A characterization of $\mathrm{Aut}(G_2(3))$

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Let p be prime and G a finite group. We say that G has characteristic p if $C_G(O_p(G)) \leq O_p(G)$ and that G has local characteristic p if all p-local subgroups of G have characteristic p. G is a $\mathcal{K}_p$–group, if any simple section of any p-local subgroup of G is a know finite simple group, that is an abelian, an alternating group, a group of Lie type or one of the 26 sporadic groups. This paper is part of a program to investigate $\mathcal{K}_p$-groups of local characteristic p. See [MeStStr1] for an overview.

Of fundamental importance to theory of groups of local characteristic p are large subgroups: A p–subgroup of a group G is called large if

(i) $C_G(Q) \leq Q$ and

(ii) $N_G(U) \leq N_G(Q)$ for all $1 \neq U \leq C_G(Q)$.

For example, if G is simple group of Lie-type in characteristic p, $S \in \mathrm{Syl}_p(G)$ and $Q = O_p(C_G(Z(S)))$, then Q is almost always a large subgroup of G. Indeed this is true exactly when $Z(S)$ is a root group, that is if G is neither $Sp_{2n}(2^k)$, $n \geq 2$, $F_4(2^k)$ nor $G_2(3^k)$.

If Q is a large subgroup of G, then it easy to see that also $O_p(N_G(Q))$ is a large subgroup of G. For a finite group L let Y_L be the unique maximal elementary abelian normal p–subgroup of L with $O_p(L/C_L(Y_L)) = 1$. Such a group exists (see for example [MeStStr1, Lemma 2.0.1(a)]).

Let G be a finite $\mathcal{K}_p$-group of local characteristic p, S a Sylow p-subgroup of G and Q a large p-subgroup of G with $Q \leq S$ and $Q = O_p(N_G(Q))$. Let M be a p-local subgroup of G with $S \leq M$ and $Q \ntrianglelefteq M$. The Structure Theorem (see [MeStStr2]) determines the pair $(M/C_M(Y_M), Y_M)$. The proof of the Structure Theorem is subdivided into the cases $Y_M \leq Q$ and $Y_M \nleq Q$. Put $M^\circ = \langle Q^M \rangle$, $\overline{M} = M/C_M(Y_M)$ and $V = [Y_M, M^\circ]$. For the case that $Y_M \nleq Q$ the Structure Theorem asserts that one of the following holds:

1. [**a**] There exists a normal subgroup K of $\overline{M}$ such that $K = K_1 \circ K_2$ with $K_i \cong SL_{m_i}(q)$, $Y_M \cong V_1 \otimes V_2$, where V_i is a natural module for K_i, and $\overline{M^\circ}$ is one of K_1, K_2 or $K_1 \circ K_2$.

2. [**b**] $(\overline{M^\circ}, p, V)$ is as in the following table:

$\overline{M^\circ}$	p	V	$\overline{M^\circ}$	p	V
$\mathrm{SL}_n(q)$	p	V_{nat}	$\mathrm{O}_4^+(2)$	2	V_{nat}
$\mathrm{SL}_n(q)$	p	$\bigwedge^2(V_{\mathrm{nat}})$	$\Omega_{10}^\pm(q)$	2	halfspin
$\mathrm{SL}_n(q)$	p	$\mathrm{S}^2(V_{\mathrm{nat}})$	$\mathrm{E}_6(q)$	p	q^{27}
$\mathrm{SL}_n(q^2)$	p	$V_{\mathrm{nat}} \otimes V_{\mathrm{nat}}^q$	M_{11}	3	3^5
$3\mathrm{Alt}(6), 3\mathrm{Sym}(6),$	2	2^6	$2M_{12}$	3	3^6
$\Gamma\mathrm{SL}_2(4), \Gamma\mathrm{GL}_2(4)$	2	V_{nat}	M_{22}	2	2^{10}
$\mathrm{Sp}_{2n}(q)$	2	V_{nat}	M_{24}	2	2^{11}
$\Omega_n^\pm(q)$	p	V_{nat}			

Here q is a power of p and V_{nat} denotes the natural module of a classical group.

A priori there is no reason why one could not have that $Y_M \not\leq Q$ and $[Y_M, M^\circ] \leq Q$. Indeed this does happen, but a corollary in [MeStStr2] states that its only possible if $M/C_M(Y_M) \cong SL_3(2)$ and $[Y_M, M^\circ]$ is a natural module. The purpose of this paper is to determine G in this case. We will show that $O_2(M) = Y_M$, Q is extra-special of order 2^5, $N_G(Q) = C_G(Z(Q))$ and $N_G(Q)/Q \cong \mathrm{Sym}(3) \times \mathrm{Sym}(3)$. This allows us to conclude that G possess a subgroup G^* of index two. A result of Aschbacher [Asch] then shows that G^* is isomorphic to $G_2(3)$. More precisely we prove:

Theorem 1. [**main**] *Let G be a finite $\mathcal{K}_2$-group, S a Sylow 2-subgroup of G and $Q \leq S \leq M \leq G$. Suppose that*

(i) [**a**] *Q is a large 2-subgroup of G and $Q = O_2(N_G(Q))$;*

(ii) [**b**] *$M/O_2(M) \cong L_3(2)$ and $[Y_M, M]$ is a natural $SL_3(2)$-module for M; and*

(iii) [**c**] *$Y_M \not\leq Q$ and $[Y_M, M] \leq Q$.*

Then G is isomorphic to $\mathrm{Aut}(G_2(3))$.

We remark that the proof of this theorem is independent from the Structure Theorem. In a forthcoming paper we will determine the structure of G in the remaining cases for $Y_M \not\leq Q$ in the Structure Theorem.

1 Preliminaries

In this section we collect some results on modules for quasisimple groups, which will be needed in the proof of the theorem.

As the three dimensional module for $SL_3(2)$ will play a prominent role, we start with collecting some facts about this module:

Lemma 1.1. [**l32**] *Let $M = SL_3(2)$ and V a corresponding natural $\mathbb{F}_2M$-module. Let W_1 be the transvection group in M to a point in V and W_2 the transvection group to a hyperplane in V.*

(a) [**i**] *Let τ_i be an element of order 3 in G normalizing W_i, $i = 1, 2$, then $[[W_1, V], \tau_1] = 1$, while $[[W_2, V], \tau_2] = [W_2, V]$.*

(b) [**ii**] *Let V_1 be a $\mathbb{F}_2M$-module with $[V_1, M] = V$, $C_{V_1}(M) = 0$ and $V_1 \neq V$. Then $|V_1/V| = 2$, $V = [V_1, W_1]$ and $[V_1, W_2] = [V, W_2]$. In particular, W_1 does not act quadratically on V_1.*

(c) [**iii**] *Let V_1 be as in (b) and $v \in V_1 \setminus V$. Then $|C_M(v)| = 21$ and M acts transitively on $V_1 \setminus V$.*

Proof. (a) is clear. To prove (b) let $t \in W_2^\sharp$. Then $[V_1, t] \leq C_V(t) = C_V(W_2) = [V, W_2]$ and so $[V_1, W_2] = [V, W_2]$. Put $V_2 = V + C_{V_1}(t)$ and note that V_2 is an $\mathbb{F}_2M$-submodule of V_1 and $|V_2/C_{V_2}(t)| = 2$. Let $U = N_M(W_2)$. So $U \cong \mathrm{Sym}(4)$ and we may generate U by three conjugates of t. Hence $|V_2/C_{V_2}(U) \leq 2^3$. Since $C_V(U) = 0$, we get $V_2 = V \oplus C_{V_2}(U)$. Gaschütz' Theorem [Hu, (I.17.4)] now shows that V_2 splits over V. Since $C_V(M) = 0$ we conclude that $V = V_2$ and $C_{V_1}(t) \leq V$. From $[V_2, t] \leq C_V(t)$ we have $|V_1/C_V(t)| = |V_1/C_{V_1}(t)| = |[V_2, t]| \leq 4$. Since $V_1 \neq V$ this implies that $|V_1/V| = 2$ and $[V_1, t] = C_V(t)$. Note that $V = \langle C_V(t) \mid t \in W_1^\sharp \rangle$ and so $V = [V, W_1]$. Thus (b) holds.

Let $v \in V_1 \setminus V$. Since $C_{V_1}(t) \leq V$, $C_M(v)$ has odd order. Thus $8 \leq |M/C_M(v)| = |v^M| \leq |v + V| = 8$ and (c) holds. □

A finite group is a $\mathcal{CK}$-group if all of its composition factors are known finite simple groups.

Lemma 1.2. [**kleinlie**] *Let H be a finite $\mathcal{CK}$-group, V a faithful $\mathbb{F}_2H$–module and x a 2-central involution in H. Put $L = F^*(H)$. Suppose that*

(i) [**a**] *L is quasisimple and V is a simple $\mathbb{F}_2L$-module; and*

(ii) [**b**] *$H = L\langle x \rangle$, $|[V, x]| \leq 4$ and x is contained in a quadratic fours group of H on V.*

Then one of the following holds:

1. [**i**] *$H \cong SL_n(2)$, $SL_n(4)$, $Sp_{2n}(2)$ $Sp_{2n}(4)$, $SU_n(2)$, $G_2(2)'$ or $\Omega^\pm_{2n}(2)$ and V is a corresponding natural module.*

2. [**ii**] *$H \cong Sp(6, 2)$, V is the spin module and x is a short root element.*

3. [**iii**] *$H \cong 3\mathrm{Alt}(6)$ and V is the 6–dimensional module.*

4. [**iv**] *$L \cong \mathrm{Alt}(n)$ and V is the permutation module. Moreover, either $H \cong \mathrm{Sym}(n)$ and x is 2-cycle or $H \cong \mathrm{Alt}(n)$ and x is a double 2-cycle.*

5. [**v**] *$H \cong \mathrm{Alt}(7)$ and V is the four dimensional module.*

Proof. Suppose first that $L/Z(L)$ is a group of Lie type in characteristic 2. Since $O_p(L) = 1$ we conclude from [Gr] that L itself is a group of Lie-type. Since x is 2-central we have $x \in L$ and so $H = L$. Then by [PaRo, 14.25] either (1) holds or $L \cong Sp_6(2)$ and V is the spin module.

Consider the latter case and let S be a Sylow 2–subgroup of L with $x \in Z(S)$. Let W be the natural module for L. Then $[Z(S), W]$ is 2–dimensional and singular. So there exists $u \in W$ such that $\langle [W, Z(S)], u\rangle$ is a 3–dimensional singular space. Denote by y the transvection to u. Then we have that $C_L(y)$ acts irreducibly on $V_y = C_V(O_2(C_G(y)))$ by [Sm1]. So V_y is the natural module for $C_L(y)/O_2(C_L(y)) \cong Sp_4(2)$. As $[V, y] \cap V_y \not\leq 1$, we see that $V_y \leq [V, y] \leq C_V(y)$. In particular, $V/C_V(y)$ involves a natural module isomorphic to V_y. Further this natural modules is not isomorphic to $O_2(C_L(y))/\langle y\rangle$ as $C_L(y)$–module. By the choice of y, we have that $Z(S) \cap O_2(C_L(y)) = 1$ and $Z(S)O_2(C_L(y))/O_2(C_L(y)) = Z(S/O_2(C_L(y))$. Since $|[V, y]| = 4$, x has to induce a transvection on V_y and so does not act as a transvection on $O_2(C_L(y))/\langle y\rangle$. Hence x is a short root element in $C_L(y)/O_2(C_L(y))$ and then also in L. Thus (1) holds.

So we may assume from now on that $L/Z(L)$ is not a group of Lie-type in characteristic 2. Since $|[V, x]| \leq 4$, [PaRo, 15.3] shows that $L/Z(L)$ is not a sporadic group.

Suppose $L \cong \mathrm{Alt}(6)$, $2F_4(2)'$ or $G_2(2)'$. Since x is 2-central either $H = L$ or $H \cong Sp_4(2)$. In the first case we are done by [PaRo, 14.29] and in the second by [PaRo, 14.25].

Suppose now $L/Z(L) \cong \mathrm{Alt}(n)$ but $Z(L) \neq 1$. Then by [Gr] $n = 6$ or 7 and $|Z(L)| = 3$. As $[V, x]| \leq 4$ this forces $[Z(L), x] = 1$. Thus $x \in L$, $H = L$ and H can be generated by three conjugates of x. Therefore $|V| \leq 64$ and so $n = 6$, the assertion (3).

Suppose next that $L \cong \mathrm{Alt}(n)$, $n = 7$ or $n \geq 9$. If V is the permutation module, then $|[V, x]| \leq 4$ implies that x is a 2-cycle or a double 2-cycle and (4) holds.

If V is not the permutation module, then since M contains a quadratic fours-group on V, V is the spin-module (see [MeiStr2]). In particular, the 3-cycles in M act fixed-point freely on V. If x is not a fixed-point free permutation, then x inverts a three cycle d and so $|V| = |[V, d]| \leq [V, x]|^2 = 16$. Thus (5) holds. So suppose that x is a fixed-point free permutation. Then n is even, $n \geq 10$ and x inverts a double 3-cycle. Since a 3-cycle is the product of two double 3-cycles we conclude that $|V| \leq |[V, x]|^4 = 2^8$, a contradiction to $n \geq 10$.

Suppose finally that $L/Z(L)$ is a group of Lie-type in odd characteristic. Since M contains a quadratic fours group, [MeiStr1] show that $L \sim 3.U_4(3)$. Since x is 2-central, $x \in L$ and since L has a unique conjugacy class of involutions, we see that x is contained subgroup K of L with $K \cong 3.\mathrm{Alt}(6)$. Let U be any composition factor for K on V. Since $Z(K) \leq Z(L)$, U is a faithful K-module. By the 3.Alt(6)-case, $|U| = 2^6$ and since $[U, x]$ is $Z(K)$-invariant, $|[U, x]| \geq 4 = |[V, x]|$. Thus U is the only composition factor for K on V and $|V| = 2^6$, a contradiction, since 3^7 divides $|L|$ but not $|GL_6(2)|$. □

Lemma 1.3. [**char irr**] *Let H be a group, $\mathbb{F}$ a field, W an $\mathbb{F}H$-module and $A \trianglelefteq B \leq H$. Suppose that there exist a simple $\mathbb{F}B$-submodule Y of W with $[W, A] \leq Y$ and $W = \langle Y^H\rangle$. Then every proper $\mathbb{F}H$-submodule of W is centralized by $\langle A^H\rangle$. In particular, $W/C_W(\langle A^H\rangle)$*

is a simple $\mathbb{F}H$*-module.*

Proof. Let U be a submodule of $\mathbb{F}H$-submodule of W with $U \neq W$. Since $W = \langle Y^H \rangle$ we have $Y \not\leq U$. Hence $[U, A] \lneq Y$ and since Y is a simple B-module, $[U, A] = 1$. Thus also $[U, \langle A^H \rangle] = 1$. □

2 Proof of the Theorem

In this section we prove Theorem 1. So let G, M, S and Q be as there. We set $\tilde{C} = N_G(Q)$, $V = [Y_M, M]$, $\tilde{M} = N_G(V)$, $M^\circ = \langle Q^M \rangle$, $Z = \Omega_1 \mathrm{Z}(S)$ and $Q_M = O_2(M)$.

Let L be minimal in $\tilde{C}$ such that L is $M \cap \tilde{C}$-invariant and $Y_M \not\leq O_2(LY_M)$. Set $W = \langle V^L \rangle$, $B = (M \cap \tilde{C})(L \cap \tilde{M})$, $M_1 = MB$, $M_2 = LB$, $Q_i = O_2(M_i)$, $H = LY_M$ and $T = O^2(M \cap \tilde{C})$. Note here that $M \trianglelefteq M_1 \leq \tilde{M}$, $L \trianglelefteq M_2 \leq \tilde{C}$ and $B \leq M_1 \cap M_2$. For $X \leq M_2$ put $\overline{X} = XQ_2/Q_2$ and for $X \leq W$ put $\hat{X} = XZ(W)/Z(W)$.

Lemma 2.1. [**M**]

(a) [**f**] $C_G(M^\circ) = 1$, $Z(M) = 1$ *and* $Y_M = \Omega_1 \mathrm{Z}(Q_M)$.

(b) [**a**] $|Z| = 2$, $M \cap \tilde{C} = C_M(Z)$, $QQ_M = O_2(M \cap \tilde{C})$, $M = M^\circ Q_M$ *and* $[Y_M, Q] = V$.

(c) [**b**] $\tilde{M} = M^\circ C_G(V)$ *and* $[M^\circ, C_G(V)] \leq O_2(M^\circ) \leq O_2(\tilde{M}) \leq Q_M$.

(d) [**g**] $M_1 = M^\circ B = M^\circ(L \cap \tilde{M})$ *and* M_1 *is a subgroup of* $\tilde{M}$.

(e) [**c**] $Y_M \trianglelefteq \tilde{M}$ *and* $C_G(V) = C_G(Y_M)$.

(f) [**d**] $O_2(M^\circ) = M^\circ \cap Q_1$, $B = (M^\circ \cap B)C_B(V)$, $C_{M_1}(V) = C_B(V)$ *and* $M_1/Q_1 = M^\circ Q_1/Q_1 \times C_B(V)/Q_1$.

(g) [**e**] $O_2(B) = Q_1 Q_2 = Q_1 Q$.

Proof. (a) If $Q \leq Q_M$, then $Y_M \leq C_G(Q) \leq Q$, a contradiction to the assumptions. Thus $Q \not\leq Q_M$. Suppose $C_G(M^\circ) \neq 1$. Then since Q is large, $M \leq N_G(C_G(M^\circ)) \leq N_G(Q) = \tilde{C}$ and so $Q = O_2(\tilde{C}) \leq O_2(M) = Q_M$, a contradiction. Hence $C_G(M^\circ) = 1$ and so also $Z(M) = 1$. Clearly $Y_M \leq \Omega_1 \mathrm{Z}(Q_M)$. Since $Q_M \leq C_M(\Omega_1 \mathrm{Z}(Q_M)) \triangleleft M$ and $M/Q_M (\cong SL_3(2))$ is simple, $C_M(\Omega_1 \mathrm{Z}(Q_M)) = Q_M$ and so $O_2(M/C_M(\Omega_1 \mathrm{Z}(Q_M)) = 1$. The definition of Y_M now implies that $Y_M = \Omega_1 \mathrm{Z}(Q_M)$.

(b) By Gaschütz' theorem, $Z \leq [Y_M, M]Z(M) = V$. Since V is a natural $SL_3(2)$-module for M we get that $|Z| = |C_V(S)| = 2$. Since $Q \not\leq Q_M$ and M/Q_M is simple, $M = M^\circ Q_M$. Since $Z \leq C_G(Q) \leq Q$ and Q is large, $C_M(Z) \leq M \cap \tilde{C}$. So $C_M(Z)$ normalizes $C_V(Q)$ and thus $C_V(Q) = Z$. Since $M \cap \tilde{C}$ normalizes $C_V(Q)$ this implies $C_M(Z) = M \cap \tilde{C}$. Thus $M \cap \tilde{C}/Q_M \cong \mathrm{Sym}(4)$ and since QQ_M/Q_M is a non-trivial normal 2-subgroup of $M \cap \tilde{C}/Q_M$, $QQ_M = O_2(M \cap \tilde{C})$. Hence by 1.1(b), $[Y_M, Q] = V$.

(c) Since M° induces $\mathrm{Aut}(V)$ on V, $\tilde{M} = M^\circ C_G(V)$.

Since Q is large, $C_G(V) \leq C_G(C_V(Q)) \leq N_G(Q)$ and thus $[Q, C_G(V)] \leq Q$. So $[Q, C_G(V)] \leq O_2(C_G(V)) \cap M^\circ \leq O_2(M^\circ)$. Conjugation under M gives, $[M^\circ, C_G(V)] \leq O_2(M^\circ)$ and so (c) holds.

(d) By (b), $M = M^\circ Q_M$ and since $Q_M \leq B$, we have $M_1 = MB = M^\circ B$. As $B = (M^\circ \cap B)(L \cap \tilde{M})$, $M_1 = M^\circ(L \cap \tilde{M}$. By (c) , $\tilde{M}$ normalizes M°. Since $B \leq \tilde{M}$, we conclude that $M_1 = M^\circ B$ is a subgroup of $\tilde{M}$.

(e) Put $D := \langle Y_M^{\tilde{M}} \rangle$. Since $\tilde{M}$ normalize both M° and V we get $[D, M^\circ] = V$ and $[D, O_2(M^\circ)] = 1$. By (a) $C_D(M^\circ) = 1$. Since $[D, V] = 1$ we have $[D, M^\circ, D] = 1$ and the Three Subgroups Lemma implies $[D, D, M^\circ] = 1$ and $D' \leq C_D(M^\circ) = 1$. So D is abelian and thus elementary abelian. Hence by 1.1, $|D/V| \leq 2$ and so $Y_M = D$. Hence $Y_M \trianglelefteq \tilde{M}$. Since $|Y_M/V| = 2$ we get $[Y_M, \tilde{M}] \leq V$ and so $[Y_M, O^2(C_G(V))] = 1$. Since $Q_M = C_S(V) \in \mathrm{Syl}_2(C_G(V))$ and $[Q_M, Y_M] = 1$, this gives $[Y_M, C_G(V)] = 1$ and so $C_G(V) = C_G(Y_M)$.

(f) Since $M^\circ \trianglelefteq \tilde{M}$ and $M_1 \leq \tilde{M}$, $O_2(M^\circ) \trianglelefteq M_1$. Also $Q_1 \cap M^\circ \trianglelefteq M^\circ$ and so $O_2(M^\circ) = Q_1 \cap M^\circ$. Since $\tilde{M} = M^\circ C_G(V)$, $M_1 = M^\circ C_{M_1}(V)$. As B normalizes $C_V(Q) = Z$ we have $B \leq N_{M_1}(Z) = (M^\circ \cap B)C_{M_1}(V)$ and so $B = (M^\circ \cap B)C_B(V)$, $M_1 = M^\circ C_B(V)$ and $C_{M_1}(V) = C_B(V)C_{M^\circ}(V) = C_B(V)$.

(g) Note that $O_2(C_{M_1}(V)) \leq Q_1$ and $O_2(M^\circ \cap B) = O_2(M^\circ)Q \leq Q_1 Q$. Since $B/Q_1 = (M^\circ \cap B)Q_1/Q_1 \times C_B(V)/Q_1$, this implies $O_2(B) = Q_1 Q$. Since $Q \leq Q_2 \leq O_2(B)$, we get $O_2(B) = Q_1 Q_2$. □

Lemma 2.2. [**elem**]

(a) [**e**] $L = O^2(L) = [L, Y_M]$ *and* $H = \langle Y_M^L \rangle = \langle Y_M^{M_2} \rangle$

(b) [**f**] $W \neq V$, $[W, L] \neq 1$ *and* $C_{Q_2}(L) = C_{Q_2}(H) = C_{Q_2}(W) \leq Q_1$.

(c) [**b**] $[Q_2', L] = 1$ *and* $[Q_2, L] \leq W$.

(d) [**z**] $WQ_1 = O_2(B)$, $[Y_M, W] = V$ *and* $V \cap Z(W) = Z$.

(e) [**a**] $[W, Q_2] = W' = Z = \Phi(W)$.

(f) [**c**] $[W, L] = W$ *and* $C_W(L) = Z(W)$.

(g) [**d**] $\hat{W}$ *is a selfdual, simple* $\mathbb{F}_2 M_2$*–module and homogeneous* $\mathbb{F}_2 H$*-module.*

Proof. By the minimal choice of L, $L = O^2(L)$ and $L = [L, Y_M]$. In particular, $\langle Y_M^L \rangle = Y_M[L, Y_M] = LY_M$. Together with 2.1(e) this is (a).

Suppose $W = V$. Then $L \leq N_G(V) = \tilde{M}$ and $Y_M \leq O_2(LY_M)$, a contradiction to the choice of L. If $[W, L] = 1$, then $W = \langle V^L \rangle = V$, a contradiction.

Thus $W \neq V$. Set $D = C_{Q_2}(L)$. Suppose $D \nleq Q_1$. Since B normalizes D and acts simply on $O_2(B)/Q_1$ we get $DQ_1 = O_2(B)$ and so by 1.1(b), $V = [Y_M, D] \leq D$ and $[V, L] = 1$, a contradiction. Thus $D \leq Q_1$ and $D \leq C_{Q_2}(LY_M) = C_{Q_2}(\langle Y_M^L \rangle)$.

Since $C_{Q_2}(V) = Q_2 \cap Q_M = C_{Q_2}(Y_M)$ we have $C_{Q_2}(W) = C_{Q_2}(\langle V^{M_2} \rangle) = C_{Q_2}(\langle Y_M^{M_2} \rangle) \leq C_{Q_2}(L) \leq D$ and so (b) holds.

As $Q'_2 \leq Q_M$, we have that $[Q'_2, Y_M] = 1$. Since $L \leq \langle Y_M^L \rangle$, we get $[L, Q'_2] = 1$. Further as $[Y_M, Q_2] \leq V \leq W$, we also get $[Q_2, L] \leq W$, which is (c).

If $[W, V] = 1$, then $W = Z(W)$. Thus (b) gives $W \leq C_{Q_2}(W) = C_{Q_2}(L)$, a contradiction. Hence $[W, V] \neq 1$ and $W \not\leq Q_1$. Since B normalizes WQ_1 this gives $WQ_1 = O_2(B)$ and so $[Y_M, W] = V$ and $V \cap Z(W) = C_V(W) = Z$. Thus (d) holds. Moreover, $[W, V] = [Q_2, V] = Z$. By (c) Z is centralized by L and so since $W = \langle V^{M_2} \rangle = \langle V^L \rangle$, $[W, W] = [W, Q_2] = Z$, which is (e).

By (b) $Z(W) = C_W(L) = C_W(H)$ and since $L = O^2(L)$, $Z(W)/Z = C_{W/Z}(L) = C_{W/Z}(\langle Y_M^{M_2} \rangle)$. Since $M \cap \tilde{C}$ acts simply on V/Z we conclude from 1.3 that $Z(W)/Z$ is the unique maximal M_2-submodule of W/Z. If $[W, L] \leq Z(W)$, then $W = \langle V^L \rangle = VZ(W)$ and W is abelian, a contradiction. Thus $[W, L] \not\leq Z(W)$ and so $W = [W, L]Z$. By (e) $Z \leq [W, L]$ and so $W = [W, L]$. So (f) is proved and $\hat{W}$ is a simple $\mathbb{F}_2 M_2$-module.

The commutator map $\hat{W} \times \hat{W} \to Z, [xZ(W), yZ(W)] \to [x, y]$ is a non-degenerate bilinear form on $\hat{W}$ and so $\hat{W}$ is a selfdual $\mathbb{F}_2 M_2$-module. Suppose that $\hat{W}$ is not homogeneous as an $\mathbb{F}_2 H$-module and let $\hat{W}_i, 1 \leq i \leq n$, be the Wedderburn components of H in $\hat{W}$. Then $\hat{W} = \bigoplus_{i=1}^n \hat{W}_i$ and so $\hat{V} = [\hat{W}, Y_M] = \bigoplus_{i=1}^n [\hat{W}_i, Y_M]$. It follows that the action of B on $\hat{V}$ is imprimitive. But $\hat{V} \cong V/V \cap Z(W) = V/Z$ as B-module and so $|\hat{V}| = 4$ and B acts transitively on $\hat{V}^\sharp$, a contradiction. □

Lemma 2.3. [**Wquad**]

(a) [**a**] *W acts quadratically on Q_M/V. In particular, any non-trivial composition factor for M on Q_M/V is a natural $SL_3(2)$-module.*

(b) [**b**] $N_{M_2}(Y_M Q_2) = N_{M_2}(V) = B$.

(c) [**c**] *If $g \in L$ with $[Y_M, Y_M^g] \leq Q_2$, then $[Y_M, Y_M^g] = 1$ and $Y_M Y_M^g$ acts quadratically on Q_2 and $\hat{W}$.*

(d) [**d**] $C_{M_2}(\hat{W}) = Q_2$.

Proof. We have that $[Q_M, W, W] \leq [W, W] = Z \leq V$, by 2.2(c),(e). So $[Q_M/V, W, W] = 1$. Since WQ_M/Q_M has order 4, W does not act quadratically on the Steinberg module. Since the only simple $\mathbb{F}_2 SL_3(2)$ modules are the trivial module, the two natural modules and the Steinberg module, we have (a).

(b) Let $g \in N_L(Y_M Q_2)$. Then g normalizes $[W, Y_M Q] = [W, Y_M] = V$ by 2.2(d).

(c) By (b) we have that $Y_M^g \leq \tilde{M}$ and by symmetry $Y_M \leq \tilde{M}^g$. Thus $R := [Y_M, Y_M^g] \leq V \cap V^g$. Suppose that $R \neq 1$. Then by 2.1(e), $[V, Y_M^g] \neq 1$. By 1.1 (applied to $V_1 = Y_M$) R is a fours group. Since $R \leq V^g$ the action of $\tilde{M}^g$ on V^g shows that there is $1 \neq x \in R$ such that $V \not\leq O_2(C_{\tilde{M}^g}(x))$. Note that $x \in V$ and so $[x, Q^m] = 1$ for some $m \in M$. Then $V \leq Q^m$ and since Q^m is large, $Q^m \leq O_2(C_G(x))$, a contradiction. So we have $R = 1$. Hence $Y_M Y_M^g$ is abelian and since Q_2 normalizes $Y_M Y_M^g$, $[Q, Y_M Y_M^g] \leq Y_M Y_M^g$ and $[[Q, Y_M Y_M^g], Y_M Y_M^g] = 1$. This is (c).

(d) Since $\hat{W}$ is a simple M_2-module, $Q_2 \leq C_{M_2}(\hat{W})$. Let $E := O^2(C_{M_2}(\hat{W}))$. Since $L \not\leq E$, the minimality of L shows that $[H \cap E, H] \leq Q_2$. Hence $\overline{H \cap E} \leq Z(\overline{H})$ and so $\overline{H \cap E}$ has odd order and $O_2\big(\overline{(H \cap E)Y_M}\big) = \overline{Y_M}$. Since $[E, H] \leq H \cap E$ we conclude that E normalizes $\overline{(H \cap E)Y_M}$ and $\overline{Y_M}$. (b) implies that $E \leq B$. Thus $[V, E] \leq V \cap Z(W) = Z$. Since $E = O^2(E)$ we get $[V, E] = 1$. Thus $[M^\circ, E] \leq C_{M^\circ}(V) \leq Q_2$. Since Q_2 normalizes E we have $O^2(EQ_2) = O^2(E) = E$ and so M° normalizes E. Note that also M_2 normalizes E. Suppose for a contradiction that $E \neq 1$. Since $C_G(Q) \leq Q$ we get $1 \neq [E, Q] \leq O_2(E)$. Since $M^\circ B = M_1$, M_1 normalizes E. So $O_2(E) \leq Q_1$ and since V is the unique minimal normal subgroup of M_1, $V \leq Z(O_2(E))$. But then also $W \leq Z(O_2(E))$ and W is abelian, which contradicts 2.2(e).

Thus $E = 1$, $C_{M_2}(\hat{W})$ is 2-group and (d) holds. □

Lemma 2.4. [**qm=ym**] *Suppose* $[Q_1, O^2(M)] \leq Y_M Q_2$. *Then* $Y_M = Q_M$.

Proof. Put $E = [Q_1, O^2(M)]$. Since $[Q_M, O^2(M)] = [Q_M, O^2(M^\circ)] \leq O_2(M^\circ) \leq Q_1$ we have $E = [Q_M, O^2(M)]$. From $E \leq Y_M Q_2$ we get $[E, W] \leq [Y_M Q_2, W] \leq V$. It follows that $E = [E, O^2(M)] \leq V$. Thus $V \not\leq \Phi(Q_M)$, Q_M is elementary abelian and $Q_M = Y_M$ □

Lemma 2.5. [**nonsolv**] *Suppose L is nonsolvable and let W_1 be a simple L-submodule of $\hat{W}$. Then $\overline{L}$ is quasisimple, $\overline{L} = F^*(\overline{H})$, H normalizes W_1, W_1 is a selfdual H-module and either $\hat{W} = W_1$ or $\hat{W} = W_1 \oplus W_2$ where W_2 is a H-submodule of $\hat{W}$ isomorphic to W_1.*

Proof. Since L is nonsolvable the minimality of L shows that $\overline{L} = \mathrm{E}(\overline{L})$. By 2.3(d), $\hat{W}$ is a faithful and simple $\overline{M_2}$-module. Let $\mathcal{L}$ be the set of components of $\overline{L}$ and $L_1 \in \mathcal{L}$. Then $\overline{L} = \langle L_1^B \rangle = \langle \mathcal{L} \rangle$. By Feit-Thompson L_1 has even order and since $\overline{Y_M} \leq Z(\overline{S})$, we get that Y_M normalizes L_1. So Y_M acts trivially on $\mathcal{L}$. As $H = \langle Y_M^{M_2} \rangle$ we conclude that all components of $\overline{L}$ are normal in $\overline{H}$. Let U be a non-trivial simple L_1-submodule of $\hat{W}$. Since L_1 is not solvable, $|U| > 4$. Let $y \in Y_M$. Since $|W/C_W(y)| \leq 4$, $U \cap U^y \neq 1$ and since L_1 normalizes $U \cap U^y$, $U = U^y$. Thus $H = \langle Y_M^{M_2} \rangle$ normalizes all non-trivial simple L_1-submodules of $\hat{W}$. Schur's Lemma together with the fact that finite division ring are commutative shows that $C_H(L_1)'$ centralizes U. Since $\hat{W}$ is a homogeneous H-module, this implies that $C_H(L_1)$ is abelian. Hence L_1 is the only component of $\overline{L}$ and $\overline{L} = L_1$. Note that $O_2(\overline{H}) \leq O_2(\overline{M_2}) = 1$ and as $\overline{H}/\overline{L}$ is a 2-group, $F^*(\overline{H}) = \overline{L}$. Since $\hat{W}$ is homogeneous and $|[\hat{W}, Y_M]| = |\hat{V}| \leq 4$, $\hat{W}$ is the direct sum of at most two simple H-submodules and all parts of the lemma are proved. □

Let U be a simple H-submodule of $\hat{W}$.

Lemma 2.6. [**Xstruk**] *Suppose L is nonsolvable. Then one of the following holds:*

1. [**i**] *$\overline{H} \cong SL_n(2)$, $SL_n(4)$, $Sp_{2n}(2)$, $Sp_{2n}(4)$, $SU_n(2)$, $\Omega_{2n}^{\pm}(2)$ or $G_2(2)'$ and U is corresponding natural module.*

2. [**ii**] *$\overline{H} \cong Sp_6(2)$, U is the spin-module and $\overline{Y_M}$ is a short root subgroup of $\overline{H}$.*

3. [**iii**] $\overline{H} \cong \mathrm{Sym}(n)$ *or* $\mathrm{Alt}(n)$*, U is the natural permutation module and $\overline{Y_M}$ is generated by a 2-cycle or double 2-cycle.*

4. [**iv**] $\overline{H} \cong \mathrm{Alt}(7)$ *and U is a spin-module.*

5. [**v**] $\overline{H} \cong 3\mathrm{Alt}(6)$ *and U is the 6–dimensional module.*

Proof. By 2.5 $F^*(\overline{H}) = \overline{L}$ and $F^*(\overline{H})$ is quasisimple. Since $\hat{W}$ is a faithful, homogeneous $\overline{H}$-module, $C_{\overline{H}}(U) = 1$. Note that $|\overline{Y_M}| = 2$ and $\overline{Y_M} \leq Z(\overline{S} \cap \overline{H})$. Thus Glauberman's Z^*-Theorem implies that there exists $g \in L$ with $\overline{Y_M} \neq \overline{Y_M}^g$ and $[\overline{Y_M}, \overline{Y_M}^g] = 1$. By 2.3(c), $Y_M Y_M^g$ induces a quadratic fours group on U. Since $[U, Y_M] \leq \hat{V}$, $[U, Y_M]$ has order at most 4. Now the assertion follows with 1.2. □

Lemma 2.7. [**Sln1**] *Suppose $\overline{L} \cong \mathrm{Alt}(n)$, $n = 5$ or $n > 6$, then U is not the natural permutation module for $\overline{L}$.*

Proof. By 2.3 for any $g \in L$ with $[Y_M, Y_M^g] \leq Q_2$, we have that $Y_M Y_M^g$ induces a quadratic group on $\hat{W}$. By 2.6(3) Y_M either corresponds to (12)(34) or (12). Since $\langle (12)(34), (13)(24) \rangle$ does not act quadratically on U, we get that $\overline{Y_M}$ is conjugate to $\langle (12) \rangle$. Since $\overline{Y_M}$ is 2-central we get $n \neq 5$ and so $n > 6$. Note that $Q_1 L \trianglelefteq LB = M_2$ and so $O_2(\overline{Q_1 L}) = 1$ and $\overline{Q_1 L} \cong \mathrm{Sym}(n)$. Thus by 2.3(c), $\overline{B \cap Q_1 L} \cong C_2 \times \mathrm{Sym}(n-2)$. Since $n - 2 > 4$ we have $O_2(\mathrm{Sym}(n-2)) = 1$ and so $O_2(B \cap Q_1 L) = Y_M Q_2$. Hence $Q_1 \leq Y_M Q_2$ and $[Q_1, W] \leq [Y_M Q_2, W] \leq V$. By 2.4 $Q_M = Y_M$ and so $|S/Y_M Q_2| = |S/Q_M Q_2| = 2$, a contradiction to $(B \cap Q_1 L)/Q_2 Y_M \cong \mathrm{Sym}(n-2)$. □

Lemma 2.8. [**orth**] *Suppose $\overline{L} \cong \Omega_{2n}^{\pm}(2)$ or $Sp_{2n}(2)'$ and U is the corresponding natural module. Then $\overline{H} \cong Sp_{2n}(2)$, $\hat{W}$ is the direct sum of two H-submodules isomorphic to U and Y_M induces a transvection on U.*

Proof. Let $\overline{P}$ be the point stabilizer of $\overline{H}$ on the natural module with $\overline{S} \cap \overline{L} \leq P$. Then $\overline{Y_M} \leq O_2(\overline{P})$ and $O_2(\overline{P})$ is abelian. Hence $\langle \overline{Y_M}^{\overline{P}} \rangle$ is abelian and (by 2.3(c)) acts quadratically on $\hat{W}$ and on the natural module. The action of $\overline{P}$ on the natural module now shows that $\overline{H} \cong Sp_{2n}(2)$, $\overline{P}$ normalizes $\overline{Y_M}$ and Y_M induces a transvection on the natural module. □

Lemma 2.9. [**Sln**] *$\overline{L}$ is none of $SL_n(2)$, $n \geq 3$, $SL_n(4), n \geq 3$, $3 \cdot \mathrm{Alt}(6)$ and $\mathrm{Alt}(7)$*

Proof. Then by 2.5, U is self-dual. Note that the natural modules for $SL_n(q), n \geq 3$, is not selfdual, the 6-dimensional module for $3. \cdot \mathrm{Alt}(6)$ is not selfdual and the 4-dimensional module for $\mathrm{Alt}(7)$ is not self dual. Hence by 2.6 we conclude that U is the orthogonal module for $\overline{H} \cong SL_4(2) \cong \Omega_6^+(2)$, but this contradicts 2.8. □

Lemma 2.10. [**elem b**]

(a) [**a**] *Let $F \trianglelefteq B$ with $[V/Z, F] \neq 1$. Then $T \leq F$.*

(b) [**b**] *Suppose that $[V/Z, L \cap B] \neq 1$. Then $T \leq L \cap B$ and $M_2 = LS$.*

Proof. (a) By 2.1(f), $B = (B\cap M^\circ)C_B(V)$ and $B\cap M^\circ/O_2(B\cap M^\circ) \cong SL_2(2)$. It follows that $C_{B\cap M^\circ}(V/Z) = O_2(B\cap M^\circ)$. Hence $B/C_B(V/Z) \cong SL_2(2)$, $R := [F, M^\circ \cap B] \not\leq C_B(V/Z)$ and $R \not\leq O_2(M^\circ \cap B)$. Since $R \trianglelefteq M^\circ \cap B$ this gives $T = O^2(M^\circ \cap B) \leq R \leq F$.

(b) By (a) applied to $F = L \cap B$ we have $T \leq L \cap B \leq L$. Thus $M_2 = L(M \cap B) = LTS = LS$. □

Lemma 2.11. [**mi**] *Suppose L is non-solvable. Then one of the following holds.*

1. [**a**] $M_1/Q_1 \cong SL_3(2) \times Sp_{2n-2}(2)$, $B/O_2(B) \cong SL_2(2) \times Sp_{2n-2}(2)$, $\overline{M_2} \cong Sp_{2n}(2) \times SL_2(2)$ *and $\hat{W}$ is the tensor product of the corresponding natural modules.*
2. [**b**] $M_1/Q_1 \cong SL_3(2)\times SL_2(2)$, $B/O_2(B) \cong SL_2(2)\times SL_2(2)$, $\overline{M_2} \cong \Gamma SU_4(2) \sim SU_4(2).2$ *and $\hat{W}$ is the corresponding natural module.*
3. [**c**] $M = M_1$, $B/O_2(B) \cong \mathrm{Sym}(3)$, $\overline{M_2} \cong \Gamma GL_2(4) \sim (C_3 \times SL_2(4)).2$ *and $\hat{W}$ is the corresponding natural module.*
4. [**d**] $M = M_1$, $B/O_2(B) \cong \mathrm{Sym}(3)$, $\overline{M_2} \cong G_2(2)$ *or* $G_2(2)'$ *and $\hat{W}$ is the corresponding natural module.*
5. [**e**] $M_1/Q_1 \cong SL_3(2)$, $B/O_2(B) \cong SL_2(2) \times SL_2(2)$, $\overline{M_2} \cong Sp_6(2)$ *and $\hat{W}$ is the spin-module.*

Proof. By 2.6-2.9 one of the following holds:

(a) [**1**] $\overline{H} \cong Sp_{2n}(2)$, $n \geq 4$ and $\hat{W}$ is the direct sum of two isomorphic natural modules and Y_M induces a transvection on these natural modules.

(b) [**2**] $\overline{H} \cong SU_n(2)$, $\hat{W}$ is a natural module and Y_M induces a $\mathbb{F}_4$-transvection on $\hat{W}$.

(c) [**3**] $\overline{H} \cong Sp_{2n}(4)$, $\hat{W}$ is a natural module and Y_M induces a $\mathbb{F}_4$-transvection on $\hat{W}$.

(d) [**4**] $\overline{H} \cong G_2(2)'$, $\hat{W}$ is the natural module and $\overline{Y_M}$ is long root element.

(e) [**5**] $\overline{H} \cong Sp_6(2)$, $\hat{W}$ is the spin-module and $\overline{Y_M}$ is a short root element.

Since by 2.3(b) $\overline{H \cap B} = C_{\overline{H}}(\overline{Y_M})$ this allows us to compute $\overline{H \cap B}$. Also $V/Z \cong [\hat{W}, Y_M]$ as a B-module and so this determines the action of $H \cap B$ on V/Z. Put $D = C_{M_2}(\overline{H})$. Note that $D \leq N_{M_2}(\overline{Y_M}) = B$ and

(*) $(M^\circ \cap B)O_2(B)/O_2(B)$ is a normal subgroup of $B/O_2(B)$ isomorphic to $SL_2(2)$.

Suppose (a) holds. Then $M_2 = DH$. Since M_2 acts simply on $\hat{W}$, but H does not, we get $\overline{D} \neq 1$. Since $W = \langle V^{M_2}\rangle$ we have $[V/Z, D] \neq 1$ and so by 2.10(a), $T \leq D$. Now (*) implies that $\overline{D} \not\leq Z(\overline{M}_2)$ and so D is not abelian. Now $C_{GL(\hat{W})}(\overline{H}) \cong SL_2(2)$ and thus $\overline{D} \cong SL_2(2)$. Moreover, $B \cap H/O_2(B \cap H) \cong Sp_{2n-2}(2)$ and we see that (1) holds in this case.

Suppose (b) holds. Then $L \cap B/O_2(L \cap B) \cong C_3 \times SU_{2n-2}(2)$ and $L \cap B/C_{L\cap B}(V/Z) \cong C_3$. In particular, $L \cap B$ acts non-trivially on V/Z and so by 2.10(b), $M_2 = LS$. Then (*) shows that $\overline{M_2} \neq \overline{L}$ and so $\overline{M_2} \cong \Gamma SU_n(2) = SU_n(2)\langle\sigma\rangle$, where σ induces a field automorphism of order 2. Thus $B/O_2(B) \cong (C_3 \times SU_{n-2}(2))\langle\sigma\rangle$ and (*) implies that $n = 4$ and $B/O_2(B) \cong SL_2(2) \times SL_2(2)$. Thus (2) holds.

Suppose (c) holds. Then $L \cap B/O_2(L \cap B) \cong Sp_{2n-2}(4)$ and $L \cap B$ centralizes V/Z. Thus $T \nleq L$ and since $\mathrm{Out}(\overline{H}) = 2$ we get $\overline{D} \neq 1$. Hence by 2.2(c), $T \leq D$. Since $C_{GL(\hat{W})}(\overline{H}) \cong C_3$ this gives $\overline{T} = \overline{D} \cong C_3$. Now (*) shows $\overline{D} \nleq Z(\overline{M_2})$ and so $\overline{M_2} \cong (C_3 \times Sp_{2n}(4))\langle\sigma\rangle$, where σ induces a field automorphism of order 2. Thus $B/O_2(B) \cong (C_3 \times Sp_{2n-2}(4))\langle\sigma\rangle$ and (*) implies that $n = 1$ and $B/O_2(B) \cong SL_2(2)$. Thus $B = M \cap B$ and (3) holds.

Suppose that (d) holds. Then $B \cap H/O_2(B \cap H) \cong SL_2(2)$ and $B \cap L$ acts non-trivally on V/Z. So 2.10(b) shows that $M_2 = LS$ and $T \leq L \cap B$. Therefore $B = M \cap B$ and (4) holds.

Suppose that (e) holds. Then $B \cap H/O_2(B \cap H) \cong SL_2(2)$ and $B \cap L$ acts non-trivally on V/Z. So 2.10(b) shows that $T \leq L \cap B$ and $M_2 = LS$. Since $\mathrm{Out}(\overline{H}) = 1$, this gives $\overline{M_2} = \overline{H}$, $B/O_2(B) \cong SL_2(2) \times SL_2(2)$ and (5) holds. □

Lemma 2.12. [**q=w**] *Suppose L is nonsolvable. Then $Q_2 = W = Q$ and $Z(W) = Z$.*

Proof. Suppose first that $C_{Q_2}(W) \neq Z$ and let $D \trianglelefteq M_2$ be minimal with $D \leq C_{Q_2}(W)$ and $D \neq Z$. By 2.2, $[D, L] = 1$ and $D \leq Q_1$. Since $M_2 = (M \cap B)L$ and $(M \cap B)/O_2(M \cap B) \cong SL_2(2)$ we get that either $[D, M_2] \leq Z$ and $|D/Z| = 2$ or $M_2/C_{M_2}(D/Z) \cong SL_2(2)$ and $|D/Z| = 4$. In any case $[D, Q_M] \leq Z$ and $\Phi(D) \leq Z$. Let $g \in M_1 \setminus B$. Then $Z \neq Z^g$.

We will now show that D is abelian. If $|D/Z| = 2$ this is obvious. So suppose $|D/Z| = 4$. Then $C_{M\cap B}(D/Z) = O_2(M \cap B)$. Since $W \cap B^g = C_W(Z^g)$ acts non-trivially on V/Z^g, we have $W \cap B^g \nleq O_2(M \cap B^g)$. Put $R := [D^g, W \cap B^g]$. It follows that $R \leq D^g$ and $R \nleq Z^g$. Since $D^g \leq Q \leq N_G(W)$, $R \leq W$. Thus by 2.1(b), $\Phi(R) \leq Z$. On the other hand $\Phi(R) \leq \Phi(D^g) \leq \Phi(W^g) = Z^g$. As $Z \cap Z^g = 1$, R is elementary abelian. Since B^g acts transitively on D^g/Z^g this implies that all non-trivial elements of D^g have order two.

Thus D is abelian. Note that $[D, D^g] \leq [D, Q_1] \cap [Q_1, D^g] \leq Z \cap Z^g = 1$ and so $E := \langle D^{M_1}\rangle$ is abelian. Suppose that $[E, W] \leq V$. Since $O^2(M) \leq \langle W^{M_1}\rangle$, we get $[E, O^2(M)] \leq V$. Since $M_1 = O^2(M)B$ and B normalizes D, $E = \langle D^{O^2(M)}\rangle \leq DV$. Hence $E = DV$, $[D, Q_M] \trianglelefteq M$ and $\Phi(D) \trianglelefteq M$. Since $[D, Q_M] \leq Z$ and $\Phi(D) \leq Z$ we conclude that $[D, Q_M] = 1$, $\Phi(D) = 1$ and $D \leq Y_M$. Thus $D \leq Y_M \cap Q_2 = V$. Since B normalizes D and $V \nleq D$ this implies $D = Z$, a contradiction.

Hence $[E, W] \nleq V$ and so $E \nleq Y_M Q_2$ and $\overline{Y_M} \lneq \overline{EY_M}$. Since EY_M is abelian and W normalizes EY_M, EY_M acts quadratically on $\hat{W}$.

In all cases of 2.11 except (3) $\overline{Y_M}$ is a maximal quadratic normal subgroup of $\overline{B \cap M_2} = C_{\overline{M_2}}(\overline{Y}_M)$ on $\hat{W}$. So $\overline{M_2} \cong \Gamma GL_2(4)$. Note that $S \cap H = Y_M Y_M^h Q_2$ for some $h \in M_2$ and $[W, S \cap H] \leq [W, Y_M Y_M^h]Z \leq Y_M Y_M^h$. By 2.3(c), $Y_M Y_M^h$ is elementary abelian and so also $[W, S \cap H]$ is elementary abelian. Since $W = [W, H]$, Gaschütz Theorem shows that $Z(W)/Z = C_W(L) \leq [W/Z, S \cap H]$ and so $Z(W) \leq [W, S \cap H]$. It follows that $Z(W)$ is elementary abelian. Since H acts transitively on $\hat{W}^\sharp$ this means that all non-trivial elements in W are involutions. Thus W is elementary abelian, a contradiction.

We have proved that $C_{Q_2}(W) = Z$. In particular, $Z(W) = Z$. Since $[W, Q_2] = Z$ we have $|Q_2/C_{Q_2}(W)| \leq |\hat{W}|$ and so $Q_2 = WC_{Q_2}(W) = WZ = W$. □

Lemma 2.13. [**g22**] *$\overline{L} \not\cong G_2(2)'$ and $\overline{L} \not\cong SL_2(4)$.*

Proof. Otherwise $\overline{L}$ acts transitively on $\hat{W}^\sharp$. Since $Z(W) = Z$ and $V \leq W$ we conclude that all elements of $W^\sharp$ have order two and W is elementary abelian, a contradiction. □

Lemma 2.14. [**e/v**] *Suppose L is nonsolvable. Then*

(a) [**a**] *$M_1/Q_1 \cong SL_3(2) \times SL_2(2)$, $Q_1 = [Q_1, M_1]Y_M$, and $[Q_1, M_1]/V$ is a tensor product of natural modules.*

(b) [**b**] *$M_2/Q_2 \cong \mathrm{SL}_2(2) \times Sp_4(2)$, Q_2 is extra special of order 2^9 and Q_2/Z is the tensor product of natural modules.*

Proof. Put $E = \langle (W \cap Q_1)^{M_1} \rangle$. By 2.13 one of 2.11(1), (2) and (5) holds. Put $m = n-1$ in the first case and $m = 1$ in the other two. Since $Z(W) = Z$ by 2.12 this implies that in all cases $W \cap Q_1 = [W, Q_1]$, $B/O_2(B) \cong SL_2(2) \times Sp_{2m}(2)$ and $W \cap Q_1/V$ is the tensor product of natural modules for $B/O_2(B)$-module. In particular, $W \cap Q_1/V$ is a simple B-module. Moreover, $[E, Q_1] = V$ and E/V is elementary abelian. Put $F/V = C_{E/V}(\langle W^{M_1} \rangle)$. Then by 1.3, E/F is a simple M_1-module and so $E/F \cong E_1 \otimes E_2$ where E_1 is a simple M°-module and E_2 is a simple $C_B(V)$-module. Since $[E_1, W] \otimes E_2 \cong [E, W]F/F \cong W \cap Q_1/V$ as an B-module we conclude that E_2 is natural $Sp_{2m}(2)$-module for $C_B(V)$and $[E_1, W]$ is a natural $SL_2(2)$-module for $B \cap M^\circ$. Thus E_1 is a natural $SL_3(2)$-module for M° dual to V. In particular, $[E, T] \leq (W \cap Q_1)F$. Since $[Q_1, W] \leq Q_1 \cap W \leq E$ we have $[Q_1, O^2(M)] \leq E$. It follows that $[Q_1, T] \leq W$. Since $O_2(B) = Q_1 W$ by 2.2(d) this implies $[O_2(B), T] \leq W \leq Q_2$. Thus T centralizes $O_2(B)/Q_2$. This rules out cases 2.11(2) and (5).

Hence 2.11(1) holds. The structure of M_2 shows that $C_B(V)$ has exactly three non-trivial composition factors on $O_2(B)$. Since $C_B(V)$ also has three non-trivial composition factors on E/F we conclude that $[E, O^2(C_B(V))] \leq V$. On the other hand, $E/V = \langle (W \cap Q_1/V_1)^{M^\circ} \rangle$ and so E/V as an $C_B(V)$-module is the direct sum of copies of the non-trivial simple $C_B(V)$-module $W \cap Q_1/V_1$. Thus $F = V$ and $E/W \cap Q_1$ is a natural $Sp_{2m}(2)$-module for $C_B(V)$. It follows that $E \cap Q_2 = W \cap Q_1$ and so EQ_2/Q_2 is a natural $Sp_{2m}(2)$-module for $C_B(V)$. Hence $n = 2$ (Indeed if $n \geq 3$ and so $m \geq 2$, the structure of M_2/Q_2 shows that $O_2(B)/Q_2$ as a $C_B(V)$-module is a non-split extension $\overline{Y_M}$ by a natural $Sp_{2m}(2)$-module).

In M_2 we see that $|O_2(B)| = 2^{1+8+3} = 2^{12}$ and so $|Q_1| = 2^{10}$. This shows that $Q_1 = Y_M E$. □

Lemma 2.15. [**solv**] *L is solvable.*

Proof. We need to show that the situation described in 2.14 does not occur. For this let D be a Sylow 3-subgroup of B, $D_1 = C_D(V)$ and $D_2 = D \cap (M^\circ Q_1)$. Then $D = D_1D_2$ and $D_1Q_1 \trianglelefteq M_1$. Put $N_1 = N_{M_1}(D_1)$. By the Frattini Argument $M_1 = N_1Q_1$ and since D_1 acts fixed-point freely on Q_1/Y_M, $N_1 \cap Q_1 = Y_M$. Hence $N_1 \sim (2^{3+1})(SL_3(2) \times SL_2(2)$ and $|O_2(N_1/D_1)| = 2^5$. Therefore 1.1(b) implies that $|Z(N_1/D_1)| = 2$. Let E_1 be the inverse image of $Z(N_1/D_1)$ in N_1 and put $F_1 = C_{N_1}(E_1)$. Then $E_1 \cong SL_2(2)$ and so $N_1 = F_1 \times E_1$, $Y_MD_2 \le F_1$ and $F_1/Y_M \cong SL_3(2)$. Put $N = N_B(D) = N_{N_1}(D_2) \cap B$. Then $|Y_M \cap N| = 4$ and $(F_1 \cap N)/(Y_M \cap N) \cong SL_2(2)$. Moreover, by 1.1(c) $[Y_M \cap N, F_1 \cap N] \ne 1$ and so $N/D \cong D_8 \times C_2$. Also $C_N(D_2)/D = (Y_M \cap N)E_1D/D \cong C_2^3$.

We now investigate the embedding of N in M_2. Since D_1 and D_2 are the only normal subgroups of order three in N we have $D_1 \le L$ and $D_2Q_2 \trianglelefteq M_2$. Thus $[O_2(B \cap F_1), D_2] \le Q_2$ and so $|C_{Q_2}(E_1)| = 2^5$. Note that $\overline{H} = O^{2'}(C_{\overline{M_2}}(D_2)) \cong Sp_4(2)$ and W/Z is a direct sum of two natural modules for $\overline{H}$. Since $[E_1, D_2] = 1$ we conclude that $\overline{E_1} \le \overline{H}$ and the involutions in E_1 act as transvections on these natural modules. It follows that $\overline{E_1} \not\le \overline{H}' \cong Sp_4(2)'$. Put $N_2 = N_{M_2}(D_2)$ and $U_2 = C_{M_2}(D_2)'$. Then $N_2/D_2 \sim 2.(Sp_4(2) \times 2)$ and $U_2Z/Z \cong Sp_4(2)'$. Since $C_N(D_2)/D$ is elementary abelian of order 2^3 we conclude that U_2Z contains a fours group and so $U_2 \cong Sp_4(2)'$. Thus $U_2 \cap N \cong SL_2(2)$ and $(U_2 \cap N)D/D \le \mathrm{Z}(N/D)$. Also $ZD/D \le \mathrm{Z}(N/D)$ and $E_1D/D \le \mathrm{Z}(N/D)$. Since $\overline{E_1} \not\le \overline{H}' = \overline{U_2Z}$ this implies $|Z(N/D)| \ge 8$, a contradiction to $N/D \cong D_8 \times C_2$. □

Proposition 2.16. [**end**] *$Q_M = Y_M$, Q is extraspecial of order* 32 *and* $\tilde{C}/Q \cong \mathrm{Sym}(3) \times \mathrm{Sym}(3)$.

Proof. By 2.15 we have that L is solvable and so by minimality $\overline{L}$ is a r-group for some odd prime r, $M \cap B$ acts simply on $\overline{L}/\Phi(\overline{L})$, Y_M inverts $\overline{L}/\Phi(\overline{L})$ and Y_M centralizes $\Phi(\overline{L})$. Thus $\Phi(\overline{L}) \le \mathrm{Z}(\langle \overline{Y_M}^{\overline{L}} \rangle) = \mathrm{Z}(\overline{H})$. By 2.2 $W = [W, L]$ and $[W/Z, Q_2] = 1$, so $C_{W/Z}(L) = 1$ and $Z(W) = Z$ by 2.2(f). Thus W is an extra-special 2-group.

Suppose for a contradiction that $\overline{L}$ is not abelian. Then $Z(\overline{L}) = Z(\overline{H}) \ne 1$. Since $W = \langle V^{\overline{H}} \rangle$ and $\overline{L}$ acts faithfully on $\hat{W}$, we get that $Z(\overline{L})$ acts faithfully on V/Z. Thus $|Z(\overline{L})| = 3$ and $\overline{L}$ is an extraspecial 3-group. Let $Z(\overline{L}) \le A \le \overline{L}$ with $|A| = 9$ and put $A_1 = [A, Y_M]$. Then $A = A_1 \times Z(\overline{L})$ and A is elementary abelian. Let $A_1, A_2, A_3, \mathrm{Z}(\overline{L})$ be the subgroups of order 3 in A. From $C_{W/Z}(Z(\overline{L})) = 1$ we have

$$W/Z = \bigoplus_{i=1}^{3} C_{W/Z}(A_i).$$

Since $\overline{L}$ acts transitively on $\{A_1, A_2, A_3\}$ we have $|W/Z| = |C_{W/Z}(A_i)|^3$. As $Z(\overline{L})$ acts non-trivially on $C_{W/Z}(A_i)$, $|C_{W/Z}(A_i)| \ge 4$. Note that Y_M does not normalizes A_2 and that $|[W/Z, Y_M]| = 4$. Hence $|C_{W/Z}(A_i)| = 4$ and so $|W/Z| = 2^6$. It follows that $|\overline{L}| = 3^3$. Since $[Z(\overline{L}), Y_M] = 1$, 2.3(b) gives $Z(\overline{L}) \le \overline{B}$. Hence $[\overline{O_2(B)}, Z(\overline{L})] = 1$. Since $C_{\mathrm{Out}(\overline{L})}(Z(\overline{L})) \cong SL_2(3)$ and $|C_{GL_{W/Z}}(\overline{L})| = 3 = |Z(\overline{L})|$ we get that $\overline{O_2(B)}$ is isomorphic to subgroup of $SL_2(3)$ and so to a subgroup of Q_8. Thus $\Omega_1(\overline{O_2(B)}) \le \overline{Y_M}$. Put $E = \langle (W \cap Q_M)^M \rangle$. Since

$\Phi(W \cap Q_M) \leq Z \leq V$ we conclude that E/V is generated by involutions. As $V \leq Q_2$ this gives $\overline{E} \leq \Omega_1(\overline{O_2(B)}) \leq \overline{Y_M}$ and $E \leq Y_M Q_2$. Hence by 2.4 $Q_M = Y_M$ and so $|S| = 2^7 = |W|$, a contradiction.

So we have shown that $\overline{L}$ is abelian. It follows that $\overline{L}$ is elementary abelian and Y_M inverts $\overline{L}$. Let R be a simple L-submodule of $\hat{W}$. Note that $C_{\overline{L}}(R)$ is normalized by $LY_M = H$ and so centralizes $\langle R^H \rangle$. Since $\hat{W}$ is a homogeneous H-module by 2.2(g), this gives that $C_{\overline{L}}(R) = 1$ and so $\overline{L}$ is cyclic. Thus $|W/Z| = |[W/Z, Y_M]|^2 = 4^2 = 16$. Hence W is extra special of order 2^4 and since $V \leq W$, $W \cong Q_8 \circ Q_8$. Thus $\mathrm{Out}(W) \cong O_4^+(2) \cong SL_2(2) \wr C_2$ and $\overline{L} \cong C_3$. Since $[T, Y_M] \leq V \leq Q_2$, $\overline{T} \not\leq \overline{L}$ and so $\overline{TL} \cong C_3 \times C_3$. Moreover, $[W, Q_M] \leq C_W(V) = V$ and so $[O^2(M), Q_M] \leq V$. Now 2.4 gives $Q_M = Y_M$ and so $|S| = 2^7$. In particular, $Q_M \cap Q_2 = V = Q_M \cap W$ and $Q_1 W = Q_1 Q_2 = O_2(B)$. Thus $Q_2 = W = Q$ and $|S/Q_2| = 2^2$. It follows that $\overline{M_2} = \overline{TLS} \cong \mathrm{Sym}(3) \times \mathrm{Sym}(3)$. Since $C_G(Q) \leq Q$ and $\mathrm{Out}(Q) \cong O_4^+(2)$ we have $|N_G(Q)/M_2| \leq 2$. Since $S \in \mathrm{Syl}_2(G)$ this forces $M_2 = N_G(Q)$. □

Proof of Theorem 1:

We are now able to prove the theorem. By 2.16 we have that M is an extension of an elementary abelian group of order 16 by $SL_3(2)$. Let $z \in Z^\sharp$. Since Q is large, $C_G(z) \leq N_G(Q)$ and so $N_G(Q) = C_G(z)$. Since Q is generated by involutions, there exists involutions in $M \setminus Y_M$ and so $M/V \ncong SL_2(7)$. Hence M has a subgroup M^* of index two, which is an extension of V by $SL_3(2)$.

Let $y \in Y_M \setminus V$. 1.1(c) implies that $C_M(y)$ is divisible by seven. Since $C_G(z) = N_G(Q)$ is not divisible by seven, y and z are not conjugate in G. Note that $V \leq Q = [Q, B] \leq M^*$. Hence every involutions in M^* is conjugate to an involution in Q. Since $M_2/Q \cong \mathrm{Sym}(3) \times \mathrm{Sym}(3)$ we see that all involutions in $Q \setminus Z(Q)$ are conjugate under M_2. Thus all involution in M^* are conjugates of z in G. This shows that y is not conjugate to any involution in M^*. By Thompson's Transfer Lemma we get that G possesses a subgroup G^* of index two. Since M^* is perfect, $M^* = M \cap G^*$. Moreover $O^2(M_2) \leq G^*$, $M_2 \cap G^* = C_{G^*}(z)$, $O^2(M_2) \cong SL_2(3) * SL_2(3)$ and $|(M_2 \cap G^*)/O^2(M_2)| = 2$. Hence [Asch] shows that $G^* \cong G_2(3)$. Since $|\mathrm{Out}(G_2(3))| = 2$ we conclude that $G \cong \mathrm{Aut}(G_2(3))$.

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