The Fitting Submodule

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Abstract

Let H be a finite group, $\mathbb{F}$ a field and V a finite dimensional $\mathbb{F}H$-module. We introduce the Fitting submodule $\mathrm{F}_V(H)$, an $\mathbb{F}H$ submodule of V which has properties similar to the generalized Fitting subgroup of a finite group.

1 Introduction

Throughout this paper $\mathbb{F}$ is a field of characteristic p, p a prime, H is a finite group, and V is a finite dimensional $\mathbb{F}H$-module.

We will use the concept of the generalized Fitting subgroup of a finite group as a model for our definition of the Fitting submodule $\mathrm{F}_V(H)$ of V. In particular, $\mathrm{F}_V(H)$ will be defined by means of components which in turn resemble components of finite groups.

Our first result can be stated without mentioning the Fitting submodule:

Theorem 1.1 *Suppose that V is faithful and $\mathrm{O}_p(H) = 1$. Then there exists an $\mathbb{F}H$-section of V that is faithful and semisimple.*

In fact 1.1 is a corollary of 1.3 and 1.4 below, which show that $\mathrm{F}_V(H)/\mathrm{rad}_{\mathrm{F}_V(H)}(H)$ has the desired properties.

To introduce the concept of a Fitting submodule we need a few basic definitions, some of them inspired by corresponding definitions in finite group theory.

Definition 1.2 *(a) H acts nilpotently on V if $[W, H] < W$ for all non-zero $\mathbb{F}H$-submodules W of V.*

(b) $\mathrm{C}^*_H(V)$ is the largest normal subgroup of H acting nilpotently on V. It is elementary to show that $\mathrm{C}^*_H(V)/\mathrm{C}_H(V) = \mathrm{O}_p(H/\mathrm{C}_H(V))$ and that $\mathrm{C}^*_H(V)$ is the largest subnormal subgroup of H acting nilpotently on V.

(c) V is H-reduced if $C^*_H(V) = C_H(V)$ (that is if any normal subgroup of H which acts nilpotently on V already centralizes V).

(d) $\mathrm{C}^*_V(H)$ is largest $\mathbb{F}H$-submodule of V on which H acts nilpotently (so $\mathrm{C}^*_V(H) = \mathrm{C}_V(\mathrm{O}^p(H))$;

(e) $\mathrm{rad}_V(H)$ is the intersection of the maximal $\mathbb{F}H$-submodules of V (so $\mathrm{rad}_V(H)$ is the smallest $\mathbb{F}H$-submodule with semisimple quotient).

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(f) *Let W be an $\mathbb{F}H$ submodule of V and $N \trianglelefteq H$. Then W is N-quasisimple if W is H-reduced, $W/\mathrm{rad}_W(H)$ is simple, $W = [W, N]$ and N acts nilpotently on $\mathrm{rad}_W(H)$. If $N = H$ we often write quasisimple rather than H-quasisimple.*

(g) *$\mathrm{S}_V(H)$ is the sum of all simple $\mathbb{F}H$-submodules of V, and $\mathrm{E}_H(V) := \mathrm{C}_{\mathrm{F}^*(H)}(\mathrm{S}_V(H))$.*

(h) *W is a component of V (or an H-component if we want to emphasize the dependence on H) if either W is a simple $\mathbb{F}H$-submodule with $[W, \mathrm{F}^*(H)] \neq 0$ or W is an $\mathrm{E}_H(V)$-quasisimple $\mathbb{F}H$-submodule. The Fitting submodule $\mathrm{F}_V(H)$ of V is the sum of all components of V.*

(i) *$\mathrm{R}_V(H) := \sum \mathrm{rad}_W(H)$, where the sum runs over all the components of V, and $\overline{\mathrm{F}_V(H)} := \mathrm{F}_V(H)/\mathrm{R}_V(H)$.*

Theorem 1.3 *The Fitting submodule $\mathrm{F}_V(H)$ is H-reduced and $\mathrm{R}_V(H)$ is a semisimple $\mathbb{F}\mathrm{F}^*(H)$-module. Moreover, $\mathrm{R}_V(H) = \mathrm{rad}_{\mathrm{F}_V(H)}(H)$; in particular $\mathrm{F}_V(H)/\mathrm{R}_V(H)$ is semisimple.*

Theorem 1.4 *Let V be faithful and H-reduced. Then also $\mathrm{F}_V(H)$ and $\mathrm{F}_V(H)/\mathrm{R}_V(H)$ are faithful and H-reduced.*

In section 3 we will discuss the relation between the Fitting submodule $\mathrm{F}_V(H)$ and the Fitting submodule $\mathrm{F}_V(N)$, where N is a subnormal subgroup of H. Finally, in section 4 the structure of the $\mathrm{F}^*(H)$-components of V is given in the case where $\mathbb{F}$ is a finite or algebraically closed field.

2 The Fitting Submodule

We will frequently use the following well-known and elementary properties of the generalized Fitting subgroup $\mathrm{F}^*(H)$ of H, see for example [1]:

Lemma 2.1 *Let $E \leq H$.*

(a) *If $E \trianglelefteq H$, then $\mathrm{F}^*(E) = E \cap \mathrm{F}^*(H)$.*

(b) *If $E = \mathrm{F}^*(E)$, then $E = \mathrm{O}_p(E)\mathrm{O}^p(E)$ and $[\mathrm{O}^p(E), \mathrm{O}_p(E)] = 1$.*

(c) *If E is a product of components of H, then $\mathrm{F}^*(H) = E\mathrm{C}_{\mathrm{F}^*(H)}(E)$.*

(d) *If E is a component of H and $N \trianglelefteq H$, then either $E \leq N$ or $[E, N] = 1$.*

□

Lemma 2.2 *Let E and P be normal subgroups of H with $[E, P] \leq \mathrm{C}_H(V)$. Suppose that $V = [V, E]$ and $V/\mathrm{C}^*_V(E)$ is a simple $\mathbb{F}H$-module. Then the following hold:*

(a) *$\mathrm{rad}_V(H) = \mathrm{C}^*_V(E)$.*

(b) *V is a semisimple $\mathbb{F}P$-module, and every simple $\mathbb{F}P$-submodule of V is isomorphic to a simple $\mathbb{F}P$-submodule of $V/\mathrm{rad}_H V$. In particular, if I_1 and I_2 are simple $\mathbb{F}P$-submodules of V, then there exists $h \in H$ such that I_1 and I_2^h are isomorphic as $\mathbb{F}P$-modules.*

(c) *If $E = \mathrm{F}^*(E)$, then $\mathrm{C}_H(V/\mathrm{rad}_V(H)) = \mathrm{C}_H(V)$, V is H-reduced and V is E-quasisimple as an $\mathbb{F}H$-module.*

(d) *If $E = \mathrm{F}^*(E)$, then E centralizes $\mathrm{rad}_V(H)$ and $\mathrm{rad}_V(H)$ is semisimple as an $\mathbb{F}EP$-module.*

(e) *Either $[V, P] = 0$, or $\mathrm{C}_V(P) = 0$ and $[V, P] = V$.*

Proof: (a): Let Y be a maximal $\mathbb{F}H$-submodule of V. If $\mathrm{C}_V^*(E) \nleq Y$ then $V = Y + \mathrm{C}_V^*(E)$. So E acts nilpotently on V/Y, contradicting $V = [V, E]$. Thus $\mathrm{C}_V^*(E) \leq Y$ and so $\mathrm{C}_V^*(E) \leq \mathrm{rad}_V(H)$. The other inclusion follows from the simplicity of $V/\mathrm{C}_V^*(E)$.

(b): Since $V/\mathrm{rad}_V(H)$ is a simple H-module, Clifford Theory shows that $V/\mathrm{rad}_V(H)$ is a semisimple $\mathbb{F}P$-module and that any two simple $\mathbb{F}P$-submodules of $V/\mathrm{rad}_V H$ are isomorphic up to conjugacy under H. Let U be the sum of all $\mathbb{F}P$-submodules of V that are isomorphic to some simple $\mathbb{F}P$-submodule of $V/\mathrm{rad}_V(H)$. It remains to show that $V = U$.

Let $x \in \mathrm{O}^p(E)$. Then

$$\phi : V \to [V, x] \text{ with } v \mapsto [v, x]$$

is an $\mathbb{F}P$-module homomorphism since $[E, P] \leq \mathrm{C}_H(V)$, so $[V, x] \cong_{\mathbb{F}P} V/C_V(x)$. As $\mathrm{rad}_V(H) \leq \mathrm{C}_V(x)$ and $V/\mathrm{rad}_V(H)$ is a semisimple $\mathbb{F}P$-module, $[V, x] \leq U$. Hence also

$$V = [V, E] = [V, \mathrm{O}^p(E)] \leq U$$

and thus $V = U$.

(c): Put $C := \mathrm{C}_H(V/\mathrm{rad}_V(H))$. By (a) $C \cap E$ acts nilpotently on $V/\mathrm{rad}_V(H)$ and on $\mathrm{rad}_V(H)$. Thus $\mathrm{O}^p(C \cap E)$ centralizes V. By 2.1(a), $C \cap E = \mathrm{F}^*(C \cap E)$, and by 2.1(b),

$$C \cap E = \mathrm{O}_p(C \cap E)\mathrm{O}^p(C \cap E) \leq \mathrm{O}_p(E)\mathrm{C}_E(V) \text{ and } [\mathrm{O}_p(E), \mathrm{O}^p(E)] = 1.$$

Thus $[\mathrm{O}^p(E), C \cap E] \leq \mathrm{C}_E(V)$. Since

$$[V, C, O^p(E)] \leq [\mathrm{rad}_V(H), \mathrm{O}^p(E)] = 0 \text{ and } V = [V, \mathrm{O}^p(E)],$$

the Three Subgroups Lemma gives $C \cap E \leq C_H(V)$ and then that $C \leq \mathrm{C}_H(V)$. Clearly $\mathrm{C}_H(V) \leq C$ and so $C = \mathrm{C}_H(V)$. Since $\mathrm{C}_H^*(V)$ acts trivially on every simple $\mathbb{F}H$-section of V, $\mathrm{C}_H^*(V) \leq C = \mathrm{C}_H(V)$ and so V is H-reduced. Together with (a) we see that V fulfills all the conditions of an E-quasisimple $\mathbb{F}H$-module.

(d): According to (c), $[\mathrm{rad}_V(H), \mathrm{O}^p(E)] = 0$ and $[V, \mathrm{O}_p(E)] = 0$. Hence E centralizes $\mathrm{rad}_V(H)$ since $E = \mathrm{F}^*(E) = \mathrm{O}_p(E)\mathrm{O}^p(E)$.

By (b) V and so also $\mathrm{rad}_V(H)$ is a semisimple $\mathbb{F}P$-module. Thus $\mathrm{rad}_V(H)$ is a semisimple $\mathbb{F}EP$-module.

(e): This is an immediate consequence of (b). □

Lemma 2.3 *Let $N \trianglelefteq H$ with $N \leq \mathrm{E}_H(V)$ and $N \nleq \mathrm{C}_H^*(V)$. Let W be an $\mathbb{F}H$-submodule of V that is minimal with respect to $N \nleq \mathrm{C}_H^*(W)$. Then*

(a) W is a H-component of V.

(b) $W = [W, N]$ and $\mathrm{C}_W^*(N) = \mathrm{rad}_W(H) = \mathrm{C}_W(\mathrm{E}_H(V)) \neq 0$.

(c) There exists a unique normal subgroup M of H minimal with respect to $M \leq \mathrm{E}_H(V)$ and $[W, M] \neq 0$.

(d) $M \leq N$, $W = [W, M]$ and M is a product of components of H transitively permuted by H.

(e) $\mathrm{C}_{\mathrm{E}_H(V)}(M) = \mathrm{C}_{\mathrm{E}_H(V)}(W)$.

Proof: The minimality of W implies that $\mathrm{C}_W^*(N)$ is the unique maximal H-submodule of W. Thus

1° $\mathrm{C}_W^*(N) = \mathrm{rad}_W(H)$ *and* $W/\mathrm{rad}_W(H)$ *is a simple* $\mathbb{F}H$*-module,*

If $[W, N] \leq \mathrm{C}_W^*(N)$, then N acts nilpotently on W, a contradiction. Thus $[W, N] \nleq C_W^*(N)$ and the minimality of W gives

2° $W = [W, N]$.

Observe that by 2.1 $N = \mathrm{F}^*(N)$. Hence by 2.2(c) applied with N and W in place of E and V, respectively:

3° *W is a H-reduced.*

Since $[W,N] \neq 0$ and $N \leq \mathrm{E}_H(V)$, W is not a simple H-module. Thus

4° $\mathrm{rad}_W(H) \neq 0$.

Choose $M \trianglelefteq H$ minimal in N with $[W,M] \nleq \mathrm{rad}_W(H)$. Then

5° $W = [W,M] = [W, \mathrm{O}^p(M)]$,

and so by the minimality of M and 2.1, $M = \mathrm{O}^p(M) = \mathrm{F}^*(M)$. Hence M is a p'-group or a product of components transitively permuted by H. As a subgroup of N, M acts nilpotently on $\mathrm{rad}_W(H)$, so

6° $[\mathrm{rad}_W(H), M] = 0$.

Assume that M is a p'-group. Then Maschke's Theorem implies $W = \mathrm{C}_W(M) \oplus [W,M]$ and $\mathrm{rad}_W(M) = \mathrm{C}_W(M) = 0$, which contradicts (4°). Thus

7° *M is a the product of components of H transitively permuted by H.*

By 2.1(c)

$$\mathrm{E}_H(V) = M\mathrm{C}_{\mathrm{E}_H(V)}(M). \tag{$*$}$$

We now apply 2.2(e) with $E := M$, $P := \mathrm{C}_{\mathrm{E}_H(V)}(M)$ and W in place of V. Since $P \leq \mathrm{E}_H(V)$, P centralizes all simple $\mathbb{F}H$-submodules of W. Thus $\mathrm{C}_W(P) \neq 0$ and 2.2(e) implies $[W,P] = 0$. Hence by (6°)

8° $[W,P] = 0$, *and* $\mathrm{E}_H(V) = MP$ *centralizes* $\mathrm{rad}_W(H)$.

Since $N \leq \mathrm{E}_H(V)$, (2°) implies

9° $W = [W, \mathrm{E}_H(V)]$.

If M_1 is any normal subgroup of H with $M_1 \leq \mathrm{E}_H(V)$ and $M_1 \nleq P$, then $1 \neq [M, M_1] \leq M \cap M_1$, and (7°) shows that $M = [M, M_1] \leq M_1$. In particular,
$[W, M_1] \neq 0$ and so by (8°)

10° $\mathrm{C}_{\mathrm{E}_H(V)}(W) = P = \mathrm{C}_{\mathrm{E}_H(V)}(M)$.

Hence M_1 is an arbitrary normal subgroup of $\mathrm{E}_H(V)$ not centralizing W. Thus $M \leq M_1$ implies (c). By (1°),(4°) (8°) and (9°) W is $\mathrm{E}_H(V)$-quasisimple and so (a) holds. Moreover, (b) follows from (1°),(2°), (3°) and (8°), and (d) follows from (5°) and (7°). Finally (e) is (10°).□

Lemma 2.4 *(a)* $\overline{\mathrm{F}_V(H)}$ *is a semisimple H-module and* $[\mathrm{F}_V(H), \mathrm{F}^*(H)] = \mathrm{F}_V(H)$.

(b) $\mathrm{F}_V(H)$ *is H-reduced.*

(c) $\mathrm{R}_V(H)$ *is a semisimple* $\mathbb{F}\mathrm{F}^*(H)$*-module.*

Proof: Let W be a component of V. Note that either $W \leq \mathrm{R}_V(H)$ or $W \cap \mathrm{R}_V(H) = \mathrm{rad}_W(H)$.

(a): Using the above observation, (a) is an immediate consequence of the definition of $\mathrm{F}_V(H)$.

(b): By definition all components are either simple or $\mathrm{E}_H(V)$-quasisimple, so they are reduced. Clearly sums of reduced modules are reduced and so (b) holds.

(c): Let W be a non-simple component of V. Then by 2.2(d) applied with $E := \mathrm{E}_H(V)$, $P := \mathrm{C}_{\mathrm{F}^*(H)}(E)$ and W in place of V, $\mathrm{rad}_W H$ is a semisimple $\mathbb{F}EP$-module. By 2.1 $\mathrm{F}^*(H) = EP$ and so (c) holds. □

Lemma 2.5 *The following hold:*

(a) $\mathrm{C}^*_{\mathrm{F}_V(H)}(\mathrm{E}_H(V)) = [\mathrm{S}_V(H), \mathrm{F}^*(H)] + \mathrm{R}_V(H) = \mathrm{C}_{\mathrm{F}_V(H)}(\mathrm{E}_H(V))$

(b) $\mathrm{C}^*_{\mathrm{F}^*(H)}(V) = \mathrm{C}_{\mathrm{F}^*(H)}(\mathrm{F}_V(H)) = \mathrm{O}_p(H)\mathrm{C}_{\mathrm{F}^*(H)}(V) \leq \mathrm{E}_H(V)$.

(c) $\mathrm{F}(H) \cap \mathrm{E}_H(V) = \mathrm{C}^*_{\mathrm{F}(H)}(V) = \mathrm{C}_{\mathrm{F}(H)}(\mathrm{F}_V(H)) = \mathrm{O}_p(H)\mathrm{C}_{\mathrm{F}(H)}(V)$.

(d) If V is faithful and H-reduced, then $\mathrm{E}_H(V)$ is the direct product of perfect simple groups.

Proof:

(a): Let W be a component of V. Then either $\mathrm{rad}W = 0$ and $W \leq [\mathrm{S}_V(H), \mathrm{F}^*(H)]$ or $\mathrm{rad}W \neq 0$, $W = [W, \mathrm{E}_H(V)]$ and $W/\mathrm{rad}_W(H)$ is simple. Thus $U := \mathrm{F}_V(H)/[\mathrm{S}_V(H), \mathrm{F}^*(H)] + \mathrm{R}_V(H)$ is a sum of simple $\mathbb{F}H$-module that are not centralized by $\mathrm{E}_H(V)$. So $\mathrm{C}_U(\mathrm{E}_H(V)) = 0$ and $\mathrm{C}^*_{\mathrm{F}_V(H)}(\mathrm{E}_H(V)) \leq [\mathrm{S}_V(H), \mathrm{F}^*(H)] + \mathrm{R}_V(H)$.

Let W be a component of V with $\mathrm{rad}_W(H) \neq 0$. Then by 2.3(b), applied with $N := \mathrm{E}_H(V)$, $[\mathrm{rad}_V(H), \mathrm{E}_H(V)] = 0$. The definition of $\mathrm{E}_H(V)$ shows that $[\mathrm{S}_V(H), \mathrm{E}_H(V)] = 0$ and so

$$[\mathrm{S}_V(H), \mathrm{F}^*(H)] + \mathrm{R}_V(H) \leq \mathrm{C}_{\mathrm{F}_V(H)}(\mathrm{E}_H(V)).$$

Clearly $\mathrm{C}_{\mathrm{F}_V(H)}(\mathrm{E}_H(V)) \leq \mathrm{C}^*_{\mathrm{F}_V(H)}(\mathrm{E}_H(V))$ and so (a) holds.

(b): Since $\mathrm{F}_V(H)$ is H-reduced by 2.4(b) , $\mathrm{C}^*_{\mathrm{F}^*(H)}(V) \leq \mathrm{C}_{\mathrm{F}^*(H)}(\mathrm{F}_V(H)) =: N$. Then $N \leq \mathrm{E}_H(V)$ since

$$\mathrm{S}_V(H) = (\mathrm{S}_V(H) \cap \mathrm{F}_V(H)) + \mathrm{C}_V(\mathrm{F}^*(H)).$$

If N does not act nilpotently on V, then 2.3 gives a component W of V with $[W, N] \neq 0$, which contradicts $[\mathrm{F}_V(H), N] = 0$. Thus N acts nilpotently on V and so $\mathrm{O}^p(N) \leq \mathrm{C}_N(V)$. By 2.1(b) $N = \mathrm{O}_p(N)\mathrm{O}^p(N) = \mathrm{O}_p(N)\mathrm{C}_N(V) \leq \mathrm{O}_p(H)\mathrm{C}_{\mathrm{F}^*(H)}(V)$.

Clearly $\mathrm{O}_p(H)\mathrm{C}_{\mathrm{F}^*(H)}(V) \leq \mathrm{C}^*_{\mathrm{F}^*(H)}(V)$ and so (b) holds.

(c): By (b), $\mathrm{C}^*_{\mathrm{F}(H)}(V) = \mathrm{C}_{\mathrm{F}(H)}(\mathrm{F}_V(H)) = \mathrm{O}_p(H)\mathrm{C}_{\mathrm{F}(H)}(V) \leq \mathrm{F}(H) \cap \mathrm{E}_H(V)$. Suppose $\mathrm{F}(H) \cap \mathrm{E}_H(V) \not\leq \mathrm{C}^*_{\mathrm{F}(H)}(V)$. Then by 2.3(d), $\mathrm{F}(H) \cap \mathrm{E}_H(V)$ is not nilpotent, a contradiction.

(d) If V is faithful and H-reduced, $\mathrm{C}^*_H(V) = 1$. So by (c), $\mathrm{F}(\mathrm{E}_H(V)) = \mathrm{F}(H) \cap \mathrm{E}_H(V) = 1$. Since $\mathrm{E}_H(V)$ is the central product of nilpotent and quasisimple groups, (d) holds. □

Lemma 2.6 *Let W be an H-submodule of V such that $W/C^*_W(\mathrm{E}_H(V))$ is a simple H-module. Choose an $\mathbb{F}H$-submodule Y in W minimal with $Y \not\leq \mathrm{C}^*_W(\mathrm{E}_H(V))$. Then*

$$Y = [W, \mathrm{O}^p(\mathrm{E}_H(V))] \text{ and } W = Y + \mathrm{C}^*_W(\mathrm{E}_H(V)),$$

and Y is a component of V.

Proof: Since $W/C^*_W(\mathrm{E}_H(V))$ is simple, $W = Y + \mathrm{C}^*_W(\mathrm{E}_H(V))$. Thus $[W, \mathrm{O}^p(\mathrm{E}_H(V))] \leq Y$. Since $[W, \mathrm{O}^p(\mathrm{E}_H(V))] \not\leq \mathrm{C}^*_W(H)$, the minimality of Y implies $Y = [W, \mathrm{O}^p(\mathrm{E}_H(V))]$. By 2.3(a), Y is a component. □

Lemma 2.7 *Put $\mathcal{C} := \{\overline{W} \mid W \text{ component of } V, \overline{W} \neq 0\}$. Then every $\mathbb{F}H$-submodule of $\overline{\mathrm{F}_V(H)}$ is the direct sum of elements of $\mathcal{C}$.*

Proof: It suffices to show the assertion for simple submodules since $\overline{\mathrm{F}_V(H)}$ is semisimple by 2.4(a).

Let $U/\mathrm{R}_V(H)$ be a simple $\mathbb{F}H$-submodule of $\overline{\mathrm{F}_V(H)}$. We need to show that $U = Y + \mathrm{R}_V(H)$ for some component Y of V. Since by 2.5(a) $\mathrm{R}_V(H) \leq \mathrm{C}_U(\mathrm{E}_H(V))$ either $\mathrm{C}^*_U(\mathrm{E}_H(V)) = \mathrm{R}_V(H)$ or $\mathrm{C}^*_U(\mathrm{E}_H(V)) = U$. In the first case the claim for U follows from 2.6.

In the second case 2.5(a) implies $U \leq [\mathrm{S}_V(H), \mathrm{F}^*(H)] + \mathrm{R}_V(H)$ and so

$$U = (U \cap [\mathrm{S}_V(H), \mathrm{F}^*(H)]) + \mathrm{R}_V(H).$$

Since $[\mathrm{S}_V(H), \mathrm{F}^*(H)]$ is the sum of simple H-components of V, so is $U \cap [\mathrm{S}_V(H), \mathrm{F}^*(H)]$. Thus, also in this case the claim holds for U. □

Lemma 2.8 *Let $N \trianglelefteq H$. Then*

$$C_{\overline{F_V(H)}}(N) = \overline{C_{F_V(H)}(N)}.$$

Proof: Let $\mathcal{C}$ be as in 2.7. Then by 2.7

$$C_{\overline{F_V(H)}}(N) = \langle \overline{W} \in \mathcal{C} \mid [\overline{W}, N] = 0 \rangle.$$

Let W be a H-component of V such that $\overline{W} \neq 0$ and $[\overline{W}, N] = 0$. Since $\overline{W}$ is semisimple and $\mathrm{rad}_W(H)$ is the unique maximal $\mathbb{F}H$-submodule of W, $\mathrm{rad}_W(H) = W \cap \mathrm{R}_V(H)$. If $\mathrm{rad}_W(H) = 0$, this shows that $[W, N] = 0$. If $\mathrm{rad}_W(H) \neq 0$, then W is E-quasisimple for $E := \mathrm{E}_H(V)$. In this case 2.2(c) with W in place of V implies that $[W, N] = 0$, so the lemma holds. □

Lemma 2.9 *Let $\mathcal{W}$ be a set of $\mathbb{F}H$-submodules of V. Then*

$$\mathrm{rad}_{\sum_{W \in \mathcal{W}} W}(H) = \sum_{W \in \mathcal{W}} \mathrm{rad}_W(H).$$

Proof: Clearly $\sum_{W \in \mathcal{W}} W / \sum_{W \in \mathcal{W}} \mathrm{rad}_W(H)$ is semisimple and so

$$\mathrm{rad}_{\sum_{W \in \mathcal{W}} W}(H) \leq \sum_{W \in \mathcal{W}} \mathrm{rad}_W(H).$$

On the other hand, for $W \in \mathcal{W}$, $W + \mathrm{rad}_{\sum_{W \in \mathcal{W}} W}(H) / \mathrm{rad}_{\sum_{W \in \mathcal{W}} W}(H)$ semisimple, so

$$\mathrm{rad}_W(H) \leq \mathrm{rad}_{\sum_{W \in \mathcal{W}} W}(H),$$

and the reverse inequality holds . □

Lemma 2.10 $\mathrm{R}_V(H) = \mathrm{rad}_{\mathrm{F}_V(H)}(H)$.

Proof: This follows immediately from 2.9 and the definition of $\mathrm{R}_V(H)$. □

Lemma 2.11 $C_H(\mathrm{F}_V(H)) = C_H(\overline{\mathrm{F}_V(H)})$

Proof: Let $N = C_H(\mathrm{F}_V(H))$. Then by 2.8, $\mathrm{F}_V(H) = C_{\mathrm{F}_V(H)}(N) + \mathrm{R}_V(H)$, and by 2.10, $\mathrm{F}_V(H) = C_{\mathrm{F}_V(H)}(N)$. □

Theorem 2.12 *Suppose that V is faithful and H-reduced. Then $\mathrm{F}_V(H)$ and $\mathrm{F}_V(H)/\mathrm{R}_H(V)$ are faithful and H-reduced $\mathbb{F}H$-modules.*

Proof: By 2.4(b) $\mathrm{F}_V(H)$ is reduced. Moreover $\overline{\mathrm{F}_V(H)}$ is semisimple and thus also H-reduced. From 2.5(b) we get $C_{\mathrm{F}^*(H)}(\mathrm{F}_V(H)) \leq C_H^*(V) = 1$. Hence $\mathrm{F}^*(C_H(\mathrm{F}_V(H))) = 1$ and so $C_H(\mathrm{F}_V(H)) = 1$. Thus $\mathrm{F}_V(H)$ is faithful. Now 2.11 implies that also $\overline{\mathrm{F}_V(H)}$ is faithful. □

The proof of the Theorems 1.1, 1.3, and 1.4: Theorem 1.3 is 2.4(a), (c) and 2.10, while Theorem 1.4 is 2.12. Moreover, Theorem 1.1 is a direct consequence of these two theorems.

3 The Fitting Submodule for Normal Subgroups

In this section we investigate the relationship between the Fitting submodules for H and for normal subgroups.

Lemma 3.1 *Let $N \trianglelefteq\trianglelefteq H$. Then* $\mathrm{S}_V(H) \le \mathrm{S}_V(N)$ *and* $\mathrm{E}_H(V) \cap N = \mathrm{E}_N(V)$.

Proof: Using induction on the subnormal defect of N in H, it suffices to treat the case $N \trianglelefteq H$.

Let W be a simple H-submodule. Then by Clifford Theory W is a semisimple N-module and so $\mathrm{S}_V(H) \le \mathrm{S}_V(N)$. Thus together with 2.1 $\mathrm{E}_N(V) \le \mathrm{E}_H(V) \cap N$.

Conversely, let Y be a simple N-submodule of V and let U be the sum of all N-submodule of V isomorphic to some H-conjugate of Y. Then U is an $\mathbb{F}H$-submodule of V and we can choose a simple H-submodule W of U. As an $\mathbb{F}N$-module, W is a direct sum of modules isomorphic to H-conjugates of Y, with each H-conjugate appearing at least once. Hence $\mathrm{E}_H(V) \cap N \le \mathrm{C}_{\mathrm{F}^*(N)}(W) \le \mathrm{C}_{\mathrm{F}^*(N)}(Y)$. Intersecting over all the possible Y gives $\mathrm{E}_H(V) \cap N \le \mathrm{E}_N(V)$. □

Lemma 3.2 *Let W be a H-component of V and $N \trianglelefteq\trianglelefteq H$. Then there exists a $\mathbb{F}N$-submodule $Y \le W$ with $W = \langle Y^H \rangle$ such that either Y is an N-component of V or Y is simple and* $[W, \mathrm{F}^*(N)] = 1$. *In particular,* $\mathrm{F}_V(H) \le \mathrm{S}_V(N) + \mathrm{F}_V(N)$.

Proof: Again we use induction on the defect $d_H(N)$ of N in H. Assume that $d_H(N) > 1$. Then there exists $N \le N_1 \trianglelefteq H$ so that $d_{N_1}(N) < d_H(N)$. By induction there exists an $\mathbb{F}N_1$-submodule Y_1 of W with $W = \langle Y_1^H \rangle$ such that either Y_1 is simple and $[W, \mathrm{F}^*(N_1)] = 0$ or Y_1 is a N_1-component of V. In the first case by 2.1 also $[W, \mathrm{F}^*(N)] = 0$ and we can choose a simple $\mathbb{F}N$-submodule Y of Y_1. In the second case induction applies to every H-conjugate of Y_1 in W, so either $[W, \mathrm{F}^*(N)] = 0$ and $Y_1 = \langle Y^{N_1} \rangle$ for a simple $\mathbb{F}N$-submodule Y of Y_1 or there exists $h \in H$ and an N-component Y of V with $W^h = \langle Y^{N_1} \rangle$. In any case $W = \langle Y^H \rangle$ and the lemma holds.

Thus it remains to treat the case $N \trianglelefteq H$. Let $Y \le W$ be a N-submodule minimal with $Y \not\le \mathrm{rad}_W(H)$. Then $Y/Y \cap \mathrm{rad}_W(H)$ is a simple N-module and $W = \langle Y^H \rangle$.

Suppose that Y is a semisimple $\mathbb{F}N$-module, then the minimality of Y shows that Y is simple. If $[Y, \mathrm{F}^*(N)] = 0$, then also $[W, \mathrm{F}^*(N)] = 0$; and if $[Y, \mathrm{F}^*(N)] \neq 0$, then Y is a N-component of V.

Suppose that Y is not a semisimple $\mathbb{F}N$-module. Then $Y \cap \mathrm{rad}_W(H) \neq 0$ and also W is not a semisimple $\mathbb{F}N$-module. By the definition of a component, W is $\mathrm{E}_H(V)$-quasisimple. Now 2.2(b), with $(N, \mathrm{E}_H(V), W)$ in place of (P, E, V), shows that $R := [\mathrm{E}_H(V), N] \not\le \mathrm{C}_H(W)$.

Since W is reduced, R does not act nilpotently on W, so $\mathrm{C}^*_W(\mathrm{E}_N(V)) \le \mathrm{C}^*_W(R) \le \mathrm{rad}_W(H)$. On the other hand $\mathrm{E}_H(V) \le \mathrm{C}^*_H(\mathrm{rad}_W(H))$ and so by 3.1 $\mathrm{rad}_W(H) \le \mathrm{C}^*_W(\mathrm{E}_N(V))$. This shows that $\mathrm{C}^*_W(\mathrm{E}_N(V)) = \mathrm{rad}_W(H)$. Hence by 2.6, applied to (N, Y) in place of (H, W), and the minimality of Y show that Y is a N-component of V. □

Corollary 3.3 $\mathrm{E}_{\mathrm{F}^*(H)}(V) = \mathrm{E}_H(V)$ *and* $\mathrm{F}_V(H) \le \mathrm{F}_V(\mathrm{F}^*(H))$.

Proof: Since $\mathrm{E}_H(V) \le \mathrm{F}^*(H)$ the first statement follows from 3.1 applied with $N := \mathrm{F}^*(H)$.

Note that $W = [W, \mathrm{F}^*(H)]$ for all components of V. The second statement then follows from 3.2 again with $N := \mathrm{F}^*(H)$. □

Proposition 3.4 *Let E and F be two distinct components of $\mathrm{E}_H(V)$. Then*

$$[\mathrm{F}_V(H)), E, F] = 0$$

Proof: By 3.3 we may assume that $H = \mathrm{F}^*(H)$. Let W be an H-component of V with $[W,E] \neq 0$. We can apply 2.3 with $N := \mathrm{E}_H(V)$. By part (c) and (d) of that lemma E is the unique normal subgroup of H minimal with $[W,E] \neq 0$. Since $[E,F] = 1$, part (e) gives $[W,F] = 0$. Since $\mathrm{F}_V(H)$ is the sum of all the components of V the lemma holds. □

Recall that a Wedderburn-component for H on V is a maximal sum of isomorphic simple $\mathbb{F}H$-submodules.

Proposition 3.5 *Let W be an H-component of V, $N \trianglelefteq H$ and $\mathrm{rad}_W(H) \leq Y_1 \leq W$ such that $Y_1/\mathrm{rad}_W(H)$ is a Wedderburn-component for N on $W/\mathrm{rad}_W(H)$. Put $Y := Y_1$ if $\mathrm{rad}_W(H) = 0$, and $Y := [Y_1, \mathrm{E}_H(V)]$ if $\mathrm{rad}_W(H) \neq 0$. Then $\mathrm{E}_H(V) \leq \mathrm{N}_H(Y_1)$, $Y = Y_1 + \mathrm{rad}_W(H)$ and Y is a $\mathrm{N}_H(Y_1)$-component of V.*

Proof: Set $L := \mathrm{N}_H(Y_1)$ and $\widetilde{W} := W/\mathrm{rad}_W(H)$. Let $D \leq H$ with $[D,N] \leq C_N(Y_1)$ and U be any simple N-submodule of $\widetilde{Y_1}$. Then $[D,N] \leq C_N(U)$, and for all $d \in D$ the map

$$U \to U^d \text{ with } u \mapsto u^d \quad (u \in U)$$

is an $\mathbb{F}N$-isomorphism. Thus U^d is in the Wedderburn-component $\widetilde{Y_1}$, and

$$(*) \qquad D \leq L$$

By 2.5(c) and 2.1(d), $\mathrm{E}_H(V) = \mathrm{C}_{\mathrm{E}_H(V)}(\mathrm{F}_V(H))(\mathrm{E}_H(V) \cap N)\mathrm{C}_{\mathrm{E}_H(V)}(N)$ and so by $(*)$, $\mathrm{E}_H(V) \leq L$. This is the first part of the claim.

Since $\widetilde{W}$ is a simple $\mathbb{F}H$-module, Clifford Theory implies that $\widetilde{Y}_1$ is a simple $\mathbb{F}L$-module. Suppose first that $\mathrm{rad}_W(H) \neq 0$. With $\mathrm{E}_H(V)$ in place of N 3.1 gives $\mathrm{E}_H(V) = \mathrm{E}_{\mathrm{E}_H(V)}(V)$. Then with $(L, \mathrm{E}_H(V))$ in place of (H,N) 3.1 gives $\mathrm{E}_H(V) \leq \mathrm{E}_L(V)$. Observe that $W = [W, \mathrm{E}_H(V)]$ since W is $\mathrm{E}_H(V)$-quasisimple, so $Y = [Y, \mathrm{E}_H(V)] \neq 0$. Hence 2.3(a) applied with $(L, \mathrm{E}_H(V), Y, V)$ in place of (H,N,W,V) shows that Y is a component for L.

Suppose now that $\mathrm{rad}_W(H) = 0$. To show that Y is an L-component of V it suffices to show that $[Y, \mathrm{F}^*(L)] \neq 0$. Observe that $[\mathrm{F}^*(H), N] \leq \mathrm{F}^*(N) \leq \mathrm{F}^*(L)$. So if $[\mathrm{F}^*(H), N]$ does not centralizes Y, we are done. If $[F^*(H), N] \leq \mathrm{C}_N(Y)$, then $(*)$ implies $\mathrm{F}^*(H) \leq L$ and so $\mathrm{F}^*(H) \leq \mathrm{F}^*(L)$. By the definition of a component, $[W, \mathrm{F}^*(H)] \neq 1$, and since W is a simple $\mathbb{F}H$-module also $[Y, \mathrm{F}^*(H)] \neq 1$. Thus $[Y, \mathrm{F}^*(L)] \neq 1$. □

Lemma 3.6 *Let $N \trianglelefteq H$. Then the following are equivalent:*

(a) $\mathrm{F}_V(H)$ *is a semisimple* $\mathbb{F}N$*-module.*

(b) *If K is a component of $\mathrm{E}_H(V)$ with $K \leq N$, then $[V,K] = 0$.*

(c) $\mathrm{E}_N(V) = \mathrm{E}_H(V) \cap N \leq \mathrm{C}^*_H(V)$.

(d) $[N, \mathrm{E}_H(V)] \leq \mathrm{C}_H(\mathrm{F}_V(H))$.

Proof: (a)$\Longrightarrow$ *(b)*: Let K be a component of $\mathrm{E}_H(V)$ with $K \leq N$. By 3.1, $K = \mathrm{E}_K(V)$ and so K centralizes all simple $\mathbb{F}K$-submodules. Since $\mathrm{F}_V(H)$ is semisimple as an $\mathbb{F}N$-module and so also as an $\mathbb{F}K$-module, $[\mathrm{F}_V(H), K] = 1$. Thus by 2.5(b), K centralizes V.

(b)$\Longrightarrow$ *(c)*: By 2.5(c), $\mathrm{F}(\mathrm{E}_H(V) \cap N) \leq \mathrm{C}^*_H(V)$ and by (b) any component of $\mathrm{E}_H(V) \cap N$ is contained in $\mathrm{C}_H(V)$. Hence $\mathrm{E}_H(V) \cap N \leq \mathrm{C}^*_H(V)$, and 3.1 shows that $E_N(V) = \mathrm{E}_H(V) \cap N$.

(c)$\Longrightarrow$ *(d)*: By (c) $[N, \mathrm{E}_H(V)] \leq \mathrm{C}^*_{\mathrm{F}^*(H)}(V)$ and by 2.5(c), $\mathrm{C}^*_{\mathrm{F}^*(H)}(V) \leq \mathrm{C}_H(\mathrm{F}_V(H))$.

(d)$\Longrightarrow$ *(a)*: Let W be a component of V. If $\mathrm{rad}_W(H) = 0$, W is a simple H-module and so a semisimple N-module. Suppose that $\mathrm{rad}_W(H) \neq 0$. Then $W = [W, \mathrm{E}_H(V)]$ and by 2.2(b) with $(W, N, \mathrm{E}_H(V))$ in place of (V,P,E), W is a semisimple N-module. □

Lemma 3.7 *Let W be component of V and $N \trianglelefteq H$ such that W is semisimple as an $\mathbb{F}N$-module. Let Y be a Wedderburn-component for N on W. Then Y is an $\mathrm{N}_H(Y)$-component of V.*

Proof: Define $\widetilde{W} := W/\mathrm{rad}_W(H)$. Then since W is a semisimple N-module, $\widetilde{Y}$ is a Wedderburn component for N on $\widetilde{W}$. If $\mathrm{rad}_W(H) = 0$ we are done by 3.5.

Suppose that $\mathrm{rad}_W(H) \neq 0$. Then $W = [W, \mathrm{E}_H(V)]$ and clearly $W = \mathrm{F}_W(H)$. Since 3.6(a) holds for W in place of V, 3.6(d) implies $[N, \mathrm{E}_H(W)] \leq \mathrm{C}_H(W)$. Thus also $[N, \mathrm{E}_H(V)] \leq \mathrm{C}_H(W)$ since $\mathrm{E}_H(V) \leq \mathrm{E}_H(W)$. This shows that $\mathrm{E}_H(V)$ normalizes every Wedderburn component for N on W.

Let $Y_1, \ldots, Y_r$ be the Wedderburn components of N on W. Then

$$W = Y_1 \oplus \cdots \oplus Y_r \text{ and } [W, \mathrm{E}_H(V)] = [Y_1, \mathrm{E}_H(V)] \oplus \cdots \oplus [Y_r, \mathrm{E}_H(V)].$$

From $W = [W, \mathrm{E}_H(V)]$ we conclude that $Y_i = [Y_i, \mathrm{E}_H(V)]$ for $i = 1, \ldots, r$. On the other hand by 2.2(d) and 3.5

$$[Y_i + \mathrm{rad}_W(H), \mathrm{E}_H(V)] = [Y_i, \mathrm{E}_H(V)] = Y_i$$

is a $\mathrm{N}_H(Y_i + \mathrm{rad}_W(H))$-component of W. Then $\mathrm{N}_H(Y_i + \mathrm{rad}_W(H)) = \mathrm{N}_H(Y_i)$, and Y_i is a $\mathrm{N}_H(Y_i)$-component. □

4 The Structure of an $\mathrm{F}^*(H)$-component of V

In this section we determine the structure the $\mathrm{F}^*(H)$-components of V for the case that $\mathbb{F}$ is finite or algebraically closed.

Lemma 4.1 *Suppose $\mathbb{F}$ is finite or algebraically closed. Let $E, P \leq H$ with $[E, P] = 1$. Suppose there exists a simple $\mathbb{F}P$-module Y and $n \in \mathbb{N}$ such that $V \cong Y^n$ has an $\mathbb{F}P$-module. Put $\mathbb{K} = \mathrm{End}_{\mathbb{F}P}(Y)$. Then $\mathbb{K}$ is a finite field extension of $\mathbb{F}$ and there exists an $\mathbb{K}E$-module X with $\dim_{\mathbb{K}} X = n$ such that*

$$V \cong_{\mathbb{F}(E \times P)} X \otimes_{\mathbb{K}} Y$$

Moreover, the following hold:

(a) If $\mathrm{End}_{\mathbb{F}EP}(V) = \mathbb{F}$, then $\mathbb{K} = \mathbb{F}$.

(b) If V is a simple $\mathbb{F}EP$-module, then X is a simple $\mathbb{K}E$-module.

(c) If V is an E-quasisimple $\mathbb{F}EP$-module, then X is a quasisimple $\mathbb{K}E$-module.

Proof: By Schur's Lemma $\mathbb{K}$ is a division ring. Since V is finite dimensional, $\dim_{\mathbb{F}} \mathbb{K}$ is finite. More precisely, if $\mathbb{F}$ is algebraically closed then $\mathbb{F} = \mathbb{K}$; and if $\mathbb{F}$ is finite then $\mathbb{K}$ is finite. In any case $\mathbb{K}$ is a field.

Let $X = \mathrm{Hom}_{\mathbb{F}P}(Y, V)$. Then X is a vector space over $\mathbb{K}$ via

$$(kx)(y) := x(ky) \text{ for all } k \in \mathbb{K}, y \in Y \text{ and } x \in X.$$

Moreover, E acts on X by $(x^e)(y) := x(y)^e$ and this action is $\mathbb{K}$-linear. Thus X is a $\mathbb{K}E$-module. We now regard $X \otimes_{\mathbb{K}} Y$ as an $\mathbb{F}(E \times P)$-module via $(x \otimes y)^{ea} = x^e \otimes y^a$ for $e \in E$, $a \in P$. Let

$$\phi : X \otimes_{\mathbb{K}} Y \to V \text{ with } x \otimes y \to x(y).$$

Then ϕ is well-defined since $\phi(kx \otimes y) = (kx)(y) = x(ky) = \phi(x \otimes ky)$. Also if $e \in E$ and $a \in P$, then

$$\phi((x \otimes y)^{ea}) = \phi(x^e \otimes y^a) = (x^e)(y^a) = x((y^a))^e = (x(y)^a)^e = x(y)^{ae} = x(y)^{ea} = \phi(x \otimes y)^{ea}.$$

So ϕ is an $\mathbb{F}(E\times P)$-module homomorphism. Note that for each submodule Z of V isomorphic to Y, there exists $x\in X$ with $x(Y)=Z$ and so $\phi(x\otimes Y)=Z$. Since V is the sum of such submodules, ϕ is surjective. As $V\cong Y^n$,

$$X=\mathrm{Hom}_{\mathbb{F}P}(Y,V)\cong \mathrm{Hom}_{\mathbb{F}P}(Y,Y^n)\cong \mathrm{End}_{\mathbb{F}P}(Y)^n=\mathbb{K}^n.$$

Hence the $\mathbb{F}$-spaces $X\otimes_{\mathbb{K}} Y$ and Y^n and thus also V have the same finite dimension. So ϕ is also injective and ϕ is an $\mathbb{F}(E\times P)$-isomorphism.

Observe that $E\times P$ acts $\mathbb{K}$-linearly on $X\otimes_{\mathbb{K}} Y$. So if we view V as a $\mathbb{K}$-space via

$$k\phi(u)=\phi(ku) \text{ for all } k\in\mathbb{K}, u\in X\otimes_{\mathbb{K}} Y,$$

then EP acts $\mathbb{K}$-linearly on V. Hence if $\mathrm{End}_{\mathbb{F}EP}(V)=\mathbb{F}$ we conclude that $\mathbb{K}=\mathbb{F}$. This proves (a).

Let X_0 be a proper $\mathbb{K}E$-submodule of X, then $\phi(X_0\otimes_{\mathbb{K}} Y)$ is a proper $\mathbb{K}EP$-submodule of V. This gives (b). In a similar way $[V,E]=V$ implies $[X,E]=X$.

Now assume that V is E-quasisimple, so $\mathrm{rad}_V(EP)$ is the unique maximal EP-submodule and $\mathrm{rad}_V(EP)=\mathrm{C}^*_V(E)$. Then

$$X_0\otimes Y\le \mathrm{rad}_{X\otimes_{\mathbb{K}}Y}(EP)=\mathrm{C}^*_{X\otimes_{\mathbb{K}}Y}(E)=\mathrm{C}^*_X(E)\otimes_{\mathbb{K}} Y.$$

This yields $X_0\le \mathrm{C}^*_X(E)$. Since X_0 was an arbitrary proper submodule, it also shows that $\mathrm{C}^*_X(E)$ is the unique maximal $\mathbb{K}E$-submodule of X and $\mathrm{C}^*_X(E)=\mathrm{rad}_X(E)$ and (c) follows. □

Proposition 4.2 *Suppose $\mathbb{F}$ is finite or algebraically closed. Let W be an $\mathrm{F}^*(H)$-component of V with $\mathrm{rad}_W(\mathrm{F}^*(H))\neq 0$, and let E be the unique component of $\mathrm{E}_H(V)$ with $[W,E]\neq 0$ (see 2.3). Put $P:=\mathrm{C}_{\mathrm{F}^*(H)}(E)$. Then $\mathrm{F}^*(H)=EP$ and there exists a finite field extension $\mathbb{K}$ of $\mathbb{F}$, a quasisimple $\mathbb{K}E$-module X and an absolutely simple $\mathbb{K}P$-module Y such that*

$$V\cong_{\mathbb{F}\mathrm{F}^*(H)} X\otimes_{\mathbb{K}} Y$$

Proof: By 2.1(c), $\mathrm{F}^*(H)=EP$. Let Y be a simple $\mathbb{F}P$-submodule of W. Since $[E,P]=1$, any $\mathrm{F}^*(H)$ conjugate of Y is isomorphic to Y. Thus by 2.2(b) applied to $\mathrm{F}^*(H)$ in place of H, $W\cong_{\mathbb{F}P} Y^n$ for some n. Now 4.1 shows that $V\cong_{\mathbb{F}\mathrm{F}^*(H)} X\otimes_{\mathbb{K}} Y$. Moreover, since $\mathbb{K}=\mathrm{End}_{\mathbb{F}P}(Y)$ and $\mathbb{K}$ is commutative, $\mathbb{K}=\mathrm{End}_{\mathbb{K}P}(Y)$, and Y is an absolutely simple $\mathbb{K}P$-module. Also since W is an E-quasisimple $\mathbb{F}EP$-module, X is E-quasisimple. □

Corollary 4.3 *Suppose $\mathbb{F}$ is finite or algebraically closed. Let W be a $\mathrm{F}^*(H)$-component of V, and let $\mathcal{K}$ be the set consisting of all the components of H and all the $O_r(H)$, r a prime divisor of $|H|$. Then there exists a finite field extension $\mathbb{K}$ of $\mathbb{F}$ and for each $K\in\mathcal{K}$ a $\mathbb{K}K$-module W_K such that*

$$W\cong_{\mathbb{F}\mathrm{F}^*(H)} \bigotimes_{K\in\mathcal{K}} W_K.$$

Moreover, either

1. $\mathrm{rad}_W(\mathrm{F}^*(H))=0$ *and W_K is absolutely simple for every $K\in\mathcal{K}$, or*
2. $\mathrm{rad}_W(\mathrm{F}^*(H))\neq 0$*, W_E is E-quasisimple and W_K is absolutely simple for every $K\in\mathcal{K}\setminus\{E\}$, where E is the unique component of $\mathrm{E}_H(V)$ with $[W,E]\neq 0$.*

Proof: We may assume that $|\mathcal{K}|>1$. If $\mathrm{rad}_W(H)=0$ put $Y:=W$, $\mathbb{K}:=\mathrm{End}_{\mathbb{F}\mathrm{F}^*(H)}(W)$ and $\mathcal{K}_0:=\mathcal{K}$. Otherwise let X,Y and $\mathbb{K}$ be as in 4.2 and put $W_E:=X$ and $\mathcal{K}_0:=\mathcal{K}\setminus\{E\}$. Observe that then W_E is E-quasisimple.

In any case Y is an absolutely simple $\mathbb{K}P$-module, where $P:=\langle\mathcal{K}_0\rangle$. If $|\mathcal{K}_0|=1$, we are done. In the other case pick $K\in\mathcal{K}_0$ and set $\mathcal{K}_1:=\mathcal{K}_0\setminus\{K\}$. Then 4.1 applies with $(\mathbb{K},K,\langle\mathcal{K}_1\rangle,Y)$ in place of $(\mathbb{F},E,P,V)$. Note that in addition $\mathbb{K}=\mathrm{End}_{\mathbb{K}K\langle\mathcal{K}_1\rangle}(Y)$, so 4.1(a) also applies. Now an easy induction finishes the proof. □

References

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