by (c) and (d), $p = 2$ and $I \leq C$. The action of C on $[V, I]$ shows that $C = I\mathrm{C}_C([V, I])$. But now again (b), this time applied to $\mathrm{C}_C([V, I])$, gives $\mathrm{C}_C([V, I]) \leq \mathrm{Z}(J)$ and thus $C = \mathrm{Z}(J)I$.

(h): Note that $K[K, U] = K^u[K, U]$ for every $u \in U$. Assume first that $U \not\leq \mathrm{N}_M(K)$. Then there exists $u \in U \setminus \mathrm{N}_U(K)$, and by (f) $[W, K] \leq \mathrm{C}_W(K^u)$. Now (e) yields

$$[W, K] = [W, K, K] \leq [W, K, K^u[K, U]] = [W, K, [K, U]] \leq [W, [K, U]].$$

Assume now that $U \leq \mathrm{N}_M(K)$, $[K, U] \neq 1$ and $[W, K] \neq 0$. Then $1 \neq [K, U] \trianglelefteq K$. By (c) and (d) K is a component, or $K \cong \mathrm{C}_3$, or $K \cong Q_8$. In the first case $K \leq [K, U]$, and (h) follows. In the other two cases by (d) $[W, K] = [V, K]$ is a faithful simple K-module, so $[V, K] = [V, [K, U]]$. □

Lemma 2.3. *Let A be a best offender of M on V and $K \in \mathcal{J}$. Then the following hold:*

(a) $[K, A] = K$ or $[K, A] = 1$.

(b) If $[K, A] \neq 1$, then there exists a best offender $A_0 \leq A$ such that $K = [K, A_0]$, $[[V, K], A_0, A] = 0$, and A_0 is quadratic on $[V, K]$.

Proof. (a) is obvious since $K \trianglelefteq J$ and by 2.2 either K is quasisimple or isomorphic to C_3 or Q_8.

(b): This is essentially [MS1, 3.3], but since our assumption is slightly weaker we repeat the proof: By (a) $[K, A] = K$ and by 2.2(e) $[V, K] = [V, K, K]$, so $[V, K, A] \neq 0$. The Timmesfeld Replacement Theorem [MS1, 2.7] with $W := [V, K]$ gives a best offender $A_0 \leq A$ satisfying $[W, A_0, A] = 0$ and $[W, A_0] \neq 0$. The first property shows that A_0 is quadratic on W. Suppose that $[K, A_0] = 1$. Then by [MS1, 2.9], $[W, A_0] = 0$, a contradiction. Thus $[K, A_0] \neq 1$ and by (a), $K = [K, A_0]$. □

Lemma 2.4. *Let $K \in \mathcal{J}$ and A be a subgroup of M such that $[V, A, A] = 0$ and $[K, A] \neq 1$. Suppose that X is a perfect K-submodule of V and $\overline{X}$ is a non-zero K-factor module of X. Then*

$$\mathrm{C}_A(X) = \mathrm{C}_A(K) = \mathrm{C}_A(\overline{X}).$$

Proof. Put $L := [K, A]$. The quadratic and faithful action of A shows that A is an elementary abelian p-subgroup. Hence $A_0 := \mathrm{C}_A(K)$ centralizes $\langle K, A\rangle$ and so also L. The quadratic action of A gives

$$[V, L] \leq [V, \langle A^K\rangle] = \langle [V, A]^K\rangle \leq \mathrm{C}_V(A_0).$$

As $[K, A] \neq 1$, 2.2(h) yields $X = [X, K] \leq [X, L] \leq \mathrm{C}_V(A_0)$ and $A_0 \leq \mathrm{C}_A(X) \leq \mathrm{C}_A(\overline{X})$. Conversely, $[X, [K, \mathrm{C}_A(\overline{X})]] \neq X$ since $\overline{X} \neq 0$. Hence again 2.2(h) implies that $\mathrm{C}_A(\overline{X}) \leq \mathrm{C}_A(K)$. □

Lemma 2.5. *Let $K \in \mathcal{J}$ and $\mathbb{K} := \mathrm{End}_K(V)$. Suppose that V is a simple K-module and M is generated by quadratic offenders on V. Then the following hold:*

(a) $\mathbb{K}$ is a finite field.

(b) M acts $\mathbb{K}$-linearly on V, or $|V| = 4$ and $M \cong \mathrm{SL}_2(2)$.

(c) $\mathrm{F}^*(M) = \mathrm{Z}(M)K$, and $\mathrm{C}_M(K) = \mathrm{Z}(M)$ if $|V| > 4$.

Proof. (a): By Schur's Lemma $\mathbb{K}$ is a finite division ring, so by Wedderburn's Theorem $\mathbb{K}$ is a field.

(b): Let $A \leq M$ be a quadratic offender and suppose A does not act $\mathbb{K}$-linearly on V. Then by [MS3, 2.14], $|A| = 2$. Since $|A|$ is an offender we get $|V/\mathrm{C}_V(A)| = 2$. Since A does not act $\mathbb{K}$-linearly, there exists $0 \neq k \in \mathbb{K}$ which is inverted by $a \in A^\sharp$; and since k acts fixed-point-freely on V, $|C_V(a)|^2 = |V|$. This implies $|\mathbb{K}| = 4 = |V|$. Hence $M \cong \mathrm{SL}_2(2)$ and (b) is proved.

(c): Suppose K is solvable. Then by 2.2 $|V| = 4$ or $|V| = 9$ and (c) is obvious. So we may assume that K is not solvable and so by 2.2 K is a component of M; in particular $\mathrm{F}^*(M) = K\mathrm{C}_{\mathrm{F}^*(M)}(K)$. By (b) M acts $\mathbb{K}$-linearly on V, so $\mathrm{C}_M(K) \leq \mathrm{Z}(M)$, and $\mathrm{F}^*(M) = K\mathrm{C}_{\mathrm{F}^*(M)}(K) = K\mathrm{Z}(M)$. □

Lemma 2.6. *Let $K \in \mathcal{J}$ and X be a perfect K-submodule of V, and let A be a best offender of M on V such that $[K, A] \neq 1$. Then A normalizes X.*

Proof. By 2.3(b) there exists a best offender $A_0 \leq A$ such that $[K, A_0] = K$, $[[V, K], A_0, A] = 0$ and A_0 is quadratic on $[V, K]$. Clearly A normalizes K since $K \trianglelefteq J$.

We will first show that A_0 normalizes X. Note that by 1.2 A_0 is a best offender on $W := \langle X^{A_0} \rangle$. Let $R := \mathrm{rad}_K(W)$, that is, the intersection of the maximal K-submodules of W, and put $\overline{W} := W/R$. Note that $W = [W, K]$ and so by 2.4 $\mathrm{C}_{A_0}(W) = \mathrm{C}_{A_0}(\overline{W}) = \mathrm{C}_{A_0}(K)$. Since A_0 is a quadratic offender on W, we conclude that A_0 is also a quadratic offender on $\overline{W}$. Thus there exists a quadratic best offender $A_1 \leq A_0$ on $\overline{W}$ such that $[\overline{W}, A_1] \neq 0$ and so by 2.4 $[K, A_1] \neq 1$.

Note that $\overline{X}$ is a semisimple K-module. Let $\overline{Y}$ be any simple K-submodule of $\overline{X}$. By [MS1, 2.10] A_1 normalizes $\overline{Y}$. Moreover, since $\overline{X}$ is a perfect K-module and $[K, A_1] \neq 1$, 2.4 gives $[\overline{Y}, A_1] \neq 0$. Now $0 \neq [\overline{Y}, A_1] \leq \mathrm{C}_{\overline{Y}}(A_0)$ shows that also A_0 normalizes $\overline{Y}$. Hence, A_0 normalizes $\overline{X}$ and $W = X + R$, so $W = X$.

Thus A_0 normalizes X. Let $a \in A$. Then $[X, A_0] \leq X \cap X^a =: D$. Since D is a KA_0-module and $[X, A_0] \leq D$, we get from 2.2(h) $X = [X, K] \leq [X, [K, A_0]] \leq D$ and thus $X^a = X$. So A normalizes X. □

Lemma 2.7. *Let $K \in \mathcal{J}$ and X be a perfect K-submodule of V, and let B be a best offender of M on V such that $[K, B] = 0$. Then $[X, B] = 0$.*

Proof. Let X be a counterexample such that $\dim_{\mathbb{F}_p} X$ is minimal, and let W be a maximal K-submodule of X. We use the following notation:

$$Y := \langle X^B \rangle,\ U := [W, K],\ \ B_0 := \mathrm{C}_B(Y),\ \overline{Y} := Y/\mathrm{C}_Y(K).$$

Note that $[Y, K] = Y$. Since $[Y, \mathrm{C}_B(\overline{Y}), K] = 0$ and $[\mathrm{C}_B(\overline{Y}), K] \leq [B, K] = 1$, the Three Subgroups Lemma gives $[Y, \mathrm{C}_B(\overline{Y})] = [K, Y, \mathrm{C}_B(\overline{Y})] = 0$. It follows that

$$\mathrm{C}_B(X) = B_0 = \mathrm{C}_B(\overline{Y}) = \mathrm{C}_B(\overline{X}).$$

As B is a best offender on Y by 1.2, B is an offender on $\overline{Y}$.

Since U is a perfect K-module, the minimality of X gives $[U, B] = 0$. Thus $[W, K, B] = 0$ and $[K, B] = 0$, and the Three Subgroups Lemma yields $[W, B, K] = 0$. Thus $[\overline{W}, B] = 0$ and so $\mathrm{C}_{\overline{X}}(b) = \overline{W}$ for every $b \in B \setminus B_0$ since $\overline{X}/\overline{W}$ is simple. Hence $[\overline{X}, b] \cong \overline{X}/\mathrm{C}_{\overline{X}}(b) = \overline{X}/\overline{W} \cong X/W := I$. This shows that $[\overline{X}, B]$ is the direct sum of, say n, copies of I.

Put $\mathbb{F} := \mathrm{End}_K(I)$. Let

$$\kappa_b : \overline{X} \to [\overline{X}, B] \text{ with } \overline{x} + \overline{W} \mapsto [\overline{x}, b]. \quad (b \in B)$$

Then $b \mapsto \kappa_b$, $b \in B$, defines to a homomorphism from B to $\mathrm{Hom}_{\mathbb{F}}(\overline{X}/\overline{W}, [\overline{X}, B]) \cong \mathbb{F}^n$ whose kernel is $\mathrm{C}_B(\overline{X}) = \mathrm{C}_B(X)$. It follows that $|B/\mathrm{C}_B(X)| \leq |\mathbb{F}|^n$. Since B is an offender on $\overline{Y}$ with $B_0 = \mathrm{C}_B(\overline{Y})$ and $\mathrm{C}_{\overline{X}}(B) = \overline{W}$,

$$|\mathbb{F}|^n \geq |B/B_0| \geq |\overline{Y}/\mathrm{C}_{\overline{Y}}(B)| \geq |\overline{X}\mathrm{C}_{\overline{Y}}(B)/\mathrm{C}_{\overline{Y}}(B)| = |\overline{X}/\overline{W}| = |I|,$$

so

$$(+) \qquad \dim_{\mathbb{F}} I \leq n.$$

According to 1.2 and (b) there exists a best offender A on V such that $[K, A] = K$ and A is quadratic on V. By 2.6 A normalizes X, Y and U and thus also W and X/W since $W/U = \mathrm{C}_{X/U}(K)$. Let $b \in B \setminus \mathrm{C}_B(\overline{X})$. Then $[X, b]$ is a perfect K-submodule of Y, and so again by 2.6 A normalizes $[X, b]$ and thus also $[\overline{X}, b]$. Since $I = X/W \cong [\overline{X}, b]$ as K-module, $D := \mathrm{Hom}_K(I, [\overline{X}, b])$ is a non-trivial p-group. Since A acts on D we get $\mathrm{C}_D(A) \neq 0$ and so $\mathrm{Hom}_{KA}(I, [\overline{X}, b]) \neq 0$. Thus $[\overline{X}, b]$ is isomorphic to I as an KA-module.

By 2.4

$$(*) \qquad \mathrm{C}_A(I) = \mathrm{C}_A(K) = \mathrm{C}_A(Y),$$

so 1.2 shows that A is a non-trivial quadratic offender on I. Hence by 2.5(b) A acts $\mathbb{F}$-linearly on I or $|I| = 4$. In the latter case $(*)$ implies $|A/C_A(I)| = 2 = |Y/\mathrm{C}_Y(A)|$, $|K| = 3$ and $|Y| = 4$. In particular $[Y, B] = 0$.

Assume now that A acts $\mathbb{F}$-linearly on I. Let $m = \dim_{\mathbb{F}} I$ and $c = \dim_{\mathbb{F}} \mathrm{C}_I(A)$. Recall that $\overline{Y} = \overline{X} + [\overline{X}, B]$ and $[\overline{X}, B]$ is the direct sum of n copies of KA-modules isomorphic to I. Hence

$$\dim_{\mathbb{F}} Y/\mathrm{C}_Y(A) \geq \dim_{\mathbb{F}} \overline{Y}/\mathrm{C}_{\overline{Y}}(A) \geq n \cdot \dim_{\mathbb{F}} I/\mathrm{C}_I(A) = n(m-c).$$

Since A acts quadratically on I, $|A/\mathrm{C}_A(I)| \leq |\mathrm{Hom}_{\mathbb{F}}(I/\mathrm{C}_I(A), \mathrm{C}_I(A))|$, so $|A/\mathrm{C}_A(I)| \leq |\mathbb{F}|^{c(m-c)}$. On the other hand, by $(*)$ $\mathrm{C}_A(I) = \mathrm{C}_A(Y)$ and so by $(+)$

$$|A/\mathrm{C}_A(Y)| = |A/\mathrm{C}_A(I)| \leq |\mathbb{F}|^{c(m-c)} < |\mathbb{F}|^{n(m-c)} \leq |Y/\mathrm{C}_Y(A)|,$$

a contradiction since A is an offender. □

Proposition 2.8. *Let $K \in \mathcal{J}$ and X be a perfect K-submodule of V. Then J normalizes X.*

Proof. This follows from 2.6 and 2.7. □

Lemma 2.9. *Let $K \in \mathcal{J}$ and let*

$$X_0 \leq Y_1 \leq X_1 \leq Y_2 \leq X_2 \ldots \leq Y_n \leq X_n \leq V$$

be a series of K-submodules such that $X_i = [X_i, K]$, X_i/Y_i is a simple K-module, and $[Y_i, K] \leq X_{i-1}$ for $i = 1, \ldots, n$. Then the following hold for $S := \oplus_{i=1}^n X_i/Y_i$:

(a) J acts on S and $\mathrm{O}_p(\widetilde{J}) = 1$, where $\widetilde{J} := J/\mathrm{C}_J(S)$.

(b) Every best offender on V is an offender on S; in particular $\widetilde{J}$ is generated by offenders on S.

(c) $\widetilde{K}$ is the unique $\mathrm{J}_{\widetilde{J}}(S)$-component of $\widetilde{J}$.

Proof. (a): By 2.8 J normalizes every X_i and Y_i since $Y_i/X_{i-1} = C_{X_i/X_{i-1}}(K)$, so J acts on S. Since X_i/Y_i, $i \geq 1$, is a simple K-module, we also get $\mathrm{O}_p(\widetilde{J}) = 1$.

(b): Let A be a best offender on V. By 2.7 $[S,A] = 0$ if $[K,A] = 1$. In the other case 2.4 shows that

$$(*) \qquad \mathrm{C}_A(K) = \mathrm{C}_A(X_i) = \mathrm{C}_A(X_i/Y_i),\ i = 1,\dots,n.$$

Hence in both cases $\mathrm{C}_A(S) = \mathrm{C}_A(K)$.

By 1.2 A is a best offender on X_n. Hence

$$|X_n/\mathrm{C}_{X_n}(A)| \leq |A/\mathrm{C}_A(X_n)| = |A/\mathrm{C}_A(K)| = |A/\mathrm{C}_A(S)|.$$

On the other hand,

$$|X_n| = |X_n/Y_n||Y_n/X_{n-1}||X_{n-1}/Y_{n-1}|\cdots|X_1/Y_1||Y_1|$$

and

$$|\mathrm{C}_{X_n}(A)| \leq |\mathrm{C}_{X_n/Y_n}(A)||Y_n/X_{n-1}||\mathrm{C}_{X_{n-1}/Y_{n-1}}(A)|\cdots|\mathrm{C}_{X_1/Y_1}(A)||Y_1|,$$

so

$$|A/\mathrm{C}_A(S)| \geq |X_n/\mathrm{C}_{X_n}(A)| \geq |X_n/Y_n/\mathrm{C}_{X_n/Y_n}(A)|\cdots|X_1/Y_1/\mathrm{C}_{X_1/Y_1}(A)| \geq |S/\mathrm{C}_S(A)|.$$

This shows that A is an offender on S.

(c): There exists a best offender A on V such that $[K,A] \neq 1$ and thus by $(*)$ also $[S,A] \neq 0$. By (b) A is an offender on S, so A contains a non-trivial best offender B on S. Again $(*)$ yields $[K,B] \neq 1$. Hence by 2.3(a), $\widetilde{K} \leq \mathrm{J}_{\widetilde{J}}(S)$ and so $\widetilde{K} \trianglelefteq \mathrm{J}_{\widetilde{J}}(S)$. Now 2.2(c) and (d) show that $\widetilde{K}$ is a $\mathrm{J}_{\widetilde{J}}(S)$-component of $\widetilde{J}$. Moreover, since $[S,\widetilde{K}] = S$, 2.2(f) implies that $\widetilde{K}$ is the unique $\mathrm{J}_{\widetilde{J}}(S)$-component of $\widetilde{J}$. □

Lemma 2.10. *Let $K \in \mathcal{J}$ and L be a normal subgroup of M with $L = \mathrm{O}^{p'}(L)$. Then either $K \leq [K,L] \leq L$ or $[K,L] = 1$.*

Proof. If K is a component of M, this is [KS, 6.5.2]. So suppose K is solvable. Then either $p = 2$ and $K \cong \mathrm{C}_3$, or $p = 3$ and $K \cong Q_8$.

We may assume that $[K,L] \neq 1$. Since $L = \mathrm{O}^{p'}(L)$, there exists a p-subgroup T of L with $[K,T] \neq 1$. If If T normalizes K, the structure of $\mathrm{Aut}(K)$ shows that $K = [K,T] \leq [K,L] \leq L$. So we may assume there exists $t \in T$ with $K \neq K^t$. Put $L_0 := KK^t \cap L$. Then $L_0 \trianglelefteq J$, and $KK^t = KL_0 = K^tL_0$ since $[K,t] \leq L$. In particular $[L_0,J] \neq 1$ since $K = [K,J] \neq K^t$. Hence, by 2.2(b) there exists a J-component $\widetilde{K} \leq L_0$, so $\widetilde{K} \leq KK^t$. If $\widetilde{K} = K$ or K^t, then $K \leq KK^t = \widetilde{K}L_0 \leq L_0 \leq L$. Suppose that $\widetilde{K}$ is different from K and K^t. Then by 2.2(e),(f)

$$[V,\widetilde{K}] = [V,\widetilde{K},\widetilde{K}] \leq [V,KK^t,\widetilde{K}] = 0,$$

a contradiction. □

Lemma 2.11. *Let $K \in \mathcal{J}$, W a K-submodule of V, $\overline{V} := V/W$ and U a K-submodule of $\overline{V}$. Then the following are equivalent:*

(a) U is a perfect K-module and $U/C_U(K)$ is a simple K-module.

(b) U is a quasisimple K-module.

(c) U is a minimal non-trivial K-submodule of $\overline{V}$.

Proof. (a) $\Longrightarrow$ (b): Let N be the inverse image of $\mathrm{O}_p(K/\mathrm{C}_K(U))$ in K. Then $U \neq [U,N]$ and since U is a perfect K-module, $N \neq K$. By 2.2 K is quasisimple or K is p'-group. In the first case $N \leq \mathrm{Z}(K)$ and since $\mathrm{O}_p(K) \leq \mathrm{O}_p(M) = 1$, N is a p'-group. So in any case N is a p'-group. Thus $N/\mathrm{C}_K(U) = 1$ and so U is a quasisimple K-module.

(b) $\Longrightarrow$ (c): Let Y be non-zero K-submodule of U. By 2.2, $K = \mathrm{O}^p(K)$ and so $\mathrm{C}_U(K) = \mathrm{C}_U(\mathrm{O}^p(K))$. Thus $U/C_U(K)$ is a simple K-module. If $Y \nleq \mathrm{C}_U(K)$ we get $U = Y + \mathrm{C}_V(K)$ and so $U = [U,K] = [Y,K] \leq Y$ and $Y = U$. Thus, either $Y = U$ or $Y \leq \mathrm{C}_U(K)$, so Y is a minimal non-trivial K-submodule of $\overline{V}$.

(c) $\Longrightarrow$ (a): Since U is non-trivial, $U \neq \mathrm{C}_U(K)$. Let Y be a proper K-submodule of U with $\mathrm{C}_U(K) \leq Y$. Then $[Y,K] = 0$ by minimality of U. Thus $Y = \mathrm{C}_U(K)$ and so $U/\mathrm{C}_U(K)$ is a simple K-module. Since $K = O^p(K)$, $[U,K,K] \neq 1$ and so $U = [U,K]$ by minimality of U. Thus U is a perfect K-module and (a) holds. □

3 Maximal Quadratic Offenders in Classical Groups

In this section $\mathbb{K}$ is a field and V is an n-dimensional vector space over $\mathbb{K}$. We assume that there exists a sesquilinear form f on V such that one of the following holds: (Recall here that f is non-degenerate if for each $0 \neq v \in V$ there exists $w \in V$ with $f(v,w) \neq 0$.)

(i) $f = 0$.

(ii) f is a non-degenerate symplectic form on V; so f is bilinear and $f(v,v) = 0$ for $v \in V$.

(iii) f is a non-degenerate unitary form; so there exists $\alpha \in \mathrm{Aut}(\mathbb{K})$ such that $\alpha^2 = \mathrm{id}_{\mathbb{K}} \neq \alpha$, f is linear in the first component, and $f(v,w) = f(w,v)\alpha$ for $v,w \in V$.

(iv) f is a symmetric bilinear form and there exists an associated non-degenerate quadratic form h on V, that is a function $h : V \to \mathbb{K}$ with

$$h(k_1 v + k_2 w) = k_1^2 h(v) + k_2^2 h(w) + k_1 k_2 f(v,w) \text{ for } k_1, k_2 \in \mathbb{K},\ v, w \in V.$$

(Recall here that h is non-degenerate if for each $0 \neq v \in V$ with $h(v) = 0$ there exists $w \in V$ with $f(v,w) \neq 0$.) Also if $\mathrm{char}\,\mathbb{K} = 2$, we assume that $\mathbb{K}$ is perfect and so for each $k \in \mathbb{K}$ there exists a unique element $\sqrt{k} \in \mathbb{K}$ with $\sqrt{k}^2 = k$.

By $\mathrm{GL}(V)$, $\mathrm{Sp}(V)$, $\mathrm{GU}(V)$, and $\mathrm{O}(V)$, respectively, we denote the group of automorphisms of V leaving invariant f (in the first three cases) and h in the fourth case. In the last three cases V is called a non-degenerate symplectic, unitary and orthogonal space, respectively.

We also use the notation $\mathrm{GL}_n(\mathbb{F})$, $\mathrm{Sp}_n(\mathbb{F})$, $\mathrm{GU}_n(\mathbb{F})$, and $\mathrm{O}_n(\mathbb{F})$, where $n := \dim V$ and either $\mathbb{F} = \mathbb{K}$ or, in the unitary case, $\mathbb{F} = \mathbb{K}_\alpha$, the subfield centralized by α. In the first three cases put $\alpha = \mathrm{id}_{\mathbb{K}}$, so $\mathbb{F} = \mathbb{K}_\alpha$. If $\mathbb{F}$ is finite, say $|\mathbb{F}| = q$, we also write $\mathrm{GL}_n(q)$, $\mathrm{Sp}_n(q)$, etc.

An element $v \in V$ is called isotropic if $f(v,v) = 0$. A subspace U of V is called isotropic if $f\,|_{U \times U} = 0$. An element $v \in V$ is called singular if v isotropic and (in the fourth case) $h(v) = 0$. A subspace is called singular if it is isotropic and all its elements are singular.

By V^* we denote the vector space dual to V, so $V^* := \mathrm{Hom}_{\mathbb{K}}(V,\mathbb{K})$ and an element $g \in GL(V)$ acts on V^* via

$$xg: \ v \mapsto (vg^{-1})x \quad (x \in V^*,\, v \in V).$$

We will use the notion of perpendicularity (and the symbol $\perp$) with respect to f.

An α-sesquilinear form on V is a function $g : V \times V \to \mathbb{K}$ such that g is $\mathbb{K}$-linear in the first coordinate and α-semilinear in the second coordinate. We denote the set of α-sesquilinear forms on V be $\mathrm{F}_\alpha(V)$. Observe that $\mathrm{F}_\alpha(V)$ is vector space over $\mathbb{K}$. Moreover, an element $t \in \mathrm{GL}_{\mathbb{K}}(V)$ acts on $\mathrm{F}_\alpha(V)$ via

$$gt : (u,v) \mapsto g(ut^{-1}, vt^{-1}) \quad u,v \in V.$$

Let $\eta \in \{\pm\}$. An (α,η)-sesquilinear form on V is an α-sesquilinear form g with $g(v,w) = \eta g(w,v)\alpha$ for all $v,w \in V$. $\mathrm{F}_{\alpha,\eta}(V)$ denotes the set all (α,η)-sesquilinear forms. Note that $\mathrm{F}_{\alpha,\eta}(V)$ is an $\mathbb{F}$-subspace of $\mathrm{F}_\alpha(V)$. $\bigwedge_2(V)$ denotes the set of symplectic forms on V and $\mathrm{S}_2(V)$ denotes the set symmetric bilinear forms on V. So $\mathrm{S}_2(V) = \mathrm{F}_{\mathrm{id},+}(V)$. Also $\bigwedge_2(V) \leq \mathrm{F}_{\mathrm{id},-}(V)$ with equality if $\mathrm{char}\,\mathbb{K} \neq 2$.

Note that, if $f \neq 0$, then f is an (α,ϵ)-sesquilinear form, where $\epsilon = +$ for $M = \mathrm{O}(V)$ or $M = \mathrm{GU}(V)$ and $\epsilon = -$ for $M = \mathrm{Sp}(V)$.

In the following $M = \mathrm{GL}(V)$, $\mathrm{Sp}(V)$, $\mathrm{GU}(V)$ and $\mathrm{O}(V)$, respectively. In this section we will write the action of M on V as right multiplication.

Lemma 3.1. *Let U be an isotropic but not singular $\mathbb{K}$-subspace of V. Let U_0 be the set of singular vectors in U. Then $G = \mathrm{O}(V)$, $p = 2$, U_0 is $\mathbb{K}$-subspace of U and $\dim_{\mathbb{K}} U/U_0 = 1$. In particular, $\dim_{\mathbb{K}} V^\perp \leq 1$.*

Proof. Since U is isotropic, $f\mid_{U\times U} = 0$, so all elements in U are isotropic. Since U is not singular, there exists a non-singular element u in U. Since u is isotropic, we conclude that $G = \mathrm{O}(V)$ and $h(u) \neq 0$. Then $4h(u) = h(2u) = h(u+u) = h(u) + f(u,u) + h(u) = 2h(u)$ and so $p = 2$. In particular, K is perfect and for every $k \in \mathbb{K}$ there exists a unique $\sqrt{k}$ such that $\sqrt{k}^2 = k$. Consider the map

$$\tau : U \to \mathbb{K} \text{ with } u \to \sqrt{h(u)}.$$

Observe that $U_0 = \ker \tau$. Since U is isotropic,

$$\tau(u+v) = \sqrt{h(u+v)} = \sqrt{h(u) + f(u,v) + h(v)} = \sqrt{h(u)} + \sqrt{h(v)} = \tau(u) + \tau(v).$$

for all $u, v \in U_0$. Also

$$\tau(ku) = \sqrt{h(ku)} = \sqrt{k^2 h(u)} = k\tau(u),$$

and so τ is $\mathbb{K}$-linear. Thus $U_0 = \ker\tau$ is $\mathbb{K}$-subspace and $\dim_{\mathbb{K}} U/U_0 = \dim_{\mathbb{K}} \mathbb{K} = 1$. $\square$

Lemma 3.2. *Suppose $f \neq 0$. Let $A \leq M$ and U be subspace of V.*

(a) $V/U^\perp$ and $U/U \cap V^\perp$ are isomorphic $\mathbb{F}N_M(U)$-modules. In particular, if f is non-degenerate, then V and V^* are isomorphic $\mathbb{F}M$-modules.

(b) $\mathrm{C}_{V/V^\perp}(A) = \mathrm{C}_V(A)/V^\perp$.

(c) $\mathrm{C}_V(A) = [V,A]^\perp$.

(d) $C_M(V/U) \le C_M(U^\perp)$*; in particular* $C_M(V/U) \le C_M(U)$ *if* U *is isotropic.*

(e) If A *acts quadratically on* $V/V^\perp$*, then* A *acts quadratically on* V *and* $[V, A]$ *is an isotropic subspace of* V*.*

Proof. (a): Replacing V by $V/V^\perp$ and U by $U + V^\perp/V^\perp$ we may assume that $V^\perp = 0$. For $w \in V$ define $w^* : U \to \mathbb{K}, u \mapsto f(u, w)$. Since f is $\mathbb{K}$-linear in the first coordinate, $w^* \in U^*$. Define

$$\phi : V \to U^*, v \mapsto v^*.$$

Since f is α-linear in the second coordinate, ϕ is α-linear and so $\mathbb{F}$-linear. Moreover, $\ker\phi = U^\perp$. Hence $\dim V/U^\perp = \dim V\phi \le \dim U^* = \dim U$. This result applied to $U^\perp$ gives $\dim V/U^{\perp\perp} \le \dim U^\perp$ and since $U \le U^{\perp\perp}$,

$$\dim U \le \dim U^{\perp\perp} \le \dim V/U^\perp \le \dim U.$$

So equality holds in the preceding inequalities. Therefore $\dim V\phi = \dim U^*$ and ϕ is surjective. For $g \in N_M(U)$ and $u \in U$:

$$u((w\phi)g) = (ug^{-1})(w\phi) = f(ug^{-1}, w) = f(u, wg) = u((wg)\phi),$$

so $(w\phi)g = (wg)\phi$. Thus (a) holds.

Put $\overline{V} := V/V^\perp$ and define $\overline{f} : \overline{V} \to \overline{V} \to \mathbb{K}, (v + V^\perp, w + V^\perp) \to f(v, w)$. Then $\overline{f}$ is a non-degenerate form on $\overline{V}$.

(b): If $V^\perp = 0$, there is nothing to prove. So suppose $V^\perp \ne 0$, that is $G = \mathrm{O}(V)$, $\operatorname{char}\mathbb{K} = 2$, and n is odd. Let $v \in V$ with $\overline{v} \in \mathrm{C}_{\overline{V}}(A)$ and $g \in A$. Then $vg = v + u$ for some $u \in V^\perp$, so

$$h(v) = h(vg) = h(v + u) = h(v) + f(u, v) + h(u) = h(v) + h(u).$$

Hence $h(u) = 0$. Since $u \in V^\perp$ and h is non-degenerate this gives $u = 0$ and so $v \in \mathrm{C}_V(g)$. Thus (b) holds.

(c): By 1.8(c) and (a) we have $\mathrm{C}_{\overline{V}}(A) = [\overline{V}, A]^\perp$. Observe that $[V, A]^\perp$ is the preimage of $[\overline{V}, A]^\perp$ in V. By (b), $\mathrm{C}_V(A)$ is the preimage of $\mathrm{C}_{\overline{V}}(A)$ in V. Thus (c) holds.

(d): Put $C := \mathrm{C}_M(V/U)$. Note that $[V, C] \le U$ and so by (c), $\mathrm{C}_V(C) = [V, C]^\perp \ge U^\perp$. Hence $C \le \mathrm{C}_M(U^\perp)$. If U is, in addition, isotropic, $U \le U^\perp$ and so $C \le \mathrm{C}_M(U)$.

(e): Suppose that A is quadratic on $\overline{V}$. Then $[\overline{V}, A] \le \mathrm{C}_{\overline{V}}(A) = \overline{\mathrm{C}_V(A)}$. Thus $[V, A, A] = 0$ and $[V, A] \le \mathrm{C}_V(A) = [V, A]^\perp$ by (c). Hence $[V, A]$ is isotropic. □

Lemma 3.3. *Suppose that* $f \ne 0$ *and* U *is an isotropic subspace of* V *with* $U \cap V^\perp = 0$*. Put* $\overline{V} := V/U^\perp$*,* $D := \mathrm{C}_{\mathrm{GL}(V)}(U^\perp) \cap \mathrm{C}_{\mathrm{GL}(V)}(V/U)$ *and*

$$f_d(\overline{x}, \overline{y}) := f(x, [y, d]) \text{ for all } d \in D,\ x, y \in V.$$

Let $d \in D$*. Then*

(a)

$$\lambda : D \to \mathrm{F}_\alpha(\overline{V}),\ d \mapsto f_d$$

is a $\mathbb{Z}\mathrm{N}_M(U)$*-module isomorphism.*

(b) $f(xd, yd) = f(x, y)$ *for all* $x, y \in V$ *if and only if* $f_d \in F_{\alpha,-\epsilon}(\overline{V})$*.*

(c) Suppose $M = \mathrm{Sp}(V)$ then $d \in M$ if and only if $f_d \in \mathrm{S}_2(\overline{V})$.

(d) Suppose $M = \mathrm{GU}(V)$, then $d \in M$ if and only if $f_d \in F_{\alpha,-}(\overline{V})$.

(e) Suppose $M = \mathrm{O}(V)$ and U is singular, then $d \in M$ if and only if $f_d \in \bigwedge_2(\overline{V})$.

(f) Suppose that $M = \mathrm{O}(V)$ and U is not singular. Then there exists a unique $\overline{w} \in \overline{V}$ such that

$$h(u) = f(w,u)^2 \quad \textit{for all } u \in U.$$

Moreover, $d \in M$ if and only if $d \in S_2(\overline{V})$ and

$$f_d(\overline{x},\overline{x}) = f_d(\overline{w},\overline{x})^2 \quad \textit{for all } \overline{x} \in \overline{V}.$$

Proof. Observe that f_d is well-defined and α-sesquilinear, so $f_d \in \mathrm{F}_\alpha(\overline{V})$. Note that $[V,D] \leq U \leq U^\perp$ and so $[\overline{V},D] = 0$. Thus λ is a homomorphism, and for $d \in D$, $g \in \mathrm{N}_M(U)$ and $h \in \mathrm{F}_\alpha(\overline{V})$

$$\begin{aligned}(f_d g)(\overline{x},\overline{y}) &= f_d(\overline{x}g^{-1},\overline{y}g^{-1}) = f(xg^{-1},[yg^{-1},d]) = f(xg^{-1},-yg^{-1}+yg^{-1}d)\\ &= f(xg^{-1},(-y+y(g^{-1}dg))g^{-1}) = f(x,-y+y(g^{-1}dg))\\ &= f_{d^g}(\overline{x},\overline{y}).\end{aligned}$$

To see that λ is a $\mathbb{Z}\mathrm{N}_M(U)$-module isomorphism it remains to show that λ is bijective. The injectivity follows from the fact that $[V,D] \leq U$ and $U \cap V^\perp = 0$.

Let $g \in \mathrm{F}_\alpha(\overline{V})$. For $u \in U$ define $\phi_u \in \overline{V}^*$ by $\overline{x}\phi_u := f(x,u)$ for all $x \in V$. Since $U \cap V^\perp = 0$, the map $U \to \overline{V}^*, u \mapsto \phi_u$, is an α-semilinear isomorphism. For $w \in \overline{V}$, the map $t \mapsto g(t,w)$ is in $\overline{V}^*$ and so there exists a unique $u_w \in U$ with $\overline{x}\phi_{u_w} = f(x,u_w) = g(\overline{x},w)$ for all $x \in V$. Define $d_g \in \mathrm{GL}(V)$ by $d_g(v) := v + u_{\overline{v}}$. Clearly $d_g \in D$, and for all $x,y \in V$:

$$f_{d_g}(\overline{x},\overline{y}) = f(x,[y,d_g]) = f(x,u_{\overline{y}}) = g(\overline{x},\overline{y}),$$

so $f_{d_g} = g$, and λ is surjective. Thus (a) holds.

To prove (b) let $d \in D$. We will determine necessary and sufficient conditions for d to be in M.

Since f is an (α,ϵ)-sesquilinear form and U is isotropic,

$$f(xd,yd) - f(x,y) = f(x+[x,d],y+[y,d]) - f(x,y) = f(x,[y,d]) + f([x,d],y) =$$

$$f(x,[y,d]) + \epsilon f(y,[x,d])\alpha = f_d(\overline{x},\overline{y}) + \epsilon f_d(\overline{y},\overline{x})\alpha.$$

Thus d preserves f if and only if

(1) $$f_d(\overline{x},\overline{y}) = -\epsilon f_d(\overline{y},\overline{x})\alpha \quad \text{for all } \overline{x},\overline{y} \in \overline{V}.$$

That is, if and only if $f_d \in \mathrm{F}_{\alpha,-\epsilon}(\overline{V})$. So (b) follows.

(c) and (d): These statements follow immediately from (b).

(d) and (e): So suppose that $G = \mathrm{O}(V)$ and let $d \in D$ such that (1) holds. Since $\epsilon = 1$ and $\alpha = \mathrm{id}_\mathbb{K}$, f_d is a skew-symmetric form. Then

(2) $$h(xd) - h(x) = h(x+[x,d]) - h(x) = f(x,[x,d]) + h([x,d]) = f_d(\overline{x},\overline{x}) + h([x,d]).$$

So

(3) $$d \in \mathrm{O}(V) \text{ if and only if } d \in \mathrm{F}_{\mathrm{id},-}(\overline{V}) \text{ and } f_d(\overline{x},\overline{x}) = -h([x,d]) \text{ for all } x \in V.$$

If U is singular, then $h([x,d]=0$ and we conclude that (d) holds. So suppose U is not singular. Then $p=2$. Define $\delta : U \to \mathbb{K}, u \mapsto \sqrt{h(u)}$, and observe that δ is $\mathbb{K}$-linear, so $\delta \in U^*$. On the other hand the map

$$\phi^* : \overline{V} \to U^*,\ \phi^*(\overline{v}) :\ u \mapsto f(v,u)$$

is an isomorphism. Thus there exists a unique $\overline{w} \in \overline{V}$ with $\phi^*(\overline{w}) = \delta$. This gives

$$h(u) = \delta(u)^2 = f(w,u)^2 \text{ for all } u \in U.$$

Together with (3) we conclude that (e) holds. □

Lemma 3.4. *Let U be an k-dimensional isotropic subspace of V and $E := \mathrm{C}_M(U) \cap \mathrm{C}_M(V/U)$.*

(a) Suppose $M = \mathrm{GL}(V)$. Then $E \cong U \otimes_{\mathbb{K}} (V/U)^*$, $|E| = |\mathbb{K}|^{k(n-k)}$ and $|V/\mathrm{C}_V(E)| = |\mathbb{K}|^{n-k}$.

(b) Suppose $M = \mathrm{Sp}(V)$. Then $E \cong \mathrm{S}_2(U^*)$, $|E| = |\mathbb{K}|^{\frac{k(k+1)}{2}}$ and $|V/\mathrm{C}_V(E)| = |\mathbb{K}|^k$.

(c) Suppose $M = \mathrm{GU}(V)$ Then $E \cong \mathrm{F}_{\alpha,-}(U^*)$, $|E| = |\mathbb{F}|^{k^2}$ and $|V/\mathrm{C}_V(E)| = |\mathbb{F}|^{2k}$.

(d) Suppose $M = \mathrm{O}(V)$ and U is singular. Then $E \cong \bigwedge_2(U^*)$, $|E| = |\mathbb{K}|^{\frac{k(k-1)}{2}}$, $|V/\mathrm{C}_V(E)| = |\mathbb{K}|^k$,

(e) Suppose $M = \mathrm{O}(V)$ and U is not singular. Put $U_0 := \{u \in U \mid h(u) = 0\}$, $E_0 := \mathrm{C}_E(V/U_0)$, and $E_1 := E \cap \Omega_n(V)$. Then $p = 2$, $E_0 \leq E_1 \leq E$, $E_1/E_0 \cong U_0$, $E_0 \cong \bigwedge_2(U_0^*)$, and $|E_1| = |\mathbb{K}|^{\frac{k(k-1)}{2}}$. If $V^\perp \cap U \neq 0$ then $|V/\mathrm{C}_V(E)| = |\mathbb{K}|^{k-1}$ and $E = E_1$. If $V^\perp \cap U = 0$ then $|V/\mathrm{C}_V(E)| = |\mathbb{K}|^k$ and $|E/E_1| = 2$.

Here all the isomorphisms are $\mathbb{Z}\mathrm{N}_M(U)$-module isomorphisms.

Proof. Suppose first that $f = 0$, so $M = \mathrm{GL}(V)$. Then clearly $E \cong \mathrm{Hom}_{\mathbb{K}}(V/U, U) \cong U \otimes_{\mathbb{K}} (V/U)^*$ and (a) holds.

Suppose next that $f \neq 0$ and $U \cap V^\perp = 0$. We apply 3.3 with the notation introduced there. Since $[V,E] \leq U$, 3.2(c) gives $\mathrm{C}_V(E) = [V,E]^\perp \geq U^\perp$ and so $E \leq D$. Thus $E = D \cap M$. So 3.3(c), (d) and (e) imply (b), (c) and (d).

Suppose that $G = \mathrm{O}(V)$ and U is not singular. Let $d \in D$. By 3.3(f) there exists $w \in V$ with

$$h(u) = f(w,u)^2 \quad \text{for all } u \in U. \tag{2}$$

and

$$d \in \mathrm{O}(V) \text{ if and only if } d \in S_2(\overline{V}) \text{ and } f_d(\overline{x},\overline{x}) = f_d(\overline{w},\overline{x})^2 \text{ for all } x \in V \tag{3}$$

Recall from the proof of 3.3 that the map $\phi^* : \overline{V} \to U^*$ with $\overline{v}\phi^* : u \mapsto f(v,u)$ is an isomorphism. For $\delta := \overline{w}\phi^*$ we get from (3) that $\ker\delta = U_0 = w^\perp \cap U$. Note that ϕ^* also induces an isomorphism $\overline{V}/\mathbb{K}\overline{w} \to (\ker\delta)^* = (U_0)^*$.

Consider the map $\tau : E \to \overline{V}^*$ defined by $\overline{x}\tau(d) := f_d(\overline{w},\overline{x})$. By (3) $\ker\tau$ consists of all $d \in D$ such that f_d is a symplectic form on $\overline{V}$ with $\overline{w} \in \mathrm{rad} f_d$. Also $f_d(\overline{w},\overline{x}) = 0$ iff $f(w,[x,d]) = 0$ and (by (2)) iff $h([x,d]) = 0$. Thus $d \in \ker\tau$ iff $[V,d] \leq U_0$. Hence $\ker\tau = E_0$. As $\overline{V}/\mathbb{K}\overline{w} \cong U_0^*$ we get

$$E_0 = \ker\tau \cong \bigwedge\nolimits_2(\overline{V}/\mathbb{K}\overline{w}) \cong \bigwedge\nolimits_2(U_0^*). \tag{5}$$

We claim that $\mathrm{Im}\,\tau = X_1 := \{\phi \in \overline{V}^* \mid \phi(\overline{w}) \in \{0,1\}\}$.

If $d \in E$ then (3) applied with $\overline{x} = \overline{w}$ gives $f_d(\overline{w}, \overline{w}) = f_d(\overline{w}, \overline{w})^2$ and so $f_w(\overline{w}, \overline{w})^2 \in \{0, 1\}$. Hence $\operatorname{Im} \tau \leq X_1$.

Conversely let $\phi \in \overline{V}^*$ with $\phi(\overline{w}) = 1$. Define $g : \overline{V} \times \overline{V}, (\overline{x}, \overline{y}) \mapsto \phi(\overline{x})\phi(\overline{y})$. Then g is a symmetric bilinear form on $\overline{V}$, so by 3.3 with $d_g := g\lambda^{-1}$

$$f_{d_g}(\overline{w}, \overline{x}) = g(\overline{w}, \overline{x}) = \phi(\overline{x})\phi(\overline{w}) = \phi(\overline{x})$$

and

$$f_{d_g}(\overline{x}, \overline{x}) = g(\overline{x}, \overline{x}) = \phi(\overline{x})^2 = g(\overline{w}, \overline{x})^2 = f_{d_g}(\overline{w}, \overline{x}).$$

Thus by (3), $d_g \in E$ and $\tau(d_g) = \phi$. Any $\phi \in \overline{V}^*$ with $\phi(\overline{w}) = 0$ can be written as a sum $\phi_1 + \phi_2$ where $\phi_i \in \overline{V}^*$ and $\phi_i(\overline{w}) = 1$. It follow that $\tau(E) = X_1$.

Put $X_0 := \{\phi \in \overline{V}^* \mid \phi(\overline{w}) = 0\}$. Then $X_0 \cong (\overline{V}/\mathbb{K}\overline{w})^* \cong U_0$. Also $|X_1/X_0| = 2$ and so (e) holds. Thus we have proved all claims in the case $V^\perp \cap U = 0$.

Suppose now that $V^\perp \cap U \neq 0$. Then V is an orthogonal space and $\dim V^\perp = 1$, so $V^\perp \leq U$. Let $\tilde{V}$ be an orthogonal space of dimension $n+1$ with $V \leq \tilde{V}$ and $\tilde{V}^\perp = 0$; in particular, $\tilde{V}^\perp \cap U = 0$. Put $\tilde{M} = \mathrm{O}(\tilde{V})$ and $\tilde{E} := \mathrm{C}_{\tilde{M}}(U) \cap \mathrm{C}_{\tilde{M}}(\tilde{V}/U)$. Then (e) holds for $\tilde{V}$, $\tilde{M}$ and $\tilde{E}$.

Note that in $\tilde{V}$, $V^{\perp\perp} = V$. Since $V^\perp \leq U$, this gives $\tilde{E} \leq \mathrm{C}_{\tilde{M}}(V^\perp) \leq \mathrm{N}_{\tilde{M}}(V)$ and we obtain a homomorphism $\beta : \tilde{E} \to E, e \mapsto e\mathrm{C}_{\tilde{M}}(V)$. Note that $\ker \beta$ has order two, indeed the only non-trivial element in $\ker \beta$ is the transvection associated to the 1-space $V^\perp$. By Witt's theorem, β is onto. Also $\ker \beta$ is not contained in $\tilde{E} \cap \Omega(\tilde{V})$. Thus (e) applied to $\tilde{M}$ shows that $E \cong \tilde{E}_0$, and (e) also holds in this case. □

Lemma 3.5. *Let U be a isotropic subspace of V, let U_0 be the subspace of all singular elements of U and put $k = \dim_{\mathbb{K}} U_0$. Suppose that $\mathbb{K}$ is finite and $k \geq 2$. Put $E := \mathrm{C}_M(U) \cap \mathrm{C}_M(V/U)$, and $P := \mathrm{O}^{p'}(\mathrm{N}_{M'}(U))$, where $p = \operatorname{char} \mathbb{K}$.*

(a) If $M = \mathrm{GL}(V)$ or $\mathrm{GU}(V)$ then E is a simple $\mathbb{F}_p P$-module.

(b) If $M = \mathrm{Sp}(V)$ and p is odd, then E is a simple $\mathbb{F}_p P$ module.

(c) If $M = \mathrm{O}(V)$ and U is singular, then one of the following holds:

1. *$k \geq 3$ and E is a simple $\mathbb{F}_p P$-module.*
2. *$k = 2$, P centralizes E and E is a simple $\mathbb{F}_p \mathrm{N}_{M'}(U)$-module.*

(d) Suppose $M = \mathrm{Sp}(V)$ and $p = 2$ or $M = \mathrm{O}(V)$ and U is not singular. Then $p = 2$. Let E_0 be the sum of the simple $\mathbb{F}_2 P$-submodules of E. Then one of the following holds:

1. *$k \geq 3$, E_0 is a simple $\mathbb{F}_2 P$-module, and $E_0 \cong \bigwedge_2 U_0^*$.*
2. *$k = 2$, $|\mathbb{K}| > 2$ or $V^\perp \not\leq U$, $E_0 = C_E(P)$. $|E_0| = |\mathbb{K}|$ and $\mathrm{N}_{M'}(U)$ acts simply on E_0.*
3. *$k = 2$, $|\mathbb{K}| = 2$, $M = \mathrm{Sp}(V)$ or $V^\perp \leq U$, and E is the direct sum of simple $\mathbb{F}_2 P$-modules of order 2 and 4.*

Proof. Let S be a Sylow p-subgroup of P and D be a simple $\mathbb{F}_p P$-submodule of E.

Assume first that $M = \mathrm{GL}(V)$ and put $S_0 := \mathrm{C}_S(V/U))$. Then S_0 induces a Sylow p-subgroup of $\mathrm{GL}_{\mathbb{K}}(U)$ on U. Hence 3.4 implies that $\mathrm{C}_E(S_0) \cong x \otimes (V/U)^*$ for some $0 \neq x \in U$. Thus $\mathrm{C}_P(U)$ acts simply on $\mathrm{C}_E(S_0)$ and so $\mathrm{C}_E(S_0) \leq D$. Since $\mathrm{C}_P(V/U)$ acts simply on U, we conclude that $E = \langle \mathrm{C}_E(S_0)^{\mathrm{C}_P(V/U)} \rangle \leq D$. Thus E is a simple $\mathbb{F}_p P$-module.

Assume next that $f \neq 0$ and $U \cap V^\perp = 0$. Put $W := V/U^\perp$ and note that $\dim W = \dim U$. By Witt's Theorem S induces a Sylow p-subgroup of $\mathrm{GL}_{\mathbb{K}}(U)$ on U and thus also on W. Thus $C_W(S)$ is 1-dimensional. By 3.4 E is embedded into $\mathrm{F}_{\alpha,-\epsilon}(W)$. Let $1 \neq x \in \mathrm{C}_D(S)$, and let $f_x \in \mathrm{F}_{\alpha,-\epsilon}(W)$, f_x as in 3.3. Then f_x is invariant under S, so $W/\mathrm{rad}\, f_x$ possesses a non-degenerate $(\alpha, -\epsilon)$ sesquilinear form invariant under a Sylow p-subgroup of $\mathrm{GL}(W/\mathrm{rad}\, f_x)$. If follows that either $W/\mathrm{rad}\, f_x$ is 1-dimensional or $\alpha = \mathrm{id}_{\mathbb{K}}$, $-\epsilon = -1$ and $\dim W/\mathrm{rad}\, f_x = 2$.

Suppose that $M = \mathrm{Sp}(V)$ and p is odd or that $M = \mathrm{GU}(V)$, so $\dim_{\mathbb{K}} U = k$. Then P induces $\mathrm{SL}_{\mathbb{K}}(U)$ on U. Moreover $\dim W/\mathrm{rad}\, f_x = 1$ and $\mathrm{N}_P(S)$ acts simply on the subspace $\mathbb{F} f_x$ of $F_{\alpha,-\epsilon}(W)$. Also for any $\psi \in \mathrm{F}_{\alpha,-\epsilon}(W)$ there exists a basis $(x_i)_{1\leq i\leq k}$ of W which is orthogonal with respect to ψ, that is, $\psi(x_i, x_j) = 0$ for $i \neq j$. It follows that ψ is a $\mathbb{F}$-linear combination of conjugates of f_x under P and so $D = E$.

Suppose that $M = \mathrm{O}(V)$ and U is singular. Then P induces $\mathrm{SL}_{\mathbb{K}}(U)$ on U. By 3.4(d) $E \cong \bigwedge_2 W$ and f_x is a symplectic form. Thus $\dim W/\mathrm{rad}\, f_x = 2$. Let $\psi \in \bigwedge_2(W)$. Then W has basis x_i, y_i, z_s, $1 \leq i \leq r$ and $1 \leq s \leq t$, where $\psi(x_i, y_i) = 1$, $\psi(y_i, x_i) = -1$, and $\psi(c, d) = 0$ for any other pair of basis elements.

Assume that $k \geq 3$. Then P acts transitively on the set of symplectic forms on W with radical of codimension 2. Hence ψ is a sum of P-conjugates of f_x. Thus $D = E$ and (c:1) holds in this case. Assume that $k = 2$. Then P centralizes $\bigwedge^2 W$. Also any scalar multiplication on W is induced by an element of $N_{M'}(U)$ and so $N_{M'}(U)$ acts simply on $\bigwedge^2 W$. Thus (c:2) holds.

Suppose that $M = \mathrm{O}(V)$ and U is not singular. Put $F = C_M(V/U_0)$. Note that $F \leq C_M(U_0^\perp)$ by 3.2(d), and so $F \leq E$ since $U \leq U_0^\perp$. By the preceding case $F \cong \bigwedge_2(U_0^*)$ and either $k = 3$ and F is a simple $\mathbb{F}_pP$-module or $k = 2$, $[F, P] = 1$ and F is a simple $N_{M'}(U)$-module. Thus $F \leq E_0$ and it suffices to show that $E_0 \leq F$. Let $\overline{w}$ be as in 3.3(f). The uniqueness of $\overline{w}$ show that $\overline{w} \in C_W(S)$. Since $\dim W = \dim U > \dim U_0 \geq 2$ and $\dim W/\mathrm{rad} f_x \leq 2$ we have $\mathrm{rad} f_x \neq 0$. Hence $C_{\mathrm{rad} f_x}(S) \neq 0$ and since $C_W(S)$ is 1-dimensional, $\overline{w} \in \mathrm{rad} f_x$. So 3.3(f) shows that f_x is symplectic and thus $f_x \in F$. Since D is simple, $D \leq F$ and $E_0 \leq F$.

Suppose $M = \mathrm{Sp}(V)$ and $p = 2$. Then by 3.4(b) $E \cong \mathrm{S}_2(U^*)$, and by 3.2(a) $W \cong U^*$, so $S_2(U^*) \cong S_2(W)$. Since $p = 2$, $\bigwedge_2(W) \leq \mathrm{S}_2(W)$. Let F be the inverse image of $\bigwedge_2(W)$ in E. Then $F \cong \bigwedge_2(W) \cong \bigwedge_2(U^*)$. As seen in the case where U is singular either $k \geq 3$ and E_0 is a simple $\mathbb{F}_pP$-module, or $k = 2$, $[F, P] = 1$ and $N_{M'}(U)$ acts simply on F. If $|\mathbb{K}| = 2$ and $k = 2$, then $|U| = 4$ and $|E| = 8$ and it is easy to see that (d:3) holds. So suppose that $|\mathbb{K}| > 2$ or $k > 2$. We will show that $D \leq F$. For this we just need to show that there exists $1 \neq u \in D$ such that f_u is a symplectic form. Fix a basis (v_i) for W and for $e \in E$ let M_e be the matrix $(f_e(v_i, v_j))$. Then M_e is symmetric and $e \in F$ if and only if all diagonal elements of M_e are zero. Moreover, $\dim W/\mathrm{rad} f_e = \mathrm{rank} M_e$. We may assume that f_x is not symplectic and so there exists $v \in V$ with $f_x(v, v) \neq 0$. Since $\mathbb{K}$ is perfect we can choose v such that $f_x(v, v) = 1$. Put $s = \dim W/\mathrm{rad} f_x$. Then either $s = 1$ and $V = \mathbb{K}v + \mathrm{rad} f_x$, or $s = 2$, there exists $w \in W$ with $f_x(v, w) = 0$ and $f_x(w, w) = 1$ and $V = \mathbb{K}v + \mathbb{K}w + \mathrm{rad} f_x$. So we can choose our basis such that $f_x(v_i, v_j) = 1$ for $1 \leq i = j \leq s$ and $f_x(v_i, v_j) = 0$ for all other i, j.

Suppose $s = 1$. Note that

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

The three matrices on the left side of the equation all are symmetric of rank 1 and so conjugate under $\mathrm{SL}_2(\mathbb{K})$ on it actions on $\mathrm{S}_2(\mathbb{K}^2)$. The matrix on the right is symplectic. Thus $\langle d^P \rangle \cap F \neq 1$ and so $D \leq F$.

Suppose that $s = 2$ and $|\mathbb{K}| > 2$. Pick $a, b \in \mathbb{K} \setminus \{0, 1\}$ with $a + b = 1$. Note that

$$\begin{pmatrix}1&0\\0&1\end{pmatrix}+\begin{pmatrix}a&b\\b&a\end{pmatrix}+\begin{pmatrix}b&a\\a&b\end{pmatrix}=\begin{pmatrix}0&1\\1&0\end{pmatrix}$$

The three matrices on the left side of the equation are symmetric, not symplectic and have determinant 1. So they are conjugate under $\mathrm{SL}_2(\mathbb{K})$ on it actions on $\mathrm{S}_2(\mathbb{K}^2)$. The matrix on the right is symplectic and so again $D \leq F$.

Suppose that $s = 2$, $|\mathbb{K}| = 2$ and $k \geq 3$. We have

$$\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}+\begin{pmatrix}1&0&1\\0&1&1\\1&1&0\end{pmatrix}=\begin{pmatrix}0&0&1\\0&0&1\\1&1&0\end{pmatrix}$$

The two matrices on the left side of the equation are symmetric, not symplectic and have rank 2. So they are conjugate under $\mathrm{SL}_3(\mathbb{K})$ on it actions on $\mathrm{S}_2(\mathbb{K}^3)$. The matrix on the right is symplectic and so again $D \leq F$.

We have proved that $D \leq F$. So $E_0 = F$ and (d:1) or (d:2) holds.

Assume finally that $M = \mathrm{O}(V)$, U is not singular and $U \cap V^{\perp} \neq 0$. Then $p = 2$ and $M \cong \mathrm{Sp}(V/V^{\perp})$. Hence the case where $M = \mathrm{Sp}(V)$ applied to $V/V^{\perp}$ and $U/V^{\perp}$ shows that (d) holds. □

4 Smith's Lemma and Ronan-Smith's Lemma

In this section we provide a few pieces from the theory of equicharacteristic representations of groups of Lie-type. The material presented here essentially comes from [GLS3, Section 2.8] except that we are looking at representations over $\mathbb{F}_p$ rather than its algebraic closure $\overline{\mathbb{F}_p}$.

Lemma 4.1 (Steinberg's Lemma). *Let M be a genuine group of Lie-type defined over a finite field of characteristic p. Let V be a simple $\mathbb{F}_pM$-module, $S \in \mathrm{Syl}_p(M)$, and $B := \mathrm{N}_M(S)$. Put $\mathbb{K} := \mathrm{End}_M(V)$. Then $\mathrm{C}_V(S)$ is 1-dimensional over $\mathbb{K}$, $\mathbb{K}$ is isomorphic to the subring of $\mathrm{End}_{\mathbb{F}_p}(\mathrm{C}_V(S))$ generated by the image of B, and $\mathrm{C}_V(S)$ is a simple $\mathbb{F}_pB$-module.*

Proof. Choose an embedding $\sigma : \mathbb{K} \to \overline{\mathbb{F}_p}$ and put $\overline{V} = \overline{\mathbb{F}_p} \otimes_{\mathbb{K}} V$. Then $\overline{V}$ is a simple $\overline{\mathbb{F}_p}M$-module. Thus by [St, Theorem 46] $\mathrm{C}_{\overline{V}}(S)$ is 1-dimensional over $\overline{\mathbb{F}_p}$ and so $\mathrm{C}_V(S)$ is 1-dimensional over $\mathbb{K}$. Define $\lambda : B \to \mathbb{K}$ by $v^b = \lambda(b)v$ for all $b \in B, v \in \mathrm{C}_V(S)$, and let $\mathbb{E}$ be the subfield of $\mathbb{K}$ generated by $\lambda(B)$. Let $\rho \in \mathrm{Aut}_{\mathbb{E}}(\overline{\mathbb{F}_p})$. Then [St, Theorem 46] shows that $\overline{V} \cong \overline{V}^{\rho}$ as a $\mathbb{K}M$-module. Thus ρ centralizes $\mathbb{K}$ and so $\mathbb{K} = \mathbb{E}$. Since $\mathrm{C}_V(S)$ is 1-dimensional over $\mathbb{K}$ this implies that $\mathrm{C}_V(S)$ is a simple $\mathbb{F}_pB$-module. □

Let $\mathbb{F}$ be a finite field of characteristic p, M a finite group, V a simple $\mathbb{F}M$-module and W a simple $\mathbb{F}_pM$-submodule. Recall that the field $\mathbb{K} := \mathrm{End}_M(W)$ is called the field of definition of the $\mathbb{F}M$-module W.

Theorem 4.2 (Smith's Lemma). *Let M be a genuine group of Lie-type defined over a finite field of characteristic p. Let V be a simple $\mathbb{F}_pM$-module, $\mathbb{K} := \mathrm{End}_M(V)$, E a parabolic subgroup of M, $L := \mathrm{O}^{p'}(E)$ and $P = \mathrm{N}_M(L)$. Then $L = O^{p'}(P)$, $\mathrm{O}_p(E) = \mathrm{O}_p(P) = \mathrm{O}_p(L)$, and P is a Lie-parabolic subgroup of M. Moreover, $\mathrm{C}_V(\mathrm{O}_p(P))$ is a simple $\mathbb{F}_pP$-module, an absolutely simple $\mathbb{K}L$-module, and an absolutely simple $\mathbb{K}E$-module.*

Proof. Let $S \in \mathrm{Syl}_p(E)$ and $B = \mathrm{N}_M(S)$. Then $P = BL = BE$ and so P is a Lie-parabolic subgroup of M. Since B/S is a p'-group we conclude that $E = \mathrm{O}^{p'}(P)$ and $\mathrm{O}_p(E) = \mathrm{O}_p(L) = \mathrm{O}_p(P)$.

Choose an embedding $\sigma : \mathbb{K} \to \overline{\mathbb{F}_p}$ and put $\overline{V} = \overline{\mathbb{F}_p} \otimes_{\mathbb{K}} V$. Then $\overline{V}$ is a simple $\overline{\mathbb{F}_p}M$-module. Put $U = \mathrm{C}_V(\mathrm{O}_p(P))$ and $\overline{U} = \mathrm{C}_{\overline{V}}(\mathrm{O}_p(P)) = \overline{\mathbb{F}_p} \otimes_{\mathbb{K}} U$. By [Ti] $\overline{U}$ is a simple $\overline{\mathbb{F}_p}P$-module.

Let Y be a simple $\overline{\mathbb{F}_p}L$-submodule of $\overline{U}$. Then $\mathrm{C}_Y(S) \neq 0$, and since by [St, Theorem 46] $\mathrm{C}_{\overline{V}}(S)$ is 1-dimensional over $\overline{\mathbb{F}_p}$, $\mathrm{C}_{\overline{V}}(S) \leq Y$. Thus

$$\overline{U} = \langle \mathrm{C}_{\overline{U}}(S)^P \rangle = \langle \mathrm{C}_{\overline{U}}(S)^{BL} \rangle = \langle \mathrm{C}_{\overline{U}}(S)^L \rangle \leq Y,$$

so $\overline{U}$ is simple $\overline{\mathbb{F}_p}L$. Thus, U is an absolutely simple $\mathbb{K}L$-module, and since $L \leq E$, U is also an absolutely simple $\mathbb{K}E$-module.

Let X be a simple $\mathbb{F}_pP$-submodule of U. Then again $0 \neq \mathrm{C}_X(S)$ is B-invariant and since $\mathrm{C}_V(S)$ is a simple $\mathbb{F}_pB$-module by 4.1, $\mathrm{C}_V(S) \leq X$. Since $\langle \mathrm{C}_V(S)^P \rangle$ is a $\mathbb{K}$-submodule of U we conclude that $X = U$. □

Theorem 4.3 (**Ronan-Smith's Lemma).** *Let M be a universal group of Lie-type defined over a finite field of characteristic p, S a Sylow p-subgroup of M, $P_1, P_2, \ldots, P_n$ the minimal Lie-parabolic subgroups of M containing S, and $L_i = \mathrm{O}^{p'}(P_i)$. Let $\mathcal{V}$ be the class of all tuples $(\mathbb{K}, V_1, V_2, \ldots V_n)$ such that*

(i) $\mathbb{K}$ is a finite field of characteristic p.

(ii) Each V_i is an absolutely simple $\mathbb{K}L_i$-module.

(iii) $\mathbb{K} = \langle \mathbb{K}_i \mid 1 \leq i \leq n \rangle$, where $\mathbb{K}_i$ is the field of definition of the $\mathbb{K}L_i$-module V_i.

Define two elements $(\mathbb{K}, V_1, V_2, \ldots V_n)$ and $(\tilde{\mathbb{K}}, \tilde{V}_1, \tilde{V}_2, \ldots \tilde{V}_n)$ of $\mathcal{V}$ to be isomorphic if there exists a field isomorphism $\sigma :\ \tilde{\mathbb{K}} \to \mathbb{K}$ such that $V_i \cong \tilde{V}_i^{\sigma}$ as an $\mathbb{K}L_i$-module for all $1 \leq i \leq n$. Then the map

$$V \to (\mathrm{End}_M(V), \mathrm{C}_V(\mathrm{O}_p(L_i)), \ldots \mathrm{C}_V(\mathrm{O}_p(L_n))) \quad (V \textit{ a simple } \mathbb{F}_pM\textit{-module})$$

induces a bijection between the isomorphism classes of simple $\mathbb{F}_pM$-modules and the isomorphism classes of $\mathcal{V}$.

Proof. Let V be a simple $\mathbb{F}_pM$-module and put $\mathbb{K} := \mathrm{End}_M(V)$ and $V_i := \mathrm{C}_V(\mathrm{O}_p(L_i))$. By Smith's Lemma 4.2, V_i is an absolutely simple $\mathbb{K}L_i$-module. Let $\mathbb{K}_i$ be the field of definition of the $\mathbb{K}L_i$-module V_i. Put $B := \mathrm{N}_M(S)$. By 4.1 $\mathbb{K}$ is generated by the image of B in $\mathrm{End}_{\mathbb{F}_p}(\mathrm{C}_V(S))$. Moreover, each $\mathbb{K}_i$ is generated by the image of $B \cap L_i$ in $\mathrm{C}_V(S)$. Since $B = \langle B \cap L_i, 1 \leq i \leq n \rangle$ we conclude that $\mathbb{K} = \langle \mathbb{K}_i \mid 1 \leq i \leq n \rangle$.

Clearly, if $\tilde{V}$ is an $\mathbb{F}_pM$-module isomorphic to V, then the corresponding elements of $\mathcal{V}$ are isomorphic.

Now let $(\mathbb{K}, V_1, V_2, \ldots V_n) \in \mathcal{V}$. Pick $0 \neq v_i \in \mathrm{C}_{V_i}(S)$ and define λ_i, n_i and μ_i as in [St, Theorem 46] applied to the $\overline{\mathbb{F}_p}L_i/\mathrm{O}_p(L_i)$-module $\overline{V}_i = \overline{\mathbb{F}_p} \otimes_{\mathbb{K}} V_i$. Since $B/S = \bigtimes_{i=1}^{n} (B \cap L_i)/S$, there exists a unique homomorphism $\lambda : B \to \overline{\mathbb{F}_p}$ with $\lambda \mid_{B \cap L_o} = \lambda_i$. Let $\overline{V}$ be the simple $\overline{\mathbb{F}_p}M$-module obtained from [St, Theorem 46]. Since $\mathrm{C}_{\overline{V}}(\mathrm{O}_p(V_i))$ is simple we conclude from [St, Theorem 46] applied to L_i that $\mathrm{C}_{\overline{V}}(\mathrm{O}_p(V_i)) \cong \overline{V_i}$. Let V be a simple $\mathbb{F}_pM$-submodule of $\overline{V}$ and put $\mathbb{E} = \mathrm{End}_M(V)$. Then $\overline{V} \cong \overline{\mathbb{F}_p} \otimes_{\mathbb{E}} V$ as an $\overline{\mathbb{F}_p}M$-module. It is now easy to see that $\mathbb{E} \cong \mathbb{K}$, that V is send to $(\mathbb{K}, V_1, V_2, \ldots V_n) \in \mathcal{V}$ and that V is unique up to isomorphism with this property. □

5 Generating Genuine Groups of Lie-type

Lemma 5.1. *Let G be a simple genuine group of Lie Type over a field of characteristic p, P^+ a Lie-parabolic subgroup of G and P^- an opposite Lie-parabolic. Then $G = \langle \mathrm{O}_p(P^+), \mathrm{O}_p(P^-) \rangle$.*

Proof. Put $L = \langle O_p(P^+), O_p(P^-)\rangle$. Since P^+ is opposite to P^-, $G = \langle P^+, P^-\rangle$ and $P^\epsilon = O_p(P^\epsilon)(P^+ \cap P^-)$. It follows that $L \trianglelefteq L(P^+ \cap P^-) = \langle P^+, P^-\rangle = G$, and since G is simple, $G = L$. □

Lemma 5.2. *Let $G \cong G_2(q)$, $p = q^k$, P a Lie-parabolic subgroup of G with $Z(O^{p'}(P)) = 1$ and $A \trianglelefteq P$ with $|A| = q^3$. Then $G = \langle A, A^t\rangle$ for some $t \in G$.*

Proof. Choose a root system Φ for G such that P is a Lie-parabolic with respect to Φ and let N/H be the corresponding Weyl-group. Let $\mathcal{R}_l$ ($\mathcal{R}_s$) be set root subgroups in G corresponding to the long (short) roots in Φ. Put $L = \langle \mathcal{R}_l\rangle$. Then L is a genuine group of Lie-type of type A_2 and $P \cap L$ is a Lie-parabolic subgroup of L with $L \cap A = O_p(P \cap L)$. Since $N/H \cong D_{12}$ we can choose $t \in N \setminus H$ with $[t, N] \leq H$. Put $K = \langle A, A^t\rangle$. Since $(P \cap L)^t$ is opposite to $P \cap L$ in L, 5.1 implies that $L = \langle L \cap A, (L \cap A)^t\rangle$. Thus $L \leq K$. Since $(N \cap L)H/H \cong D_6$ we have $N = (L \cap H)\langle t\rangle H$ and so N normalizes K. Since N acts transitive $\mathcal{R}_s$ and there exists $R \in \mathcal{R}_s$ with $R \leq A$, $\langle \mathcal{R}_s\rangle \leq K$. Hence $G = \langle \mathcal{R}_l, \mathcal{R}_s\rangle \leq K$ and $G = K$. □

Lemma 5.3. *Let $G \cong SL_n(\mathbb{K})$. Then G is generated by n root subgroups.*

Proof. Let $I = \{1, \ldots, n\}$ and $\Phi = \{e_i - e_j \mid i, j \in I, i \neq j\}$ by the root system for G and for $\phi \in \Phi$ let Z_ϕ be the corresponding root subgroup. Then

$$(*) \qquad [Z_{e_i-e_j}, Z_{e_j-e_k}] = Z_{e_i-e_k} \text{ for all distinct } i, j, k \text{ in } I.$$

Put $U := \langle Z_{e_i-e_{i+1}} \mid n \neq i \in I\}\rangle$ and $L := \langle U, Z_{e_n-e_1}\rangle$. Let $i, j \in I$ with $i < j$.

We will first show by induction on $j - i$ that $Z_{e_i-e_j} \in U$. If $j - i = 1$, this holds by definition of U. So suppose $j - i > 1$ and by induction that $Z_{e_i-e_{j-1}} \leq U$. Thus using $(*)$,

$$Z_{e_i-e_j} = [Z_{e_i-e_{j-1}}, Z_{e_{j-1}-e_j}] \leq U.$$

Next we will show by downwards induction on $j - i$, then $Z_{e_j-e_i} \leq L$. If $j - i = n - 1$, then $j = n$ and $i = 1$ and so this holds by definition on L. So suppose $j - i < n - 1$.

Assume that $i > 1$ and by induction that $Z_{e_j-e_{i-1}} \leq L$. Then by $(*)$

$$Z_{e_j-e_i} = [Z_{e_j-e_{i-1}}, Z_{e_{i-1}-e_i}] \leq U.$$

Assume that $i = 1$. Then $j < n$ and by induction $Z_{e_{j+1}-e_i} \leq U$. So by $(*)$

$$Z_{e_j-e_i} = [Z_{e_j-e_{j+1}}, Z_{e_{j+1}-e_i}] \leq U.$$

Thus L contains all $Z_\phi, \phi \in \Phi$ and so $L = M$. □

Lemma 5.4. *Let H be quasisimple with $H/Z(H) \cong \mathrm{Alt}(6)$ and $|Z(H)|\big|3$. Let $S \in \mathrm{Syl}_2(H)$, $B = N_H(S)$, and M_1 and M_2 be the two maximal subgroups of H containing B. Let $\mathbb{K}$ be a field of characteristic 2, V be a faithful $\mathbb{K}H$-module, U a simple $\mathbb{K}B$-submodule of V and put $U_i := \langle U^{M_i}\rangle$. Suppose that*

(i) $V = \langle U^M\rangle$,

(ii) $U = U_1$, *and*

(iii) $\dim_{\mathbb{K}} U_2 = 2\dim_{\mathbb{K}} U$.

Then the following hold:

(a) *Suppose* $H \cong \mathrm{Alt}(6)$*, then* V *is a quotient of the natural even permutation module for* H *over* $\mathbb{K}$*. In particular,* $V/\mathrm{C}_V(H)$ *is a natural* $\mathbb{K}\mathrm{Alt}(6)$*-module for* H*,* $\dim_{\mathbb{K}} \mathrm{C}_V(H) \leq 1$ *and* $\mathrm{C}_V(H) \leq \langle U_2^{M_1} \rangle$*.*

(b) *Suppose* $H \sim 3^{\cdot}\mathrm{Alt}(6)$*. Let* $\mathbb{E}$ *be subring of* $\mathrm{End}_{\mathbb{K}H}(V)$ *generated by the images of* $\mathbb{K}$ *and* $\mathrm{Z}(H)$*. Then* $\mathbb{E}$ *is a field,* $\mathbb{E} = \mathbb{K}(\xi)$ *for* $\xi \in \mathbb{E}$ *with* $|\xi| = 3$*,* $\dim_{\mathbb{E}} U = 1$ *and* $\dim_{\mathbb{E}} V = 3$*.*

Proof. Since $S \trianglelefteq B$ and U is a simple $\mathbb{F}_2B$-module, $[U, S] = 0$. As the Sylow 2-subgroups of $\mathrm{Alt}(6)$ are self-normalizing, $B = S\mathrm{Z}(H)$, and so U is a simple $\mathbb{K}\mathrm{Z}(H)$-module.

Since $V = \langle U^M \rangle$, $\mathrm{Z}(H)$ acts homogeneously on V and so the subring $\mathbb{E}$ of $\mathrm{End}_{\mathbb{K}H}(V)$ generated by the images of $\mathbb{K}$ and $\mathrm{Z}(H)$ is a field. Moreover, $\mathbb{E} = \mathbb{K}$ if $\mathrm{Z}(H) = 1$ or $\mathbb{K}$ contains a non-trivial third root of unity; in the other case $\mathbb{E} = \mathbb{K}(\xi)$ where $\xi \in \mathbb{E} \setminus \mathbb{K}$ with $\xi^3 = 1$. Also $\dim_{\mathbb{E}} U = 1$ and since $\dim_{\mathbb{K}} U_2 = 2\dim_{\mathbb{K}} U$, $\dim_{\mathbb{E}} U_2 = 2$.

Let A be the natural $\mathbb{F}_2\mathrm{Alt}(6)$-module for H with $\mathrm{C}_A(M_1) \neq 0$. Then there exists an M-equivariant bijection $A^\sharp \to U_1^M, a \to U_a$. We now use the fact that $Alt(6) \cong Sp_4(2)'$ and A is also a natural $Sp_4(2)'$-module for H, so there exists an H-invariant non-degenerate symplectic form on A.

For $B \subseteq A$ define $U_B := \langle U_b \mid b \in B^\sharp \rangle$ and $W_B := U_{B^\perp}$, where $B^\perp$ is the $\mathbb{F}_2$-subspace of A perpendicular to B with respect to the above mentioned symplectic form on A.

Let B be a singular 2-subspace of A. By Witt's Theorem H acts transitively on the singular 2-subspaces of A and so U_B is a conjugate of U_2. In particular,

$$(*) \qquad U_B = U_b + U_c \text{ and } U_{a+c} \leq U_a + U_c \text{ for } B = \langle a, c \rangle.$$

Now let $a \in A^\sharp$. Since $\dim_{\mathbb{F}_2} A = 4$, $a^\perp = \langle a \rangle \oplus B$, where B is a non-singular 2-subspace. Then $\langle a, b \rangle$ is singular for every $b \in B$. Thus by $(*)$

$$(**) \qquad W_a = \Sigma_{b \in B^\sharp} U_{\langle a, b \rangle} = U_a + U_B.$$

Since $|B^\sharp| = 3$, $\dim_{\mathbb{E}} U_B \leq 3$ and so $\dim_{\mathbb{E}} W_a \leq 4$.

Now let $d \in A \setminus a^\perp$ and put $B := a^\perp \cap d^\perp$. Then B is a non-singular 2-space, and by $(**)$ applied to a and d, $W_a + W_d = U_a + U_B + U_d$. Thus $\dim_{\mathbb{E}} W_a + W_d \leq 5$.

Put $W := W_a + W_d$. We will show that $V = W$, that is $U_b \leq W$ for all $b \in A^\sharp$. Certainly $U_b \leq W$ if $b \in a^\perp \cup d^\perp$. So suppose $b \notin a^\perp$ and $b \notin d^\perp$.

Assume first that $b \neq a + d$. Then $\langle b, d \rangle \neq \langle a, d \rangle$ and so also $b^\perp \cap a^\perp \neq b^\perp \cap d^\perp$. Choose $e \in b^\perp \cap a^\perp \setminus d^\perp$; in particular $U_e \leq W_a$. Then $e + b \leq b^\perp \cap d^\perp$, so $U_{e+b} \leq W_d$, and by $(*)$ $U_b \leq U_e + U_{e+b} \leq W_a + W_d = W$.

Assume next that $b = a + d$. Pick $\tilde{b} \in A \setminus (a^\perp \cup d^\perp)$ with $\tilde{b} \neq b$. Put $c = b + \tilde{b}$. By the previous case $U_{\tilde{b}} \leq W$. Note that $\tilde{b} \in b^\perp$ and $c \in a^\perp$. Thus $U_c \leq W$ and by $(*)$ $U_b \leq U_{\tilde{b}} + U_c$. Hence $U_b \leq W$.

We have shown that $U_b \leq W$ for all $b \in A^\sharp$ and so $W = V$; in particular $\dim_{\mathbb{E}} V \leq 5$.

Suppose now that $H \cong \mathrm{Alt}(6)$. Then $\mathrm{Z}(H) = 1$ and $\mathbb{E} = \mathbb{K}$. Let $\check{V}$ be the $\mathbb{K}H$-module induced from the trivial $\mathbb{K}M_1$-module U_1, and let $\check{U}_1$ be the image of U_1 in $\check{V}$. Put $\check{U}_2 := \langle \check{U}_1^{M_2} \rangle$. Then $\check{U}_2/\mathrm{C}_{\check{U}_2}(M_2)$ has dimension 2 over $\mathbb{K}$. It follows that $\hat{V} := \check{V}/\langle C_{\check{U}_2}(M_2)^H \rangle$ fulfills the assumptions of (a).

Choose a faithful action of H on $I := \{1, 2, 3, 4, 5, 6\}$ with

$$M_1 = \mathrm{N}_H(\{1, 2\}) \text{ and } M_2 = \mathrm{N}_H(\{\{\{1, 2\}, \{3, 4\}, \{5, 6\}\}.$$

Let $\tilde{V}$ be the corresponding permutation module for H over $\mathbb{K}$ with $\mathbb{K}$ basis $\{b_i \mid i \in I\}$, and let $\tilde{V}_0 := \{\sum_{i \in I} k_i b_i \mid k_i \in \mathbb{K}, \sum_{i \in I} k_i = 0\}$ be the even permutation module. For $J \subseteq I$ put

$b_J = \sum_{j\in J} b_j$. Then M_1 centralizes $\mathbb{K}b_{3456}$, $\langle \mathbb{K}b_{3456}^{M_2}\rangle = \mathbb{K}\langle b_{3456}, b_{1234}\rangle$ and $\tilde{V}_0 = \mathbb{K}\langle b_{3456}^H\rangle$. Thus $\tilde{V}_0$ and V are $\mathbb{K}H$-quotients of $\hat{V}$. Since $\dim_{\mathbb{K}} \tilde{V}_0 = 5$ and $\dim_{\mathbb{K}} \hat{V} \leq 5$ we conclude that $\hat{V}$ is isomorphic to $\tilde{V}_0$. Thus V is isomorphic to a quotient of $\tilde{V}_0$. Observe that $\mathrm{C}_{\tilde{V}_0}(H) = \mathbb{K}\langle b_{123456}\rangle$ and $b_{123456} = b_{1234} + b_{1235} + b_{1245} + b_{3456} \in \mathbb{K}\langle b_{3456}^{M_1}, b_{1234}^{M_1}\rangle$. So (a) holds.

Suppose next that $H \sim 3{}^{\cdot}\mathrm{Alt}(6)$. Let R be a Sylow 3-subgroup of H. The R is extraspecial of order 27. Let Y be any R-chief-factor of V. Then $Z(H) = Z(R)$ acts non-trivially on Y and so $\dim_{\mathbb{E}} Y = 3$. Thus $\dim_{\mathbb{E}} V$ is a multiple of three and since $\dim_{\mathbb{E}} V \leq 5$, $\dim_{\mathbb{E}} V = 3$. So (b) holds. □

6 Module Decompositions

Lemma 6.1. *Let H be a finite group, V an $\mathbb{F}_pH$-module, and $\mathbb{K} := \mathrm{End}_H(V)$. The following table lists the dimension $d := \dim_{\mathbb{K}}(H^1(H, V))$ for various pairs (H, V).*

H	p	V	*Conditions*	d
$\Omega_n^\epsilon(p^k), n \geq 3$	p	V_{nat}^*	$n = 3, p^k = 2$	1
”	”	”	$n = 3, p^k = 5$	1
”	”	”	$n = 4, \epsilon = -, p^k = 3$	2
”	”	”	$n = 5, p^k = 3$	1
”	”	”	$n = 6, \epsilon = +, p^k = 2$	1
”	”	”	*all others*	0
$Sp_{2n}(p^k)$	p	V_{nat}	$p = 2, (2n, p^k) \neq (2, 2)$	1
”	”	”	*all others*	0
$SL_n(p^k)$	p	V_{nat}	$n = 2, p = 2, k > 1$	1
”	”	”	$n = 3, p = 2, k = 1$	1
”	”	”	*all others*	0
$SU_n(p^k), n \geq 3$	p	V_{nat}	$n = 4, p^k = 2$	1
”	”	”	*all others*	0
$\mathrm{G}_2(2^k)'$	2	$\mathbb{K}^6$	–	1
$\mathrm{G}_2(p^k)'$	$p \neq 2$	$\mathbb{K}^7$	–	0
${}^3D_4(p^k)$	p	$\mathbb{K}^8$	–	0
$Spin_n^\epsilon(p^k)$	p	*(Half)-Spin*	$n \geq 7$	0
$3.\mathrm{Alt}(6)$	2	$\mathbb{K}^3$	–	0
$\mathrm{Alt}(n), n \geq 5$	2	V_{nat}	n *even*	1
”	”	”	n *odd*	0
$SL_n(p^k), n \geq 5$	p	$\bigwedge^2(V_{\mathrm{nat}})$	–	0
$SL_n(p^k), n \geq 3$	*odd*	$\mathrm{Sym}^2(V_{\mathrm{nat}})$	–	0
$SL_n(p^{2k}), n \geq 3$	p	$V_{\mathrm{nat}} \otimes V_{\mathrm{nat}}^{p^k}$	$n = 3, p^{2k} = 4$	2
”	”	”	*all others*	0
$E_6(p^k)$	p	$\mathbb{K}^{27}$	–	0
$\mathrm{Mat}_n, 22 \leq n \leq 24$	2	*Todd*	$n = 24$	1
”	”	”	$n = 22, 23$	0
$\mathrm{Mat}_n, 22 \leq n \leq 24$	2	*Golay*	$n = 22$	1
$\mathrm{Mat}_n, 22 \leq n \leq 24$	2	*Golay*	$n = 23, 24$	0
$3.\mathrm{Mat}_{22}$	2	$\mathbb{F}_4^6$	–	0
Mat_{11}	3	*Todd*	–	0
Mat_{11}	3	*Golay*	–	1
$2.\mathrm{Mat}_{12}$	3	*Todd*	–	0
$2.\mathrm{Mat}_{12}$	3	*Golay*	–	0

Proof. Let $T \in \mathrm{Syl}_p(H)$ and W be an $\mathbb{F}_pH$-module with $[W, H] \leq V$ and $\mathrm{C}_W(H) \leq V$. Note that by Gaschütz's Theorem, $\mathrm{C}_W(T) \leq V$.

1°. *Let $C \leq H$ and A and B be normal p-subgroups of C with $A \leq B$, and let X, Y, Z be C-submodules of W with $X \leq Y \leq Z$. Suppose that*

(i) B centralizes Z/Y and Y/X.

(ii) A *centralizes* Z/X.

(iii) $\Phi(B) \leq A$.

Put $U/X := \mathrm{C}_{Z/X}(B)$. *Then* Z/U *is isomorphic to a* C*-submodule of* $\mathrm{Hom}_{\mathbb{F}_p}(B/A, Y/X)$. *If in addition* C *centralizes* Z/U, *then* Z/U *embeds into* $\mathrm{Hom}_{\mathbb{F}_pC}(B/A, Y/X)$.

For $z \in Z$ define
$$\tilde{z} : B/A \to Y/X \text{ with } bA \to [b, z] + X.$$
Since B/A and Y/X are $\mathbb{F}_pC$-modules, for $c \in C$ the element $\tilde{z}^c := c^{-1}\tilde{z}c \in \mathrm{Hom}_{\mathbb{F}_p}(B/A, Y/X)$ is defined, and

$$(bA)\tilde{z}^c = bA(c^{-1}\tilde{z}c) = (b^{c^{-1}}A\tilde{z})c = ([b^{c^{-1}}, z] + X)^c = [b, z^c] + X = bA\widetilde{z^c}.$$

Thus, the map
$$Z \to \mathrm{Hom}_{\mathbb{F}_p}(B/A, Y/X) \text{ with } z \to \tilde{z}$$
is C-equivariant with kernel U. So the first statement holds. The second follows from the first.

Case 1. *V is the dual of a natural module for* $H \cong \Omega_n^\epsilon(q)$, $n > 2$ *and* $q = p^k$.

This case is covered by [Po] and [JP].

Case 2. *V is a natural module for* $H = \mathrm{Sp}_{2n}(q)$.

See [JP].

Case 3. *V is a natural module for* $H = \mathrm{SL}_n(q)$, $q = p^k$.

See [JP].

Case 4. *V is a natural module for* $H = \mathrm{SU}_n(q)$, $q = p^k$, *and* $n \geq 3$.

If $q > 3$ see [JP]. So assume that $q \leq 3$. If H is solvable, then $H = \mathrm{SU}_3(2)$, and Maschke's Theorem shows that the lemma holds. Thus, assume in addition that $H \neq \mathrm{SU}_3(2)$. Let V_1 be a 1-dimensional singular $\mathbb{K}$-subspace of V, $V_2 = V_1^\perp \leq V$, $L = \mathrm{C}_H(V_1)$, and $L^* = \mathrm{N}_H(V_1)$.

Suppose for a contradiction that $[V, \mathrm{O}_p(L)] \not\leq V_2$. Since L centralizes W/V and V/V_2 we conclude that $\mathrm{O}_p(L) \not\leq \mathrm{O}^p(L)$ and so $n = 3$ and $q = 3$. In particular, $L = \mathrm{O}_3(L)$ is extraspecial of exponent 3 and $[W, \Phi(L)] \leq V_2$. Hence, there exists $g \in L \setminus \Phi(L)$ with $[W, g] \not\leq V_2$. Note that $[v, g, g] \neq 0$ for every $v \in V \setminus V_2$. On the other hand $|g| = 3$, so g acts cubically on W. This shows that $[W, g] \leq V_2$, which contradicts the choice of g. Thus

2°. $[W, \mathrm{O}_p(L)] \leq V_2$.

Since $[V_2, \mathrm{O}_p(L)] \leq V_1$ we conclude that $[W, \mathrm{O}_p(L)'] \leq V_1$. Let W_2 be maximal in W with $[W_2, \mathrm{O}_p(L)] \leq V_1$. In addition we use the following notation:

$$K^* := \mathrm{C}_{L^*}(L/\mathrm{O}_p(L)),\ K := \mathrm{C}_{L^*}(V_2/V_1),\ X/V_2 := \mathrm{C}_{W/V_2}(K^*).$$

Then $K \leq K^*$, $K^*/\mathrm{O}_p(L)$ has order $q^2 - 1$ and $K/\mathrm{O}_p(L)$ has order $q - 1$. We will prove next:

3°. $[W, L] \leq V_2$.

By Maschke's Theorem and (2°), $W/V_2 = X/V_2 \oplus V/V_2$. Since $[X, L^*] \leq X \cap V = V_2$ we conclude that $[W, L] \leq V_2$.

4°. *Either* $W = W_2 + V$ *or* $q = 2$, $n = 4$ *and* $|W/W_2 + V| \leq 4$.

Suppose that $q \neq 2$. Then $\mathrm{O}_p(L) = [\mathrm{O}_p(L), K]$ and so $K = \mathrm{O}^p(K)$. Since $[X, K] \leq V_2$ and $[V_2, K] \leq V_1$ we have $[X, K] = [X, \mathrm{O}^p(K)] \leq V_1$. Thus $X \leq W_2$. Since $W = X + V$, (4°) holds in this case.

So we may assume that $q = 2$. Then $n > 3$ since we are assuming that $H \neq \mathrm{SU}_3(2)$. Put $Z := \mathrm{O}_2(L)'$. Then $[Z, L] = 1$ and by (2°), $[W, \mathrm{O}_2(L), Z] \leq [V_2, Z] = 0$. Since by (3°) $[W, L] \leq V_2$, we conclude from (1°) that W/V_2 embeds into $\mathrm{Hom}_L(\mathrm{O}_2(L)/Z, V_2/V_1)$.

Suppose that $n > 4$. Then L acts simply on $\mathrm{O}_p(L)/Z$ and on V_2/V_1 and thus

$$q^2 = |V/V_2| \leq |W/V_2| \leq |\mathrm{Hom}_L(\mathrm{O}_p(L)/Z, V_2/V_1)| = q^2.$$

We conclude that $V = W$, so (4°) holds in this case.

Suppose that $n = 4$. Since $V_2 \leq W_2$ and L^* centralizes X/V_2, L^* centralizes $X + W_2/W_2$. So by (1°) $X + W_2/W_2$ embeds into $\mathrm{Hom}_{L^*}(\mathrm{O}_p(L)/Z, V_2/V_1)$. Since L^* acts simply on $\mathrm{O}_p(L)/Z$ and on V_2/V_1 we conclude as above that $|X/X \cap W_2| = |X + W_2/W_2| \leq q^2 = 4$. Now $W/V_2 = X/V_2 \oplus V/V_2$ and $V_2 \leq W_2$ imply

$$|W/(X \cap W_2) + V| = |X + V/(X \cap W_2) + V| = |(X/V_2)/(X \cap W_2/V_2)| = |X/X \cap W_2| =\leq 4,$$

so (4°) also holds in this case.

5°. *Put* $W_1 := \mathrm{C}_{W_2}(\mathrm{O}_p(L))$. *Then* $W_2 = W_1 + V_2$ *and* $W_2 + V = W_1 + V$.

Since $[W_2, \mathrm{O}_p(L)] \leq V_1 \leq \mathrm{C}_V(\mathrm{O}_p(L))$ the Three Subgroups Lemma gives that $[W_2, Z] = 0$. So by (1°) W_2/W_1 embeds into $\mathrm{Hom}_{\mathbb{F}_p}(\mathrm{O}_p(L)/Z, V_1)$. As an L-module $\mathrm{Hom}_{\mathbb{F}_p}(\mathrm{O}_p(L)/Z, V_1)$ is a direct sum of copies of the dual of $\mathrm{O}_p(L)/Z$. If $n > 3$ we conclude that $W_2/W_1 = [W_2/W_1, L]$ and so by (3°) $W_2 = W_1 + V_2$. Thus (5°) holds in this case. So suppose $n = 3$. Let $Y/V_1 = \mathrm{C}_{W_2/V_1}(L^*)$. Then by Maschke's Theorem, $W_2 = Y + V_2$.

Suppose that $Y \nleq W_1$. Then $\mathrm{O}_p(L)/Z \cong V_1$ as an L^*-module. Since $n = 3$ we have $q > 2$, and so L^* acts simply on $\mathrm{O}_p(L)/Z$ and on V_1. It follows that there exists $0 \leq l < 2k$ with $\lambda^{2-p^k} = \lambda^{p^l}$, for all $0 \neq \lambda \in \mathbb{F}_{p^{2k}}$. Thus $p^{2k}-1$ divides $p^l + p^k - 2$. Hence either $p^l + p^k - 2 \leq 0$ or $p^l + p^k - 2 \geq p^{2k} - 1$. Since $p^k = q > 2$ we have $p^l + p^k - 2 > 0$. Moreover,

$$p^l + p^k - 2 \leq p^{2k-1} + p^{2k-1} - 2 \leq p^{2k} - 2 < p^{2k} - 1,$$

a contradiction. Thus $Y \leq W_1$, and (5°) also holds for $n = 3$.

6°. $W_1 = V_1$ *and* $W_2 + V = V$.

Let $g \in H$ such that V_1 is not perpendicular to V_1^g in V, so $V_1 \nleq V_2^g$. Then by (3°), $[W_1, L \cap L^g] \leq W_1 \cap V_2^g \leq (W_1 \cap V) \cap V_2^g = V_1 \cap V_2^g = 0$. Thus W_1 is centralizes by $\mathrm{O}_p(L)(L \cap L^g) = L$ and so $W_1 \leq \mathrm{C}_W(T) \leq V$. Thus $W_1 = V_1$, and (5°) implies (6°).

From (4°) and (6°) we see that the lemma holds in (Case 4).

Case 5. $H = \mathrm{G}_2(q)'$, $q = p^k$, *and either* $p = 2$ *and* $V = \mathbb{K}^6$ *or* $p \neq 2$ *and* $V = \mathbb{K}^7$.

See [JP].

Case 6. V *is a natural module for* $H = {}^3D_4(q)$, $q = p^k$.

Fix a root system Φ. With respect to Φ, let C be the Cartan subgroup, N/C the Weyl-group, and L be the subgroup of H generated by the long root subgroups. Then $L \cong \mathrm{SL}_3(q)$ and C normalizes L.

Let $K \leq H$ be the centralizer of a field automorphism of order 3 in H such that $K \cong \mathrm{G}_2(q)$, each root subgroup with respect to Φ intersects K in a root subgroup of K, and $N = (N \cap K)C$. Then $L \leq K$ and $\langle K, C\rangle$ contains all the root subgroups from Φ. So $\langle K, C\rangle = H$. In the case $q = 2$, the action of C on the Lie-parabolic subgroups of H shows that also $\langle \mathrm{O}^2(K), C\rangle = H$.

Note that $V/\mathrm{C}_V(K)$ is a 7-dimensional K-module (over $\mathbb{K}$), which is a natural module for p odd and a non-split central extension of a natural module for $p = 2$. By (Case 5), $W = \mathrm{C}_W(\mathrm{O}^p(K)) + V$. Moreover, the action of K on V shows that $\mathrm{C}_V(\mathrm{O}^p(K)) = \mathrm{C}_V(L(N \cap \mathrm{O}^p(K)))$. So also $\mathrm{C}_W(\mathrm{O}^p(K)) = \mathrm{C}_W(L(N \cap \mathrm{O}^p(K))$. Note that C acts fixed-point freely on $\mathrm{C}_V(L)$. Since C is a p'-group we get $\mathrm{C}_W(L) = \mathrm{C}_V(L) \oplus \mathrm{C}_W(LC)$. Thus also $W = V \oplus \mathrm{C}_W(LC)$. Since N normalizes $\mathrm{C}_W(LC)$ we have

$$\mathrm{C}_W(LC) = \mathrm{C}_W(LN) \leq \mathrm{C}_W(L(N \cap \mathrm{O}^p(K))) \leq \mathrm{C}_W(\mathrm{O}^p(K)).$$

Thus $\mathrm{C}_W(LC) \leq \mathrm{C}_W(\langle C, \mathrm{O}^p(K)\rangle) = \mathrm{C}_W(H) = 0$ and so $V = W$.

Case 7. *V is the (half)-spin-module for $H = \mathrm{Spin}_n^{\epsilon}(q), q = p^k$, $n \geq 7$.*

See [JP].

Case 8. *$H = 3.\mathrm{Alt}(6)$ and $V = \mathbb{K}^3$.*

Since $[V, \mathrm{Z}(H)] \neq 0$, Maschke's Theorem implies that $V = W$.

Case 9. *V is a natural module for $H \cong \mathrm{Alt}(n), n \geq 5$, $p = 2$.*

See [As, page 74].

Case 10. *V is the symmetric square of a natural module for $H \cong \mathrm{SL}_n(q)$, $q = p^k$, p odd, $n \geq 3$.*

Let $V_2 := [V, T]$, $L^* := \mathrm{N}_H(V_2)$, $L_1 := \mathrm{C}_{L^*}(V/V_2)$ and $L := \mathrm{O}^{p'}(L^*)$. Then $L/\mathrm{O}_p(L) \cong \mathrm{SL}_{n-1}(q)$ and $|L_1/L| = 2$. Note that $L = \mathrm{O}^p(L)$ unless $n = 3 = q$, in which case $L_1/\mathrm{O}_p(L_1) \cong \mathrm{GL}_2(3)$. So in any case $L_1 = \mathrm{O}^p(L_1)$ and thus

7°. $[W, L_1] = V_2 = [W, L]$.

Let $V_1 := \mathrm{C}_V(\mathrm{O}_p(L)) = [V_2, \mathrm{O}_p(L)]$. Then V_2/V_1 is a natural $\mathrm{SL}_{n-1}(q)$-module for $L/\mathrm{O}_p(L)$ isomorphic to $\mathrm{O}_p(L)$. Hence $|\mathrm{Hom}_L(\mathrm{O}_p(L), V_2/V_1)| = q$. Let $W_2/V_1 := \mathrm{C}_{W/V_1}(\mathrm{O}_p(L))$. Then by (1°) W/W_2 embeds into $\mathrm{Hom}_L(\mathrm{O}_p(L), V_2/V_1)$. Since $|V/V_2| = q$ we conclude that

8°. $W = W_2 + V$.

Let $W_1/V_1 := \mathrm{C}_{W_2/V_1}(L)$. By (Case 3) $\mathrm{H}^1(L/\mathrm{O}_p(L), V_2/V_1) = 0$ and so by (8°)

9°. $W_2 = W_1 + V_2$ *and* $W = W_1 + V$.

Note that V_1 is the symmetric square of a natural module for $L/\mathrm{O}_p(L)$. In particular, V_1 and $\mathrm{O}_p(L))$ are non-isomorphic simple $L/\mathrm{O}_p(L)$-modules and so $[W_1, \mathrm{O}_p(L)] = 1$. Let $W_0 = \mathrm{C}_{W_1}(L)$. Suppose that $W_1 \neq W_0 \oplus V_1$. By induction on n and with (Case 1) we conclude that $n = 3$ and $q = 5$. (Note here that for $n = 3$ V_1 is an orthogonal $\Omega_3(q)$-module for $L/\mathrm{O}_p(L)$.)

Since $T/\mathrm{O}_5(L)$ is cyclic, the Jordan Form for T on V shows that T does not act cubically on W_1. Pick $g \in H$ with $T = \mathrm{O}_5(L)(\mathrm{O}_5(L)^g \cap T)$. By (9°), $\mathrm{O}_5(L)$ acts cubically on V and so T acts cubically in W_1, a contradiction.

Thus $W_1 = W_0 + V_1$. As $W_0 \leq \mathrm{C}_W(T) \leq V$ we have $W_1 \leq V$, and by (9°) $V = W$.

Case 11. *V is the alternating square of a natural module for $H \cong \mathrm{SL}_n(q)$, $q = p^k$, $n \geq 5$.*

See [JP].

Case 12. *$H \cong \mathrm{E}_6(q)$, $q = p^k$, and $V = \mathbb{K}^{27}$.*

See [JP].

Case 13. *$H \cong \mathrm{SL}_n(q^2)$, $q = p^k$, and V is a simple $\mathbb{F}_q H$-submodule of $N \otimes_{\mathbb{F}_{q^2}} N^\sigma$, where N is the natural $\mathbb{F}_{q^2} H$-module and σ is the field automorphism of order 2 of $\mathbb{F}_{q^2}$.*

Let $N_1 := \mathrm{C}_N(T)$, $L^* := \mathrm{N}_H(N_1)$, and $L := \mathrm{C}_H(N_1)$, and let $J \leq L^*$ with $L^* = \mathrm{O}_p(L)J$ and $N = N_1 \oplus [N, J \cap L]$. Then $J \cap L \cong \mathrm{SL}_{n-1}(q^2)$ and $J \cong \mathrm{GL}_{n-1}(q^2)$. Let $V_1 = \mathrm{C}_V(L)$ and $V_2 = [V, \mathrm{O}_p(L)]$. Then V_2/V_1 is a natural $\mathrm{SL}_{n-1}(q^2)$-module for $L/\mathrm{O}_p(L)$ isomorphic to N/N_1 and dual to $\mathrm{O}_p(L)$. Also V/V_2 is isomorphic to a simple $\mathbb{F}_q L/\mathrm{O}_p(L)$ submodule of $N/N_1 \otimes_{\mathbb{F}_{q^2}} N^\sigma/N_1^\sigma$. We first show:

10°. *Suppose $n = 3$ and $q \neq 2$. Then $\mathrm{Z}(J)$ acts fix-point freely on V/V_2, and $\mathrm{O}_p(L)$ and V_2/V_1 are not isomorphic as $\mathbb{F}_p\mathrm{Z}(J)$-modules.*

Let $j \in \mathrm{Z}(J)$, then j acts as an $\mathbb{F}_{q^2}$-scalar λ on N/N_1. It follows that j acts as λ^{-2} on N_1, as λ^{-3} on $\mathrm{O}_p(L)$, as λ^{q-2} on V_2/V_1 and as λ^{q+1} on V/V_2. Since $q > 2$ we conclude that $\mathrm{Z}(J)$ is fixed-point free on V/V_2. Suppose that V_2/V_1 and $\mathrm{O}_p(L)$ are isomorphic as $\mathbb{F}_p\mathrm{Z}(J)$-modules. Then there exists $0 \leq l < 2k$ with $\lambda^{-3p^l} = \lambda^{q-2}$ for all $0 \neq \lambda \in \mathbb{F}_{q^2}$ and so

$$p^{2k} - 1 \big| 3p^l + p^k - 2.$$

Since $p^k = q > 2$, the right side is positive and so

$$p^{2k} - 1 \leq 3p^l + p^k - 2 \leq 3p^{2k-1} + p^k - 2 \leq 4p^{2k-1} - 2.$$

Thus $p \leq 3$. If $p = 3$ we have

$$3^{2k} \leq 3^{l+1} + 3^k - 1 \leq 2 \cdot 3^m - 1,$$

where $m = \max\{l+1, k\}$. Hence $m = l + 1 = 2k$. and so

$$3^{2k} - 1 \big| 3 \cdot 3^{2k-1} + 3^k - 2 = (3^{2k} - 1) + 3^k - 1.$$

Therefore $3^{2k} - 1 \big| 3^k - 1$, a contradiction.

Thus $p = 2$. If $l = 0$ we get $2^{2k} - 1 \leq 2^k + 1$ and $q = 2^k = 2$, contradiction. Hence $l > 0$ and since $2^{2k} - 1$ is odd,

$$2^{2k} - 1 \big| 3 \cdot 2^{l-1} + 2^{k-1} - 1.$$

So

$$2^{2k} \leq 3 \cdot 2^{l-1} + 2^{k-1} = 2^l + 2^{l-1} + 2^{k-1}.$$

It follows that $k = 1 = l$ and $q = 2$, a contradiction.

11°. *Suppose $n = 3$ and $V \neq W$. Then $q = 2$ and $|W/V| \leq 4$.*

Since $\mathrm{O}_p(L)$ and V/V_2 are non-isomorphic simple L-modules, $[W, \mathrm{O}_p(L)] \leq V_2$. Let $W_2/V_2 = \mathrm{C}_{W/V_2}(L)$. If $q \neq 2$, then by (10°) $\mathrm{Z}(J)$ acts fixed-point-freely on V/V_2, and if $q = 2$, then by (Case 1), $\mathrm{H}^1(L/\mathrm{O}_p(L), V/V_2) = 0$. So in any case $W = W_2 + V$.

Let $W_1/V_1 = \mathrm{C}_{W_2/V_1}(\mathrm{O}_p(L))$. Then W_2/W_1 embeds into $\mathrm{Hom}_{L^*}(\mathrm{O}_p(L), V_2/V_1)$. By (10°) this group is trivial for $q \neq 2$. For $q = 2$ it has order 4. So $W_2 = W_1$ if $q \neq 2$ and $|W_2/W_1| \leq 4$ if $q = 2$. It remains to show that $W_1 \leq V$.

Let $W_0 = \mathrm{C}_{W_1}(\mathrm{O}_p(L))$. Then W_1/W_0 embeds into $\mathrm{Hom}_{\mathbb{F}_p}(\mathrm{O}_p(L), V_1)$. The latter group is as an L-module isomorphic to a direct sum of copies of the dual of $\mathrm{O}_p(L)$. Hence $[W_1/W_0, L] = W_1/W_0$ and so $W_1 = W_0 + V_2$. Since $W_0 \cap V = V_1$ and $L = \mathrm{O}^p(L)$ we have $[W_0, L] = 0$ and so $W_0 \leq \mathrm{C}_V(T) \leq V$. Thus also $W_1 \leq V$, and (11°) is proved.

12°. *Suppose $n = 3$ and $q = 2$. Then $|\mathrm{H}^1(H, V)| = 4$, and $\mathrm{GL}_3(4)$ acts fixed-point freely on $\mathrm{H}^1(H, V)$.*

By (11°) $|\mathrm{H}^1(H, V)| \leq 4$. Let I be the simple 11-dimensional Golay code-module for $M = \mathrm{Mat}_{24}$ over $\mathbb{F}_2$. Let $\tilde{H} = \mathrm{Mat}_{21} \cong \mathrm{PSL}_3(4)$. Then $[I, \tilde{H}]$ is simple of $\mathbb{F}_2$-dimension 9 and $\mathrm{C}_I(\tilde{H}) = 0$. Moreover, $\mathrm{N}_M(\tilde{H}) \cong \mathrm{PGL}_3(4)$ acts fixed-point freely on $I/[I, \tilde{H}]$, so (12°) holds.

13°. *Suppose $n > 3$. Then $V = W$.*

Note that W/V_2 and $\mathrm{O}^{p'}(L^*/\mathrm{O}_p(L))$ satisfy (Case 13) for $n-1$, and note further that $L^*/\mathrm{O}_p(L) \cong \mathrm{GL}_{n-1}(q^2)$. Moreover, for $n-1 = 3$ the case described in (12°) does not occur since $[W, L^*] = V$. Hence induction shows that $\mathrm{H}^1(L^*/\mathrm{O}_p(L), V/V_2) = 0$. By (Case 3), also $\mathrm{H}^1(L^*/\mathrm{O}_p(L), V/V_2) = 0$. Since $n > 3$, V/V_2 and V_2/V_1 are simple L^*-modules not isomorphic to $\mathrm{O}_p(L)$. Also since $L = \mathrm{O}^p(L)$, $\mathrm{H}^1(L, V_1) = 0$. Thus $\mathrm{H}^1(L^*, V) = 0$ and $V = W$.

By (11°), (12°) and (13°) the Lemma holds in case (Case 13).

Case 14. *$p = 2$, and V is the simple Todd- or Golay code-module for $H = \mathrm{Mat}_n$, $n = 22, 23$, or 24.*

Let $P := \mathrm{Mat}_{n-1} \leq H$. Suppose first that $H = \mathrm{Mat}_{22}$ and V is the Todd-module. Put $V_1 := \mathrm{C}_V(T)$ and $L := \mathrm{C}_H(V_1)$. Then $L/\mathrm{O}_2(L) \cong \mathrm{Sym}(5)$, and $\mathrm{O}_2(L)$ is a natural $\Gamma\mathrm{SL}_2(4)$-module for L. Put $V_2 := [V, \mathrm{O}_2(L)]$. Then $\mathrm{O}_2(L)$ centralizes V_2/V_1, and V_2/V_1 is an non-split extension of a 1-dimensional module by a natural $\Gamma\mathrm{SL}_2(4)$-module for $L/\mathrm{O}_2(L)$. Moreover, V/V_2 is a natural $\mathrm{O}_4^-(2)$-module for L. Since V/V_2 is not isomorphic to $\mathrm{O}_2(L)$ as an L-module, $[W, \mathrm{O}_2(L)] \leq V_2$. Put $W_2/V_2 := \mathrm{C}_{W/V_2}(L)$. By (Case 1) $W = W_2 + V$. Since V_2/V_1 is indecomposable, $\mathrm{Hom}_L(\mathrm{O}_2(L), V_2/V_1) = 0$ and so $[W_2, \mathrm{O}_2(L)] \leq V_1$. Let $W_1 = \mathrm{C}_{W_2}(\mathrm{O}_2(L))$. Then W_2/W_1 embeds into $\mathrm{Hom}_{\mathbb{F}_2}(\mathrm{O}_2(L), V_1)$. The latter is isomorphic to the dual of $\mathrm{O}_2(L)$ and so $W_2 = W_1 + V_2$. Note that $[W_1, \mathrm{O}^2(L)] = 1$ and $W_1 \cap V$ has order 4 with $L/\mathrm{O}^2(L)$ acting non-trivial on $W_1 \cap V$. It follows that $W_1 = \mathrm{C}_{W_1}(L) + (W_1 \cap V)$ and so $W_1 \leq \mathrm{C}_W(T) + V \leq V$. Hence also $W_2 \leq V$ and $W = V$.

Suppose next that $H = \mathrm{Mat}_{22}$ and V is the Golay code -module. Then $|[V, P]| = 2^9$ and $\mathrm{C}_V(P) = 0$, so V is a non-split extension for P as in case (Case 13). Thus (Case 13) shows that $|W/V + \mathrm{C}_W(P)| \leq 2$. Let $L_0 = \mathrm{Mat}_{20} \leq P$ and $L = \mathrm{N}_H(L_0) \sim 2^4\mathrm{Sym}(5)$. Then $\mathrm{C}_V(L_0) = 0$ and so $\mathrm{C}_W(P) \leq \mathrm{C}_W(L_0) \leq \mathrm{C}_W(L)$. Since L contains a Sylow 2-subgroup of H, $\mathrm{C}_W(L) \leq V$ and so $\mathrm{C}_W(P) = 0$ and $|W/V| \leq 2$.

Suppose next that $H = \mathrm{Mat}_{23}$. Then P contains a Sylow 2-subgroup of H and so $\mathrm{C}_W(P) \leq V$. If V is the Todd-module, then $V = [V, P]$ and $V/\mathrm{C}_V(P)$ is the Todd-module for $P = \mathrm{Mat}_{22}$. Since $P = \mathrm{O}^2(P)$, the Mat_{22}-case implies that $W = \mathrm{C}_W(P) + V = V$.

If V is the Golay code-module, then $C_V(P) = 0$ and $[V, P]$ is the 10 dimensional Golay code module for P. Thus by the Mat_{22}-case, $W = C_W(L) + V = V$.

Suppose that $H = \mathrm{Mat}_{24}$. Then V is simple as a P-module, so by the Mat_{23}-case, $W = C_W(P) + V$. Let $w \in C_W(P)$. Then $\langle w^H \rangle$ is a quotient of the natural permutation module of Mat_{24}. If V is the Golay code-module, we conclude that $[w, H] = 0$ and so $V = W$. If V is the Todd module and $w \neq 0$, we conclude that $\langle w^H \rangle = \langle w \rangle + V$ is uniquely determined as an $\mathbb{F}_2H$-module. Since $|\mathbb{K}| = 2$ this implies $|W/V| \leq 2$.

Case 15. *$V = \mathbb{F}_4^6$ and $H = 3.\mathrm{Mat}_{22}$.*

Since $Z(H) \neq 1$, we have $V = W$.

Case 16. *$p = 3$, V is the simple Todd- or Golay code-module for $H = \mathrm{Mat}_{11}$ or $2.\mathrm{Mat}_{12}$.*

If $H = 2.\mathrm{Mat}_{12}$, we have $W = C_W(Z(H)) \oplus V$ and so $V = W$. Suppose $H = \mathrm{Mat}_{11}$.

Assume first that V is the Golay code-module. Let $L_0 = \mathrm{Mat}_{10}$ and $L = L_0' \cong L_2(9)$. Then $[V, L]$ is the natural $\Omega_4^-(3)$-module for L and $C_V(L) = 0$. Thus by (Case 1), $|W/V + C_W(L)| \leq 3$. Since L contains a Sylow 3-subgroup of H, $C_W(L) \leq V$ and so $|W/V| \leq 3$.

Suppose next that V is the Todd-module. Let $L = N_H(T)$. Then L/T is semidihedral of order 16. Let $K \in \mathrm{Syl}_2(L)$ and put $V_2 = [V, T]$ and $V_1 = C_V(T)$. Then $|V/V_1| = 3$ with $D := C_K(V/V_1)$ dihedral of order 8. Moreover, V_2/V_1 has order 9 with K acting faithfully on V_2/V_1, and V_1 has order 9 with $|C_K(V_1)| = 2$. Since $T = [T, D]$, we have $[W, T] \leq V_2$. Let $W_2/V_1 = C_{W/V_1}(T)$. Then W/W_2 embeds into $\mathrm{Hom}_D(T, V_2/V_1)$. Since D acts simply on T and V_2/V_1, we conclude that $\mathrm{Hom}_D(T, V_2/V_1)$ has order 3. Thus $W = W_2 + V$. Let $W_1/V_1 = C_{W_2/V_1}(L)$. By Mascke's Theorem, $W_2 = W_1 + V_2$. Since V_1 is not isomorphic to T as an L-module, $[W_1, T] = 0$ and so $W_1 \leq V$ and $V = W$. □

Definition 6.2. *Let H be a finite group, V an $\mathbb{F}_pH$-module and Q a p-subgroup of H. Then V is called a $Q!$-module for H if Q is not normal in H and*

$$(Q!) \qquad Q \trianglelefteq N_H(A) \text{ for all } 1 \neq A \leq C_V(Q).$$

Lemma 6.3. *Let $M \cong \mathrm{SL}_n(q)$, q a power of p, $n \geq 2$, and let V be an $\mathbb{F}_pM$-module. Suppose that there exists an M-submodule I in V such that the following hold:*

(i) $W := V/I$ is a natural $\mathrm{SL}_n(q)$-module for M.

(ii) $I \cong \Lambda^2_{\mathbb{K}}W$ as an $\mathbb{F}_pM$-module, where $\mathbb{K} := \mathrm{End}_M(W)$.

(iii) If H is a $\mathbb{K}$-hyperplane in W and $A := C_M(H) \cap C_M(W/H)$, then $C_V(A) \not\leq I$.

Then there exists $x \in V \setminus W$ with $C_M(x) = C_M(x + I/I)$. Moreover, V is not a $Q!$-module for any p-subgroup Q of M.

Proof. Put $U := C_V(A)$, $L = N_M(H) \cap C_M(W/H)$ and $T \in \mathrm{Syl}_p(L)$. Note $T \in \mathrm{Syl}_p(M)$. We will first show:

1°. $C_V(T) \not\leq I$.

The proof is by induction on n. If $n = 2$ then $A = T$ and (1°) follows from (iii). Suppose that $n \geq 3$. Note that $L/A \cong \mathrm{SL}_{n-1}(q)$, $H \cong U/U \cap I$ is a natural module for L/A and $U \cap I \cong \Lambda^2_{\mathbb{K}}H$. Let $g \in M$ with $H^g \neq H$ and put $R_0 := L \cap A^g$ and $R := A(L \cap A^g)$.

Assume that $n = 3$. Then $T = R$ and $I \cong W^*$. In particular

$$[U \cap (U^g + I), R] = [I, R_0] \cap I \cap U = 0.$$

Since $|U \cap (U^g + I)| = q^2$ while $|U \cap I| = q$, we conclude that $C_U(R) = \mathrm{C}_U(T) \not\leq I$, and (1°) holds.

Suppose now that $n > 3$. Then $\mathrm{C}_I(R) = \mathrm{C}_{U\cap I}(R_0)$ and so $\mathrm{C}_I(R)$ has order $q^{\binom{n-2}{2}}$. On the other hand, $\mathrm{C}_V(A)$ has index q^n in V. Hence $\mathrm{C}_V(\langle A, A^g\rangle)$ has index at most q^{2n} in V. Thus also $|V/\mathrm{C}_V(R)| \leq q^{2n}$. Note that

$$|V/\mathrm{C}_I(R)| = q^{n+\binom{n}{2}-\binom{n-2}{2}} = q^{3n-3} > q^{2n},$$

where the last inequality holds since $n > 3$.

Thus $\mathrm{C}_V(R) \not\leq \mathrm{C}_I(R)$ and since $\mathrm{C}_V(R) \leq U$, $\mathrm{C}_U(R) \not\leq U \cap I$. Thus $(U, U\cap I, L/A, H\cap H^g, R/A)$ in place of (V, I, M, H, A) fulfills the assumptions (i)-(iii) and so by induction $\mathrm{C}_U(T/A) \not\leq U \cap I$. Thus (1°) holds.

Put $Y := I + \mathrm{C}_V(T)$ and $F_1 := \mathrm{C}_M(Y/I)$. Then $\dim_{\mathbb{K}} Y/I = 1$, so $F_1 = \mathrm{C}_M(x + I/I)$ for $x \in \mathrm{C}_V(T)\setminus I$. Since $T \in \mathrm{Syl}_p(F_1)$, Gaschütz' Theorem implies that $Y = I \oplus X$ for some F_1-invariant subspace X of Y. Then $[X, F_1] \leq X \cap I = 0$. Let $0 \neq x \in X$. Then $F_1 \leq \mathrm{C}_{F_1}(x) \leq \mathrm{C}_M(x+I/I) = F_1$, and so the first statement in 6.3 is proved.

Suppose V is a Q!-module. If $n = 2$, then $[I, M] = 0$ and so $Q \trianglelefteq \mathrm{C}_M(I) = M$, a contradiction. Thus $n \geq 3$. Without loss $Q \leq T$. Thus $X \leq \mathrm{C}_V(Q)$ and so by Q! we get that $Q \trianglelefteq F_1$. Similar $Q \trianglelefteq F_2 := \mathrm{N}_M(\mathrm{C}_I(T))$. Since F_2 is the normalizer of a 2-dimensional subspace of W, we have $M = \langle F_1, F_2\rangle$ and so $Q \trianglelefteq M$, a contradiction to the definition of a Q!-module. □

Lemma 6.4. *Let $M = \mathrm{SL}_2(\mathbb{F})$, $\mathbb{F}$ a field, and let Z be a maximal unipotent subgroup of M and $B := \mathrm{N}_M(Z)$. Suppose that X is an $\mathbb{Z}M$-module with $[X, Z, Z] = 0$ and Y is a B-submodule of $\mathrm{C}_X(Z)$ with $X = \langle Y^M\rangle$. Then for every $h \in M \setminus B$*

$$X = Y + Y^h + \mathrm{C}_X(M) = Y + Y^h + [Y^h, Z] \text{ and } \mathrm{C}_X(Z) = Y + [Y^h, Z] = Y + \mathrm{C}_X(M);$$

in particular $\mathrm{C}_X(M) \leq Y + [Y^h, Z]$.

Proof. Note that Z acts transitively on $Z^M \setminus \{Z\}$ and so $Z^M = \{Z\} \cup Z^{hZ}$ and $Y^M = \{Y\} \cup Y^{hZ}$ for all $h \in M \setminus B$. Thus

$$(*) \qquad X = \langle Y^M\rangle = Y + \langle Y^{hZ}\rangle = Y + Y^h + [Y^h, Z].$$

By the quadratic action of Z, $[Y^h, Z] \leq \mathrm{C}_X(Z)$. By assumption also $Y \leq \mathrm{C}_X(Z)$ and so $\mathrm{C}_X(Z) = Y + [Y^h, Z] + \mathrm{C}_{Y^h}(Z)$. Note that $M = \langle Z^M\rangle = \langle Z, Z^{hZ}\rangle = \langle Z, Z^h\rangle$ and so $\mathrm{C}_{Y^h}(Z) \leq \mathrm{C}_X(\langle Z^h, Z\rangle) \leq \mathrm{C}_X(M)$. Hence $\mathrm{C}_{Y^h}(Z) \leq \mathrm{C}_{Y^h}(M) \leq Y$ and so $\mathrm{C}_X(Z) = Y + [Y^h, Z]$.

Now by $(*)$ $X = Y^h + \mathrm{C}_X(Z)$ and thus $\mathrm{C}_X(Z^h) = Y^h + \mathrm{C}_X(Z) \cap \mathrm{C}_X(Z^h) = Y^h + \mathrm{C}_X(M)$. Hence $\mathrm{C}_X(Z) = Y + \mathrm{C}_X(M)$ and $X = Y^h + Y + \mathrm{C}_X(M)$. □

Notation 6.5. *Let*

$$\mathcal{CL}(p) := \{\mathrm{SL}_n(q),\ \mathrm{SU}_n(q),\ \mathrm{Sp}_{2n}(q)\ (q \text{ odd}),\ \Omega_n^\epsilon(q),\ \mathrm{O}_n^\epsilon(q)\ (q \text{ even})\},$$

where q is a power of p. Let $H \in \mathcal{CL}(p)$ and $\tilde{A}$ be the corresponding natural $\mathbb{F}_pH$-module. Put $A := \tilde{A}/\mathrm{C}_{\tilde{A}}(H)$. Note that A is a simple $\mathbb{F}_pH$-module. Also $\mathrm{C}_{\tilde{A}}(H) = 0$ unless $H = \Omega_{2m+1}(2^k))$, in which case $\mathrm{C}_{\tilde{A}}(H)$ is 1-dimensional, $H \cong \mathrm{Sp}_{2m}(2^k)$, and A is the natural $\mathrm{Sp}_{2m}(2^k)$-module for H.

Furthermore set $K := \mathrm{O}^p(H)$ and $\mathbb{K} := \mathrm{End}_H(A)$. Then A is also a $\mathbb{K}H$-module, and A is equipped with a natural sesquilinear form f if A is not the natural $\mathrm{SL}_n(q)$-module.

The groups $\mathrm{Sp}_{2n}(2^k)$ have been excluded from the list in 6.5, since it will be more convenient for us to treat $\mathrm{Sp}_{2n}(2^k)$ as $\Omega_{2n+1}(2^k)$.

Lemma 6.6. *Let $H \in \mathcal{CL}(p)$, V be a faithful $\mathbb{F}_pH$-module with H-submodules $A_0 \leq B \leq V$, and let $D \leq H$. Suppose that*

(i) $[B, K] \leq A_0$, $A \cong A_0$ and $V/B \cong A$ or A^*,

(ii) D is a non-trivial quadratic best offender on V.

Then there exists a KD-submodule C in V such that $A_0 \not\leq C$ and $V = B + C$.

Proof. Let D^* be any non-trivial quadratic best offender on V such that $KD^* < H$. Then we may assume by induction on H that $V = B + C$ for a KD^*-submodule C with $A_0 \not\leq C$. Since V/B is a perfect K-module and $K = O^p(K)$, also $V = B + [C, K]$ and $[C, K] = [C, K, K]$. Hence 2.6 shows that C is D-invariant, and we are done. Thus, we may assume

1°. *$H = KD^*$ for every non-trivial quadratic best offender D^* on V; in particular $H = KD$.*

Note that by 1.2 D is a best offender on $[V, K]$ and that D is a quadratic offender on $V/\mathrm{C}_V(K)$, so D contains a best offender on $V/\mathrm{C}_V(K)$. Hence we may assume that

2°. *$V = [V, K]$ and $\mathrm{C}_V(K) = 0$.*

We will now compare the action of H on V with that on the natural module $\tilde{A}$. According to (1°) we can choose D such that $U := [\tilde{A}, D]$ is minimal with respect to (ii). Observe that U is a $\mathbb{K}$-subspace. Put $P := \mathrm{N}_H(U)$ and $E = \mathrm{C}_H(U) \cap \mathrm{C}_H(\tilde{A}/U)$. Note that D acts quadratically on A_0 and so also on A. By 3.2(e), D acts quadratically on $\tilde{A}$ and U is isotropic. Thus $D \leq E$. Since E acts quadratically on $\tilde{A}$, E is an elementary abelian p-group.

Pick $D_1 \leq E$ such that first $|D_1||\mathrm{C}_V(D_1)|$ is maximal among all subgroups of E and then that $|D_1|$ is maximal with that property. Since $D \leq E$, $|D_1||\mathrm{C}_V(D_1)| \geq |D||\mathrm{C}_V(D)| \geq |V|$ and so D_1 is a non-trivial best offender on V. By [MS1, 2.6] D_1 is uniquely determined in E and so $D_1 \trianglelefteq P$. By the Timmesfeld Replacement Theorem, $D_2 := \mathrm{C}_{D_1}([V, D_1])$ is a non-trivial quadratic best offender on V. Since $[\tilde{A}, D_2] \leq [\tilde{A}, E] \leq U$, the minimal choice of U and (1°) imply $[\tilde{A}, D_2] = U$, and so we may assume

3°. *$D \trianglelefteq P$.*

By our hypothesis

$$|D| \geq |A/\mathrm{C}_A(D)||V/B/\mathrm{C}_{V/B}(D)|.$$

Since A is self-dual if A is not the natural $\mathrm{SL}_n(q)$-module, we get:

4°. *$|D| \geq |A/\mathrm{C}_A(D)||A^*/\mathrm{C}_{A^*}(D)|$ and A is the natural $\mathrm{SL}_n(q)$-module, or $|D| \geq |A/\mathrm{C}_A(D)|^2$.*

Let CL be the type of H, so $\mathrm{CL} \in \{\mathrm{SL}, \mathrm{Sp}, \mathrm{SU}, \Omega^\epsilon, \mathrm{O}^\epsilon\}$ and $H = \mathrm{CL}_n(\mathbb{K})$.

Case 1. *Suppose* $\mathrm{CL} = \mathrm{SL}, \mathrm{SU}$ *or* Sp.

Recall that in these cases $A = \tilde{A}$ and $U = [A, D]$. If $\dim_{\mathbb{K}} U = 1$ we get $|A/\mathrm{C}_A(D)| \geq |D|$, a contradiction to (4°). Thus $\dim_{\mathbb{K}} U \geq 2$. By 3.5 and since by assumption p is odd in the symplectic case, P acts simply on E and so $D = E$. Let U_1 be a 1-dimensional subspace of U. If $H = \mathrm{SL}_n(\mathbb{K})$ let U_{n-1} be a hyperplane of A containing U, $Z := \mathrm{C}_H(A/U_1) \cap \mathrm{C}_H(U_{n-1})$ and $L := \mathrm{C}_H(U_1) \cap C_H(U/U_{n-1})$. In the other cases let $U_{n-1} := U_1^{\perp}$, $Z := \mathrm{C}_H(U_1^{\perp})$ and $L := \mathrm{C}_H(U_1)$. In either case put $\overline{W} := U_{n-1}/U_1$. Then Z is a transvection group, $Z \leq \mathrm{Z}(L) \cap D$, $\mathrm{O}_p(L) = \mathrm{C}_L(\overline{W})$ and $L/\mathrm{O}_p(L)$ induces $\mathrm{CL}_{n-2}(\overline{W})$ on $\overline{W}$. Moreover, if $\mathrm{CL} = \mathrm{SL}$, $\mathrm{O}_p(L)/Z$ is as an $L/\mathrm{O}_p(L)$-module isomorphic to the direct sum of $\overline{W}$ and its dual. And if $\mathrm{CL} = \mathrm{Sp}$ or SU, then $\mathrm{O}_p(L)/Z \cong \overline{W}$ as an L-module. Let $S \in \mathrm{Syl}_p(L)$ and note that $S \in \mathrm{Syl}_p(H)$.

5°. $[V, Z, L] = 0$.

Note that $D = E$ induces $\mathrm{C}_{\mathrm{CL}_{n-2}(\overline{W})}(\overline{U}) \cap \mathrm{C}_{\mathrm{CL}_{n-2}(\overline{W})}(\overline{W}/\overline{U})$ on $\overline{W}$. Since $\dim U \geq 2$ we have $\overline{U} \neq 0$. It follows that either $L = \mathrm{O}_p(L)\langle D^L \rangle$ or $D \leq \mathrm{O}_p(L)$, $\mathrm{CL} = \mathrm{SL}$ and $U = U_{n-1}$.

In the first case $O_p(L)/Z$ is a perfect L-module and $Z \leq \Phi(O_p(L))$, so $L = \langle D^L \rangle$. Since D is quadratic on V and $Z \leq D$ we have $[V, Z, D] = 0$, and since $Z \leq \mathrm{Z}(L)$, this implies $[V, Z, \langle D^L \rangle] = 0$ and so $[V, Z, L] = 0$.

Now suppose $\mathrm{CL} = \mathrm{SL}$ and $U = U_{n-1}$, so $|D| = q^{n-1}$. Since $\dim U \geq 2$, $n \geq 3$. If $V/B \cong A^*$, then $|V/B/\mathrm{C}_{V/B}(D)| = q^{n-1} = |D|$, a contradiction to (4°). Thus $V/B \cong A$. Suppose for a contradiction that $A_0 \neq B$. Then by 6.1 $n = 3$ and $q = 2$. So $|D| = 4$. From

$$|V/B/\mathrm{C}_{V/B}(D)||B/C_B(D)| \leq |V/C_V(D)| \leq |D| = 4$$

we conclude that $|B/C_B(D)| = 2$. Since H ($\cong \mathrm{GL}_3(2)$) is generated by three conjugates of D, this gives $|B/C_B(H)| \leq 2^3 = |A_0|$. Hence $|A_0| < |B|$ implies $C_B(H) \neq 0$, which contradicts (2°).

Hence $A_0 = B$ and thus $|V/\mathrm{C}_V(D)| = q^2$. In particular $|[V, z]| = q^2$ for $1 \neq z \in Z$. Let $h \in H$ with $Z^h \leq L$, but $Z^h \not\leq D$. Note that $\mathrm{C}_V(D) + B/B = \mathrm{C}_{V/B}(Z)$ and $|[\mathrm{C}_A(D), z^h]| = q$. Since B and V/B are isomorphic to A we conclude that $|[\mathrm{C}_V(D), z^h]| = q^2$. Since $|[V, z]| = q^2$ we get $[V, z^h] = [\mathrm{C}_V(D), z^h] \leq \mathrm{C}_V(D)$, so $\langle D^{L^h} \rangle \leq \mathrm{C}_H([V, Z^h])$. In $\mathrm{C}_H([A, Z^h]) = \mathrm{C}_H(U_1^h) \sim q^{n-1}\mathrm{SL}_{n-1}(q)$ we see that $\langle D^{L^h} \rangle = \mathrm{C}_H(U_1^h)$. Since $L^h \leq \mathrm{C}_H(U_1^h)$, also $L^h \leq \langle D^{L^h} \rangle \leq \mathrm{C}_H([V, Z^h])$, and so $[V, Z^h, L^h] = 0$ and again (5°) holds.

Put $\widetilde{L} := \mathrm{C}_H([V/B, Z])$. Observe that $[V/B, Z]$ is a 1-dimensional $\mathbb{K}$-subspace of V/B and $S \leq L \leq \widetilde{L}$. Thus by (5°), $[V, Z] + B = \mathrm{C}_V(S) + B = Y^* \oplus B$ for some $Y^* \leq \mathrm{C}_V(S)$. By Gaschütz' Theorem there also exists a $\tilde{L}$-invariant complement Y to B in $B + \mathrm{C}_V(S)$, in particular $[Y, \tilde{L}] \leq Y \cap B = 0$. Let $W := \langle Y^H \rangle$ and $h \in H$.

6°. $[Y^h, Z] \leq Y$.

If $Z \leq \widetilde{L}^h$, then $[Y^h, Z] = 0$. So assume that $Z \not\leq \widetilde{L}^h$. Note that there exists $h^* \in H$ with $Y^h = Y^{h^*}$ and $T := \langle Z^{h^*}, Z \rangle \cong \mathrm{SL}_2(q)$. Without loss $h = h^*$. Put $X := \langle Y^T \rangle$. Then 6.4 and (5°) give

$$Y + \mathrm{C}_X(T) = Y + [Y^h, Z] \leq \mathrm{C}_V(L).$$

Note that T normalizes neither U_1 nor U_{n-1}, so T and L are not contained in a proper parabolic subgroup. Hence $H = \langle L, T \rangle$ and $\mathrm{C}_V(H) = 0$. Since $\mathrm{C}_X(T) \leq \mathrm{C}_V(L)$, this gives $\mathrm{C}_X(T) = 0$, and we conclude that $Y = [Y^h, Z]$.

From (6°) we get $[W, Z] = Y$. In particular $A \not\leq W$, and the lemma holds in (Case 1).

Case 2. *Suppose* $\mathrm{CL} = \Omega^\epsilon$ *or* O^ϵ.

7°. *If* $0 \neq \tilde{A}^{\perp} \leq U$, *then* $\dim U \geq 4$ *and* $n \geq 7$. *In the other cases* $\dim U \geq 5$ *and* $n \geq 10$.

Put $k := \dim U$. Suppose first that $0 \neq \tilde{A}^{\perp} \leq U$. By 3.4, $|D| \leq |E| \leq q^{\frac{k(k-1)}{2}}$ and $|A/\mathrm{C}_A(D)|^2 = |\tilde{A}/U^{\perp}|^2 \geq q^{2(k-1)}$. Thus by (4°) $\frac{k}{2} \geq 2$ and so $k \geq 4$.

Suppose next that $\tilde{A}^{\perp} = 0$ or $\tilde{A}^{\perp} \not\leq U$. By 3.4, $|D| \leq |E| \leq 2q^{\frac{k(k-1)}{2}} \leq q^{\frac{k(k-1)}{2}+1}$ and $|A/\mathrm{C}_A(D)|^2 = |\tilde{A}/U^{\perp}|^2 \geq q^{2k}$. Thus by (4°) $\frac{k(k-1)}{2} + 1 \geq 2k$, $k(k-5) \geq -2$ and $k \geq 5$.

By (7°), U contains a singular 2-space U_2. Put

$$Z := \mathrm{C}_H(\widetilde{A}/U_2),\ L := \mathrm{C}_{H'}(U_2),\ \text{ and } \overline{W} := U_2^{\perp}/U_2.$$

Then $|Z| = q$, Z is a long root subgroup of H in $\mathrm{Z}(L)$, and L induces $\Omega^{\epsilon}_{n-4}(\overline{W})$ on $\overline{W}$. Moreover, $\mathrm{C}_L(\overline{W}) = \mathrm{O}_p(L)$, and $\mathrm{O}_p(L)/Z$ is as an L-module the direct sum of two copies of $\overline{W}$. Let U_0 be the singular radical of U and $E_0 := \mathrm{C}_H(\tilde{A}/U_0)$. Then $Z \leq E_0$ and by 3.5, $E_0 \leq D$. In particular, $Z \leq D$. If $E \neq E_0$, we have $[\tilde{A}, E_0] = U_0 \neq U$ and so $E_0 < D$.

8°. $L = \langle D^L \rangle$.

From 3.5 and (7°) we see that D acts non-trivially on $\overline{W}$. Suppose $n \geq 9$. Then $n - 4 \geq 5$ and so $L/\mathrm{O}_p(L)$ is simple and $\overline{W} = [\overline{W}, L]$. It follows that $L = \langle D^L \rangle \mathrm{O}_p(L)$ and then $L = \langle D^L \rangle$.

So suppose $n < 9$. Then (7°) implies that $n = 7$, $0 \neq \tilde{A}^{\perp} \leq U$, $\dim U = 4$. By 3.4(e), $E/E_0 \cong U_0$, and since $E_0 < D \trianglelefteq P$, 3.5 implies that $D = E$. Thus $\mathrm{C}_H(U_2^{\perp}) \leq D$. Also $L/\mathrm{O}_p(L) \cong \mathrm{SL}_2(q)$ and so $L = \langle D^L \rangle \mathrm{O}_p(L)$. Since $\mathrm{O}_p(L)/\mathrm{C}_H(U^{\perp})$ is a direct sum of two copies of the natural $\mathrm{SL}_2(q)$-module $\overline{W}/\overline{W}^{\perp}$ we again get that $L = \langle D^L \rangle$.

9°. $[V, Z, L] = 0$.

This follows immediately from $[V, Z, D] = 0$ and (8°).

Note that we can can embed $[\tilde{A}, Z]$ in a non-degenerate subspace U_4 of $\tilde{A}$ of dimension 4. Put $K := \mathrm{O}^{p'}(\mathrm{N}_{H'}(U_4) \cap \mathrm{C}_{H'}(U_4^{\perp}))$, $\hat{L} := \mathrm{O}^{p'}(\mathrm{N}_H(Z))$, and let U_1 be a 1-subspace of U_2.

Then $Z \leq K$ and $K \cong \mathrm{O}^{p'}(\Omega_4^+(q)) \cong \mathrm{SL}_2(q) * \mathrm{SL}_2(q)$. Moreover $T^* := \langle Z^K \rangle \cong \mathrm{SL}_2(q)$. Since $\dim \tilde{A} \geq 7$, $\mathrm{N}_H(U_4)$ induces $\mathrm{O}_4^+(U_4)$ on U_4 and there exists $h \in \mathrm{N}_H(U_4) \cap \mathrm{N}_H(U_1)$ with $T := T^{*h} \neq T^*$. Then

$$K = TT^*, T \cong \mathrm{SL}_2(q), \hat{L} = TL, \text{ and } [T, T^*] = 1.$$

Note that $U_1 = U_2 \cap U_2^h = [\widetilde{A}, Z, Z^h] \neq 0$. Put $\widetilde{P} := \mathrm{N}_H(U_1)$, so $\widetilde{P}$ is the stabilizer of a 1-dimensional singular subspace of $\tilde{A}$.

Since $U_1 \neq 0$ also $V_1 := [V, Z, Z^h] \neq 0$. Note that V_1 is centralizes by LZ^h and thus by a Sylow p-subgroup of $\widetilde{P}$. Again Gaschütz' Theorem gives a $\widetilde{P}$-invariant complement Y to B in $B + V_1$.

Let $s \in T^* \setminus \mathrm{N}_{T^*}(Z)$. Then $U_1 + U_1^s$ is a singular 2-space normalized by T^* and $U_1^s \not\leq U_2^{\perp}$. Since $\mathrm{O}_p(L)$ is transitive on the singular 1-spaces of $U_2^{\perp} + U_1^s$ not contained in $U_2^{\perp}$, and T is transitive on $\widetilde{A}/U_2^{\perp}$, we get that TL is transitive on the conjugates of $\widetilde{P}$ that do not contain Z. As in the previous case, this gives

$$[\langle Y^H \rangle, Z] = [\langle Y^{sTL} \rangle, Z] = \langle [Y^s, Z]^T \rangle.$$

Observe that $\langle L, T^* \rangle = H$. Hence, 6.4 implies $\langle Y^{T^*} \rangle = Y + Y^s$. Since $U_1^h = U_1$ we have $Y^h = Y$. Hence also $\langle Y^T \rangle = Y + Y^{sh}$ since $T^h = T^*$, and so $[\langle Y^H \rangle, Z] = Y + Y^{sh}$. Then as in the previous case $[A_0, Z] \not\leq [\langle Y^H \rangle, Z]$, so $A \not\leq \langle Y^H \rangle$, and the lemma also follows in (Case 2). □

7 Quadratic Modules

In this section M is a finite group, and V is a finite dimensional $\mathbb{F}_pM$-module.

Lemma 7.1. *Let V be faithful. Suppose that p is odd, $A \leq M$ with $[V, A, A] = 0$, and R is an A-invariant p'-subgroup of M satisfying $R = [R, A] \neq 1$. Then $p = 3$ and R is a non-abelian 2-group. If in addition $|\Phi(R)| = 2$ and $|A| = 3$, then $RA \cong \mathrm{SL}_2(3)$.*

Proof. Observe that by coprime action for every prime divisor r of R there exists an A-invariant Sylow r-subgroup S_r in R. If $[S_r, A] \neq 1$ then [KS, 9.1.3] implies that $p = 3$, $r = 2$ and S_r is not abelian. It follows that $R = \mathrm{C}_R(A)S_2$ and so $R = [R, A] = [S_2, A] \leq S_2$.

Suppose now that $|\Phi(R)| = 2$ and $|A| = 3$. Then A acts fixed-point freely on $\overline{R} := R/\Phi(R)$. Since A centralizes $\mathrm{Z}(R)$, this gives $\mathrm{Z}(R) = \Phi(R)$ and R is an extraspecial 2-group. Assume that there exists an involution $t \in R \setminus \Phi(R)$. Then $F := \langle t^A \rangle$ has order at most 8. Since $|\overline{F}| = 4$ and F contains an involution, we conclude that F is abelian. But, as we have already seen, $[F, A]$ has to be non-abelian.

This contradiction shows that there are no involutions in $R \setminus \Phi(R)$, and so $R \cong Q_8$ and $RA \cong \mathrm{SL}_2(3)$. □

Lemma 7.2. *Let $p = 2$ and V be a faithful indecomposable M-module with $\mathrm{C}_V(M) = 0$ and $[V, M] = V$. Suppose that $M = \mathrm{Alt}(n)$, $n \geq 5$, and that $A = \langle (12)(34), (13)(24) \rangle$ acts quadratically on V. Then $\langle (123) \rangle$ acts fixed-point freely on V. Moreover, one of the following holds:*

1. *V is the (simple) spin module for M.*
2. *4 divides n and there exists an $\mathbb{F}_2M$-submodule in W such that W and V/W spin modules for M and $V/W \cong W^h$, where $h \in \mathrm{Sym}(n) \setminus \mathrm{Alt}(n)$.*

Proof. Let $E = \langle 123 \rangle$ and $B = AE \cong \mathrm{Alt}(4)$ and for $5 \leq i \leq n$ let $D_i = C_M(\{1, 2, 3, 4, i\})$. Then $B \leq D_i$, $D_i \cong \mathrm{Alt}(5)$ and

$$(*) \qquad M = \langle D_5, D_6, \ldots, D_n \rangle.$$

Suppose there exists $0 \neq w \in V$ with $[w, B] = 0$. Then $\langle w^{D_i} \rangle$ is a quotient of the natural permutation module for $D_i \cong \mathrm{Alt}(5)$ over $\mathbb{F}_2$, and the quadratic action of A forces $[w, D_i] = 0$. So by (*) $[w, M] = 0$, which contradicts $\mathrm{C}_V(M) = 0$.

Thus $\mathrm{C}_V(B) = 0$. Since $B/A \cong E$ is a $2'$-group,

$$\mathrm{C}_V(A) = \mathrm{C}_V(B) \oplus [\mathrm{C}_V(A), B] = [\mathrm{C}_V(A), B] = [\mathrm{C}_V(A), E],$$

and so E acts fixed-point freely on $\mathrm{C}_V(A)$. This result applied to the dual of V shows that E acts fixed-point freely on $V/[V, A]$. Since A is quadratic, $[V, A] \leq \mathrm{C}_V(A)$ and so E acts fixed-point freely on V. Now [Me, Theorem 2] shows that (1) or (2) holds. □

Corollary 7.3. *Let $p = 2$ and $M \cong \mathrm{Alt}(6)$. Suppose that all fours groups in M act quadratically on V. Then $[V, M] = 0$.*

Proof. Since $M = \mathrm{O}^2(M)$ we may assume for a contradiction that V is a non-trivial simple module. By 7.2, (123) acts fix-point freely on V. Since there exists an automorphism of $\mathrm{Alt}(6)$ sending (123) to (123)(456), the same results shows that (123)(456) acts fixpoint freely. So all non-trivial elements of order three in the non-cyclic abelian 3-group $\langle (123), (456) \rangle$ act fixed-point freely on V, a contradiction to coprime action. □

Lemma 7.4. *Let $p = 2$ and V be faithful and simple, and let $A \leq M$ with $[V, A, A] = 0$ and $|A| > 2$. Put $L := \mathrm{F}^*(M)$. Suppose that $M = \langle A^M \rangle$, L is quasisimple, $\mathrm{Z}(L) \neq 1$, and $L/\mathrm{Z}(L) \cong \mathrm{Alt}(n)$, $n \geq 5$. Then one of the following holds:*

1. *$M \sim 3.\mathrm{Alt}(6)$ and $|V| = 2^6$.*

2. *$M \sim 3.\mathrm{Alt}(7)$, $|V| = 2^{12}$, and $A\mathrm{Z}(L)/\mathrm{Z}(L)$ is conjugate to $\langle (12)(34), (13)(24) \rangle$.*

Proof. Since V is a faithful simple M-module, $\mathrm{O}_2(M) = \mathrm{O}_2(L) = 1$. From [Gr] we get that $n = 6$ or 7 and $|\mathrm{Z}(L)| = 3$. Put $Z := \mathrm{Z}(L)$ and let $\mathbb{F}$ be the subring of $\mathrm{End}(V)$ generated by the image of Z in $\mathrm{End}(V)$. Then $\mathbb{F}$ is a field of order four and M acts semilinear on the $\mathbb{F}$-module V. Now $[V, A, A] = 0$ and $|A| > 2$ imply that A acts $\mathbb{F}$-linearly on V, see for example [MS3, 2.15]. Thus $[Z, A] = 1$ and $Z = \mathrm{Z}(M)$. Hence $M = L$ or $M/Z \cong \mathrm{Mat}_{10}$. But $M = \langle A^M \rangle$ is generated by involutions while Mat_{10} is not, so $M = L$. Since A is elementary abelian and $|A| > 2$ we have $|A| = 4$.

Note that there are two conjugacy classes of fours groups in L. In any case we can choose a series of subgroups $A \leq B \leq D \leq H \leq L$ with $B \cong \mathrm{Alt}(4)$, $D \cong \mathrm{Alt}(5)$ and $H \sim 3.\mathrm{Alt}(6)$. Let $E \in \mathrm{Syl}_3(B)$. Then $E \cong \mathrm{C}_3$ and $B = AE$. By Gaschütz' Theorem, the Sylow 3-subgroups of L are not abelian and so the subgroups $E = E_1$, E_2, E_3 of order three in EZ other than Z are conjugate. Since Z acts fixed-point freely on V we have $V = [V, Z] = \bigoplus_{i=1}^{3} \mathrm{C}_V(E_i)$ and so $|V| = |\mathrm{C}_V(E)|^3$. In particular, $\mathrm{C}_V(E) \neq 0$.

We claim that $\mathrm{C}_V(B) \neq 0$ or $[V, B] \neq V$. If $\mathrm{C}_V(E) \leq \mathrm{C}_V(A)$, then $0 \neq \mathrm{C}_V(E) \leq \mathrm{C}_V(B)$. So suppose $\mathrm{C}_V(E) \not\leq \mathrm{C}_V(A)$ and put $\overline{V} = V/\mathrm{C}_V(A)$. Then $0 \neq \overline{\mathrm{C}_V(E)} \leq \mathrm{C}_{\overline{V}}(E)$. By coprime actions, $\overline{V} = \mathrm{C}_{\overline{V}}(E) \oplus [\overline{V}, E]$ and so $\overline{V} \neq [\overline{V}, E]$. Since A centralizes $\overline{V}$, this give $\overline{V} \neq [\overline{V}, B]$ and so $V \neq [V, B]$, proving the claim. Note further that by 1.8(d) A is also quadratic on the dual module V^*. So replacing V by its dual, if necessary, we may assume that $\mathrm{C}_V(B) \neq 0$.

Let W be 1-dimensional $\mathbb{F}$-subspace of $\mathrm{C}_V(B)$. Then $\langle W^D \rangle$ is a quotient of the natural permutation module for $D \cong \mathrm{Alt}(5)$ over $\mathbb{F}$. The quadratic action of A forces $[W, D] = 0$. Put $U = \langle W^H \rangle$. Then $U \cong \hat{V}/\hat{X}$, where $\hat{V}$ is the $\mathbb{F}H$-module induced from the $\mathbb{F}ZD$-module W and $\hat{X}$ is a $\mathbb{F}H$-submodule of $\hat{V}$. Note that $\dim_{\mathbb{F}} \hat{V} = 6$. Since A has a regular orbit on H/ZD, A does not act quadratically on $\hat{V}$. Thus $U \neq \hat{V}$. Since H acts faithfully on $\hat{V}/\hat{X}$ and on $\hat{X}$ and since H has no faithful module of dimension less than 3, we conclude that $\dim_{\mathbb{F}} \hat{V}/\hat{X} = 3 = \dim_{\mathbb{F}} \hat{X}$.

If $n = 6$, then $H = L$, $V = U$ and (1) holds. So suppose that $n = 7$. Choose a transitive action of L on $I := \{1, \ldots, 7\}$. Suppose first that A has an orbit J on I with $|J| = 2$. Put $K := \mathrm{C}_L(J)'$. Then $K \cong \mathrm{Alt}(5)$ and $AK \cong \mathrm{Sym}(5)$. Note that K is contained in a conjugate of H and that all composition factors for $\mathbb{F}H$ on V are 3-dimensional. It follows that all non-trivial composition factor for $\mathbb{F}K$ on V are 2-dimensional. Since $A \cap K \neq 1$, the quadratic action of A in V shows that also the non-trivial composition factors for $\mathbb{F}KA$ on V are 2-dimensional, a contradiction since $|KA| > |K| = |\mathrm{SL}_2(4)|$.

Thus A has no orbits of length 2 and so A has three fixed-points on I. Then D has two fixed-points, say i and j. Put $D^* := \mathrm{O}^{2'}(\mathrm{N}_L(\{i, j\})$. Then $D^* \cong \mathrm{Sym}(5)$ and $D \trianglelefteq D^*$. Recall from above that W is a 1-dimensional subspace of $\mathrm{C}_V(D)$, so $\mathrm{C}_V(D) \neq 0$ and thus also $\mathrm{C}_V(D^*) \neq 0$. Hence we may and do choose W such that $[W, D^*] = 0$. For $k \neq l \in I$ and $g \in G$ with $\{k, l\} = \{i, j\}^g$ put $W_{kl} = W_{lk} = W^g$. Since $\mathrm{N}_L(\{i, j\}) = ZD^* \leq \mathrm{N}_L(W)$, W_{kl} is well-defined. Let i be the fixed-point of H. Since $\langle W^H \rangle$ is 3-dimensional and H acts triple transitively on $\{W_{ij} \mid j \in I \setminus i\}$ we conclude that for any distinct $a, b, c, d \in I$, $\langle W^H \rangle = W_{ab} + W_{ac} + W_{ad}$. Since $V = \langle W^L \rangle$ is now easy to see that $V = \langle W_{kl} \mid 1 \leq k < l \leq 4 \rangle$. Thus V is at most 6-dimensional. By the action of H on V, $\dim_{\mathbb{F}} V$ is a multiple of 3, so $\dim_F V = 3$ or 6. Since $\frac{|\mathrm{L}_3(4)|}{|\mathrm{Alt}(7)|} = 8$ and $\mathrm{L}_3(4) \not\cong \mathrm{Alt}(8)$, $\mathrm{Alt}(7)$ is not involved in $\mathrm{L}_3(4)$. We conclude that $\dim_{\mathbb{F}} V > 3$ and so $\dim_{\mathbb{F}} V = 6$, and (2) holds. □

We remark that 3.Alt(7) has indeed a 6-dimensional quadratic module over $\mathbb{F}_4$. One way to see this is to use the embedding $3.\mathrm{Alt}(7) \leq 3.\mathrm{Mat}_{22} \leq \mathrm{SU}_6(2)$ (thanks to J. Hall for pointing out this embedding to us): Consider the block normalizer $P \sim 3.2^4.\mathrm{Alt}(6)$ in $3.\mathrm{Mat}_{22}$. Then P has a unique proper submodule on $\mathbb{F}_4^6$, namely a 3-dimensional one. In particular, $\mathrm{O}_2(P)$ acts quadratically. Alt(7) has orbits of length 7 and 15 on the 22 points. Any three points from the 7 lie in a unique block and so we can choose P to intersect 3.Alt(7) in $B \sim 3.(\mathrm{Alt}(4) \times \mathrm{Alt}(3)).2$. It follows that $\mathrm{O}_2(B) \leq \mathrm{O}_2(P)$ and so $\mathrm{O}_2(B)$ is a quadratic fours group.

Lemma 7.5. *Let* $M = \mathrm{Alt}(n)$ *or* $\mathrm{Sym}(n)$, $n \geq 5$, $n \neq 6, 8$, *and* V *be a simple spin module for* $\mathbb{F}_2M$. *Suppose that* A *is a maximal quadratic subgroup of* M *on* V *with* $|A| > 2$. *Then* $|V| = |\mathrm{C}_V(A)|^2$ *and* $[V, a] = [V, A] = \mathrm{C}_V(A) = \mathrm{C}_V(a)$ *for all* $1 \neq a \in A$. *Moreover, one of the following holds:*

1. *A is conjugate to* $\langle (12)(34), (13)(24)\rangle$.
2. $M \cong \mathrm{Alt}(9)$, $|A| = 8$, $|A|$ *has a regular orbit of length 8 on* $\{1, 2, \ldots, 9\}$ *and, up to conjugation, A is unique in M, with the conjugacy class depending on the isomorphism type of V.*

Proof. Let $I = \{1, 2, \ldots, n\}$ with M acting transitively on I. Let $K \leq M$ with $K \cong \mathrm{Alt}(5)$ and K fixing $n-5$ points of I. Then V is a direct sum of natural $\mathrm{SL}_2(4)$-modules. From this we get for $B \in \mathrm{Syl}_2(K)$: B is a quadratic fours group, and

$$|V| = |\mathrm{C}_V(B)|^2 \text{ and } [V, b] = [V, B] = \mathrm{C}_V(B) = \mathrm{C}_V(b) \text{ for all } 1 \neq b \in B.$$

Moreover, the non-trivial elements of odd order in K act fixed-point-freely on V.

Let $1 \neq z \in B$ and let D be a quadratic subgroup with $z \in D$. Then $\mathrm{C}_V(B) = \mathrm{C}_V(z) = \mathrm{C}_V(D)$ and so DB is quadratic. In particular, DB is elementary abelian.

Let W be a simple $\mathbb{F}_2M'$-submodule of V. Since $A \cap M' \neq 1$, then $0 \neq [W, A \cap M'] \leq \mathrm{C}_W(A)$. Thus A normalizes W.

If $n = 5$ or 7 then all involutions in M' are conjugate. Thus we may assume that $z \in A$. If $n = 5$, then $A \leq \mathrm{C}_M(B) = B$. If $n = 7$, then Sym(7) does not act on W and so $A \leq M'$. Also B is a Sylow 2-subgroup of $\mathrm{C}_{M'}(B)$ and again $A \leq B$. So the lemma holds for $n = 5$ and 7.

Suppose next that $n \geq 9$. As in Section 4 of [MeSt2] define $L_z := \mathrm{O}^2(\mathrm{C}_M(z))$ and $A_z := \mathrm{O}_2(\mathrm{C}_L(z))$. Moreover, for $t \in M$ with $|t| = 2$ let K_t be the subgroup generated by the quadratic subgroups of M containing t. Observe that $[V, t, K_t] = 0$, so every fours group of K_t containing t is quadratic on V. Note further that $A_z = B$ and $L_z \cong \mathrm{Alt}(n-4)$.

According to [MeSt2, Lemma (4.3)] we have that $L_z \not\leq K_z$. Since $K_z \trianglelefteq \mathrm{C}_M(z)$ and L_z is simple this implies $[L_z, K_z] = 1$. Since $B = \mathrm{C}_M(L_z)$ we conclude that $K_z \leq B$.

If $z \in A$ we conclude that $A = B$, and case (1) of the lemma holds. So suppose $z^M \cap A = \emptyset$. Let $1 \neq a \in A$. Then $A \leq K_a$. If $z \in K_a$, then by the above observation, $a \in K_z = B$ and so $a \in z^M$, contrary to the assumption. Thus $z^M \cap K_a = \emptyset$.

Let $k := |\mathrm{C}_I(a)|$, $J = I \setminus \mathrm{C}_I(a)$ and $m := \frac{|J|}{2}$. We now choose $1 \neq a \in M' \cap A$ and so m is even and $m \geq 4$. Let D be the largest subgroup of M' which has the same orbits as a on I. Put $X = \mathrm{C}_M(I \setminus J)$ and $Y = \mathrm{C}_M(J)$. Then D is elementary abelian of order 2^{m-1} and $Y \leq \mathrm{C}_M(a)$. Suppose that $Y \cap A \neq 1$ and let $1 \neq b \in A \cap Y$. Then $\mathrm{Alt}(J) \cong \langle a^{\mathrm{C}_M(b)}\rangle \leq K_b$ and $z^M \cap K_b \neq 1$, a contradiction. Thus $A \cap Y = 1$ and $A \not\leq \langle a\rangle Y$. In particular, $K_a \not\leq \langle a\rangle Y$. Since $D \cap z^M \neq \emptyset$ we have $D \not\leq K_a$. Also $D = [D, X] = [DY, X]$ and so $D \not\leq K_aY$ and $DY \cap K_aY = \langle a\rangle Y$.

Hence $DY/\langle a\rangle Y$ is not the only minimal normal subgroup of $\mathrm{C}_M(a)/\langle a\rangle Y$. Since

$$\mathrm{C}_M(a)/\langle a\rangle Y \sim 2^{m-1}\mathrm{Sym}(m) \text{ or } 2^{m-2}\mathrm{Sym}(m)$$

(with $k \leq 1$ and $M = \mathrm{Alt}(n)$ in the latter case) we conclude that $m = 4$, $\mathrm{C}_M(a)/\langle a\rangle Y \sim 2^2\mathrm{Sym}(4)$ and $M \cong \mathrm{Alt}(9)$. Moreover, $|K_a/\langle a\rangle| = 4$ and $\mathrm{C}_M(a)$ acts transitively on $(K_a/\langle a\rangle)^\sharp$. Thus K_a is elementary abelian of order 8 and since $K_a \cap z^M = \emptyset$, K_a acts regularly on J. It follows that $\mathrm{N}_M(K_a)$ acts transitively on $K_a^\sharp$. Since $[V, a, K_a] = 0$ we conclude that K_a acts quadratically on V. Thus $A = K_a$ by the maximality of A. In particular, A is unique up to conjugacy. Also if $t \in \mathrm{C}_{\mathrm{Sym}(9)}(a) \setminus \mathrm{Alt}(8)$, then $A^t \neq A = K_a$. So A^t will not act quadratically on V, and A^M depends on the isomorphism type of V. Let $F \in \mathrm{Syl}_5(K)$. As seen above F acts fixed-point freely on V, and F is inverted by a conjugate of a. Thus $\mathrm{C}_V(a) = [V, a]$ and the quadratic action of A forces $\mathrm{C}_V(a) = [V, A] = \mathrm{C}_V(A)$; in particular $|V| = |\mathrm{C}_V(a)|^2$. □

Lemma 7.6. *Let $M = \mathrm{G}_2(2)$ or $\mathrm{G}_2(2)'$, and let V be a non-trivial simple $\mathbb{F}_2M$-module. Suppose there exists $A \leq M$ with $|A| > 2$ and $[V, A, A] = 0$. Then V is a natural $\mathrm{G}_2(2)$- and $\mathrm{G}_2(2)'$-module, respectively.*

Proof. Since $|A| > 2$, there exists $1 \neq z \in A \cap M'$, and since M' has a unique class of involutions, z is 2-central. Put $P_1 := \mathrm{C}_M(z)$, let $S \in \mathrm{Syl}_2(P_1)$, and let P_2 be the other minimal parabolic subgroup containing S. Suppose for a contradiction that $\mathrm{C}_V(P_2) = 0$.

Let $\Gamma = P_1^G \cup P_2^G$ be the generalized hexagon associated to M. Let (P_1, P_2, P_3, P_4) be a path of length 4 in Γ. Put $Z := \langle z\rangle$. Then

$$Z \leq P_4,\ Z \not\leq \mathrm{O}_2(P_4),\ T := Z\mathrm{O}_2(P_4) \in \mathrm{Syl}_2(P_4),\ \text{and } P_4 = \langle Z^{P_4}\rangle \mathrm{O}_2(P_4).$$

Since $\mathrm{C}_V(P_2) = 0$ and P_2 and P_4 are conjugate, we also have $\mathrm{C}_V(P_4) = 0$, so

$$X := [\mathrm{C}_V(\mathrm{O}_2(P_4)), Z] \neq 0.$$

Note that T centralizes X, and since T is a maximal subgroup of P_4, $\mathrm{C}_{P_4}(X) = T$. Since P_4 and P_3 are the only maximal subgroups of M containing T, it follows that $\mathrm{C}_M(X) \leq P_3$. From $Z \leq A$ and $[V, A, A] = 0$ we get $A \leq \mathrm{C}_M(X) = P_3$. So A fixes all vertices of distance two from P_1. But the stabilizer in P_1 of these vertices is cyclic, a contradiction since $|A| > 2$ and A is elementary abelian.

Thus $\mathrm{C}_V(P_2) \neq 0$. Let $M \leq M^*$ with $M^* \cong \mathrm{G}_2(2)$, and let V^* be a simple quotient of the induced $\mathbb{F}_2M^*$-module V^{M^*} and identify V with its image in V^*. Let $S^* \in \mathrm{Syl}_2(M^*)$ with $S \leq S^*$. Put $P_i^* = P_iS^*$. Since $|P_2^*/P_2| \leq 2$ we get that $\mathrm{C}_{V^*}(P_2^*) \neq 0$. By Smith's lemma 4.2 $V_i := \mathrm{C}_{V^*}(\mathrm{O}_2(P_i^*))$ is a simple P_i^*-module. It follows that $V_2 = \mathrm{C}_V(P_2^*) = \mathrm{C}_V(S^*)$ has order two, $\mathrm{C}_{V^*}(P_1^*) = 0$, and V_1 is the unique non-trivial simple $P_1^*/\mathrm{O}_2(P_1^*)$-module, namely the natural $\mathrm{SL}_2(2)$-module. Thus by Ronan-Smith's Lemma 4.3 V^* is uniquely determined, and so V^* is the natural $\mathrm{G}_2(2)$-module for M^*. Hence $V = V^*$ and the lemma is proved. □

Remark 7.7. *Let $L := \mathrm{F}^*(M)$ and suppose that $\mathrm{O}_2(M) = 1$, L is quasisimple and $L/\mathrm{Z}(L) \cong \mathrm{U}_4(3)$. Let $\overline{M} = M/\mathrm{Z}(L)$, $S \in Syl_2(M)$, and $Z = \Omega_1\mathrm{Z}(S)$. In the following we use some information about the structure of M which can be found for example in [ATLAS]. More precisely we use the following facts:*

There exists exactly two elementary abelian subgroups Q_1 and Q_2 of order 2^4 in S, and for

$$P_1 = \mathrm{C}_L(Z),\ Q_1 := \mathrm{O}_2(P_1),\ P_2 := \mathrm{N}_L(Q_2),\ \textit{and } P_3 := \mathrm{N}_L(Q_3)$$

the following hold:

(a) For $i = 1, 2, 3$, $\overline{P}_i$ is a maximal subgroup of $\overline{M}$ and has characteristic 2.

(b) $\overline{P}_1/\overline{Q}_1 \cong \mathrm{Sym}(3) \times \mathrm{Sym}(3)$, Q_1 is extraspecial of order 2^5, and Q_1/Z is a simple P_1-module.

(c) For $i = 1, 2$, $\overline{P}_i/\overline{Q}_i \cong \mathrm{Alt}(6)$, and Q_i is a natural $\mathrm{Alt}(6)$-module for P_i.

(d) All involutions in L are conjugate.

(e) Suppose in addition that $|\mathrm{Z}(L)| = 3$, $M \neq L$, $[\mathrm{Z}(L), M] = 1$, $M = \mathrm{N}_M(Q_2)L$, and that $\mathrm{N}_M(Q_2)$ induces inner automorphisms on $\overline{P_2/Q_2}$. Put $P_i^* = \mathrm{N}_M(Q_i)$ and $Q_i^* = \mathrm{O}_2(P_i^*)$. Then

   (a) M is unique up to isomorphism and $|M/L| = 2$.

   (b) M has two classes of involutions in $M \setminus L$ with representatives a and b in Q_2 such that $\mathrm{C}_{\overline{L}}(a) \cong \mathrm{U}_4(2)$ and $\mathrm{C}_{\overline{L}}(b) \sim 2^4.3^2.2^2$.

   (c) $P_2^*/Q_2^* \cong 3{}^\cdot\mathrm{Alt}(6)$, and Q_2^* is the dual of the natural $\Omega_5(2)$-module for P_2^*.

   (d) $Q_3^* = Q_2$ and $P_2^*/Q_2 \cong \mathrm{C}_3 \times \mathrm{Sym}(6)$.

Lemma 7.8. *Let $p = 2$ and V be faithful $\mathbb{F}_2M$-module, and let $Z \leq M$ with $|Z| = 2$. Suppose that*

(i) M is quasisimple, $\mathrm{O}_2(M) = 1$ and $M/\mathrm{Z}(M) \cong \mathrm{U}_4(3)$.

(ii) $\mathrm{C}_M([V, Z]) \not\leq Z$.

(iii) $\mathrm{C}_V(M) = 0$, $V = [V, M]$ and V is indecomposable, that is, V is not the sum of two proper (non-zero) $\mathbb{F}_2M$-submodules.

Put $P_1 := \mathrm{N}_M(Z)$ and $Q_1 := \mathrm{O}_2(P_1)$, and let $S \in \mathrm{Syl}_2(P_1)$ and Q_i, $i = 2, 3$, be the two elementary abelian subgroup of order 16 in S. Put $P_i := \mathrm{N}_M(Q_i)$, $L_i := \mathrm{O}^{2'}(P_i)$, $L_{12} := \langle Q_3^{P_1}\rangle$, $L_{13} := \langle Q_2^{P_1}\rangle$, and $\mathbb{F} := \mathrm{End}_M(V)$. Then we can choose $\{i, j\} = \{2, 3\}$ such that the following hold :

(a) V is a simple M-module, $|\mathbb{F}| = 4$ and $\dim_{\mathbb{F}} V = 6$.

(b) $C_V(L_i) = 0$ and $C_V(L_j) \neq 0$.

(c) V is uniquely determined as a $\mathbb{F}_2M$-module.[3]

(d) There exists a non-degenerate M-invariant unitary $\mathbb{F}$-form on V.

(e) $Q_1 \leq L_{1k}$, $L_{1k}/Q_1 \cong \mathrm{Sym}(3)$, $k = 2, 3$, and $L_1/Q_1 = L_{12}/Q_1 \times L_{13}/Q_1 \cong \mathrm{Sym}(3) \times \mathrm{Sym}(3)$.

(f) $L_{1j} = \mathrm{C}_M([V, Z])$, $\mathrm{C}_V(Z) = [V, Q_1] = [V, L_{1j}]$ and $[V, Z] = \mathrm{C}_V(Q_1) = \mathrm{C}_V(L_{1j})$.

(g) $1 \leq [V, Z] \leq \mathrm{C}_V(Z_1) \leq V$ is the unique chiefseries for P_1 on V, each of the factors is 2-dimensional over $\mathbb{F}$, L_{1i} centralizes $\mathrm{C}_V(Z)/[V, Z]$ and L_{1j} centralizes $[V, Z]$ and $V/[V, Z]$.

(h) $P_i = L_i$ and L_i/Q_i is quasisimple of shape $3{}^\cdot\mathrm{Alt}(6)$.

(i) Q_i acts quadratically on V and $\mathrm{C}_V(Q_i) = [V, Q_i]$.

(j) $1 \leq [V, Q_i] \leq V$ is the unique chiefseries for P_i on V, each of the factors is 3-dimensional over $\mathbb{F}$ and faithful for P_i/Q_i. Moreover, $V/[V, Q_i]$ is as an $\mathbb{F}_2P_i$-module isomorphic to the dual of $[V, Q_i]$.

(k) L_j/Q_j is isomorphic to $\mathrm{Alt}(6)$.

[3] Note that $3^2{}^\cdot\,\mathrm{U}_4(3)$ has two quotients isomorphic to M and so has two modules which fulfill the hypothesis of this lemma, except that the modules are not faithful.

(l) $\mathrm{C}_V(S) = \mathrm{C}_V(Q_j) = \mathrm{C}_V(L_j)$ *and* $[V, S] = [V, Q_j] = [V, L_j]$.

(m) $1 \le \mathrm{C}_V(Q_j) \le [V, Q_j] \le V$ *is the unique chiefseries for* P_j *on* V*, where* $\mathrm{C}_V(Q_j)$ *and* $V/[V, Q_j]$ *are* 1*-dimensional over* $\mathbb{F}$ *and centralized by* L_j *while* $[V, Q_j]/\mathrm{C}_V(Q_j)$ *is a 4-dimensional natural* $\mathbb{F}\mathrm{Alt}(6)$*-module for* L_j.

Proof. Let $\overline{M} := M/\mathrm{Z}(M)$, $\{k, l\} = \{2, 3\}$ and $P_{1k} := P_1 \cap P_k$.

1°. *V is an homogeneous $\mathbb{F}_2\mathrm{Z}(M)$-module and $\mathrm{Z}(M)$ is cyclic.*

Since $\mathrm{O}_2(M) = 1$, $\mathrm{Z}(M)$ is an abelian $2'$-group. Thus V is a semisimple $\mathbb{F}_2\mathrm{Z}(M)$-module. Since V is indecomposable, we conclude that V is an homogeneous $\mathbb{F}_2\mathrm{Z}(M)$ module and so $\mathrm{Z}(M)$ is cyclic. Thus (1°) holds.

In the following we will only use (1°) but no longer that V is indecomposable. Moreover, we make use of the properties listed in 7.7.

2°. *$[V, Z, Q_1] = 0$.*

By (1°) $\mathrm{Z}(M) \cap \mathrm{C}_M([V, Z]) = 1$ and so by (ii) $\overline{\mathrm{C}_M([V, Z])} \not\le \overline{Z}$. Note that $\overline{P_1}/\overline{Q_1} \cong \mathrm{Sym}(3) \times \mathrm{Sym}(3)$, $\overline{Q_1}$ is extra special of order 2^5 and $\overline{P_1}$ acts simply on $\overline{Q_1}/\overline{Z}$. Hence $\overline{Q_1}/\overline{Z}$ is the unique minimal normal subgroup of $\overline{P_1}$ and we conclude that $\overline{Q_1} \le \overline{\mathrm{C}_M([V, Z])}$. Thus $Q_1 \le \mathrm{C}_M([V, Z])$ and (2°) holds.

3°. *$[V, Q_k, Q_k, L_k] = 1$.*

Observe that $\overline{P_k}/\overline{Q_k} \cong \mathrm{Alt}(6)$, $\mathrm{C}_{\overline{M}}(\overline{Q_k}) = \overline{Q_k}$ and $\overline{Q_k}$ is a natural $\mathrm{Alt}(6)$-module for $\overline{P_k}$. Since $P_{1k} = \mathrm{N}_{P_k}(Z)$ we conclude that $\overline{P_{1k}}/\mathrm{O}_2(\overline{P_{1k}}) \cong \mathrm{Sym}(3)$ and $[Q_k, P_{1k}]$ is a hyperplane of Q_k. The structure of P_1 shows that $[\mathrm{O}_2(P_{1k}), P_{1k}] \le Q_1$ and so $[Q_k, P_{1k}] \le Q_1$ and $|Q_k/Q_k \cap Q_1| \le 2$. In particular, P_{1k} normalizes $[V, Z, Q_1Q_k]$, and by (2°) $[V, Z, Q_1Q_k] = [V, Z, Q_k]$.

Note that Q_1 does not contain an elementary abelian subgroup of order 2^4. So $Q_k \not\le Q_1$ and $Q_1 \cap Q_k = [Q_k, P_{1k}]$. Pick $g \in P_k$ with $Q_k = (Q_1 \cap Q_k)Z^g$. Then by (2°)

$$[V, Z, Q_k] = [V, Z, (Q_1 \cap Q_k)Z^g] = [V, Z, Z^g] \le [V, Z^g] \le \mathrm{C}_V(Q_1^g).$$

It follows that $[V, Z, Q_k]$ is normalized by $\langle P_{1k}, Q_k^g \rangle = P_k$. Thus $[V, Z, Q_k] = [V,$
$< Z^{P_k} \rangle, Q_k] = [V, Q_k, Q_k]$ and $[V, Q_k, Q_k]$ is centralized by $\langle Q_1^{gP_k} \rangle = L_k$.

4°. *$[\mathrm{C}_V(Q_k), Q_1, Q_1] = 0$.*

Let $h \in P_1 \setminus P_1 \cap P_k$. Then $Q_1 = (Q_1 \cap Q_k)(Q_1 \cap Q_k^h)$. Since Q_1 normalizes $C_V(Q_k)$, (3°) implies

$$[\mathrm{C}_V(Q_k), Q_1, Q_1] = [\mathrm{C}_V(Q_k), (Q_1 \cap Q_k^h), (Q_1 \cap Q_k^h)] \le \mathrm{C}_V(Q_k) \cap [V, Q_k, Q_k]^h \le \mathrm{C}_V(Q_k) \cap \mathrm{C}_V(L_k^h).$$

Since $\overline{L_k}$ is a maximal subgroup of $\overline{M}$ and $Q_k \not\le L_k^h$ we have $M = \langle Q_k, L_k^h \rangle$. So

$$\mathrm{C}_V(Q_1) \cap \mathrm{C}_V(Q_k) \le \mathrm{C}_V(M) = 0,$$

and (4°) is proved.

In the next step we regard Q_k is a 4-dimensional symplectic space for $\overline{L}_k/\overline{Q}_K \cong \mathrm{Sp}_4(2)'$.

5°. *$|Q_kQ_l/Q_k| = 4$ and $Q_kQ_l \ne Q_kQ_1$. Moreover, $Q_k \cap Q_l$ is a singular subgroup of order 4 in Q_k (and Q_l), and $Q_k \cap Q_l$ acts quadratically on V.*

Since Q_l is elementary abelian of order 2^4 and no element in L_k acts as a transvection on Q_k,

$$|Q_kQ_l/Q_k| = |Q_l \cap Q_k| = 4,\ Q_k \cap Q_l = [Q_k, Q_l] = C_{Q_k}(Q_1).$$

Hence 3.2(c) shows that $Q_l \cap Q_k$ is a singular subspace of Q_k. Moreover, $Z \le Q_k \cap Q_l \le Q_k \cap Q_1$ and so by (2°), $[V, Z, Q_k \cap Q_l] = 1$. Since $|Q_k \cap Q_l| = 4$ and $Z \le Q_k \cap Q_l$, this shows that $Q_k \cap Q_l$ is quadratic on V, and (5°) holds.

6°. $[\mathrm{C}_V(Q_k), Q_l, Q_l] = 1$

By (5°) $Q_l = (Q_l \cap Q_k)(Q_l \cap Q_k)^g$ for a suitable $g \in P_l$ and $(Q_l \cap Q_k)^g$ acts quadratically on V. Thus

$$[\mathrm{C}_V(Q_k), Q_l, Q_l] = [\mathrm{C}_V(Q_k), (Q_l \cap Q_k)^g, (Q_l \cap Q_k)^g] = 0,$$

and (6°) holds.

Since $\mathrm{C}_V(M) = 0$, $M = \langle L_2, L_3\rangle$ and $\mathrm{C}_V(S) \le \mathrm{C}_V(Q_2) \cap \mathrm{C}_V(Q_3)$ we can choose $i \in \{2,3\}$ such that $[\mathrm{C}_V(Q_i), L_i] \ne 0$. Let $\{2,3\} = \{i,j\}$.

7°. $P_i = L_i$, $\mathrm{Z}(M) = \mathrm{Z}(L_i) \cong C_3$. *$L_i/Q_i$ is quasisimple of shape* $3{}^{\cdot}\mathrm{Alt}(6)$ *and* $\mathrm{C}_V(L_i) = 0$.

By (4°),(5°), (6°) all the fours groups in L_i/Q_i act quadratically on $\mathrm{C}_V(Q_i)$. Since $[\mathrm{C}_V(Q_i), L_i] \ne 0$, 7.3 shows that $L_i/Q_i \not\cong \mathrm{Alt}(6)$. Hence $\mathrm{Z}(M) \cap L_i \ne 1$. By [Gr] and since $\mathrm{Z}(M)$ is a cyclic $2'$-group, $\mathrm{Z}(M) \cong C_3$ and so $\mathrm{Z}(M) \le L_i$. So $P_i = L_i$, and $\mathrm{C}_V(L_i) \le \mathrm{C}_V(\mathrm{Z}(M)) = 0$. Thus L_i/Q_i is quasisimple of shape 3.Alt(6), and (7°) is proved.

In particular, (h) holds.

8°. *Q_i acts quadratically on V.*

By (3°) and (7°), $[V, Q_k, Q_k] \le \mathrm{C}_V(L_k) = 0$.

9°. $[\mathrm{C}_V(Q_i), Q_j] \le \mathrm{C}_V(L_j) = C_V(Q_j)$ *and* $L_j/Q_j \cong \mathrm{Alt}(6)$*; in particular* $C_V(L_j) \ne 0$.

Let $g \in L_j$ with $Z^g \not\le Q_i \cap Q_j$. Then $Z^g \le L_i$ and $Z^g \not\le Q_i$. Since L_i/Q_i is quasisimple, $L_i = \langle Z^{gL_i}\rangle Q_i$ and so $[\mathrm{C}_V(Q_i), Z^g] \ne 0$. On the other hand $[\mathrm{C}_V(Q_i), Z^g]$ is centralized by $\langle Q_i, Q_1^g\rangle = L_j$ and we conclude that $0 \ne [\mathrm{C}_V(Q_i), Q_j] \le \mathrm{C}_V(L_j)$. In particular, $Z(M) \not\le L_j$ and so $L_j/Q_j \cong \mathrm{Alt}(6)$.

Thus $\mathrm{C}_V(L_j) \ne 0$. If $[\mathrm{C}_V(Q_j), L_j] \ne 0$ we could apply (7°) to j in place of i and conclude that $\mathrm{C}_V(L_j) = 0$, a contradiction. Thus $[\mathrm{C}_V(Q_j), L_j] = 0$ and (9°) holds.

In particular, (k) holds. Since $\mathrm{C}_V(L_j) \ne 0$, (b) is proved.

10°. $V = \langle \mathrm{C}_V(L_j)^M\rangle$.

By (9°) $[\mathrm{C}_V(Q_i), Q_j] \le \mathrm{C}_V(L_j)$. It follows that

$$[\mathrm{C}_V(Q_i), L_i] = [\mathrm{C}_V(Q_i), \langle Q_j^{L_j}\rangle] \le \langle \mathrm{C}_V(L_j)^{L_i}\rangle.$$

On the other hand, by (7°) $\mathrm{Z}(M) \le \mathrm{Z}(L_i)$, so by (1°) L_i does not have any central chieffactor in $\mathrm{C}_V(Q_i)$. Hence $\mathrm{C}_V(Q_i) = \langle \mathrm{C}_V(L_j)^{L_i}\rangle$.

Since $V = [V, M]$ and $M = \langle Q_i^M\rangle$, $V = \langle [V, Q_i]^M\rangle$. As Q_i acts quadratically we conclude that $V = \langle C_V(Q_i)^M\rangle$, and as $\mathrm{C}_V(Q_i) = \langle \mathrm{C}_V(L_j)^{L_i}\rangle$, this gives (10°).

11°. $\mathrm{C}_V(L_1) = 0$.

By (9°) $\mathrm{C}_V(L_1) \le \mathrm{C}_V(L_j)$. Since $\mathrm{C}_V(M) = 0$ and $M = \langle L_1, L_j\rangle$, (11°) follows.

12°. $[V,Z,L_{1j}]=0$, $L_{1k}Q_k=O^{2'}(P_1\cap P_k)$*, and (e) holds.*

Put $P^* := \mathrm{C}_{P_1}([V,Z])$. Since P_1 normalizes $[V,Z]$, $P^*\trianglelefteq P_1$. Moreover, by (11°) $L_j\le \mathrm{C}_M([V,Z]\cap \mathrm{C}_V(S))$ and so $\mathrm{C}_M([V,Z]\cap \mathrm{C}_V(S))\le P_j$, since $\overline{L}_j$ is a maximal subgroup of $\overline{M}$. It follows that $P^*\le P_1\cap P_j$.

Since Q_i acts quadratically on V and $Z\le Q_i$, $[V,Z,Q_i]=0$. Hence $L_{1j}=\langle Q_i^{P_1}\rangle\le P^*$, so $[V,Z,L_{1j}]=0$. Moreover, since $L_{1j}\trianglelefteq P_1$, and P_1 acts simply on Q_1/Z, also $Q_1\le L_{1j}$. Since $L_j\cap P_1/Q_j\cong \mathrm{Sym}(4)$ and $L_{1j}=\langle Q_i^{L_{1j}}\rangle$, we conclude that $L_{1j}/Q_1\cong\mathrm{Sym}(3)$ and $L_{1j}Q_j=\mathrm{O}^{2'}(P_1\cap P_j)$. In particular $[L_{1j},Q_j]\le Q_1$ and so $[L_{1j},L_{1i}]\le Q_1$. Hence also $L_{1i}/\mathrm{O}_2(L_{ij})\cong\mathrm{Sym}(3)$ and again by the simple action of P_1 on Q_1/Z, $\mathrm{O}_2(L_{1i})=Q_1$. In addition, $P_{1i}\le \mathrm{N}_{P_1}(Q_i)$ and so $L_{1i}=\mathrm{O}^{2'}(P_1\cap P_i)$ since by (7°) $P_1\cap P_i/Q_i\cong \mathrm{C}_3\times\mathrm{Sym}(4)$. Hence (12°) and (e) has been proved.

13°. *Let $\mathbb{E}$ be the subring of $\mathbb{F}$ generated by the image of $\mathrm{Z}(M)$. Then $\mathbb{E}\cong\mathbb{F}_4$ and $[V,Z]$ is a direct sum of 2-dimensional simple $\mathbb{E}L_1$-modules.*

Since $\mathrm{Z}(M)\cong \mathrm{C}_3$, $\mathbb{E}\cong\mathbb{F}_4$. The second statement follows from (12°) (and (e)) since $L_{1j}=\mathrm{C}_{L_1}([V,Z])$, $\mathrm{C}_V(L_1)=0$ and $L_1/L_{1j}\cong\mathrm{Sym}(3)$.

Let U_j be a 1-dimensional $\mathbb{E}$-subspace of $\mathrm{C}_V(L_j)$. In the following we use the fact that (e) has already been proved, so we know that $L_{1j}=\mathrm{C}_{L_1}([V,Z])\trianglelefteq P_1$ and

$$L_1/Q_1=L_{12}/Q_1\times L_{13}/Q_1\cong\mathrm{Sym}(3)\times\mathrm{Sym}(3);$$

in particular $L_1/\mathrm{C}_{L_1}([V,Z])\cong\mathrm{Sym}(3)$.

Put $U_1:=\langle U_j^{P_1}\rangle$ and $U_i:=\langle U_j^{P_i}\rangle$, so $[U_j,L_{1j}]=0$ since $L_{1j}\le L_j$, and

$$U_1=\langle U_j^{L_{1i}}\rangle=\langle U_j^{P_1\cap P_i}\rangle$$

since U_j is an $\mathbb{E}$-space. As $L_1/\mathrm{C}_{L_1}([V,Z])\cong\mathrm{Sym}(3)$ and $\mathrm{C}_V(L_1)=0$ we conclude that $\dim_{\mathbb{E}}U_1=2$. Since $P_i\cap L_j$ centralizes U_j and $U_1=\langle U_j^{P_i\cap P_1}\rangle$, (7°) and 5.4 imply that $\dim_{\mathbb{E}}U_i=3$. In particular,

$$U_i=\langle U_1^{P_i\cap P_j}\rangle.$$

Put $W_1:=\langle U_i^{L_1}\rangle$ and $W_j:=\langle U_1^{L_j}\rangle$. Since $[U_i,L_{1i}]\le U_1$ and $L_{1i}\trianglelefteq L_1$ we have

$$[W_1,L_{1i}]\le U_1 \text{ and } W_1=\langle U_i^{L_{1j}}\rangle=\langle\langle U_1^{P_i\cap P_j}\rangle^{L_{1j}}\rangle\le W_j.$$

Put $Y_j:=C_{W_j}(L_j)$ and $\overline{W}_j:=W_j/U_j$. Then $\dim_{\mathbb{E}}\overline{U}_1=1$, $\dim_{\mathbb{E}}\overline{U}_i=2$, and $\overline{U}_i=\langle\overline{U}_1^{P_i\cap L_j}\rangle$. Thus, we can apply 5.4 (and (9°)) with $U=\overline{U}_1$. This shows that $\overline{W}_j/C_{\overline{W}_j}(L_j)$ is a natural $\mathbb{E}\mathrm{Alt}(6)$-module and $C_{\overline{W}_j}(L_j)\le\langle\overline{U}_i^{L_{1j}}\rangle=\overline{W}_1$; in particular $\dim_{\mathbb{E}}\overline{W}_j/C_{\overline{W}_j}(L_j)=4$. Since $L_j=O^2(L_j)$ and $[U_j,L_j]=0$, we also have $C_{\overline{W}_j}(L_j)=\overline{Y_j}$.

Since $Y_j\le W_1$ $[Y_j,L_{1i}]\le[W_1,L_{1i}]\le U_1$. From $L_{1i}L_{1j}=L_1$ we conclude that $[Y_jU_1,L_1]\le U_1$. Note that $[Y_jU_1,Q_1]=0$ and $\mathrm{O}^2(L_1)/Q_1$ is a $2'$-group. So coprime action implies

$$Y_jU_1=\mathrm{C}_{Y_jU_1}(\mathrm{O}^2(L_1))[Y_jU_1,\mathrm{O}^2(L_1)].$$

Since $\mathrm{C}_V(L_1)=0$ also $\mathrm{C}_V(\mathrm{O}^2(L_1))=0$ and so $Y_jU_1=U_1$. Thus $Y_j\le\mathrm{C}_{U_1}(Q_j)=U_j$. Hence $\dim_{\mathbb{E}}W_j/U_j=4$ and since $W_1\le W_j$, $\dim_{\mathbb{E}}W_1/U_1=2$. It follows that $\dim_{\mathbb{E}}W_j/W_1=1$ and $W_j=\langle W_1^{P_i\cap P_j}\rangle$.

Put $W := \langle W_1^{L_i} \rangle$ and $\check{W} = W/U_i$. Then $W_j \leq W$, $\dim_{\mathbb{E}} \check{W}_1 = 1$ and $\dim_{\mathbb{E}} \check{W}_j = 2$. Hence (7°) and 5.4 give $\dim_{\mathbb{E}} \check{W} = 3$; in particular $\dim_{\mathbb{E}} W/W_j = 1$. Since P_{1i} does not normalize W_j, $W = \langle W_j^{P_i \cap P_1} \rangle$. Since $\dim_{\mathbb{E}} W_j/W_1 = 1$, $[W_j, L_{1j}] \leq W_1$ and so $[W, L_{1j}] \leq W_1 \leq W$. Thus W is normalized by L_i and $L_{1j}L_{1i} = L_1$. Hence W is an $\mathbb{E}M$ submodule of V, $\dim_{\mathbb{E}} W = 6$ and $W = \langle U_j^M \rangle$.

Note that $[U_j, L_j] = 0$ and U_j is the (up to isomorphism) unique non-trivial simple $\mathbb{F}_2\mathrm{Z}(M)$ module. So U_j is uniquely determined as an $\mathbb{F}_2 P_j$-module. Let $\hat{W}$ be the $\mathbb{F}_2 M$-module induced from the $\mathbb{F}_2 P_j$ module U_j. Put $\widetilde{W} := \hat{W}/\langle [\hat{W}, Z, Q_1]^M \rangle$ and let $\hat{U}_j$ be the image of U_j in $\hat{W}$. Note that $Z(M)$ acts fixed-point freely on $\hat{W}$ and so also on $\widetilde{W}$. In particular, $\mathrm{C}_{\widetilde{W}}(M) = 0$, $\widetilde{W} = [\widetilde{W}, M]$ and $[\widetilde{W}, Z, Q_1] = 0$. Thus $\widetilde{W}$ fulfills the assumption on W in this proof. Since $\widetilde{W} = \langle \widetilde{U}_j^M \rangle$ we conclude that $\dim_{\mathbb{E}} \widetilde{W} = 6$. On the other hand W is as an $\mathbb{F}_2 M$-module an homomorphic image of $\hat{W}$ and so also of $\widetilde{W}$. It follows that $W \cong \widetilde{W}$ as an $\mathbb{F}_2 M$-module and so W is unique up to isomorphism.

Up to now we only used (1°) to determine W. Suppose now that V is indecomposable. Then by (10°) we can choose U_j such that $V = \langle U_j^M \rangle$. Thus $V = W$ and $\dim_{\mathbb{E}} V = 6$. Any non-trivial $\mathbb{F}_2 M$ quotient of V fulfills the same assumption and so is 6-dimensional over $\mathbb{E}$. Thus V is a simple $\mathbb{F}_2 M$-module.

Let V^* be the $\mathbb{F}$-dual of V. Then $V^* = [V^*, \mathrm{Z}(M)] = [V^*, M]$ and $0 = \mathrm{C}_{V^*}(\mathrm{Z}(M)) = \mathrm{C}_{V^*}(L_i^*) = \mathrm{C}_{V^*}(M) = 0$. By 1.8(c) Q acts quadratically on V^* and so $C_M([V^*, Z]) \not\leq Z$. Thus V^* and i fulfill the same assumption as V and i, and V and V^* are isomorphic $\mathbb{F}_2 M$-modules. Hence by 1.9(a) there exists a M-invariant non-degenerate symmetric, symplectic or unitary $\mathbb{F}$-form on V^*. In the symmetric or symplectic case, V would be selfdual as an $\mathbb{F}M$-module and so also an $\mathbb{E}\mathrm{Z}(M)$-module, a contradiction. Thus (d) holds.

Since L_i acts simply on U_i and V/U_i, $\mathrm{C}_V(Q_i) = U_i = [V, Q_i]$ and (i) and (j) hold. Note that $Z = Q_1'$ centralizes $V/[V, Q, Q]$. Since Q_1 centralizes V/W_1 and W_1/U_1 we conclude that $[V, Q, Q] = W_1 = [V, Z]$ and $[V, Q] = W_1$. By a dual argument, $\mathrm{C}_V(Z) = W_1$ and $\mathrm{C}_V(Q_1) = U_1$. Also $[U_1, L_{1j}] = 1$ and dually $[V, L_{1j}] \leq W_1$. Thus (f) and (g) are proved.

$\mathrm{C}_V(Q_j) \leq \mathrm{C}_V(Z) = W_1 < W_j$ and since W_j/U_j is a simple $\mathbb{E}L_j$-module, $\mathrm{C}_V(Q_j) = U_j$. Dually $[V, Q_j] = W_j$ and so (l) and (m) hold. Since $|U_j| = 4$ and $\mathrm{C}_V(Q_j)$ is an $\mathbb{F}$-subspace, $|\mathbb{F}| \leq 4$ and so $\mathbb{F} = \mathbb{E}$. Since W is unique up to isomorphism we conclude that (a) and (c) hold.

□

Lemma 7.9. *Put $L := \mathrm{F}^*(M)$ and suppose that*

(i) V is faithful and indecomposable $\mathbb{F}_2 M$-module, $\mathrm{C}_V(L) = 0$ and $V = [V, L]$.

(ii) $M = \langle D \leq M \mid [V, D, D] = 0, |D| > 2 \rangle$; and

(iii) L is quasi-simple and $\mathrm{Z}(L) \cong \mathrm{U}_4(3)$.

Put $\mathbb{F} := \mathrm{End}_M(V)$ and let A be a maximal quadratic subgroup of M on V. Then

(a) V is a simple $\mathbb{F}_2 L$-module and (L, V) fulfills the assumptions on (M, V) and so also the conclusions in 7.8.

(b) $M = LA$.

(c) $|A/A \cap L| \leq 2$, $|A \cap L| = 2^4$ and $C_M(A) = \mathrm{C}_M(A \cap L) = A\mathrm{Z}(M)$.

(d) $\mathrm{N}_M(A) = \mathrm{N}_M(A \cap L)$ and so $\mathrm{N}_M(A)/A$ is a quasisimple group of shape $3.\mathrm{Alt}(6)$.

(e) $C_V(A \cap L) = C_V(A) = [V, A] = [V, A \cap L]$ *is a 3-dimensional. simple module for* $N_M(A)$.

(f) *A is unique up to conjugation under L, with the conjugacy class depending on the isomorphism type of V.*

(g) *Let* $1 \neq B \leq M$ *such that B acts quadratically on V. Then B is conjugate under L to an subgroup of A and assuming that* $B \leq A$ *one of the following holds:*

(a) $|B| = 2$, $B \leq L$ *and* $\dim_{\mathbb{F}}[V, B] = \dim_{\mathbb{F}} V/C_V(B) = 2$.

(b) $|B| = 2$, $\dim_{\mathbb{F}}[V, B] = \dim_{\mathbb{F}} V/C_V(B) = 1$. *and* $C_V(B)/[V, B]$ *is natural* $\mathbb{F}SU_4(2)$*-module for* $C_L(B)$.

(c) $|B| = 4$, $B \not\leq L$, $\dim_{\mathbb{F}}[V, B] = \dim_{\mathbb{F}} V/C_V(B) = 2$ *and* $\dim_{\mathbb{F}}[V, b] = 1$ *for all* $b \in B \setminus L$.

(d) $C_V(B) = [V, B] = C_V(A)$ *and A is the unique maximal quadratic subgroup of M containing B.*

Proof. Put $\overline{M} = M/Z(L)$. Among all $A \leq M$ with $[V, A, A] = 0$ and $|A| > 2$ let A be maximal. Let $S \in \mathrm{Syl}_2(M))$ with $A \leq S$. Since $\mathrm{Out}(\overline{L}) \cong \mathrm{Dih}_8$, M/L is isomorphic to a subgroup of Dih_8. In particular, $M = LS$. Let Y be non-trivial indecomposable $\mathbb{F}_2L$-submodule of V.

By [MeSt1, 2.3] we have $C_{S\cap L}([V, Z]) \not\leq Z$ and so (L, Y) fulfills the hypothesis of 7.8 in place of (M, V). It follows that Y is a simple $\mathbb{F}_2L$-module and so V is a semisimple $\mathbb{F}_2L$-module.

Let W be a maximal homogeneous $\mathbb{F}_2L$-submodule of V and suppose that A does not normalizes W. Then by [MS3, 2.11]$|A/C_A(W)| = 2$ and so $C_A(W) \neq 1$. Since L is quasisimple we conclude that $L = [L, C_A(W)] \leq C_L(W)$, a contradiction to $C_V(L) = 0$. Hence A normalizes W. As A was an arbitrary maximal quadratic subgroup of order larger than 2, (ii) shows that M normalizes every maximal homogeneous $\mathbb{F}_2L$-submodule W. Since V is indecomposable as an $\mathbb{F}_2M$-module and semisimple as an $\mathbb{F}_2L$-module we conclude that $V = W$ and so V is a homogeneous $\mathbb{F}_2L$-module. In particular, $C_L(Y) = C_L(V) = 1$, $Z(L) \cong C_3$ and the subring $\mathbb{E}$ of $\mathrm{End}_{\mathbb{F}_2L}(V)$ generated by the image of $Z(L)$ is a field isomorphic to $\mathbb{F}_4$.

Put $\mathbb{F}_0 := Z(\mathrm{End}_{\mathbb{F}_2L}(V))$ and note that $\mathbb{F}_0$ is field isomorphic to $\mathrm{End}_{\mathbb{F}_2L}(Y)$ and so to $\mathbb{F}_4$. Thus $\mathbb{F}_0 = \mathbb{E}$. Since $|A| \geq 4$, we conclude from [MS3, 2.15], that A and so also M acts $\mathbb{F}_0$-linear on V. Hence $Z(L) = Z(M)$ and $\mathbb{F}_0 = \mathbb{F}$.

Let $Z = Z(S \cap L)$, $P_1 = N_L(Z)$, $Q_1 = O_2(P_1)$, Q_i, $i = 2, 3$, the two elementary abelian subgroups of order 16 in $S \cap L$, $P_i = N_{L_i}(Q_i)$ and for $i \in \{1, 2, 3\}$, $P_i^* = N_M(Q_i)$, $L_i = O^{2'}(P_i)$, and $Q_i^* = O_2(P_i^*)$. Choose notation such that $C_Y(L_2) = 0$ and so $C_Y(L_3) \neq 0$. In the following we will use the properties of P_i, $i = 1, 2, 3$, given in 7.8.

Since V is a homogeneous $\mathbb{F}_2L$-module we conclude that also $C_V(L_2) = 0$ and $C_V(L_3) \neq 0$. Thus S normalizes L_2 and L_3 and so $S \leq P_i^*$ for all $1 \leq i \leq 3$. In particular, $|M/L| \leq 4$. Since $P_2/Q_2 \sim 3{\cdot}\mathrm{Alt}(6)$ and P_2^* centralizes $Z(L)$ we conclude that P_2^* induces inner automorphisms on P_2/Q_2, so $P_2^* = Q_2^*P_2$. Thus $|M/L| \leq 2$. Since $|A| \geq 4$ we get $A \cap L \neq 1$, and since L has unique class of involutions and $|Z| = 2$, we may assume that $Z \leq A \cap L$. In particular, $0 \neq [Y, A \cap L] \leq C_Y(A)$ and since Y is a simple $\mathbb{F}_2L$-module, A normalizes Y. Thus Y is an $\mathbb{F}_2M$ submodule. As this holds for all simple $\mathbb{F}_2L$-submodules on V and V is a semisimple $\mathbb{F}_2L$-module and an indecomposable $\mathbb{F}_2M$-module, $V = Y$. Thus V is a simple $\mathbb{F}_2L$-module and (a) holds. By 7.8(d), there exists an L-invariant non-degenerate quadratic form on V and by 1.9(f), this form is invariant under M.

Let $D \leq Q_2$ with $|D| \geq 4$ and let $a, b \in D^\sharp$ with $a \neq b$. Note that P_2 acts simple on $[V, Q_2]$ and $\langle C_{P_2}(a), C_{P_2}(b)\rangle = P_2$. Since $0 \neq [V, a] < [V, Q_2]$ we conclude that $[V, a] \neq [V, b]$. Since $\dim_{\mathbb{F}}[V, a] = 2$ and $\dim_{\mathbb{F}}[V, Q_2] = 3$ this gives $[V, D] = [V, a] + [V, b] = [V, Q_2]$ We have proved

(∗) $[V,D]=[V,Q_2]$ for all $D\le Q_2$ with $|D|>2$.

Put $L_{13}:=\langle Q_2^{P_1}\rangle$. Then $Q_1\le L_{13}$, $L_{13}\le P_1\cap P_3$, $L_{13}/Q_1\cong \mathrm{Sym}(3)$ and $L_{13}=C_L([V,Z])$. Put $L_{13}^*:=\mathrm{C}_M([V,Z])$. Then $A\le L_{13}^*$ and so $M=L_{13}^*L$ and $P_1^*=L_{13}^*P_1$. Since $|L_{13}/L_{13}^*|\le 2$ we conclude that $\mathrm{O}_2(L_{13}^*)=Q_1^*$, $L_{13}^*=L_{13}Q_1^*$ and $L_1^*=L_1Q_1^*$.

Put $Z^*:=\mathrm{Z}(Q_1^*)$. Since L_1 acts simply on Q_1/Z, we have $[Q_1,Q_1^*]\le Z$ and conclude that $Q_1^*=Z^*Q_1$. Note that $[Z^*,L_1]\le Z$ and so $[Z^*,\mathrm{O}^2(L_1)]=1$. Since $V/\mathrm{C}_V(Z)$ and $\mathrm{C}_V(Z^*)/[V,Z]$ are non-isomorphic as $\mathrm{O}^2(L_1)$-modules, $[V,Z^*]=[V,Z]$ and similarly $\mathrm{C}_V(Z^*)=\mathrm{C}_V(Z)$. It follows that $[V,Z^*]\le[V,Z]\le[V,A]\le \mathrm{C}_V(A)\le \mathrm{C}_V(Z)=\mathrm{C}_V(Z^*)$ and so Z^*A is quadratic on V. Thus by maximality of A, $Z^*\le A$ and $A=Z^*(A\cap L)$. We will show that A is contained in a conjugate of Q_2^* under P_1. Since $A=Z^*(A\cap L)$ it suffices to show that $A\cap L$ is contained in a conjugate of Q_2 under P_1.

Suppose $A\cap L\le Q_1$. Note that P_1 acts transitively on fours groups of Q_1 containing Z and so we may assume $|A\cap Q_2|\ge 4$. Thus using (∗),

$$A\le C_M([V,A\cap Q_2])=C_M([V,Q_2])\le Q_2^*.$$

Suppose next that $A\cap L\not\le Q_1$. Since $L_{13}/Q_1\cong\mathrm{Sym}(3)$ we may assume that $A\cap L\le Q_1Q_2$. Let $\widetilde{P}_1:=P_1/Z$ and let $q\in Q_2\setminus Q_1$. Then $C_{\widetilde{Q}_1}(q)=[\widetilde{Q}_1,q]=\widetilde{Q_1\cap Q_2}$. It follow that all involutions in $\widetilde{Q}_1\widetilde{Q}_2\setminus\widetilde{Q}_1$ are conjugate and so Q_2 is the unique maximal elementary subgroup of Q_1Q_2 not contained in Q_1. Thus $A\cap L\le Q_2$.

We proved that A is conjugate to a subgroup of Q_2^* and we may assume that $A\le Q_2^*$. Since $\mathrm{C}_V(Q_2)$ is the unique non-zero proper $\mathbb{F}_2L_2$ submodule of V, $\mathrm{C}_V(Q_2^*)=[V,Q_2^*]=\mathrm{C}_V(Q_2)$ and so Q_2^* is quadratic on V. This gives $A=Q_2^*$, and all maximal quadratic subgroups of M of order at least 4 are conjugate to Q_2^*.

It remains to proof (g). So let B be any quadratic subgroup of M. Suppose first that $|B|=2$. If $B\le L$ then B is conjugate to $|Z|$ and so (g:a) holds. If $B\not\le L$ then either $\mathrm{C}_{\overline{L}}(B)\cong U_4(2)$ or $\mathrm{C}_{\overline{L}}(B)\sim 2^4.3^2.2$.

Suppose that $\mathrm{C}_{\overline{L}}(B)\sim 2^4.3^2.2$. Then $\mathrm{O}_2(\mathrm{C}_L(B))$ is conjugate to $A\cap L$ and we may assume that $B\le A$ and $C_M(B)\le P_2$. Note that $\mathrm{C}_M(B)$ contains a Sylow 3-subgroups of P_2. Since the Sylow 3-subgroups of P_2 are extraspecial of order 3^3 they act simply on $[V,A]$ and we conclude that $[V,B]=\mathrm{C}_V(B)=[V,A]=\mathrm{C}_V(A)$ and so (g:d) holds.

Suppose $\mathrm{C}_{\overline{L}}(B)\cong U_4(2)$. Let $y\in Z^*\setminus Z$. Then $[V,y]\le[V,Z]$. The preceding paragraph shows that $\mathrm{C}_{\overline{L}}(B)\not\sim 2^4.3^2.2$ and thus $\langle y\rangle$ is conjugate to B. So we may assume that $B\le Z^*$. Thus $V/\mathrm{C}_V(B)$ and $[V,B]$ have dimension at most two over $\mathbb{F}$ and so are centralized by $\mathrm{C}_L(B)$. Thus $\mathrm{C}_L(B)$ acts faithfully on $\mathrm{C}_V(B)/[V,B]$. Since $[V,B]\le\mathrm{C}_V(B)=[V,B]^\perp$, the L-invariant unitary form on V gives raises to an $\mathrm{C}_L(B)$-invariant unitary form on $\mathrm{C}_V(B)/[V,B]$. It follows that $\dim_{\mathbb{F}}\mathrm{C}_V(B)/[V,B]=4$ and $\mathrm{C}_V(B)/[V,B]$ is a natural $\mathrm{SU}_4(2)$-module for $\mathrm{C}_L(B)$. Thus $\dim_{\mathbb{F}}V/\mathrm{C}_V(B)=1=\dim_{\mathbb{F}}[V,B]$ and (g:b) holds.

Suppose next that $|B|>2$. Then B is contained in a maximal quadratic subgroup of order at least 4 and so we may assume that $B\le A$. If $[V,B]=[V,A]$, then $\mathrm{C}_V(B)=[V,B]^\perp=[V,A]^\perp=\mathrm{C}_V(A)$ and (g:d) holds. So suppose $[V,B]<[V,A]$. Then (∗) implies that $|B\cap L|=2$ and so $|B|=4$. If $d\in B\setminus L$, then $\dim_{\mathbb{F}}[V,d]\le\dim_{\mathbb{F}}[V,B]\le 2$ and so (g:b) must hold for $\langle d\rangle$ in place of B. Thus (g:c) holds. □

Lemma 7.10. *Let $M=\mathrm{O}^\epsilon_{2n}(q)$, $q=2^k$, and V be the corresponding natural module over $\mathbb{F}_q$. Let $a\in M$ with $|a|=2$. Then $a\in\Omega^\epsilon_{2n}(q)$ if and only if $\dim_{\mathbb{F}_q}[V,a]$ is even.*

Proof. This is well known, but a reference seems to be hard to come by. So here is a proof: If $n = 1$, this is obvious. Suppose there exists an a-invariant proper subspace W of V with $V = W \oplus W^\perp$. Then the claim follows by induction on n. So we may assume that no such W exists. In particular $v \perp v^a$ for all $v \in V$ and so $[V, a]$ is a singular subspace. Let $\mathrm{C}_V(a) = [V, a] \oplus W$ for some $\mathbb{F}_q$-subspace W. Since $\mathrm{C}_V(a) = [V, a]^\perp$, $V = W \oplus W^\perp$ and so $W = 0$ and $[V, A] = \mathrm{C}_V(a)$ is maximal singular subspace of V. Thus $\epsilon = +$. Since a normalize a maximal singular subspace, $a \in \Omega^+_{2n}(q)$. Consider the map $s_a : V/\mathrm{C}_V(a) \times V/\mathrm{C}_V(a) \to \mathbb{F}_q$ define by $s_a(v + \mathrm{C}_V(a), w + \mathrm{C}_V(a)) = s(v, [w, a]))$, where s is the symmetric form on V invariant under M. Then s_a is a non-degenerate bilinear form. From $v \perp v^a$ we get $v \perp [v, a]$ and so s_a is a symplectic form. Thus $\dim[V, a] = \dim V/\mathrm{C}_V(a)$ is even. □

Lemma 7.11. *Let q be a power of p and $K \trianglelefteq M$ such that $K \cong \mathrm{Spin}^\epsilon_n(q)$, $n \geq 3$, and $\mathrm{C}_M(K) = \mathrm{Z}(K)$. Let V_{nat} be the natural $\mathbb{F}_q\Omega^\epsilon_n(q)$-module for K, $S \in \mathrm{Syl}_p(M)$, $U := \mathrm{C}_{V_{\mathrm{nat}}}(S \cap K)$, $L := \mathrm{C}_K(U)$ and $Q := \mathrm{O}_p(L)$. Then the following hold:*

(a) Suppose that W is a non-trivial simple $\mathbb{F}_pK$-module with $[W, Q, Q] = 0$. Then W is a (half-)spin module for K.

(b) Suppose that $p = 2$, n even, $n \geq 6$, W is a simple $\mathbb{F}_2M$-module with $[W, K] \neq 0$ and that there exists $A \leq S$ with $[W, A, A] = 0$, $M = \langle A^M \rangle$, $|A| > 2$, and $A \not\leq K$. Then $M \cong \mathrm{O}^\epsilon_n(q)$ and W is the natural $\mathrm{O}^\epsilon_n(q)$-module for M.

Proof. Put $T := S \cap K$, so $T \in \mathrm{Syl}_p(K)$, and $\overline{\mathrm{N}_M(Q)} := \mathrm{N}_M(Q)/Q\mathrm{Z}(K)$, and let U_0 be the unique 1-dimensional singular subspace of U. Then $[U^\perp, Q] = U_0$. Moreover $U = U_0$, if n is even or p is odd, and $U = U_0 + V^\perp$ if n is odd and $p = 2$. Hence

1°. *$U^\perp/U_0$ and Q are natural $\Omega^\epsilon_{n-2}(q)$-modules for $\overline{L}$.*

Assume that $n \geq 5$. Then there exists $g \in K$ such that $Y := U_0 + U_0^g$ is a 2-dimensional singular subspace of $U^\perp$ normalized by T. Put $H := \langle Q, Q^g \rangle$ and $Z := Q \cap Q^g$. Then $H/\mathrm{C}_H(Y) \cong \mathrm{SL}_2(q)$, and H acts transitively on the 1-dimensional subspaces of Y. Thus $H = \langle Q^{\mathrm{N}_K(Y)} \rangle$; in particular, T normalizes H. Moreover, $Q\mathrm{O}_p(HT) = T \in \mathrm{Syl}_p(HT)$, and using (1°):

2°. *If $n \geq 5$, then $\overline{\mathrm{C}_{Q^g}(Y)} = \mathrm{O}_p(\mathrm{C}_{\overline{L}}(Y/U_0))$, and Z is a 1-dimensional singular subspace of Q.*

(a): Put $\mathbb{K} := \mathrm{End}_K(W)$. By Smith's Lemma 4.2 applied to W and its dual, $\mathrm{C}_W(Q)$ and $W/[W, Q]$ are simple $\mathbb{K}L$-modules. Since $[W, Q] \leq \mathrm{C}_W(Q)$ we conclude that $[W, Q] = \mathrm{C}_W(Q)$. Suppose that $n = 3$ or 4. Then $Q = T$ and so $\mathrm{C}_W(Q)$ and $W/[W, Q]$ are 1-dimensional over $\mathbb{K}$. Thus $\dim_{\mathbb{K}}(W) = 2$.

If $n = 3$ or $(n, \epsilon) = (4, +)$ then W is a natural $\mathrm{SL}_2(q)$-module. If $(n, \epsilon) = (4, -)$, then W is a natural $\mathrm{SL}_2(q^2)$-module. These are the (half-)spin modules for these groups, so (a) follows in this case.

Suppose now that $n \geq 5$, so we are allowed to use the subgroups Y, H and Z constructed above. Since $[W, Z, H] = 0$ and $Z \neq 0$ we conclude that $\mathrm{C}_W(HT) \neq 0$. By Smith's Lemma 4.2 $\mathrm{C}_W(T)$ is 1-dimensional over $\mathbb{K}$ and so $\mathrm{C}_W(T) = \mathrm{C}_W(TH)$. Since $K = \langle L, HT \rangle$ and W is simple, we have $[\mathrm{C}_W(T), L] \neq 0$, so $[\mathrm{C}_W(Q), L] \neq 0$. Now again Smith's Lemma 4.2 and (2°) show that $\mathrm{C}_W(Q)$, $\overline{L}$ and $\overline{\mathrm{C}_{Q^g}(Y)}$ satisfy the hypothesis in place of W, K, and Q. Thus by induction $\mathrm{C}_W(Q)$ is a (half-)spin module for $\overline{L}$. Together with $[\mathrm{C}_W(T), HT] = 0$, this determines W up to isomorphism (see 4.3) and so W is a (half)-spin-module.

(b): Note that $K \cong \Omega^\epsilon_n(q)$ since $p = 2$, that S normalizes L, and that by (1°) Q is a natural $\Omega^\epsilon_{n-2}(q)$-module for L. Thus there exists an L-invariant quadratic form h (over $\mathbb{F}_q$) on Q.

3°. *There exist $a, b \in A^\sharp$ with $\mathrm{C}_Q(a) \neq \mathrm{C}_Q(b)$.*

Assume first that A does not act $\mathbb{F}_q$-linearly on Q. Since $\mathrm{Aut}(\mathbb{F}_q)$ is cyclic and A is elementary abelian with $|A| \geq 4$, we conclude that there exists $1 \neq a \in A$ acting $\mathbb{F}_q$-linearly on Q and $b \in A$ acting not $\mathbb{F}_q$-linearly on Q. Hence $C_Q(a)$ is an $\mathbb{F}_q$-subspace of Q while $C_Q(b)$ is not; in particular $\mathrm{C}_Q(a) \neq \mathrm{C}_Q(b)$.

Assume now that A acts $\mathbb{F}_q$-linearly on Q. Then $\overline{AL} \cong \mathrm{O}^\epsilon_{n-2}(q)$, and there exists $a \in A \setminus K$ and $1 \neq b \in A \cap K$. By 7.10 we conclude that $\mathrm{C}_Q(a)$ is odd dimensional and $\mathrm{C}_Q(b)$ is even dimensional over $\mathbb{F}_q$. Hence again $\mathrm{C}_Q(a) \neq \mathrm{C}_Q(b)$.

4°. *There exists $D \leq LA$ with $D \cap A \not\leq Q$, $[W, D, D] = 0$, and $D \cap Q \neq 1$.*

Clearly $A \not\leq Q$ since $A \not\leq K$, so if $A \cap Q \neq 1$ we can choose $D = A$. Suppose $A \cap Q = 1$. Let $a, b \in A$ as in (3°) and without loss $\mathrm{C}_Q(a) \not\leq \mathrm{C}_Q(b)$. Then there exists $1 \neq d \in [\mathrm{C}_Q(a), b] \leq \langle b^{\mathrm{C}_Q(a)}\rangle$, so

$$[W, a, d] \leq [W, a, \langle b^{\mathrm{C}_Q(a)}\rangle] = \langle [W, a, b]^{\mathrm{C}_Q(a)}\rangle = 0.$$

Since A is elementary abelian, $d \in \langle b^{\mathrm{C}_Q(a)}\rangle \leq \mathrm{C}_L(a)$ and so $[a, d, W] = 0$. Hence by the Three Subgroups Lemma also $[W, d, a] = 0$, and $D := \langle a, d\rangle$ satisfies (4°).

5°. *There exists $B \leq Q$ and $1 \neq e \in B$ such that $[W, B, B] = 0$, $h(e) = 0$ and $B \not\leq \mathbb{F}_q e$.*

Let D be as in (4°). Pick $1 \neq b \in D \cap Q$, and put $E := \langle D^{\mathrm{C}_L(b)}\rangle$ and $C := \mathbb{F}_q b$. Then $[W, b, E] = 0$.

Suppose that $b^\perp \leq E \cap Q$. Note that there exists $u \in E \cap Q \setminus C$ such that $h(u) = 0$ if $h(b) \neq 0$. Pick such an element u and put $B := \langle b, u\rangle$. Since $[W, b, B] = 0$, B acts quadratically on W. Thus (5°) holds with $e = b$ if $h(b) = 0$ and $e = u$ if $h(b) \neq 0$.

Suppose now that $b^\perp \not\leq E \cap Q$. By the action of $C_L(b)$ on Q, any $C_L(b)$-submodule of Q, which contains b, either contains $b^\perp$ or is contained in C. In particular $E \cap Q \leq C$ and $[Q, E] \leq E \cap Q \leq C$. Since Q is a natural $\Omega^\epsilon_{n-2}(q)$-module for L, 3.4 shows $h(b) \neq 0$ and $|DQ/Q| = |EQ/Q| = 2$. Thus $[D, \mathrm{C}_L(b)] \leq C$, and since $\mathrm{C}_L(b)$ centralizes C, $[D, \mathrm{O}^2(C_L(b)] = 1$. The structure of $\mathrm{O}_{n-2}(q)$ shows that

$$[Q, D] = C \text{ and } \mathrm{C}_{LD}(b)/Q \cong C_2 \times \mathrm{Sp}_{n-4}(q)$$

Put $D^* = \mathrm{C}_{DL}(\mathrm{O}^2(\mathrm{C}_L(b)))$. It follows that $D \leq D^*$, $|D^*Q/Q| = 2$, $D^* \cap Q = C$, and the q elements in $D^* \setminus Q$ are the transvections on V_{nat} corresponding to the q non-singular 1-spaces in the isotropic 2-space $[V_{\mathrm{nat}}, b]$. Pick $d \in D \cap A \setminus Q$. Then $F := \mathrm{C}_{DK}(d) \cong \mathrm{C}_2 \times \mathrm{Sp}_{n-2}(q)$. In particular $F = \langle D^F\rangle$. From $[W, d, D] = 0$ we get $[W, d, F] = 0$ and so $[W, d, \mathrm{C}_Q(d)] = 0$. Pick $e \in \mathrm{C}_Q(d) \setminus C$. Then $\langle e, d\rangle$ is quadratic on W and satisfies (4°) in place of D. Moreover $[Q, d] \not\leq \mathbb{F}_q e$. Hence the arguments of the previous paragraph apply to $\langle e, d\rangle$ in place of D, and (5°) holds.

6°. $[W, Z, \mathrm{C}_Q(Y)] = 0$.

Let B and e be as in (5°). Since L is transitive on the singular elements of Q and since by (2°) Z is a singular subspace of Q, we may assume that $e \in Z$. Put $Q_e := e^\perp$ in Q. Note that $Q_e = \mathrm{C}_Q(Y)$, so we have to show that $[W, Z, Q_e] = 0$.

Since $B \not\leq Z = \mathbb{F}_q e$ we get $Q_e \leq \langle B^{\mathrm{C}_L(e)}\rangle$, so $[W, e, Q_e] = 0$. As $\mathrm{N}_L(Q_e)$ acts transitively on Z, we conclude that $[W, Z, Q_e] = [W, \langle e^{\mathrm{N}_L(Q_e)}\rangle, Q_e] = 0$.

7°. *Put $\mathbb{K} := \mathrm{End}_K(W)$. Then W is a simple $\mathbb{F}_2K$-module, and M acts $\mathbb{K}$-linearly on W.*

Let X be a simple $\mathbb{F}_2K$-submodule of W and $\mathbb{E} := \mathrm{End}_K(X)$, and pick D as in (4°). Then $0 \neq [X, D \cap Q] \leq \mathrm{C}_X(D)$ and so X is D-invariant. Hence $0 \neq [X, D \cap A] \leq \mathrm{C}_X(A)$ and so X is A-invariant. Since $D \cap Q$ acts $\mathbb{E}$-linearly on X, $[X, D \cap Q]$ is a non-trivial $\mathbb{E}$-subspace centralized by D, so D acts $\mathbb{E}$-linearly on X. Hence $[X, D \cap A]$ is a non-trivial $\mathbb{E}$-subspace centralized by A, and A acts $\mathbb{E}$-linearly on X. This also holds for each conjugate of A under M. Since $M = \langle A^M \rangle$ and W is a simple $\mathbb{F}_2M$-module, $X = W$, $\mathbb{K} = \mathbb{E}$, and M acts $\mathbb{K}$-linearly on W.

8°. $[W, Q, Q] \neq 0$.

Suppose $[W, Q, Q] = 0$. Then by (7°) and (a), W is a (half)-spin module. If $\epsilon = -$, then $\mathbb{K} \cong \mathbb{F}_{q^2}$ and since A acts $\mathbb{K}$-linearly on W, we conclude that $A \leq K$, a contradiction. If $\epsilon = +$, then $\mathbb{K} = \mathbb{F}_q$ and so A induces a graph automorphism on K. But graph automorphisms interchange the two half-spin modules and so do not act on W, again a contradiction.

9°. *W is a natural $\Omega_n^\epsilon(q)$-module for K.*

Put $Q_Z = \mathrm{C}_Q(Y)\mathrm{C}_{Q^g}(Y)$, where g is as in the definition of Y. Then by (6°) $[W, Z, Q_Z] = 0$. Let $l \in L$ with $Z^l \not\leq \mathrm{C}_Q(Y)$, so $Q = \mathrm{C}_Q(Y)Z^l$. Note that $L = \langle Q_Z, Q_Z^l \rangle$. Since $[W, Q, Q] \neq 0$ by (8°) and $\langle Z^L \rangle = Q$, also $[W, Z, Q] \neq 0$. Now $[W, Z, \mathrm{C}_Q(Y)] = 0$ gives

$$0 \neq [W, Z, Q] = [W, Z, \mathrm{C}_Q(Y)Z^l] = [W, Z, Z^l].$$

Since $[Z, Z^l] = 1$, we get

$$0 \neq [W, Z, Z^l] = [W, Z^l, Z] \leq [W, Z] \cap [W, Z^l] \leq \mathrm{C}_W(Q_Z) \cap \mathrm{C}_W(Q_Z^l) = \mathrm{C}_W(L).$$

Thus $\mathrm{C}_W(L) \neq 0$, and with Smith's Lemma 4.2 $[\mathrm{C}_W(S \cap K), L] = 0$.

By (6°) Z and thus also Z^l acts quadratically on W. On the other hand

$$Z^l\mathrm{O}_2(HT) = Q\mathrm{O}_2(HT) \in \mathrm{Syl}_2(HT).$$

Hence, T acts quadratically on $\mathrm{C}_W(\mathrm{O}_2(HT))$. So by (a) $\mathrm{C}_W(\mathrm{O}_2(HT))$ is a natural $\mathrm{SL}_2(q)$-module for HT. Thus by Ronan-Smith's Lemma 4.3 W is unique up to isomorphism, and (9°) holds.

From (9°) we conclude that $\mathbb{K} = \mathbb{F}_q$. Since A acts $\mathbb{K}$-linearly on W we infer that $KA \cong \mathrm{O}_{2n}^\epsilon(q)$, W is the natural module, and $M = KA$. □

8 The FF-Module Theorems

In this section we use the same hypothesis and notation as in Section 2; that is, M is a finite group with $\mathrm{O}_p(M) = 1$, V is a finite, faithful $\mathbb{F}_pM$-module such that $J = \mathrm{J}_M(V) \neq 1$, and $\mathcal{J}$ is the set of $\mathrm{J}_M(V)$-components of M on V.

Recall that a finite group H is p-minimal if $S \in \mathrm{Syl}_p(H)$ is contained in a unique maximal subgroup of H and $S \ntrianglelefteq H$.

Lemma 8.1. *Suppose that M is p-minimal and $T \in \mathrm{Syl}_p(M)$. Then there exist subgroups $E_1, \dots, E_r$ such that the following hold:*

(a) $J = E_1 \times \cdots \times E_r$ and $\mathcal{J} = \{E_1', \dots, E_r'\}$.

(b) $V = \mathrm{C}_V(J) + \sum_{i=1}^r [V, E_i]$ and $[V, E_i, E_j] = 0$ for $i \neq j$.

(c) $[\mathrm{C}_V(T), \mathrm{O}^p(M)] \neq 0$.

(d) T *is transitive on* $E_1, \dots, E_r$.

(e) *There are no over-offenders on* V *in* M.

(f) $E_i \cong \mathrm{SL}_2(q)$, $q = p^n$, *and* $[V, E_i]/\mathrm{C}_{[V,E_i]}(E_i)$ *is a natural* $\mathrm{SL}_2(q)$*-module for* E_i, *or* $p = 2$, $E_i \cong \mathrm{Sym}(2^n + 1)$, *and* $[V, E_i]$ *is a natural* $\mathrm{Sym}(2^n + 1)$*-module for* E_i.

(g) *If* $A \leq M$ *is an offender on* V, *then* $A = (A \cap E_1) \times \ldots \times (A \cap E_r)$, *and each* $A \cap E_i$ *is an offender on* V.

Proof. Using [BHS, 5.6] we see that (c) holds. Hence M and V satisfy the hypothesis of [BHS, 5.5]. This result gives subgroups $E_1, \dots, E_r$ satisfying (b),(d), (f) and (g). Moreover, [BHS, 2.16] shows that every best offender on V induces inner automorphisms in E_i and is not an over-offender on $[V, E_i]$. The first property gives (a) and the second one (e). □

The proof of Theorem 2:

Let $K \in \mathcal{J}$, $\mathbb{K} := \mathrm{End}_K(V)$, and $A \in \mathcal{D}$. From 2.8 we get:

1°. *V is a simple K-module, and K is the unique J-component of M.*

If K is solvable, then 2.2(d) shows that Theorem 2(1) holds for $q = 2$ or 3 and $n = 2$. Thus, we assume from now on that K is not solvable, so K is a component by 2.2(d).

By the definition of $\mathcal{D}$ there exists $1 \neq B \leq A$ such that B is an offender on V with

$$(*) \qquad [V, B, A] = 0.$$

We choose such an offender B with $|B|$ minimal. Then B is a minimal offender and thus a quadratic best offender on V, so $B \leq J$.

By (1°) and 2.2(b) $[K, B] \neq 1$. Hence

2°. *$K = [K, B]$ and $[V, B, A] = 0$.*

Since K is not solvable, we get from 2.5, applied to BK, that BK acts $\mathbb{K}$-linearly on V. In particular, $[V, B]$ is a $\mathbb{K}$-subspace of V. Thus $(*)$ shows that A centralizes a $\mathbb{K}$-subspace of V, so also A acts $\mathbb{K}$-linearly on V. Since this holds for every $A \in \mathcal{D}$, we conclude:

3°. *M acts $\mathbb{K}$-linearly on V, and $\mathrm{C}_M(K) = \mathrm{Z}(M)$.*

We will now prove Theorem 2 by using the information given in [GM2, Theorem B]. Observe that the bounds on the dimension of V in the cases (3) and (4) of Theorem 2 follow from 3.4.

Suppose that (KB, V) or (K, V) is one of the possibilities (1) − (12) given in Theorem 2 for (M, V). Since by (3°) $M \leq \mathrm{N}_{\mathrm{GL}_{\mathbb{K}}(V)}(K)$, then also (M, V) is on the list. Moreover, if there exists a non-trivial offender on V in K, then (3°) and [GM2] show that (K, V) is on the list. Thus, we may assume:

4°. *B is a minimal best offender on V, $M = KB$, and there is no non-trivial offender on V in K. In particular $K \neq M$.*

Case 1. *Suppose that p is odd.*

In [Ch, Corollary C] all possibilities for M are given under the hypothesis that $|V/\mathrm{C}_V(B)| \leq |B|^2$ for some non-trivial quadratic subgroup $B \leq M$. It turns out that $p = 3$ and $M \cong \mathrm{SL}_2(5)$, or M is a genuine group of Lie type in characteristic p. In the first case $|V/\mathrm{C}_V(B)| > |B|$, and B is not an offender contradicting (4°). In the second case (4°) shows that $M \cong {}^2\mathrm{G}_2(3) \sim \mathrm{SL}_2(8).3$. But then M has abelian Sylow 2-subgroups, which contradicts [KS, 9.1.4].

Case 2. *Suppose that* $|B| = 2$.

Then B acts as a transvection on V, and [McL] shows that (M, V) is on the list.

Case 3. *Suppose that* $p = 2$, $|B| > 2$, *and* K *is not a genuine group of Lie-type in characteristic* p.

Then [MeSt1], [MeSt2] and 7.4 together with (4°) show that

$$K \cong \mathrm{Alt}(n),\ n \geq 6,\ n \neq 8,\ \mathrm{U}_3(3),\ 3.\mathrm{U}_4(3),\ {}^2\mathrm{F}_4(2)',\ ,\mathrm{Mat}_{12},\ \text{or } \mathrm{Mat}_{22}.$$

Except in the case $K \cong \mathrm{Alt}(n)$ the corresponding module V is uniquely determined.

Suppose $K \cong \mathrm{Alt}(n)$. Then [MeSt2] offers two possibilities for V. If V is the natural module for $\mathrm{Alt}(n)$, then $M \cong \mathrm{Sym}(n)$ and V is the natural module for $\mathrm{Sym}(n)$. Hence (M, V) are on the list.

If V is not a natural module, then V is the (half-)spin module and $n > 6$. So 7.5 shows that $B \leq \mathrm{Alt}(n)$ contradicting (4°).

Suppose that $K \cong \mathrm{U}_3(3)$. Then $M \cong \mathrm{G}_2(2)$, and 7.6 shows that (M, V) is on the list.

Suppose $K \cong {}^2\mathrm{F}_4(2)'$. Then $M \cong {}^2\mathrm{F}_4(2)$ and so $M \setminus K$ does not contain any involution, a contradiction.

Suppose $K \cong 3.\mathrm{U}_4(3)$. Then $\mathbb{K} = \mathbb{F}_4$ and $\dim_{\mathbb{K}} V = 6$. Since M acts $\mathbb{K}$-linearly we get $|M/K| = 2$, and there exists $B \leq R \leq M$ such that $R \sim 2^{4+1}3.\mathrm{Alt}(6)$. Observe that every non-zero R-section of V is at least 3-dimensional over $\mathbb{K}$. Hence $I_R := \mathrm{C}_V(\mathrm{O}_2(R)) = \mathrm{C}_V(\mathrm{O}_2(R) \cap K)$ is 3-dimensional over $\mathbb{K}$ and $V = [V, R]$.

Clearly B is not an over-offender on I_R since $|B\mathrm{O}_2(R)/\mathrm{O}_2(R)| \leq 4$ and I_R is an $\mathbb{F}_4R$-module. Thus, by 1.3 either $V = I_R + \mathrm{C}_V(B)$ or $B \leq \mathrm{O}_2(R)$. In the first case $[V, R] \leq I_R$, a contradiction. In the second case [MS1, 2.6] implies that there exists an offender $1 \neq D \leq \mathrm{O}_2(R)$ with $D \trianglelefteq R$. Since I_R and V/I_R are simple R-modules we get $\mathrm{C}_V(D) = I_R$ and $2^5 = |\mathrm{O}_2(R)| \geq |D| \geq |V/\mathrm{C}_V(D)| = |V/I_R| = 2^6$, a contradiction.

Suppose next that $K \cong \mathrm{Mat}_{12}$ or Mat_{22}. Then $M \cong \mathrm{Aut}(\mathrm{Mat}_{12})$ and $\mathrm{Aut}(\mathrm{Mat}_{22})$, respectively, and [MeSt2] shows that $|B| = 4$. But then $|V/\mathrm{C}_V(B \cap K)| \leq |V/\mathrm{C}_V(B)| \leq |B| = 4$, which contradicts the action of K on V.

Case 4. *Suppose* $p = 2$, $|B| > 2$, *and* K *is a genuine group of Lie type defined over a field of characteristic* 2.

Recall that $B \leq T \in \mathrm{Syl}_2(M)$. Let $V_0 := \mathrm{C}_V(T \cap K)$. Note that M is generated by the 2-minimal subgroups containing T. Hence there exists $T \leq P \leq M$ such that P is 2-minimal and $[V_0, \mathrm{O}^2(P)] \neq 0$.

5°. $B \leq \mathrm{O}_2(P)$.

Suppose that $P = M$. Then by 8.1 (KB, V) is on the list, contrary to the assumptions. Thus $P \neq M$.

Put $V_P := \mathrm{C}_V(\mathrm{O}_2(P) \cap K)$. Then $V_0 \leq V_P$. Put $\tilde{P} = N_K(\mathrm{O}^{2'}(P \cap K))$. Then $\tilde{P}$ is a Lie-parabolic subgroup of K, $\mathrm{O}_2(P) \cap K = \mathrm{O}_2(\tilde{P})$ and $\mathrm{O}^{2'}(\tilde{P}) = \mathrm{O}^{2'}(P \cap K)$. Thus by Smith's Lemma 4.2 V_P

is a simple $\mathbb{K}(P \cap K)$-module. By (4°) $O^2(P) \le P \cap K$, so $C_V(O^2(P)) = 0$ and $V_P = C_V(O_2(P))$. Moreover, since P is 2-minimal, $C_T(V_P) = O_2(P)$.

Suppose that $B \not\le O_2(P)$, so $[V_P, B] \ne 0$. By 1.2 B is a non-trivial best offender on V_P, and by 8.1 B is not an over-offender on V_P. Hence 1.3 shows that $C_B(V_P) = 1$ and $V = V_P + C_V(B)$. Again by 8.1 there exists $O_2(P)B \le H \le P$ such that $H/O_2(P) \cong SL_2(|B|)$, $U := [V_P, H]$ is a natural $SL_2(|B|)$-module, and $V = U + C_V(B)$.

Put $D := \langle B^H \rangle$. Then $[V, D] \le U$, so every subgroup of V containing U is D-invariant. Since K is of local characteristic 2 and $P \ne M$, there exists a minimal normal subgroup N of D in $O_2(D) \cap K$. Then $[V, D, N] \le [U, N] = 0$ and $[V, N, O^2(D)] = U$. Hence, the Three Subgroups Lemma shows that $[O^2(D), N, V] \ne 0$ and so $[N, O^2(D)] \ne 1$. As $SL_2(|B|)$ has no non-trivial simple $\mathbb{F}_2$-module of order less than $|B|^2$, we get $|N| \ge |B|^2$.

On the other hand for every $1 \ne x \in N$, $U \le C_V(x)$ and so $C_V(x)$ is D-invariant. Since $N = \langle x^D \rangle$ it follows that $C_V(N) = C_V(x)$. Now choose $y \in N$ and $b \in B$ with $x := [y, b] \ne 1$. Then $x \in N \cap \langle B, B^y \rangle$ and $C_V(B) \cap C_V(B^y) \le C_V(x)$ and so

$$|V/C_V(N)| = |V/C_V(x)| \le |V/C_V(B)|^2 \le |B|^2 \le |N|.$$

Hence, N is a non-trivial offender on V in K. But this contradicts (4°), and so (5°) holds.

Since by (5°) $B \le O_2(P)$ and since $P = (P \cap K)B$, also $P \cap K$ is 2-minimal. Thus $P \cap K$ is a minimal parabolic subgroup of K fixed by B.

Let Δ be the Dynkin diagram of K and i be the node corresponding to $P \cap K$. Among all B-invariant proper $\Gamma \subset \Delta$ with i in Γ and Γ connected we choose Γ maximal. Let $T \cap K \le \widetilde{L}$ be the parabolic subgroup of K corresponding to Γ and put $L := O^{2'}(\widetilde{L})$, $Q := O_2(L)$, and $V_L := C_V(Q)$. Note that B normalizes L and thus also V_L. So by 1.2 B is a best offender on V_L. By Smith's Lemma 4.2 V_L is a simple $\mathbb{F}_2\widetilde{L}$-module. Let W be a simple $\mathbb{F}_2L$-submodule of V_L. By 2.6 and 1.2 B normalizes W and is a best offender on W.

6°. *Either $B \le LO_2(LB)$, or the following hold:*

(a) $LB/C_{LB}(W) \cong O^{\epsilon}_{2n}(q)$, $n \ge 3$, and W is the corresponding natural module.

(b) $|B/C_B(W)| \ge 4$.

Suppose that $B \not\le LO_2(LB)$. Note that $[V_0, O^2(L)] \ne 0$ since $O^2(P) \le L$ and $[V_0, O^2(P)] \ne 0$. Since Γ is connected, $C_B(W) \le O_2(LB)$. Thus B is a non-trivial best offender on W. If $|B/C_B(W)| = 2$, then B is not an over-offender on W, and by 1.3 $|B| = 2$, a contradiction to the assumptions of (Case 4).

Hence $|B/C_B(W)| \ge 4$, and by induction $LB/C_{LB}(W) \cong O^{\epsilon}_{2n}(q)$ and W is the corresponding natural module. Moreover (5°) shows that LB is not 2-minimal, so $n \ge 3$.

7°. *B acts transitively on $\Delta \setminus \Gamma$.*

There exists a node $j \in \Delta \setminus \Gamma$ such that j is adjacent to some node in Γ. Now the maximality of Γ shows that $\Delta = \Gamma \cup j^B$.

We now discuss the possibilities for $K/Z(K)$. Suppose first that $K/Z(K)$ is an untwisted group of Lie type defined over $\mathbb{F}_q$. Then (5°) shows that no element of B induces a field automorphism or graph-field automorphism in Δ. Thus B induces a graph automorphism on Δ, so Δ is of type A_m, D_m, F_4, or E_6. Since M is not 2-minimal by (5°), $m \ge 3$.

If Δ is of type D_m, then (M, V) is in the list by 7.11(b). Assume now that Δ is not of type D_m, so $m \ge 4$ if Δ is of type A_m. Since B induces a graph automorphism, (7°) yields one of the following possibilities:

(i) $|\Gamma| = m-2$, and Δ is of type A_m.

(ii) $|\Gamma| = 2$, and Δ is of type F_4.

(iii) $|\Gamma| = 4$ or 5, and Δ is of type E_6.

In all cases B acts non-trivially on Γ; in particular $B \not\leq L\mathrm{O}_2(LB)$. Hence (6°) shows that Γ is of type D_n. This rules out case (ii). Moreover, in case (i) $m = 5$ and Γ is of type D_3; and in case (iii) Γ is of type D_4. In particular, by (6°) in each of the remaining cases P is uniquely determined, $\mathrm{C}_V(\mathrm{O}_2(P))$ is a natural $\mathrm{SL}_2(q)$-module for P, and $[V_0, R] = 0$ for every other minimal Lie-parabolic subgroup R of K containing $T \cap K$. By Ronan-Smith's Lemma 4.3 this determines the module V uniquely.

If Δ is of type A_5, then V is the exterior cube of a natural $\mathrm{SL}_6(q)$-module. But then there exists an L-composition factor of V that is a natural $\mathrm{SL}_4(q)$-module. This contradicts 2.8 and 7.11(b).

If Δ is of type E_6, then V is the adjoint module for $\mathrm{E}_6(q)$. But then V has an L-composition factor isomorphic to the adjoint module for $\Omega_8^+(q)$, a similar contradiction as above.

Suppose now that $K/\mathrm{Z}(K)$ is a twisted group of Lie type over $\mathbb{F}_{q^\nu}$. Then $|\Delta \setminus \Gamma| = 1$ and B induces a field automorphism of order 2 on $\mathbb{F}_{q^\nu}$ with fixed field $\mathbb{F}_q$, so $\nu = 2$. Since M is not 2-minimal by (5°), K has Lie rank at least 2.

In all cases (5°) shows that $P/\mathrm{O}_2(P) \cong \mathrm{SL}_2(q)$, and this excludes that K is of type ${}^2F_4$, ${}^3D_4$ or ${}^2A_m$, m even. So K is of type ${}^2A_m$, m odd, ${}^2D_m$, or ${}^2E_6$.

If K is of type ${}^2D_m$, we are done by 7.11(b). Suppose that K is of type ${}^2A_m$, m odd. Since ${}^2A_3 = {}^2D_3$ we may assume in addition that $m \geq 5$, so by (7°) $|\Gamma| \geq 2$. In particular L contains a minimal parabolic subgroup R with $R/\mathrm{O}_2(R) \cong \mathrm{SL}_2(q^2)$, so $B \not\leq L\mathrm{O}_2(LB)$. Hence (6°) implies that K is of type ${}^2A_5$. Now as in the A_5-case, V is the exterior cube of the natural $\mathrm{SU}_5(q)$-module and L has a composition factor which is a natural $\mathrm{SU}_4(q)$-module. Since $\mathrm{SU}_4(q) \cong \mathrm{Spin}_6^-(q)$ this contradicts 7.11(b).

Suppose that K is of type ${}^2E_6$. Then $|\Gamma| = 3$ and with the same argument as in the previous paragraph using (6°) L is of type ${}^2D_4$. So Γ, P and V_P are uniquely determined. Now as in the E_6-case V is the adjoint module for K, and L has a composition factor isomorphic to the adjoint module for $\Omega_8^-(q)$, which contradicts 7.11(b).

The proof of Theorem 3:

Let B be a minimal offender in A and note that B is a quadratic best offender on V.

Case 1. *The case $M \cong \mathrm{G}_2(q)$, $q = 2^n$, V a natural $\mathrm{G}_2(q)$-module.*

We will use the following facts about the action of K on V and the structure of K, where i-subspace means $\mathbb{K}$-subspace of dimension i in V:

There exists an M-invariant non-degenerate symplectic form on V (since V is self-dual and $p = 2$). Let M_1 and M_2 be the pair of maximal parabolic subgroups of M with $T \leq M_i$ and such that M_i normalizes an i-subspace V_i in V. Note that V_i is singular and the graph with vertices $V_1^M \cup V_2^M$ and inclusion as incidence relation is a generalized hexagon. Since M acts transitively on $V^\sharp$, V_1^M consists of all the 1-dimensional subspaces of V.

Put $P_i := \mathrm{O}^{2'}(M_i)$, and $Q_i := \mathrm{O}_2(P_i)$. There exist exactly two classes of involutions in M with representatives $z, t \in T$ such that

(i) $t \notin \mathrm{Z}(Q_1)$, $P_1 = Q_1\mathrm{C}_M(t)$, and $P_2 = \mathrm{C}_M(z)$.

(ii) t and z and do not fix any vertex of distance larger than 3 from V_1 and V_2, respectively.

(iii) t and z fix all vertices of distance at most 3 from V_1 and V_2, respectively.

We will use these properties to show 3(a).

1°. $|\mathrm{C}_V(z)| = q^4$. *More precisely, z centralizes exactly the 1-subspaces of distance 1 and 3 from V_2.*

There are precisely $q+1$ 1-spaces of distance 1 and $q^2(q+1)$ 1-spaces of distance 3 from V_2. Hence by (ii) and (iii) $\mathrm{C}_V(z)$ has exactly $q+1+q^2(q+1) = q^3+q^2+q+1$ 1-spaces.

2°. $|\mathrm{C}_V(t)| = q^3$. *More precisely, t centralizes exactly the 1-dimensional subspaces of distance 0 and 2 from V_1.*

There is one 1-space of distance 0 and $q(q+1)$ 1-spaces of distance 2. Thus, as in (1°), $\mathrm{C}_V(t)$ contains exactly $1+q(q+1) = q^2+q+1$ 1-spaces.

3°. *Suppose $t \in B$. Then $|B| = |\mathrm{C}_V(B)| = |[V,B]| = q^3$, $\mathrm{C}_T(B) = B$, and B is uniquely determined in M_1.*

Since $\mathrm{C}_V(B) \le \mathrm{C}_V(t)$ and by (2°) and the quadratic action of B,

$$q^3 = |[V,t]| = |[V,B]| \text{ and } \mathrm{C}_V(B) = \mathrm{C}_V(t);\ \text{in particular } |B| \ge q^3.$$

By (2°) $\mathrm{C}_V(t)$ is uniquely determined by M_1, so also $B^* := \mathrm{O}^{p'}(\mathrm{C}_{M_1}(\mathrm{C}_V(t)))$ is uniquely determined. To prove the uniqueness of B in M_1, it suffices to show that $|B^*| \le q^3$ since then $B = B^*$.

Note that $[V_2^g, B^*] = 0$ for every $g \in M_1$, and so $B^* \le Q_1 \cap Q_2$. Let $x \in P_2 \setminus M_1$ and $D := B^* \cap B^{*x}$. Then $|B^*/D| \le q^2$ and $|D| \ge q$ since $|Q_2| = q^5$ and $|B^*| \ge q^3$. On the other hand, D fixes a path of length 6 with V_2 as midpoint, and (ii) yields $|D| \le q$. This shows that $|D| = q$ and consequently $|B^*| \le q^3$.

It remains to show that $B = \mathrm{C}_T(B)$. Assume that $B_0 =: \mathrm{C}_T(B) > B$. By Smiths' Lemma, $\mathrm{C}_V(Q_1) = V_1$ and so $[\mathrm{C}_V(t), Q_1] \ne 1$. From $[V_2, Q_1] \le V_1$ we get $\mathrm{C}_V(t) = \langle V_1^{P_1} \rangle$ and $[\mathrm{C}_V(t), Q_1] = V_1$. Thus $Q_1/B = Q_1/\mathrm{C}_{Q_1}(\mathrm{C}_V(t))$ is dual to the natural $\mathrm{SL}_2(q)$-module $\mathrm{C}_V(t)/V_1$. We claim that $\mathrm{C}_{Q_1}(B) \not\le B$. If $B_0 \le Q_1$ this is obvious. And if $B_0 \not\le Q_1$ we get $[Q_1, B_0] \not\le B$ and so again $\mathrm{C}_{Q_1}(B) \not\le B$. Since $\mathrm{C}_{Q_1}(B) \trianglelefteq P_1$ we conclude that $Q_1 = \mathrm{C}_{Q_1}(B)$ and $t \in \mathrm{Z}(Q_1)$, which contradicts (i).

4°. $t^M \cap B \ne \emptyset$.

Assume that $t^M \cap B = \emptyset$. Then we may assume that $z \in B$, so $\mathrm{C}_V(B) \le \mathrm{C}_V(z)$ and by (1°) $q^2 \le |V/\mathrm{C}_V(B)| \le |B|$. On the other hand, by (ii) and (1°) the non-trivial elements of $\mathrm{C}_T(\mathrm{C}_V(z))$ centralize every 1-subspace of distance at most 3 from V_2 but no singular 2-space of distance 4. Hence $|\mathrm{C}_T(\mathrm{C}_V(z))| = q$. It follows that there exists $z^g \in B$ with $\mathrm{C}_V(z) \ne \mathrm{C}_V(z^g)$ and so also $[V,z] \ne [V,z^g]$. Since $[V,z] + [V,z^g] \le \mathrm{C}_V(B) \le \mathrm{C}_V(z) \cap \mathrm{C}_V(z^g)$ and $|[V,z]| = q^2$, we conclude that

$$|\mathrm{C}_V(B)| = q^3,\ |B| = q^3 \text{ and } \mathrm{C}_V(B) = \mathrm{C}_V(z) \cap \mathrm{C}_V(z^g).$$

But then V_2 and V_2^g are of distance 2, and we may assume that $V_1 = V_2 \cap V_2^g$. Now (2°) shows that t centralizes $\mathrm{C}_V(B)$ and so $\mathrm{C}_V(B) = \mathrm{C}_V(t)$. Hence also $B\langle t \rangle$ is a quadratic offender, and (3°) yields $t \in B$, a contradiction.

5°. *Case (a) of Theorem 3 holds.*

According to (4°) we may assume that $t \in B$, and according to (3°) $\mathrm{C}_T(B) = B$ and so $A = B$. So 3(a) follows from (3°).

Case 2. *The case* $M \cong \mathrm{SL}_n(q)/\langle -\mathrm{id}^{n-1}\rangle$, $n \geq 5$, *and* V *the exterior square of a natural* $\mathbb{K}\mathrm{SL}_n(q)$*-module* W.

Let U be a T-invariant $\mathbb{K}$-hyperplane in W. Put $R := \mathrm{C}_M(W/U)$ and $I_R := \mathrm{C}_V(\mathrm{O}_p(R))$. Recall that $R/\mathrm{O}_p(R) \cong \mathrm{SL}_{n-1}(q)$ and $\mathrm{O}_p(R)$ is an natural $\mathrm{SL}_{n-1}(q)$-module for R isomorphic to U.

We will use the following properties of the exterior square:

6°. U, $\mathrm{O}_p(R)$ *and* V/I_R *are isomorphic natural* $\mathrm{SL}_{n-1}(q)$*-modules for* R.

7°. I_R *is as an* $\mathbb{F}_pR$*-module isomorphic to the exterior square of* U.

If $n \geq 6$, then by (7°) and induction B is not an over-offender on I_R. If $n = 5$, then $\mathrm{SL}_4(q) \cong \Omega_6^+(q)$ and I_R is the natural orthogonal module. Again by 3.4 B is not an over-offender. Hence, in both cases 1.3 shows that either $B \cap \mathrm{O}_p(R) = 1$ or $B \leq \mathrm{O}_p(R)$.

In the first case $|I_R/\mathrm{C}_{I_R}(B)| = |B|$ and $V = I_R + \mathrm{C}_V(B)$; in particular $[V, B] \leq I_R$. But this contradicts (6°). Thus we have $B \leq \mathrm{O}_p(R)$. Pick $b \in B^\sharp$ and put $C := \mathrm{C}_R(b)$. Then C acts as a point stabilizer on $\mathrm{O}_p(R)$ and thus by (6°) also as a point stabilizer on V/I_R. It follows that $\mathrm{C}_V(b) = I_R$ or $|\mathrm{C}_V(b)/I_R| = q$.

If $\mathrm{C}_V(B) = I_R$, then $|B| \geq |V/I_R| = q^{n-1}$ and $B = \mathrm{O}_p(R)$. Since $\mathrm{C}_T(\mathrm{O}_p(R)) = \mathrm{O}_p(R)$ we get $A = B$, and case (b) of Theorem 3 follows.

Assume now that $|\mathrm{C}_V(B)/I_R| = q$. Then $\mathrm{C}_V(B) = \mathrm{C}_V(b)$ for all $1 \neq b \in B$. Also $q^{n-2} = |V/\mathrm{C}_V(B)| \leq |B|$. Since $n \geq 5$ this gives $|B| > q$, so there exists $1 \neq b, \tilde{b} \in B$ with $\mathrm{C}_R(b) \neq \mathrm{C}_R(\tilde{b})$. Hence, $\mathrm{C}_V(B) = \mathrm{C}_V(b) = \mathrm{C}_V(\tilde{b})$ is normalized by $R = \langle \mathrm{C}_R(b), \mathrm{C}_R(\tilde{b})\rangle$, a contradiction.

Case 3. *The case* $M \cong \mathrm{Spin}_7(q)$ *or* $\mathrm{Spin}_{10}^+(q)$ *and* V *a corresponding spin module.*

We will use the following facts about the action of M on V and the structure of M. Recall that $\mathrm{P}\,\Omega_5(q) \cong \mathrm{PSp}_4(q)$. There exists $T \leq R \leq M$ such that for $I_R := \mathrm{C}_V(\mathrm{O}_p(R))$ the following hold:

(i) $\mathrm{Spin}_n^\epsilon(q)/\langle -\mathrm{id}_V\rangle \cong \Omega_n^\epsilon(q)$.

(ii) $R/\mathrm{O}_p(R) \cong \mathrm{Spin}_5(q)$ resp. $\mathrm{Spin}_8^+(q)$.

(iii) $\mathrm{O}_p(R)$ is a natural $\Omega_5(q)$- resp. $\Omega_8^+(q)$-module for R.

(iv) $I_R = [V, \mathrm{O}_p(R)]$.

(v) If $n = 7$, then V/I_R and I_R are isomorphic natural $\mathrm{Sp}_4(q)$-modules for R, but I_R is not isomorphic to $\mathrm{O}_p(R)/\mathrm{O}_p(R) \cap \mathrm{Z}(R)$; while if $n = 10$, $\mathrm{O}_p(R)$, V/I_R and I_R are pairwise non-isomorphic natural $\Omega_8^+(q)$-modules for R.

(vi) $\mathrm{O}_p(R)$ acts quadratically on V.

(vii) If $n = 7$ and Z is a 1-dimensional singular subspace of $\mathrm{O}_p(R)$, then $\mathrm{C}_M(Z)/\mathrm{O}_p(\mathrm{C}_M(Z)) \cong \mathrm{Spin}_4^+(q)$, and $V/[V, Z]$ is a natural $\Omega_4^+(q)$-module for $\mathrm{C}_M(Z)$.

Put $\delta = 1$ if $n = 7$ and $\delta = 2$ if $n = 10$. We first show:

8°. $\mathrm{C}_V(x) = I_R$ *for every non-singular* $x \in \mathrm{O}_p(R)$, *and* $|V/\mathrm{C}_V(x)| = q^{2\delta}$ *for every non-trivial singular* $x \in \mathrm{O}_p(R)$.

Let $1 \neq x \in \mathrm{O}_p(R)$. Suppose first that x is singular in $\mathrm{O}_p(R)$. Then $\mathrm{C}_M(x) \not\leq R$ and so $\mathrm{C}_V(x) \neq I_R$. Moreover, $\mathrm{C}_R(x)$ normalizes a unique proper submodule of V/I_R. This submodule has order $q^{2\delta}$ and so (8°) holds.

Suppose next that x is not singular. Then there exists $g \in M$ such that R^g and R^{gx} are opposite Lie-parabolics of M. So by 5.1 $M = \langle \mathrm{O}_p(R^g), \mathrm{O}_p(R^{gx}\rangle \leq \langle \mathrm{O}_p(R^g), x\rangle$. Thus $\mathrm{C}_V(\mathrm{O}_p(R^g)) \cap \mathrm{C}_V(x) = 0$ and $V = [V, \mathrm{O}_p(R^g)] + [V, x]$. Since $[V, \mathrm{O}_p(R^g)] \leq \mathrm{C}_V(\mathrm{O}_p(R^g))$ and $[V, x] \leq \mathrm{C}_V(x)$, this implies $[V, x] = \mathrm{C}_V(x)$ and so $\mathrm{C}_V(x) = C_V(\mathrm{O}_p(R)) = I_R$.

9°. *B is conjugate to a subgroup of* $\mathrm{O}_p(R)$.

Suppose not. Then $B \not\leq \mathrm{O}_p(R)$. Let $Z = \mathrm{O}_p(R) \cap B$. If Z contains a non-singular element b, then by (8°) $[V, B] \leq \mathrm{C}_V(B) \leq \mathrm{C}_V(b) = I_R$. But then $\langle B^R\rangle$ centralizes V/I_R, a contradiction to (v). Thus all elements in Z are singular. By 1.3 either $V = I_R + \mathrm{C}_V(B)$ and $[V, B] \leq I_R$, or B is an over-offender on I_R. The first possibility contradicts (v), so B is an over-offender on I_R. Then by 3.4

$$C_{I_R}(B) = [I_R, B],\ |\mathrm{C}_{I_R}(B)| = q^{2\delta} \text{ and } q^{2\delta} < |B/Z| = |B/B \cap \mathrm{O}_p(R)| \leq q^{3\delta}.$$

Put $\overline{V} = V/I_R$. Then B acts quadratically on $\overline{V}$. From $|B/Z| > q^{2\delta}$ and 3.4 we conclude that $|\overline{V}, B]| = q^{2\delta}$ and so also $|\overline{V}/C_{\overline{V}}(B)| = q^{2\delta}$. Thus $|V/C_V(B)| \geq q^{4\delta}$ and so $|Z| \geq q^{\delta}$. Let $1 \neq x \in Z$. Note that $[V, B] + I_R \leq C_V(x)$. Since x is singular in $\mathrm{O}_p(R)$ (8°) gives $|V/\mathrm{C}_V(x)| = q^{2\delta}$. Thus $\mathrm{C}_V(x) = [V, B] + I_R$ and $\mathrm{C}_R(x)$ normalizes $[V, B] + I_R$. But $R = \langle C_R(x), C_R(y)\rangle$ for any singular $x, y \in \mathrm{O}_p(R)$ with $\mathbb{F}_q x \neq \mathbb{F}_q y$ and since R does not normalizes $[V, B] + I_R$ we conclude that $Z \leq \mathbb{F}_q x$. Since $|Z| \geq q^{\delta}$, we conclude that Z is a 1-dimensional singular subspace of $\mathrm{O}_p(R)$. Also $\delta = 1$ and so $n = 7$.

Put $P := \mathrm{C}_M(Z)$. By (vii) $P/\mathrm{O}_p(P) \cong \mathrm{Spin}_4^+(q)$, and $\mathrm{C}_V(Z)/[V, Z]$ is the natural $\Omega_4^+(q)$-module for P. Thus every singular 1-space of $\mathrm{C}_V(Z)/[V, Z]$ is contained in a P-conjugate of $I_R/[V, Z]$, and the conjugates of $I_R/[V, Z]$ are TI-subgroups in $\mathrm{C}_V(Z)/[V, Z]$.

Since B acts quadratically on V, $[V, B]/[V, Z]$ is a 2-dimensional isotropic subspace and thus contains a 1-dimensional singular subspace. Hence there exists $g \in P$ such that $[V, B] \cap I_R^g \not\leq [V, Z]$. The TI-property of $I_R/[V, Z]$ implies that B normalizes I_R^g, so $B \leq R^g$.

If $B \not\leq \mathrm{O}_p(R^g)$, then the above also applies to B and R^g in place of B and R, so $[V, B] \cap I_R^g$ is 2-dimensional and so $[V, B] \cap I_R^g = [V, Z]$, a contradiction. Thus, we have that $B \leq \mathrm{O}_p(R^g)$, and B is not a counterexample. Hence (9°) is proved.

According to (9°) we may assume that $B \leq \mathrm{O}_p(R)$. If B does not contain a non-singular element of $\mathrm{O}_p(R)$, then $|B| \leq q^{2\delta}$. So also $|V/\mathrm{C}_V(B)| \leq q^{2\delta}$ and by (8°) $\mathrm{C}_V(B) = \mathrm{C}_V(b)$ for every $1 \neq b \in B$. On the other hand, for every such b, $\mathrm{C}_{R/\mathrm{O}_p(R)}(b)$ is contained in a unique maximal parabolic subgroup of $R/\mathrm{O}_p(R)$. It follows that B is has order at most q, a contradiction.

Hence B contains a non-singular element b. Then by (8°)

$$(+) \qquad I_R = \mathrm{C}_V(b) = [V, b] = \mathrm{C}_V(B) = [V, B] \text{ and } |B| \geq |V/\mathrm{C}_V(B)| = q^{4\delta}$$

If $M \cong \mathrm{Spin}_{10}^+(q)$, then $|\mathrm{O}_p(R)| = |I_R| = q^8 = q^{2\delta}$ and so by (+) $B = \mathrm{O}_p(R)$. Thus $A \leq \mathrm{C}_T(\mathrm{O}_p(R)) = \mathrm{O}_p(R)$ and $A = B$. Since $\mathrm{O}_p(R)$ is weakly closed in T, we see that case (d) of Theorem 3 follows from (+).

So suppose $M \cong \mathrm{Spin}_7(q)$. If $A \leq \mathrm{O}_p(R)$, then case (c) Theorem 3 follows. So assume for a contradiction that $A \not\leq \mathrm{O}_p(R)$. Observe that $[B, A] = 1$, $|B| \geq q^{2\delta} = q^4$ and $\mathrm{O}_p(R)$ is a natural $\Omega_5(q)$-module for $R/\mathrm{O}_p(R)$. We conclude that $p = 2$, $|B| = q^4$, $B = A \cap \mathrm{O}_p(B) = \mathrm{C}_{\mathrm{O}_p(R)}(A)$ and $|A/B| \leq q$. Thus $|A| \leq q^5$. Since $\mathrm{O}_p(R)/\mathrm{O}_p(R) \cap \mathrm{Z}(R)$ is not isomorphic to I_R, we get that $|I_R/C_{I_R}(A)| = q^2$ and so $|V/\mathrm{C}_V(A)| = q^6 > q^5 = |A|$. This contradiction completes (Case 3).

Case 4. *The case $M \cong 3.\mathrm{Alt}(6)$ and $|V| = 2^6$.*

Then $\mathbb{K} = \mathbb{F}_4$, $|A| = 4$, and $\mathrm{C}_V(A)$ is a $\mathbb{K}$-hyperplane, so case (e) Theorem 3 follows.

Case 5. *The case $K \cong \mathrm{Alt}(n)$, $n \geq 5$, and V the natural $\mathrm{Alt}(n)$-module for K.*

Let W be the natural permutation module for $\mathrm{Sym}(n)$ over $\mathbb{F}_2$ with basis w_i, $i \in \Omega := \{1, \ldots, n\}$, and $W_0 := \langle \sum_\Omega w_i \rangle$. For $\Psi \subseteq \Omega$ put $W_\Psi = \langle w_i + w_j \mid i, j \in \Psi \rangle$ and $\overline{W_\Psi} = W_\Psi + W_0/W_0$. Then $V \cong \overline{W_\Omega}$.

10°. *If A is a best offender, then case (g) or case (h) of Theorem 3 holds.*

Suppose that A acts transitively Ω. Then $n = 2^k$, and since $n \geq 5$, $k \geq 3$. Note that $|A| = 2^k$, $\mathrm{C}_{W_\Omega}(A) = W_0$, and $|\overline{W_\Omega}| = 2^{2^k - 2}$. The commutator map

$$\mathrm{C}_{\overline{W_\Omega}}(A) \times A \to W_0 \text{ with } (w + W_0, a) \mapsto [w, a]$$

shows that

$$|\mathrm{C}_{\overline{W_\Omega}}(A)| = |\mathrm{C}_{\overline{W_\Omega}}(A)/\overline{\mathrm{C}_{W_\Omega}(A)}| \leq |A| = 2^k,$$

and so

$$2^k = |A| \geq |V/\mathrm{C}_V(A)| = |\overline{W}_\Omega/\mathrm{C}_{\overline{W}_\Omega}(A)| \geq 2^{2^k - k - 2}.$$

Thus $2^{k-1} \leq k + 1$, so $k = 3$ and $|A| = |V/\mathrm{C}_V(A)| = 8 = |\mathrm{C}_V(A)|$. Since V is self-dual, also $|[V, A]| = 8$ and since $[V, A] \leq \mathrm{C}_V(A)$, $[V, A] = \mathrm{C}_V(A)$. Hence case (h:4) of Theorem 3 holds.

So we may assume from now on that A does not act transitively on Ω. Let Ψ be an orbit of A on Ω of length say 2^k. Since A is a best offender, A is an offender on $\overline{W_\Psi}$, and since $\Psi \neq \Omega$, $W_0 \nleq W_\Psi$ and so $\overline{W}_\Psi \cong W_\Psi$. Thus A is an offender on W_Ψ. Note that $|A/\mathrm{C}_A(W_\Psi)| = |A/\mathrm{C}_A(\Psi)| = 2^k$, $|W_\Psi| = 2^{2^k - 1}$, and $|\mathrm{C}_{W_\Psi}(A)| = |2|$. Thus $2^{2^k - 1 - 1} \leq 2^k$, $2^k \leq k + 2$ and $k \leq 2$.

Suppose A has two orbits Ψ_1 and Ψ_2 of length four and put $\Lambda := \Psi_1 \cup \Psi_2$. Assume for a contradiction that $\Lambda = \Omega$ and put $H := \mathrm{N}_M(\{\Psi_1, \Psi_2\})$. Then $H \cong \mathrm{Sym}(4) \wr C_2$ and $A \leq \mathrm{O}_2(H)$. So H acts simple on $\mathrm{O}_2(H)$. [MS1, 2.6] shows that $\mathrm{O}_2(H)$ is an offender, and the Timmesfeld Replacement theorem implies that $\mathrm{O}_2(H)$ acts quadratically on V, a contradiction. Hence $\Lambda \neq \Omega$ and so $W_\Lambda \cong \overline{W}_\Lambda$. Note that $|A/\mathrm{C}_A(W_\Lambda)| = |A/\mathrm{C}_A(\Lambda)| \leq 16$, $|W_\Lambda| = 2^7$ and $|\mathrm{C}_{W_\Lambda}(A)| = 4$. Thus $2^7/4 \leq 16$, a contradiction.

Suppose Ψ is an orbit of length 4 for A on Ω and A has a fixed-point i on Ω. Put $V_{\Psi i} := \langle w_i + w_j \mid j \in \Psi \rangle$. Then $V_{\Psi,i}$ is isomorphic to the permutations module for A on Ψ and is also isomorphic to $\overline{V_{\Psi,i}}$. Thus A is a best offender on $V_{\Psi,i}$. But $|A/\mathrm{C}_A(V_{\Psi,i})| = 4$ and $|V_{\Psi,i}/\mathrm{C}_{W_\Psi}(A)| = 8$, a contradiction.

We have proved that either all orbits of A on Ω have length 1 or 2, or A has a unique orbit of length four and all other orbits have length two.

Assume for a contradiction that $\mathrm{C}_{\overline{W_\Omega}}(A) \neq \mathrm{C}_{W_\Omega}(A)/W_0$. Then there exists $w \in W_\Omega$ such that $0 \neq [w, A] \in W_0$; in particular $A_0 := \mathrm{C}_A(w)$ has index 2 in A. Let $X \subseteq \Omega$ with $w = \sum_{i \in X} w_i$ and $|X|$ even. Then there exists $a \in A$ such that $\{X, X^a\}$ is a partition of Ω, and A_0 normalizes X and X^a. Note that $\mathrm{C}_{\overline{W_X}}(A) = \langle \overline{w} \rangle$ and that $|X| \geq 4$ since $n \geq 5$ and $|X|$ is even. Thus

$$4 \leq |\overline{W}_X/C_{\overline{W_X}}(A)| \leq |V/\mathrm{C}_V(A)| \leq |A|.$$

Thus $A_0 \neq 1$, and since $\mathrm{C}_{A_0}(X) = \mathrm{C}_{A_0}(X \cap X^a) = 1$, A_0 acts non-trivially on X. Since A has at most one orbit of length four on Ω we conclude that $|X \setminus \mathrm{C}_X(A_0)| = 2$. Thus $|A_0| = 2$ and $|A| = 4$. The Timmesfeld Replacement Theorem shows that A acts quadratically on V. But $[\overline{W_X}, A_0, a] \neq 0$, a contradiction.

We have proved that $C_{W_\Omega/W_0}(A) = C_{W_\Omega}(A)/W_0$, so $|V/C_V(A)| = |W_\Omega/C_{W_\Omega}(A)|$. If follows that A is an offender on W_Ω. Let k be the number of orbits of length 2. Assume that A has an orbit of length four, then A has no fixed-point, $n = 2k+4$, $|C_{W_\Omega}(A)| = 2^{k+1}$, $|A| \leq 2^k \cdot 4 = 2^{k+2}$, and

$$|V/C_V(A)| = |W_\Omega/C_{W_\Omega}(A)| = 2^{n-1-(k+1)} = 2^{k+2}.$$

Since A is an offender, this implies $|A| = 2^{k+2}$, and since V is self-dual, $|[V,A]| = |V/C_V(A)| = 2^{k+2} = |A|$. As A has on orbit of length 4, A is not quadratic on W_Ω and since $C_{W_\Omega/W_0}(A) = C_{W_\Omega}(A)/W_0$ also not quadratic on V. Hence case (h:3) of Theorem 3 holds.

Assume now that A does not have any orbit of length 4. Then $[V,A] \leq C_V(A)$ and $|A| \leq 2^k$. Suppose A has a fixed-point in Ω. Then $|V/C_V(A)| = 2^k = |[V,A]|$ and so $|A| = 2^k$ and case (g) or (h:1) of Theorem 3 holds. So suppose A has no fixed-points and so $n = 2k$ and $|V/C_V(A)| = 2^{k-1} = |[V,A]|$. Thus $2^{k-1} \leq |A|$.

Let $t_1, \ldots, t_k$ be the transpositions corresponding to the non-trivial orbits of orbits of A on Ω, say $t_i \in A$ if and only if $i > l$. If $l = 0$, then again case (h:1) of Theorem 3 holds. Suppose $l > 0$. Let $1 \leq r < s < l$ and put $A_{rs} = C_A(C_\Omega(\langle t_r, t_s\rangle))$. Then $|A/A_{rs}| \leq 2^{k-2}$ and so $A_{rs} \neq 1$. Since $A_{ts} \leq \langle t_r, t_s\rangle$ and neither t_r nor t_s are in A we conclude that $A_{rs} = \langle t_r t_s\rangle$. It follows that

$$A = \langle t_1t_2, t_2t_3, \ldots, t_{l-1}t_l, t_{l+1}, t_{l+2}, t_k\rangle.$$

Thus case (h:3) of Theorem 3 holds.

11°. *Every offender in M on V is a best offender.*

Let X be an offender and let $Y \leq X$ with $|C_V(Y)||Y|$ maximal and then Y minimal. By the Timmesfeld Replacement Theorem, Y is quadratic. If $|Y||C_V(Y)| = |V|$, then $|Y||C_V(Y)| = |X||C_V(X)|$ and so X is a best offender. If $|Y||C_V(Y)| > |V|$, then (10°) shows that Y is generated by a maximal set of commuting transpositions. So $X \leq C_M(Y) = Y$, $X = Y$, and X is a best offender.

Observe that (11°) together with (10°) completes (Case 5).

Case 6. *The case $M \cong \mathrm{Alt}(7)$ and $|V| = 2^4$.*

Choose $T \leq R \leq M$ with $R \cong \mathrm{Alt}(6)$. Then the previous case applies to R, and we are done. □

Theorem 8.2. *Let M be a finite $\mathcal{CK}$-group and V a faithful $\mathbb{F}_pM$-module. Suppose that there exists $K \in \mathcal{J}_M(V)$ such that $V = [V,K]$ and V is a semisimple but not simple $\mathbb{F}_pK$-module. Then one of the following holds, where q is a power of p and $J := J_M(V)$:*

1. *$J \cong \mathrm{SL}_n(q)$, $n \geq 3$, and $V \cong N^r \oplus N^{*s}$, where N is a natural $\mathrm{SL}_n(q)$-module, N^* its dual, and r, s are integers with $0 \leq r, s < n$ and $\sqrt{r} + \sqrt{s} \leq \sqrt{n}$.*
2. *$J \cong \mathrm{Sp}_{2m}(q)$, $m \geq 3$, and $V \cong N^r$, where N is a natural $\mathrm{Sp}_{2m}(q)$-module and r is a positive integer with $2r \leq m+1$.*
3. *$J \cong \mathrm{SU}_n(q)$, $n \geq 8$, and $V \cong N^r$, where N is a natural $\mathrm{SU}_n(q)$-module and r is a positive integer with $4r \leq n$.*
4. *$J \cong \Omega_n^\epsilon(q)$ with p odd if n is odd, or $M \cong \mathrm{O}_n^\epsilon(q)$ with $p = 2$ and n even. Moreover $n \geq 10$ and $V \cong N^r$, where N is a corresponding natural module and r is a positive integer with $4r \leq n-2$.*

In particular, if V is not a homogeneous $\mathbb{F}_pJ$ module, then (1) holds with $r \neq 0 \neq s$ and $n \geq 4$.

Proof. By 2.2(f) K is the unique J-component of M; in particular $K \trianglelefteq M$. Since V is a semisimple K-module we have

1°. *$V = N_1 \oplus \cdots \oplus N_m$, $m \geq 2$, where N_i is a perfect simple $\mathbb{F}_p K$-module.*

By 2.8 J normalizes N_i and by 1.2 every best offender on V is also a best offender on N_i. Moreover, $\mathrm{O}_p(J/\mathrm{C}_J(N_i)) = 1$ since N_i is simple. Hence

2°. *$J/\mathrm{C}_J(N_i)$ and N_i satisfy the hypothesis of Theorem 2.*

By 2.2 K is not solvable since $m \geq 2$, so K is a component of M. Now 2.5 shows that J acts $\mathbb{F}_i$-linearly on N_i, where $\mathbb{F}_i = \mathrm{End}_K(N_i)$. In particular $[J, \mathrm{C}_J(K)] \leq \mathrm{C}_J(N_i)$. Since K is the unique J-component and $K \not\leq \mathrm{C}_J(N_i)\mathrm{C}_J(K)$, we get from 2.2(b) $\mathrm{C}_J(N_i)\mathrm{C}_J(K) \leq \mathrm{Z}(J)$. Another application of Theorem 2 shows that $J/K\mathrm{C}_J(N_i)$ is a p-group. Hence J/K is nilpotent, and since J is generated by p-elements and $\mathrm{O}_p(\mathrm{Z}(J)) \leq \mathrm{O}_p(M) = 1$, we get that $\mathrm{Z}(J) \leq K$. It follows:

3°. *$\mathrm{C}_J(N_i) \leq \mathrm{C}_J(K) = \mathrm{Z}(J) = \mathrm{Z}(K)$.*

From now on we fix a non-trivial best offender $A \leq M$. By 2.3(b) there exists a minimal best offender $B \leq A$ such that $[V, B, A] = 0$; in particular B is quadratic on V.

Note that by (3°) $\mathrm{C}_A(N_i) = 1$, since $\mathrm{Z}(J)$ is a p'-group, and that B is a best offender on N_i by 1.2. Now (1°) implies

$$|V/\mathrm{C}_V(B)| = \prod_{i=1}^{m} |N_i/\mathrm{C}_{N_i}(B)| \leq |B|.$$

Since $m \geq 2$ there exists $N \in \{N_1, \ldots, N_r\}$ such that

4°. *$|N/\mathrm{C}_N(B)| \leq |B|^{\frac{1}{2}}$.*

Put $\mathbb{F} := \mathrm{End}_K(N)$. Then (2°) and Theorems 2 and 3 imply:

5°. *$J/\mathrm{C}_J(N) \cong \mathrm{SL}_n(q)$, $\mathrm{Sp}_n(q)$, $\mathrm{SU}_n(q)$, $\Omega_n^\epsilon(q)$ or $\mathrm{O}_n^\epsilon(q)$ (and $p = 2$), $n := \dim_{\mathbb{F}} N$ where $q := |\mathbb{F}|$ if $J/\mathrm{C}_J(N) \not\cong \mathrm{SU}_n(q)$ and $q = |\mathbb{F}|^{\frac{1}{2}}$ if $J/\mathrm{C}_J(N) \cong \mathrm{SU}_n(q)$. Moreover, N is the corresponding natural module.*

Let N^* be the $\mathbb{F}K$-module dual to N. We first treat the cases where each N_i is isomorphic to N or N^*, say $V \cong N^r \oplus N^{*s}$, $r + s = m$.

By 1.8(d) B is quadratic on N^*. Put

$$D := \mathrm{C}_J(\mathrm{C}_N(B)) \cap \mathrm{C}_J(\mathrm{C}_{N^*}(B)),\ k := \dim_{\mathbb{F}} N/\mathrm{C}_N(D),\ l = \dim_{\mathbb{F}}[N, D].$$

By 1.8(c) $l = \dim_{\mathbb{F}} N^*/\mathrm{C}_{N^*}(D)$, and by 1.8(d) $B \leq D$, $\mathrm{C}_V(D) = \mathrm{C}_V(B)$, $[V, D] = [V, B]$, and D is a quadratic offender on V. Moreover by 1.8(f) $k + l \leq n$. We get

6°. *$|V/\mathrm{C}_V(D)| = q^{rk+sl} \leq |D|$.*

Recall from 3.2 that N and N^* are isomorphic $\mathbb{F}J$-modules, if $J/\mathrm{C}_J(N)$ is not isomorphic to $\mathrm{SL}_n(q)$. We now treat the cases given in (5°) separately.

Case 1. *Suppose that $M \cong \mathrm{SL}_m(q)$ and $V \cong N^r \oplus N^{*s}$ with $r + s \geq 2$. Then (1) holds.*

By 3.4 $|D| = q^{kl}$, and (6°) gives $|V/\mathrm{C}_V(D)| = q^{rk+sl}$. Thus V is an FF-module if and only if there exists $0 < k, l < n$ with $rk + sl \leq kl$, that is $\frac{r}{l} + \frac{s}{k} \leq 1$. Increasing l decreases $\frac{r}{l} + \frac{s}{k}$. So we may assume that $k + l = n$. Put $g(k) = \frac{r}{n-k} + \frac{s}{k}$. We will determine the minimal value of $g(k)$ on the open interval $(0, n)$. If k approaches 0 or n, $g(k)$ approaches $+\infty$. So f obtains a minimum value at some point m in $(0, n)$ with $g'(m) = 0$. We have $g'(m) = \frac{r}{(n-m)^2} - \frac{s}{m^2}$. Straightforward calculations show that $m = \frac{\sqrt{s}}{\sqrt{r}+\sqrt{s}}n$, $n - m = \frac{\sqrt{r}}{\sqrt{r}+\sqrt{s}}n$ and $g(m) = \frac{(\sqrt{r}+\sqrt{s})^2}{n}$. Thus $g(m) \leq 1$ if and only if $\sqrt{r} + \sqrt{s} \leq \sqrt{n}$. So if V is an FF-module, then $\sqrt{r} + \sqrt{s} \leq \sqrt{n}$. (We remark that with a little more effort it can be shown that there even exists an integer k in $(0, n)$ with $g(k) \leq 1$, so V is an FF-module if and only if $\sqrt{r} + \sqrt{s} \leq \sqrt{n}$.)

In the remaining cases $M \cong \mathrm{Sp}_n(q)$, $\mathrm{SU}_n(q)$, $\Omega_n^\epsilon(q)$ or $\mathrm{O}_n^\epsilon(q)$ we get from 3.2(a) that $N \cong N^*$. Hence $k = l$. Recall that $[N, D]$ is an isotropic subspace of N by 3.2(e) since D is quadratic on N.

Case 2. *Suppose that $M \cong \mathrm{Sp}_n(q)$ and $V \cong N^r$ for some $r \geq 2$. Then (2) holds.*

By 3.4 $|D| = q^{\binom{k+1}{2}}$ and so as in the case (Case 1) $rk \leq \frac{k(k+1)}{2}$ and $2r \leq k + 1$. Since $[V, D]$ is isotropic and the maximal dimension of an isotropic subspace is $\frac{n}{2}$ we get $2r \leq \frac{n}{2} + 1$. Now $r \geq 2$ implies $n \geq 6$, and (2) holds.

Case 3. *Suppose that $M \cong \mathrm{SU}_n(q)$ and $V \cong N^r$ with $r \geq 2$. Then (3) holds.*

In this case $|N| = q^{2n}$. By 3.4 $|D| = q^{k^2}$ and as in the previous cases $2rk \leq k^2$ and $2r \leq k$. Moreover, since $k + l \leq n$ and $k = l$, also $2k \leq n$ and so $4r \leq n$. Now $r \geq 2$ implies $n \geq 8$.

Case 4. *Suppose that $M \cong \Omega_n^\epsilon(q)$ or $\mathrm{O}_n^\epsilon(q)$ and $p = 2$, with n even if $p = 2$, and $V \cong N^r$ for some $r \geq 2$. Then (4) holds.*

Suppose first that $[N, D]$ is singular. Then by 3.4 $|D| = q^{\binom{k}{2}}$ and so $rk \leq \binom{k}{2}$ and $2r \leq k - 1$. Since $k + l = 2k \leq n$, we get $4r \leq 2n - 2$. Now $r \geq 2$ implies (4).

Suppose next that $[N, D]$ is not singular. Then $p = 2$ and so n is even, and 3.4 yields $|D| \leq 2q^{\binom{k}{2}}$ and as in the previous cases $q^{rk} \leq 2q^{\binom{k}{2}}$. In addition, $r \geq 2$ implies $k \geq 2$. Then

$$rk \leq \log_q 2 + \binom{k}{2} \text{ and } 2r \leq \frac{2\log_q 2}{k} + k - 1.$$

If $\frac{2\log_q 2}{k} \geq 1$, then $q = 2 = k$ and $r = 1$, a contradiction. Thus $\frac{2\log_q 2}{k} < 1$ and $2r \leq k - 1$. Now again $2k \leq n$ implies that $4r \leq 2k - 2 \leq n - 2$. Since $r \geq 2$, $n \geq 10$, and (4) holds.

Case 5. *Suppose V is not a direct sum of copies of N and N^*.*

Without loss N_2 is neither isomorphic to N nor to N^*. We will show that this leads to a contradiction.

By (4°) B is an offender on $N \oplus N$. Hence we can apply the previous cases to $N \oplus N$ in place of V and get that $\dim N \geq 3$, 6, 8, and 10, respectively.

Suppose that $M/\mathrm{C}_M(N) \cong \mathrm{SL}_n(q)$ and N is the corresponding natural module. Since N_2 is not a natural module, Theorem 2 shows that N_2 is the exterior square of a natural module. For $n = 3$, $N_2 \cong N^*$ or N, which is not the case. Hence $n \geq 4$. Since B is an over-offender on N_2, Theorem 3(b) shows that $n = 4$. In this case N_2 is a natural $\Omega_6^+(q)$-module for $J/\mathrm{C}_J(N_2)$. Hence 3.4 gives

$$|N_2/\mathrm{C}_{N_2}(B)| = q^s < |B| \leq q^{\binom{s}{2}},$$

where s is the $\mathbb{F}_q$-dimension of a maximal singular subspace of N_2 centralized by B. On the other hand $2s \leq 6$ and so $s \leq 3$. But then s does not satisfy the above inequality.

Suppose $M/\mathrm{C}_M(N) \cong \mathrm{Sp}_{2n}(q)$. Then by Theorem 2 $n = 3$ and N_2 is a spin module. So we get $|B| \leq q^5$ and $|N_2/\mathrm{C}_{N_2}(B)| = q^4$. It follows that $|N/\mathrm{C}_N(B)| \leq q$, a contradiction to $|B| \geq q^4$.

Suppose that $K/\mathrm{C}_K(N) \cong \mathrm{SU}_n(q)$, $n \geq 8$, or $\Omega_n^\epsilon(q)$, $n \geq 10$. Then Theorems 2 and 3 show that every FF-module for J with an over-offender is a natural module, a contradiction.

Suppose now that V is not homogeneous as an $\mathbb{F}_2 J$-module. Then (1) holds with $r \neq 0 \neq s$. Thus $\sqrt{n} \geq \sqrt{1} + \sqrt{1} = 2$, $n \geq 4$ and all parts of the theorem are proved. □

Theorem 8.3. *Let M be a finite $\mathcal{CK}$-group with $\mathrm{O}_p(M) = 1$ and V a faithful $\mathbb{F}_p M$-module. Put $\mathcal{J} := \mathcal{J}_M(V)$, $J := \mathrm{J}_M(V)$ and $W := [V, \mathcal{J}] + \mathrm{C}_V(\mathcal{J})/\mathrm{C}_V(\mathcal{J})$. Then the following hold:*

(a) Let $K \in \mathcal{J}$. Then K is either quasisimple, or $p = 2$ or 3 and $K \cong \mathrm{SL}_2(p)'$.

(b) $[V, K, L] = 0$ for all $K \neq L \in \mathcal{J}$, and $W = \bigoplus_{K \in \mathcal{J}} [W, K]$.

(c) $J^p J' = \mathrm{O}^p(J) = \mathrm{F}^*(J) = \bigtimes \mathcal{J}$.

(d) W is a faithful semisimple $\mathbb{F}_p J$-module.

(e) $\mathrm{C}_J([W, K]) = \mathrm{C}_J([V, K])$.

Proof. (a) and the first part of (b) follow from 2.2. For the proof of the second part of (b) note that $\mathrm{C}_W(K) = \mathrm{C}_{[V,\mathcal{J}]}(K) + \mathrm{C}_V(\mathcal{J})/\mathrm{C}_V(\mathcal{J})$ since $K = \mathrm{O}^p(K)$. Thus, by the first part $\mathrm{C}_W(K) \cap [W, K] \leq \mathrm{C}_W(\mathcal{J}) = 0$.

(c): Put $J_0 := J' J^p$. First we prove:

1°. *Let $K \in \mathcal{J}$. Then J_0 induces inner automorphism on K.*

Let X be a quasisimple K-submodule of V and $Y = \mathrm{C}_X(K)$. Then we can apply 2.9 to $0 \leq Y \leq X \leq V$ and $S := X/Y$. By 2.9(a) $\widetilde{J} := J/\mathrm{C}_J(S)$ and S satisfy the hypothesis of Theorem 2. We conclude that $|\tilde{J}/\tilde{K}| \leq p$ and so $\tilde{J}_0 \leq \tilde{K}$. Since $\mathrm{C}_J(\tilde{K}) = \mathrm{C}_J(K)$ by 2.2(c), (d), (1°) holds.

Let $D := \langle \mathcal{J} \rangle$, so $D = \bigtimes \mathcal{J}$ and $D \leq J_0$ by 2.2. Moreover, $\mathrm{Z}(J) \leq J_0$ since $\mathrm{Z}(J)$ is a p'-group. By (1°) J_0 induces inner automorphisms on D. Hence $J_0 \leq D\mathrm{C}_J(D)$, and by 2.2(g) $J_0 = D\mathrm{Z}(J)$. Since J/J_0 is an elementary abelian p-group, J/D is nilpotent, and since J is generated by p-elements J/D is a p-group and so $D = J_0$.

(d): Since $\mathrm{O}^p(J) \leq \langle \mathcal{J} \rangle$, J acts nilpotently on $V/[V, \mathcal{J}]$ and $\mathrm{C}_V(\mathcal{J})$. Hence $\mathrm{C}_J(W)$ acts nilpotently on V and so $\mathrm{C}_J(W) \leq \mathrm{O}_p(M) = 1$. Thus W is faithful J-module.

By 2.8 every perfect simple K-submodule is also a simple J-submodule. Hence (d) follows if $[W, K]$ is a semisimple K-module. So suppose for a contradiction that $[W, K]$ is not semisimple K-module. We will use the bar-convention for the images of subgroups of V in W, so $\overline{X} = X + \mathrm{C}_V(D)/\mathrm{C}_V(D)$ for $X \leq V$.

Let $X_2 \leq V$ be a K-submodule of W that is minimal such that $X_2 = [X_2, K]$ and $\overline{X}_2$ is not a semisimple K-module. The minimality of X_2 implies that X_2 has a unique maximal K-submodule Y_2 such that $[Y_2, K] \neq 0$ and X_2/Y_2 is a simple K-module.

Recall that $[U, K, K] = [U, K]$ for every K-section of W since K is a J-component and thus is generated by p'-elements. It follows that $\mathrm{C}_{Y_2/\mathrm{C}_{Y_2}(K)}(K) = 0$. Hence there exists a K-submodule Y_1 of Y_2 that is maximal such that $Y_1 \neq Y_2$ and $\mathrm{C}_{Y_2/Y_1}(K) = 0$. Put $X_1 := [Y_2, K] + Y_1$. Let Z_1 be a K-submodule of Y_2 with $Y_1 < Z_1 < Y_2$. Then by maximality of Y_1, $\mathrm{C}_{Y_2/Z_1}(K) \neq 0$. Let Z_2 be the inverse image of $C_{Y_2/Z_1}(K)$ in Y_2. Then $C_{Y_2/Z_2}(K) = 0$ and so by maximality of Y_1, $Z_2 = Y_2$. Hence $X_1 = [Y_2, K] + Y_1 \leq Z_1$. It follows that X_1/Y_1 is the unique minimal K-submodule and Y_2/Y_1 is the unique maximal K-submodule of X_2/Y_1, while X_1/Y_1 and X_2/Y_2 are simple K-modules, and X_2/X_1 is a quasisimple K-module. In particular, K and $X_0 = Y_1 \leq X_1 \leq Y_2 \leq X_2$ satisfy the hypothesis of 2.9. This result shows that $S := X_1/Y_1 \oplus X_2/Y_2$ and $\widetilde{J} := J/C_J(S)$ satisfies the hypothesis of 8.2 in place of V and M. We conclude that

$$\tilde{K} \cong \mathrm{SL}_n(q),\ n \geq 3,\ \mathrm{Sp}_{2n}(q),\ n \geq 3,\ \Omega_n^{\epsilon}(q),\ n \geq 10,\ \text{ or } \mathrm{SU}_n(q),\ n \geq 8,$$

$N := X_1/Y_1$ is a corresponding natural module, and X_2/Y_2 is either isomorphic or dual to N. In particular, $C_K(N) = C_K(S) = C_K(X_2/Y_1)$. Put $\mathbb{F} := \mathrm{End}_K(N)$. Note that there exists a J-invariant symplectic, orthogonal or unitary form on N, which is non-degenerate with the exception of the natural $\mathrm{SL}_n(q)$-module, where it is the zero-form.

Let $B \leq J$ be a nontrivial quadratic best offender on $T := X_2/Y_1$ with $E := [N, B]$ minimal. Since B is quadratic on T, by 3.2 E is an isotropic subspace of N. Put $P := N_{KB}(E)$ and $Q = \langle B^P \rangle$. Then $[N, Q] \leq E \leq C_N(Q)$ and so Q is quadratic on N. In particular

$$Q' \leq C_Q(N) \cap (KB)' \leq C_K(N) = C_K(T).$$

Since $C_K(T) \leq Z(K)$ is a p'-group, this implies that Q is abelian, so $Q/C_Q(T)$ is elementary abelian. As Q contains an offender, [MS1, 2.6] and the Timmesfeld Replacement Theorem show that there exists $R \leq Q$ with $R \trianglelefteq P$ such that R is a quadratic best offender on T. The minimality of $[N, B]$ gives $[N, R] = E$.

Put $\overline{J} := J/C_J(N)$ and $U := C_K(E) \cap C_K(N/E)$. We will show next:

2°. *$\overline{U}$ does not possess any central $\overline{P}$-chief factor.*

Note that $\overline{R} \cap \overline{K} \leq \overline{U} \trianglelefteq \overline{P}$. If $\widetilde{K} \cong \mathrm{SL}_n(\mathbb{F})$ or $\mathrm{SU}_n(\mathbb{F})$, then $[\overline{U}, \overline{P}] \neq 1$ and $\overline{P}$ acts simply on $\overline{U}$, so (2°) holds.

Suppose that $\widetilde{K} \cong \mathrm{Sp}_{2n}(\mathbb{F})$ or $\Omega_{2n}^{\epsilon}(\mathbb{F})$. Let $l := \dim_{\mathbb{F}} E$. By 3.4

$$|T/C_T(R)| = q^{2l} \leq |\overline{R}| \leq q^{\binom{l+1}{2}} \text{ resp. } 2q^{\binom{l}{2}}.$$

It follows that $l \geq 3$ in the first case and $l \geq 5$ in the second case. Hence 3.5 shows that $\overline{P}$ has no central chief-factors on $\overline{U}$ and again (2°) holds.

3°. $C_{KR}(N) = C_{KR}(T)$.

Put $C := C_{KR}(N)$ and $R_0 := R \cap KC$. Note that $R_0 \leq UC$. It follows that

$$R_0C/C \leq UC/C \cong_P \overline{U}.$$

On the other hand $O^p(\overline{P})$ centralizes $R_0C/(K \cap R)C$. Hence (2°) gives $R_0 \leq (R \cap K)C$, so $R_0 = (R \cap K)C_R(N)$. This shows that

$$KC \cap KR = KR_0 = KC_R(N).$$

By 2.4 $C_R(N) = C_R(K) = C_R(T)$ and, as seen above, $C_K(N) \leq C_K(T)$, so $C_{KR}(N) = C_{KR}(T)$.

By (3°) $(KR/C_{KR}(T), T)$ satisfies the hypothesis of 6.6. It follows that there exists a K-submodule U of T with $T = Y_2/Y_1 + U$ and $N \not\leq U$, a contradiction since N is the unique minimal K-submodule of T. Thus (d) is proved.

To proof (e) put $C = \mathrm{C}_J([W, K])$. Since K acts faithfully on $[W, K]$, $C \cap K = 1$ and so $[C, K] = 1$. Since $[V, K] = [V, K, K]$ we have $[W, K] = [V, K] + \mathrm{C}_V(\mathcal{J})/\mathrm{C}_V(\mathcal{J})$ and $[V, K, C] \leq \mathrm{C}_V(\mathcal{J}))$. In particular, $\mathrm{C}_J([V, K]) \leq C$. Let $c \in C$. Then $[V, K, c] \cong [V, K]/\mathrm{C}_{[V,K]}(c)$ as a K-module. But any quotient of $[V, K]$ is a perfect K module, while any submodule of $\mathrm{C}_V(\mathcal{J})$ is a trivial K-module. So $[V, K, c] = 0$ and $C \leq \mathrm{C}_J([V, K])$. □

The proof of Theorem 1, apart from statement (e): The first four statements (a) – (d) follow from 8.3. The statements (f) and (g) follow from 8.2.

Theorem 1 (e) will be proved at the very end of the paper.

Lemma 8.4. *Let M be a finite $\mathcal{CK}$-group with $\mathrm{O}_p(M) = 1$ and V a faithful $\mathbb{F}_pM$-module. Suppose that*

(i) $M = \mathrm{J}_M(V)$ and there exists a unique $\mathrm{J}_M(V)$-component K,

(ii) $\mathrm{C}_V(K) \leq [V, K]$ and either $\mathrm{C}_V(K) \neq 0$ or $V \neq [V, K]$.

Let $A \leq M$ be a best offender on V and put $W := [V, K]$ and $\overline{V} := V/\mathrm{C}_V(K)$. Then $p = 2$, and one of the following holds:

(a) $M = K \cong \mathrm{SL}_3(2)$, $V = W$, $|\mathrm{C}_V(K)| = 2$, $\overline{V}$ is a natural $\mathrm{SL}_3(2)$-module, $|A| = 4$, $[\overline{V}, A]| = 2$ and $\mathrm{C}_V(A) = [V, A]$ has order 4.

(b) $M = K \cong \mathrm{SL}_3(2)$, $|V/W| = 2$, $\mathrm{C}_V(K) = 0$, W is a natural $\mathrm{SL}_3(2)$-module, $|A| = 4 = |\mathrm{C}_W(A)|$ and $\mathrm{C}_V(A) = [V, A] = \mathrm{C}_W(A)$.

(c) $M = K \cong \mathrm{SU}_4(2)$, $V = W$, $2 \leq |\mathrm{C}_V(K)| \leq 4$, $\overline{V}$ is a natural $\mathrm{SU}_4(2)$-module, A is the centralizer of a singular 2-subspace of $\overline{V}$, and $\mathrm{C}_V(A) = [V, A]$.

(d) $M \cong \mathrm{G}_2(q)$, $q = 2^k$, $V = W$, $2 \leq |\mathrm{C}_V(K)| \leq q$, $\overline{V}$ is a natural $\mathrm{G}_2(q)$-module, $|A| = q^3$, and $\mathrm{C}_V(A) = [V, A]$.

(e) $K \cong \mathrm{Alt}(2m)$ and $M \cong \mathrm{Sym}(2m)$ or $\mathrm{Alt}(2m)$. For $\Omega = \{1, 2, \ldots, 2m\}$ let $N = \{n_\Sigma \mid \Sigma \subseteq \Omega\}$ be the $2m$-dimensional natural permutation module and $\tilde{N}$ be the $\mathbb{F}_2M$-module defined by $\tilde{N} = N$ as an $\mathbb{F}_2$-space and

$$n_\Sigma^g = n_{\Sigma^g} \text{ if } |\Sigma| \text{ is even or } g \in \mathrm{Alt}(\Omega), \text{ and } n_\Sigma^g = n_{\Sigma^g} + n_\Omega \text{ if } |\Sigma| \text{ is odd and } g \notin \mathrm{Alt}(\Omega).$$

Then one of the following holds, where $t_1, t_2, \ldots, t_m$ is a maximal set of commuting transpositions:

1. *$M = \mathrm{Sym}(n)$, V is isomorphic to N or $N/\mathrm{C}_N(K)$, and $A = \langle t_1, t_2, \ldots, t_k\rangle$ for some $1 \leq k \leq m$.*
2. *$M = \mathrm{Sym}(n)$, $V \cong \tilde{N}$ and $A = \langle t_1, t_2, \ldots, t_m\rangle$.*
3. *$V \cong [N, K]$ and A fulfills one of the cases (h:1) – (h:3) of Theorem 3.*

(f) $M = K \cong \mathrm{Sp}_{2m}(q)$, $m \geq 1$, $q = 2^k$, $(m, q) \neq (1, 2), (2, 2)$, and $\overline{W}$ is the direct sum of r natural $\mathrm{Sp}_{2n}(q)$-modules.[4] Moreover, the following hold:

[4] Observe that for $m = 1$, $\mathrm{Sp}_2(q) \cong \mathrm{SL}_2(q)$ and a natural $\mathrm{Sp}_2(q)$-module is also a natural $\mathrm{SL}_2(q)$-module.

(a) $2r \le m+1$*, and if* $V \ne W$ *then* $m > 1$ *and* $2r < m+1$*.*

(b) Let X *be the* $2m+2$*-dimensional* $\mathbb{F}_q M$*-module obtained from the embedding* $\mathrm{Sp}_{2m}(q) \cong \Omega_{2m+1}(q) \le \Omega^{\pm}_{2m+2}(q)$*. Then* V *is isomorphic to an* $\mathbb{F}_p M$*-section of* X^r*.*

Proof. Suppose K is not quasisimple. Then K is a p'-group and $V = [V,K] \oplus \mathrm{C}_V(K)$. Since $\mathrm{C}_V(K) \le [V,K]$ this gives $\mathrm{C}_V(K) = 0$ and $V = [V,K]$, contrary to the assumptions.

Thus K is quasisimple. By 8.3, $\overline{W}$ is a semisimple K-module and we conclude that there exists simple K-submodule of $\overline{U}$ of $\overline{W}$ such that $\mathrm{H}^1(K,\overline{U}) \ne 0$ or $\mathrm{H}^1(K,\overline{U}^*) \ne 0$.

Let $B := \mathrm{C}_A([V,A])$. By the Timmesfeld Replacement Theorem, B is a non-trivial quadratic best offender on V. Note that by 2.4 and 1.2 A and B are offenders on $\overline{U}$ and $\overline{W}$. Comparing 6.1 with Theorem 1(g) we see that $p = 2$ and the following holds:

1°. $M \cong \mathrm{SL}_3(2)$, $\mathrm{SU}_4(2)$, $\mathrm{G}_2(q)$, $\mathrm{Alt}(2m)$, $\mathrm{Sym}(2m)$ *or* $\mathrm{Sp}_{2m}(q)$*, and* $\overline{W}$ *is the corresponding natural module, with the exception of the* $\mathrm{Sp}_{2m}(q)$*-case, where* $\overline{W}$ *is the direct sum of* r *natural modules for some integer* r *with* $2r \le m+1$*.*

We now discuss the cases given in (1°) (and 6.1) separately.

Case 1. *Suppose* $M \cong \mathrm{SL}_3(2)$ *and* $\mathrm{C}_W(K) \ne 0$*.*

Let $1 \ne a \in A$. Since $W = [W,K]$ has order 2^4 and K is generated by three conjugates of a, $|[W,a]| = |W/\mathrm{C}_W(a)| = 4$. Since A is an offender we conclude that

$$A = B,\ |V/\mathrm{C}_V(A)| = |A| = |\mathrm{C}_W(A)| = 4.$$

In particular $\mathrm{C}_W(A) = [W,A]$, $V = \mathrm{C}_V(A) + W$ and $|[\overline{V},A]| = 2$. The latter fact shows that $V = W + \mathrm{C}_V(K)$ and thus $W = V$. Hence (a) holds in this case.

Case 2. *Suppose* $M \cong \mathrm{SL}_3(2)$ *and* $\mathrm{C}_W(K) = 0$*.*

Then W is a natural module and $V \ne W$. As above, for $1 \ne a \in A$, $|V/\mathrm{C}_V(a)| = |A| = 4$, and $\mathrm{C}_V(a) = \mathrm{C}_W(a) = \mathrm{C}_V(A)$. Hence (b) holds.

Case 3. *Suppose* $M \cong \mathrm{SU}_4(2)$*.*

Then $[\overline{W},B]$ is a singular subspace of $\overline{W}$, and 3.4 shows that $|B| = 2^4 = |\overline{W}/\mathrm{C}_{\overline{W}}(B)|$. Thus $A = B$ and $|V/\mathrm{C}_V(A)| = 2^4$. Moreover, by 5.1 M is generated by two conjugates of A and so $|V/\mathrm{C}_V(K)| = 2^8$ and $V = W + \mathrm{C}_V(K)$. Hence $V = W$. As $[V,A]/[V,A] \cap \mathrm{C}_V(K)$ has order 2^4 and M is generated by two conjugates of A, $\mathrm{C}_V(K) \le [V,A]$. Since $\mathrm{C}_{\overline{V}}(A) = [\overline{V},A]$ this gives $\mathrm{C}_V(A) = [V,A]$, and (c) holds.

Case 4. *Suppose* $M \cong \mathrm{G}_2(q)$*.*

Then $|A| = q^3$, $\mathrm{C}_{\overline{W}}(A) = [\overline{W},A]$ has order q^3, $|\overline{W}| = q^6$, and by 5.2 M is generated by two conjugates of A A similar argument as in the $\mathrm{SU}_4(2)$ case now shows that (d) holds.

Case 5. *Suppose* $M \cong \mathrm{Alt}(2m)$ *or* $\mathrm{Sym}(2m)$*.*

Since K is perfect, V is as an $\mathbb{F}_2K$-module isomorphic to a section of the $2m$-dimensional permutation module N. If $V = W$ or $\mathrm{C}_V(K) = 0$ we have $\mathrm{C}_{\mathrm{GL}(V)}(K) = 1$ and so V is also an $\mathbb{F}_2M$-module isomorphic to N.

If $H = \mathrm{Sym}(n)$ and $|V| = 2^{2m}$, there are two possible isomorphism types for V, namely N and $\tilde{N}$ as described in (e). Note that if t is a transposition, and $V \cong \tilde{N}$, then $\mathrm{C}_V(t) \le W$. Since A is an offender on $\overline{W}$ we can apply Theorem 3(h).

Suppose that $C_V(A) \not\leq W$. Then there exists a proper subset Σ of $\Omega = \{1, 2, \ldots, 2m\}$ such that $|\Sigma|$ is odd and $|A|$ normalizes $\{\Sigma, \Omega \setminus \Sigma\}$. If Σ is A invariant, then A has a fixed-point on Σ. It follows from Theorem 3(h) that A is generated by transpositions, $V \not\cong \tilde{N}$, and (e:1) holds. So suppose for a contradiction that $\Sigma^a = \Omega \setminus \Sigma$ for some $a \in A$. Then $|\Sigma| = m$ is odd. So Theorem 3(h:4) does not hold. Put $A_0 := N_A(\Sigma)$. Note that $\mathrm{Supp}(b) = \Omega$ for all $a \in A \setminus A_0$ and so $b \in A_0$ for all $b \in A$ with with $|\mathrm{Supp}(b)| \leq 4$. In the first three cases of Theorem 3(h), A is generated by such elements, so $A = A_0$, a contradiction.

Suppose that $C_V(A) \leq W$. If $W \neq V$ we conclude that A is an over-offender on W. Thus by Theorem 3(h) A is generated by a maximal set of commuting transpositions. Hence (e:1) or (e:2) holds.

Assume that $W = V$. Then $W \cong [N, K]$. If $2m = 8$ and A acts transitively on Ω, then $C_V(A) = C_V(K)$ and $|V/C_V(A)| = 2^6 \geq 2^3 = |A|$, a contradiction. This excludes case (h:4) of Theorem 3, and (e:3) holds.

Case 6. *Suppose* $M \cong \mathrm{Sp}_{2m}(q)$.

Since K is perfect we conclude from 6.1, (1°) and 8.2(2) that it remains to prove the second statement of (f:a). Since A is an offender on $\overline{V}$ we may assume that $C_V(K) = 0$ and so $V \neq W$.

Suppose that there exists $v \in C_V(A) \setminus W$. Then $C_K(v)$ is contained in a subgroup isomorphic to $O^{\epsilon}_{2m}(V)$, and 8.2(4) shows that $4r \leq 2m - 2$. Thus $2r \leq m - 1 < m + 1$.

Suppose next that $C_V(A) \leq W$. Since $V \neq W$ we conclude that A is an over-offender on W. The proof of 8.2(Case 2) now shows that $r < m + 1$. □

Corollary 8.5. *Assume the hypothesis of 8.4. Then every best offender in M on V is a best offender on $[V, \mathcal{J}] + C_V(\mathcal{J})/C_V(\mathcal{J})$.*

Proof. According to 1.2 we may assume that $V = [V, \mathcal{J}]$. Put $\overline{V} := V/C_V(\mathcal{J}) =: W$ and $X := C_V(\mathcal{J})$. Let A be a best offender in M on V. Choose $1 \neq B \leq A$ such that $|B||C_W(B)|$ is maximal and then B minimal. Since A is an offender on W, B is a quadratic best offender on W.

Suppose that $C_W(B) = \overline{C_V(B)}$. Since A is a best offender on V, $|C_V(B)||B| \leq |C_V(A)||A|$ and since $B \leq A$, $C_X(B) \geq C_X(A)$. Thus

$$|C_W(B)||B| = \frac{|C_V(B)||B|}{|C_X(B)|} \leq \frac{|C_V(A)||A|}{|C_X(A)|} = |\overline{C_V(A)}||A| \leq |C_W(A)||A|,$$

and so A is a best offender on W.

Suppose that $C_W(B) \neq \overline{C_V(B)}$. Since $\overline{V}$ is J-semisimple by 8.3, there exists a perfect J-submodule Y of V such that $\overline{Y}$ is simple and $C_{\overline{Y}}(B) \neq \overline{C_Y(B)}$. Note that there exists a unique J-component K with $[Y, K] \neq 0$. Moreover, $Y = [Y, K]$ and $Y \cap X = C_Y(K) \neq 0$. Put $\tilde{J} := J/C_J(Y)$. The Three Subgroups Lemma implies that $O_p(\tilde{J})$ centralizes Y and so we can apply 8.4 to $(\tilde{J}, \tilde{K}, Y)$ in place of (H, K, V).

In Case 8.4(d),(f) we have $C_J(v) = C_J(\overline{v})$ for all $v \in V$, a contradiction.

In Case 8.4(c) we get $\tilde{A} = \tilde{B}$ and $C_{\overline{V}}(B) = [\overline{V}, A] = \overline{C_V(A)} = \overline{C_V(B)}$, contradiction.

Suppose 8.4(e) holds. Then A is generated by elements of support at most 4 and so $C_{\overline{V}}(A) = \overline{C_V(A)}$.

Suppose that 8.4(a) holds. Then $|\tilde{A}| = 4$ and $C_{\overline{Y}}(A) = [\overline{Y}, A] = \overline{C_Y(A)}$. Thus $\tilde{B} \neq \tilde{A}$ and $|\tilde{B}| = 2 = |\overline{Y}/C_{\overline{Y}}(B)|$. Put $B_0 = C_B(\overline{Y})$. Then $|C_W(B)||B| = |C_W(B_0)||B_0|$. The minimal choice of B implies $B_0 = 1$ and so $|B| = 2$. Thus $|C_W(B)||B| = |W|$. Since A is an offender on W, this gives $|C_W(B)||B| \leq |C_W(A)||A|$. Thus A is a best offender on W.

Finally Case 8.4(b) does not apply, since $C_V(K) \neq 0$. □

The proof of Theorem 1(e): This is 8.5.

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