The Big Book of Small Modules

Barbara Baumeister
Ulrich Meierfrankenfeld
Gernot Stroth

May 23, 2002

Chapter 1

Introduction

In this book we classify modules for finite groups fullfiling certain properties which forces the module to be ”small” in some sense or annother. The main motivation for the book is provide the information about modules necessary in the local classification of finite groups of local characterisic p [LGCP].

Chapter 2

Some Group Theory

Lemma 2.0.1 [**three subgroup lemma**]

Proof:

Lemma 2.0.2 [**nilpotent groups**] *Let M be a nilpotent group and A a proper subgroup of M. Then A is a proper subgroup of $\mathrm{N}_M(A)$ and $\langle A^M \rangle$ is a proper subgroup of M.*

Chapter 3

Some elementary representation theory

Lemma 3.0.3 *Let G be a finite group and V an irreducible $\mathbb{K}G$-module. If* $\operatorname{char}\mathbb{K} = p$, *$p$ a prime and* $\mathrm{O}^p(G)$ *acts homogenously on* V, $\mathrm{O}^p(G)$ *acts irreducible on* V.

Proof: Comment: ref? any extra assumptions on $\mathbb{K}$?

Chapter 4

Same Characteristic Representations

This chapter is devoted to $\mathbb{K}G(\mathrm{F})$ modules, where $\mathbb{K}$ and F are field in the same characteristic and $G(\mathrm{F})$ is a group of Lie type over field $\mathbb{K}$.

4.1 Root Systems

[**root systems**]

Definition 4.1.1 *A root system is set Φ together with vectorspace V_Φ over $\mathbb{Q}$ and a non-degenerate, postive definite, symmetric form $(\ ,\ )$ on V_Φ such that*

(RS1) *Φ is a finite set of non zero vectors in V_Φ and Φ spans V_Φ.*

(RS2) *For all $\alpha, \beta \in \Phi$, $<\!\alpha, \beta\!> := 2\frac{(\alpha,\beta)}{(\beta,\beta)} \in \mathbb{Z}$.*

(RS3) *For all $\alpha, \beta \in \Phi$, $\omega_\alpha(b) =\in \Phi$, where*

$$\omega_\alpha : V_\Phi \to V_\Phi, v \to v - <\!v, \alpha\!>a$$

is the reflection associated to α.

(RS4) *If $\alpha, \beta \in \Phi$ are linearly dependent over $\mathbb{Q}$ then $\alpha = \pm\beta$.*

Let Φ be a root system. The elements of Φ are called *roots*. Put $W := \langle \omega_\alpha \mid \alpha \in \Phi \langle \leq \mathrm{O}(V_\mathbb{Q}, (\ ,\ ))$. Note that (RS3) just says that Φ is invariant under W. Since Φ is finite and spans $V_\mathbb{Q}$, W is finite.

Lemma 4.1.2 [**dual roots system**] *Let Φ be a root system. For $\alpha \in \Phi$ define $\alpha^* := \frac{2}{(\alpha,\alpha)}\alpha$. Let $\Phi^* = \{\alpha^* \mid a \in \Phi\}$ Then for all $\alpha, \beta \in \Phi$.*

(a) $<\alpha,\beta> = (\alpha,\beta^*)$.

(b) $<\alpha,\beta> = <\beta^*,\alpha^*>$.

(c) $\omega_\alpha = \omega_{\alpha^*}$

(d) $\omega_{\alpha^*}(\beta^*) = (\omega_\alpha(\beta))^*$

(e) Φ^* *(together with* V_Φ *and* $(\ ,\ ))$ *is a root system.*

Proof: (a)-(d) are readily verified and (e) follows from (c) and (d). $\square$

Definition 4.1.3 Φ *be a root system. A* system of simple roots *for* Φ *is a linearly independent subset* Π *of* Φ *such that* $\Phi = \Phi^+ \cup \Phi^-$ *where* $\Phi^+ = \Phi \cap \mathbb{Q}^+\Pi$ *and* $\Phi^- = \Phi \cap \mathbb{Q}^-\Pi = -\Phi^+$.

Lemma 4.1.4 [**existence of simple roots**] *Let* Φ *be a roots system.*

(a) Φ *has a system of simple roots.*

(b) *Any two systems of simple roots are conjugate under* W.

(c) *If* Π *is any system of simple roots, than* $\Phi^+ = \Phi \cap \mathbb{Z}^+\Pi$.

Definition 4.1.5 *A root* α *in a roots system* Φ *is called long (short) if* $(\alpha,\alpha) \geq (\beta,\beta)$ $((\alpha,\alpha) \leq (\beta,\beta))$ *for all* $\beta \in \Phi$.

Note here that f all roots in Φ have the same length, then all roots are long and short.

Lemma 4.1.6 [**dual fundamental roots**] *Let* Φ *be a roots system with fundamental roots* Π. *Then* $\Pi^* := \{\alpha^* \mid \alpha \in \Pi$ *is a system of fundamental roots for* Φ^*.

Proof: Since for all $\alpha \in \Phi$, α and α^* only differ by a positive rational factor, $\mathbb{Q}^+\Pi = \mathbb{Q}^+\Pi^*$ and $\alpha \in Q^+\Pi$ if and only if $\alpha^* \in \mathbb{Q}^+\Pi^*$. Hence the lemma follows from the definition of a fundamental system. $\square$

$$\Lambda := \{\lambda \in V_\Phi \mid (\lambda,\alpha^*) \in \mathbb{Z}\, \forall \alpha^* \in \Phi^*\}.$$

Note that by (RS2) $\Phi \subseteq \lambda$. Let $(\lambda_\alpha \mid \alpha \in \Pi)$ be the basis of $V_\mathbb{Q}$ dual to Π^* so $(\lambda_\alpha,\beta^*) = \begin{cases} 1 & \text{if } \alpha = \beta \\ 0 & \text{if } \alpha \neq \beta\} \end{cases}$. Then $(\lambda_\alpha \mid \alpha \in \Pi)$ is a $\mathbb{Z}$ basis for Λ.

For $\alpha,\beta \in \Phi$ and let $r,s \in \mathrm{N}$ be maximal such that

$$\beta - r\alpha, \beta - (r-1)\alpha, \dots, \beta - \alpha, \beta, \beta + \alpha, \dots \beta + s\alpha)$$

all are roots. We call this sequence of roots the α-string through β. r will be denoted by $r_{\alpha\beta}$ and s by $s_{\alpha\beta}$.

Definition 4.1.7 *Let* Φ *and* $\Psi \subseteq \Phi$*.*

(a) Ψ *is a* root subsystem *of* Φ *if* $(\Psi, \mathbb{Q}\Psi)$ *is a roots system.*

(b) Ψ *is a closed root subsystem of* $\Psi = \Phi \cap \mathbb{Q}\Psi$*.*

Lemma 4.1.8 [**covering root systems**] *Let* Φ *be a root system.*

(a) *Let* Ψ *be a root subsystem on* Φ*,* $\alpha \in \Psi$ *and* $\beta \in \Phi \setminus \Psi$*. The* $\omega_\alpha(\beta) \notin \Psi$*. If in addition* $(\alpha, \beta) \neq 0$ *and* Ψ *is closed, then* $\omega_\beta(\alpha) \notin \Psi$*.*

(b) *Suppose that* $\Phi \subseteq X \cup Y$ *where* X *and* Y *are proper roots subsytems of* Φ*. If* X *is closed, then* Φ *is disconnected.*

(c) *Supose that* Φ *is connected and* $\alpha, \beta \in \Phi$*. Then there exists* $\gamma \in \Phi$ *such that* γ *is neither perpendicular to* α *nor to* β*. In particular* α *and* β *are contained in a connected subroot system of rank at most 3.*

Proof: (a) If $\omega_\alpha(\beta) \in \Psi$, then $\beta = \omega_\alpha(\omega_\alpha(\beta)) \in \Psi$ a contradiction. If $(\alpha, \beta) \neq 0$, Ψ is closed and $\omega_\beta(\alpha) \in \Psi$, then $\beta =< \alpha, \beta >^{-1} (\alpha - \omega_\beta(\alpha)) \in \mathbb{Q}\Psi$. Since Ψ is closed, $\beta \in \Phi$, a contradiction.

(b) Choose X and Y as in (a) with $|X \cap Y$ minimal.Let $A = \Phi \setminus Y$, $B = \Phi \setminus X$ and $C = \Phi \cap X \cap Y$. Let $a \in A$ and $b \in B$. Suppose that $(a, b) \neq 0$. By $(a)\omega_b(a)$ is neither contained in X nor in Y, a contradiction. So $A \perp B$. Let $\tilde{X} = B^\perp \cap X$ and $\tilde{Y} = A^\perp \cap Y$. Then $\tilde{X}$ and $\tilde{Y}$ are subsystems with $\tilde{X}$ closed. Also $A \subseteq X$ and $B \subseteq Y$. Let $c \in C$ and suppose that $c \notin \tilde{X}$. Then $(c, a) \neq 0$ for some $a \in A$. Since $c \in \tilde{Y}$ and a is not, (a) implies $\omega_c(a) = a- < a, c > c \in A$. Thus $\omega_c(a)$ and a both perpendicular to B. Hence $c \perp B$ and $c \in \tilde{Y}$. We conclude that $C = \tilde{X} \cup \tilde{Y}$. The minimal choice of $X \cap Y$ implies $X \cap Y = \tilde{X} \cap \tilde{Y}$. Hence $C \subseteq \tilde{X} \cap \tilde{Y} \leq A^\perp \cap B^\perp$. Since also $A \perp B$, $A \cup B \cup C$ an decompostion of Φ into pairwise orthorgonal subsets.

(c) By (a) there exists $\gamma \in \Phi \setminus (\alpha^\perp \cup \beta^\perp)$. Also $\Phi \cap \mathbb{Q}\langle \alpha, \beta, \gamma \rangle$ is connected root system of rank at most 3. Thus (b) holds. □

Lemma 4.1.9 [**generation by non perpendicular roots**] *Let* Φ *be a connected root system, and* α *a short root.*

(a) *Then* $\mathbb{Q}\Phi = \mathbb{Q}\Phi_{long} = \mathbb{Q}\Phi_{Short}$*.*

(b) *Let* Ψ *be the roots subsystem generated by* α *and the long roots, then* $\Psi = \Phi$*.* **Comment: false for** F_4

(c) *Let* Ψ *be the roots subsystem generated by* α *and the long roots which are not perpendicular to* α*. If* Φ *is not of type* $B_n, n \geq 3$*, then* $\Psi = \Phi$*.* **Comment: maybe false for** F_4

Proof:

(a) Let $\{i,j\} = \{long, short\}$. Since Φ is connected there exists $\alpha\Phi_i$ and $\beta \in \Phi_j$ with $<\alpha, \beta> \neq 0$. If $b \notin \mathbb{Q}\Phi_i$ then 4.1.8(a) implies $\omega_\beta(\alpha) \notin \mathbb{Q}\Phi_i$ a contradiction. Thus $\beta \in \mathbb{Q}\Phi_i$ and the transitivity of W_Φ on Φ_j implies $\mathbb{Q}\Phi_j \subseteq \mathbb{Q}\Phi_i$.

For (b) and (c) note that if Φ has rank two, then every subsystem containing a long and a short system equals Φ (**Comment: false for G_2, it contains a** $A_1(long) \times A_1(short)$ Also $\mathbb{Q}\Phi_{long} = \mathbb{Q}\Phi$ and so Ψ contains a long root. So we may assume that Φ has rank at least two. Let Σ be the subsystem generated by the long root.

(b) Without loss α is the highest short root. Let β be any short root. By (a) there exists a long root δ with $<\delta, \beta> < 0$. Then $\omega_\delta(\beta)$ has larger height than β **Comment: this is false if β is negative** and so by induction $\omega_\delta(\beta) \in \Psi$. Hence also $\beta \in \Psi$.

(c) We may assume and Φ is not of type B_n. Thus Σ is connected. By definition of Ψ, $\Sigma = (\Sigma \cap \Psi) \cup (\Sigma \cap \alpha^\perp)$. Since $\Sigma \cap \alpha^\perp$ is closed in Σ, 4.1.8(b) implies that $\Sigma \subseteq \Psi$. So (c) follows from (b). □

4.2 Lie Algebras

Let Φ be a root system. We continue to use the notation introduced in 4.1.

Definition 4.2.1 *Let $\mathbb{K}$ be a field and $\mathfrak{g}$ a Lie-algebra over $\mathbb{K}$. A* Chevalley basis *for $\mathfrak{g}$ is a basis*

$$(\mathfrak{G}_\alpha, \alpha \in \Phi; \mathfrak{H}_\gamma, \gamma \in \Pi^*)$$

such that for all $\alpha, \beta \in \Phi$, $\gamma, \delta \in \Pi^*$:

(CB1) $[\mathfrak{H}_\gamma, \mathfrak{H}_\delta] = 0.$

(CB2) $[\mathfrak{H}_\gamma, \mathfrak{G}_\alpha] = (\alpha, \gamma)\mathfrak{G}_\alpha$

(CB3) $[\mathfrak{G}_\alpha, \mathfrak{G}_{-\alpha}] = \mathfrak{H}_{\alpha^*}$
where $\mathfrak{H}_\rho$ for $\rho = \sum_{\gamma\in\Pi^*} m_\gamma\gamma \in \Phi^*$ is define by $\mathfrak{H}_\rho := \sum_{\gamma\in\Pi^*} m_\gamma \mathfrak{H}_\gamma$.

(CB4) $[\mathfrak{G}_\alpha, \mathfrak{G}_\beta] = \pm r_{\alpha\beta}\mathfrak{G}_{\alpha+\beta}$ *if $\alpha + \beta \in \Phi$.*

(CB5) $[\mathfrak{G}_\alpha, \mathfrak{G}_\beta] = 0$ *if $0 \neq \alpha + \beta \notin \Phi$.*

Lemma 4.2.2 [**nilpotent action for lie algebras**] *Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{K}$ and V be a finite dimensional $\mathfrak{g}$ module.*

(a) *Then there exists unique maximal ideal $\mathfrak{u}_v(\mathfrak{g})$ which acts nilpotently on V.*

(b) *Let $\mathfrak{d}$ be an ideal in $\mathfrak{g}$, X a $\mathfrak{d}$ submodule of V and $\mathfrak{G} \in \mathfrak{g}$.*

  (ba) *Define $T : X \to V/X, x \to \mathfrak{G}x + X$. Then T is a $\mathfrak{d}$-equivariant. Inparticular $\mathfrak{G}X + X$ is a $\mathfrak{d}$ submodule of V.*

(bb) *If V is irreducible for $\mathfrak{g}$ then all compostion factors for $\mathfrak{d}$ on V are isomorphic.*

(bc) *If X is irreducible for $\mathfrak{d}$ and $\mathfrak{G}X \not\leq X$ then $\mathfrak{G}X \cap X = 0$ and $\mathrm{Ann}_X(\mathfrak{G}) = 0$.*

Proof: (a) $\mathfrak{u}_V(\mathfrak{g})$ is just the intersection of the annhilators of the composition factors of $\mathfrak{g}$ on V.

(b) Let $\mathfrak{D} \in \mathfrak{d}$ and $x \in X$. Then $[\mathfrak{G}, \mathfrak{D}]x \in \mathfrak{d}x \leq X$ and so

$$T(\mathfrak{D}x) = \mathfrak{G}\mathfrak{D}x + X = (\mathfrak{D}\mathfrak{G} + [\mathfrak{G}, \mathfrak{D}])x + X = \mathfrak{D}(\mathfrak{G}x + X) = \mathfrak{D}(T(x))$$

So (ba) holds.

For (bb) let Y be a $\mathfrak{d}$ submodule maximal such that all compostion factors for $\mathfrak{d}$ on Y are isomorphic. By (ba) applied to Y, all compostion factors of $\mathfrak{d}$ on $\mathfrak{G}Y + Y/Y$ are isomorphic to a comostion factor of Y. Hence by maximality of Y, $\mathfrak{G}Y \leq Y$. Since $\mathfrak{G} \in \mathfrak{g}$ was arbitray and $\mathfrak{g}$ acts irreducibly, $V = Y$.

For (bc) not that the irreducibilty of X and (ba) imply $\ker T = 0$. □

We remark that under the assumption of part (bb) of the preceeding lemma, V does not need to completely reducible for $\mathfrak{g}$. For example let $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{K})$ with $\mathrm{char}\,\mathbb{K} = 2$ and V the natural 2-dimensional module. Then $\mathbb{K}\langle \mathfrak{G}_\alpha, \mathfrak{H}_\alpha \rangle$ is an ideal in $\mathfrak{sl}_2(\mathbb{K})$ and has a unique proper submodule (namely $\mathfrak{G}_\alpha V$). Thus example also shows that an ideal does not need to act faithfully on its proper submodules.

Lemma 4.2.3 **[X+bX]** *Let $\mathfrak{g}$ be a Lie algebra, $\mathfrak{a}$ and $\mathfrak{b}$ subspaces of $\mathfrak{g}$ with $\mathfrak{g} = \mathfrak{a} + \mathfrak{b}$. Let X be an $\mathfrak{a}$ invaraint subspace of V.*

(a) *For all $n \in \mathbb{N}$, $\sum_{i=0}^{n} \mathfrak{b}^i X$ is $\mathfrak{a}$ invariant.*

(b) *$\sum_{i=0}^{\infty} \mathfrak{b}^i X$ is $\mathfrak{g}$ invariant.*

(c) *If $X \neq 0$ and V is irreducible as $\mathfrak{g}$-module, then $V = \sum_{i=0}^{\infty} \mathfrak{b}^i X$.*

Proof: (a) By induction on i it suffices to show that $X + \mathfrak{b}X$ is $\mathfrak{a}$ invariant. Note that $\mathfrak{g}X = (\mathfrak{a} + \mathfrak{b})X \leq X + \mathfrak{b}X$. Let $\mathfrak{A} \in \mathfrak{a}$ and $\mathfrak{B} \in \mathfrak{b}$. Then

$$(\mathfrak{A}\mathfrak{B})X = (\mathfrak{B}\ fA + [\mathfrak{A}, \mathfrak{B}])X \leq \mathfrak{B}(\ fAX) + \mathfrak{g}X \leq X + \mathfrak{b}X.$$

So (a) holds.

(b) By (a)

$$\sum_{i=0}^{\infty} \mathfrak{b}^i X = \bigcup_{n=1}^{\infty} (\sum_{i=0}^{n} \mathfrak{b}^i X)$$

is $\mathfrak{a}$ invariant. Clealry it is also $\mathfrak{b}$ invarint and so (b) follows from $\mathfrak{g} = \mathfrak{a} = \mathfrak{b}$.

(c) Follows from (b). □

Proposition 4.2.4 [**smith's lemma**] *Let* $\mathfrak{g}$ *be a Lie algebra,* $\mathfrak{l}, \mathfrak{q}_+$ *and* $\mathfrak{q}_-$ *sub algebras and* V *an irreducible* $\mathfrak{g}$ *module. Suppose that*

(i) $\mathfrak{g} = \mathfrak{q}_+ + \mathfrak{l} + \mathfrak{q}_+$

(ii) $[\mathfrak{l}, \mathfrak{q}_+] \leq \mathfrak{q}_+$ *and* $[\mathfrak{l}, \mathfrak{q}_-] \leq \mathfrak{q}_-]$.

(iii) $\mathfrak{q}_+$ *and* $\mathfrak{q}_-$ *both act nilpotently on* V.

Then

(a) $\mathfrak{l}$ *acts irreducible on* $\mathrm{Ann}_V(q_+)$.

(b) $V = \mathrm{Ann}_V(\mathfrak{q}_+) \oplus \mathrm{Ann}^*_V(\mathfrak{q}_-)$, *where* $\mathrm{Ann}^*_V(\mathfrak{q}_-)$ *is smallest* q_- *submodule of* V *containing* $\mathfrak{q}_- V$.

Proof: Since $\mathfrak{q}_+$ acts nilpotently on V, $\mathrm{Ann}_V(\mathfrak{q}_+) \neq 0$. By (ii) $\mathrm{Ann}_V(\mathfrak{q}_+)$ is a $\mathfrak{l}$ submodule. Let X be any non-zero $\mathfrak{l}$ submodule of $\mathrm{Ann}_V(\mathfrak{q}_+$ and $Y = \sum_{i=1}^\infty \mathfrak{q}_-^i X$. Then X is an $\mathfrak{q}_+ + \mathfrak{l}$ submodule of $\mathrm{Ann}_V(\mathfrak{q}_+)$ and $Y \leq \mathrm{Ann}^*_V(\mathfrak{q}_-)$. By 4.2.3,

(*) $V = X + Y$

Suppose that $\tilde{X} := \mathrm{Ann}_V(\mathfrak{q}_+) \cap \mathrm{Ann}^*_V(\mathfrak{q}_-) \neq 0$. Since $\tilde{X}$ is $\mathfrak{l}$ invariant, (*) applied to X yields $V = \tilde{X} + \mathrm{Ann}^*_V(\mathfrak{q}_-) \leq \mathrm{Ann}^*_V(\mathfrak{q}_-)$. Since $\mathfrak{q}_-$ acts nilpotently this implies $\mathrm{Ann}^*_V(\mathfrak{q}_-)) = 0$, a contradiction to $\tilde{X} \neq 0$.

Thus $\tilde{X} = 0$. Hence also $\mathrm{Ann}_V(\mathfrak{q}_+) \cap Y = 0$ and so using (*)

$$\mathrm{Ann}_V(\mathfrak{q}_+) = X + (\mathrm{Ann}_V(\mathfrak{q}^+) \cap Y) = X$$

Since X was an arbitray $\mathfrak{l}$ submodule of $\mathrm{Ann}_V(\mathfrak{q}_+)$ we concldue that (a) and (b) hold.□

Lemma 4.2.5 [**q- quadratic**] *Let* $\mathfrak{g}$ *be a Lie algebra,* $\mathfrak{l}, \mathfrak{q}_+$ *and* $\mathfrak{q}_-$ *sub algebras and* V *an irreducible* $\mathfrak{g}$ *module. Suppose that*

(i) $\mathfrak{g} = \mathfrak{q}_+ + \mathfrak{l} + \mathfrak{q}_+$

(ii) $[\mathfrak{l}, \mathfrak{q}_+] \leq \mathfrak{q}_+$ *and* $[\mathfrak{l}, \mathfrak{q}_-] \leq \mathfrak{q}_-]$.

(iii) $\mathfrak{q}_-^2 V = 0$ *and* $\mathfrak{q}_- V \neq 0$.

(iv) $\mathfrak{q}^+$ *acts niloptently on* V.

Then

(a) $V = \mathrm{Ann}_V(\mathfrak{q}_+) \oplus \mathrm{Ann}_V(\mathfrak{q}_-)$.

(b) $\mathfrak{l}$ *acts irreducibly on* $\mathrm{Ann}_V(\mathfrak{q}_+)$ *and* $\mathrm{Ann}_V(\mathfrak{q}_-)$.

(c) $\mathfrak{q}_+^2 V = 0$ *and* $\mathfrak{q}_- V \neq 0$.

(d) $\mathrm{Ann}_V(\mathfrak{q}_+) = \mathfrak{q}_+ V$ *and* $\mathrm{Ann}_V(\mathfrak{q}_-) = \mathfrak{q}_- V$.

Proof: Note that $\mathfrak{q}_- V \leq \mathrm{Ann}_V(\mathfrak{q}^-)$. By 4.2.4(a) (applied with the roles of $+$ and $-$ interchanged, $\mathrm{Ann}_V(\mathfrak{q}_-)$ is an irreducible fl module. Thus

$$\mathfrak{q}_- V = \mathrm{Ann}_V(\mathfrak{q}_-) = \mathrm{Ann}_V^*(\mathfrak{q}_-)$$

Thus by 4.2.4(b) implies that (a) holds. In particular $\mathfrak{q}_+ + \mathfrak{l}$ acts irreducible on $V/\mathrm{Ann}_V(\mathfrak{q}_+))$. Hence $\mathfrak{q}_+$ annhilates $V/\mathrm{Ann}(\mathfrak{q}_+)$ and the remaiing parts of the lemma are readily verified. □

Comment: The preceeding lemma could be also used to in some later places to avoid the use of the graph automorphism for A_n

4.3 Groups of Lie Type and Irreducible Rational Representaions

Let Φ be a connected root system, $\mathbb{K}$ a field, $\mathbb{E}$ the algebraic closure of $\mathbb{K}$ and $G_\Phi(K)$ the corresponding unversial group of Lie type. Then $G_\Phi(K)$ is generated by elements $\chi_\alpha(t), \alpha \in \Phi, t \in \mathbb{K}$ fulfilling the Steinberg Relations: For $t \in \mathbb{K}^\#$ define $\omega_\alpha(t) = \chi_\alpha(t)\chi_\alpha(t^{-1})\chi_\alpha(t)$ and $h_\alpha(t) := \omega_\alpha(t)\omega_\alpha(1)^{-1}$.

(St1) $\chi_\alpha(t)\chi_\alpha(s) = \chi_\alpha(t+s)$

(St2) $h_\alpha(u)h_\alpha(v) = h_\alpha(uv)$

(St3) If $\alpha^* = \sum_{i=1}^n n_i\beta_i^*$ for some $n_i \in \mathbb{Z}, \beta_i \in \Phi$ then $h_\alpha(u) = \prod_{i=1}^n h_{\beta_i}(u^{n_i})$.

(St4) $h_\alpha(u)\chi_\beta(t)h_\alpha(u)^{-1} = \chi_\alpha(u^{(\beta,\alpha^*)}t)$

(ST5) $\omega_\alpha(1)\chi_\beta(t)\omega_\alpha(1)^{-1} = \chi_{\omega_\alpha(\beta)}(\epsilon_{\alpha\beta}t)$ for some $\epsilon_{\alpha\beta} = \pm$.

(ST6) If $\alpha+\beta$ is not a root, and $\alpha \neq -\beta$ then $[\chi_\alpha(t), \chi_\beta(s)] = 1$.

(ST7) If $\alpha+\beta$ is a root then

$$[\chi_\alpha(t), \chi_\beta(s)] = \chi_{\alpha+\beta}(N_{\alpha\beta}ts) \prod_{i,j>1} \chi_{i\alpha+j\beta}(C_{\alpha\beta ij}t^i s^j)$$

Let $H_\alpha = \{h_\alpha(u) \mid u \in \mathbb{K}^\#\}$, $X_\alpha = \{\chi_\alpha(t) \mid t \in \mathbb{K}^\#\}$, $U = \prod_{\alpha\in\Phi^+} X_\alpha$, $H = \prod_{H_\alpha} \mid \alpha \in \Pi$ and $B = HU$.

Let V be a finite dimensional rational $\mathbb{E}G_\Phi(\mathbb{E})$ module. Let $g \in G_\Phi(\mathbb{E})$ let g^V denote the image of $g \in \mathrm{End}_\mathbb{E}(V)$. Slighty abusing notation we will often just write g for g^V. Since V is rational and finite dimensional we have

$$\chi_\alpha(t) = \sum_{i=0}^{d_\alpha} t^i \mathfrak{G}_{\alpha,i}$$

for some $d_\alpha \in \mathrm{N}$ and some $\mathfrak{G}_{\alpha,i} \in \mathrm{End}_{\mathbb{E}}(V)$. Note that $\mathfrak{G}_{\alpha,0} = \chi_\alpha(0) = 1$.

(We remark that, if V is obtained from a module in characteristic zero via an admissible lattice and taking tensor products, then $\mathfrak{G}_{\alpha,i} = (\frac{1}{i!}\mathfrak{G}_\alpha^i) \otimes 1$.)

Comment: It might be interesting to figure out what (ST1) means for the $\mathfrak{G}_{\alpha,i}$

Since $\mathbb{E}$ is infinite (and so $|E| > d_\alpha$) it is easy to see that the subalgebra of $\mathrm{End}_{\mathbb{E}}(V)$ generated by X_α contains all of the $\mathfrak{G}_{\alpha,i}$. Let $\mathfrak{G}_\alpha^V = \mathfrak{G}_{\alpha,1}$ and $\mathfrak{g}^V$ the Lie subalgebra of $\mathfrak{gl}(V)$ generated by the G_α^V. Let A^V be the subalgebra of $\mathrm{End}_{\mathbb{E}}(V)$ generated by all the $\mathfrak{G}_{\alpha,i}$ (As usual we will ommit the superscript V). Then every $G_\Phi(\mathbb{E})$ submodule of V is also an $\mathfrak{g}$ submodule and $\mathfrak{G}_\Phi(E)$ and A have the same submodules. **Comment: Maybe One should define $\mathfrak{H}_\alpha^V$ and verify the remaining relation for the Lie algebra**

(6) and (7)

$$[\mathfrak{G}_\alpha, \mathfrak{G}_\beta] = 0$$

if $\alpha + \beta$ is not a root and

$$[\mathfrak{G}_\alpha, \mathfrak{G}_\beta] = N_{\alpha\beta}]\mathfrak{G}_{\alpha+\beta}$$

if $\alpha + \beta$ is a root. By (ST4)

$$h_\alpha(u)\mathfrak{G}_{\beta,i}h_\alpha(u)^{-1} = u^{i(\alpha^*,\beta)}\mathfrak{G}_\beta$$

Let $\mu \in \Lambda$ and $v \in V$. We say that v is a weight vector for μ if

$$h_\alpha(u)v = u^{(\alpha^*,\mu)}v$$

for all $u \in \mathbb{K}^\#$ and $\alpha \in \Phi$. Since $\mathbb{E}$ is infinite and every polynomial as at most finitely many roots, two weights with a common non zero weight vector are equal. Let V_μ be the set of all weights for μ.

We observe

$$(**) \quad \mathfrak{G}_{beta,i}V_\mu \leq V_{\mu+i\beta}$$

Indeed let $v \in V_\mu$ then

$$h_\alpha(u)\mathfrak{G}_{\beta,i}v = u^{i(\alpha^*,\beta)}\mathfrak{G}_{\beta,i}h_\alpha(u)v = u^{i(\alpha^*,\beta)}\mathfrak{G}_{beta,i}u^{(\alpha^*,\mu)}v = u^{(\alpha^*,\mu+i\beta)}\mathfrak{G}_{\beta,i}v$$

Since the different weight spaces are linear independent (that is the sum of the weight spaces is a direct sum) for a wieght vector v that X_α fixes v if and only if $\mathfrak{G}_{\alpha,i}v = 0$ for all $1 \leq i \leq infty$.

A weight vector is called a highest weigth vector if $uv = v$ for all $u \in U$. In the view of the preceeding this means $\mathfrak{G}_{\alpha,i}v = 0$ for all $\alpha \in \Phi^+$. If V is irreducible there exists a non-zero weight vector. Indeed, since U acts unipotenly **Comment: why?** $C_V(U) \neq 0$.

Since H is abelian and $\mathbb{E}$ is algebraicly closed, there exists a one dimensional $\mathbb{E}H$ submodule $\mathbb{K}v$ in $C_V(U)$. Since V is rational it is easy to see that v is a weight vector for some weight $\lambda \in \Lambda$. Now $V = Av$ and so $(**)$ implies that V is the direct sum of its weight spaces.

4.4 Translation from the group to the Lie algebra

Comment: This is taking from Tim's file, needs to be adapted

Lemma 4.4.1 *Let $K \subseteq k$ be a subfield of k and λ a dominant weight with $\lambda(\alpha) < |K|$, for all $\alpha \in \Sigma$. Then $A(\lambda)$ is irreducible as a $kG(K)$-module.*

Proof: □

Let λ be a dominant integral weight.
$A = A(\lambda)$ be an irreducible $kG(K)$-module with highest weight λ.
Order Π in some way and then order the set of weights lexicographically. **Comment: mention positive, by carter we can choose the order to be compatible with the height function**

Define the following:

- $U_\alpha^+ = \langle X_\beta \mid \beta \leq \alpha \rangle$
- $U_\alpha^- = \langle X_\beta \mid \beta < \alpha \rangle$. Note that $U_\alpha^+ = X_\alpha U_\alpha^-$.
- A_μ a weight space (as usual)
- $A_\mu^+ = \bigoplus_{\gamma \leq \mu} A_\gamma$
- $A_\mu^- = \bigoplus_{\gamma < \mu} A_\gamma$

Let $P \leq U$.
Let $\Phi = \{\alpha \in \Sigma^+ \mid P \cap U_\alpha^+ \not\leq U_\alpha^-\}$.
For $\alpha \in \Phi$, pick $g_\alpha \in (P \cap U_\alpha^+) \setminus U_\alpha^-$. Then $g_\alpha = x_\alpha(t) u_\alpha$ for some $u_\alpha \in U_\alpha^-$ and $t \neq 0$.
Let $D = \Sigma k X_\alpha^k \leq \mathcal{L}^k$ - note, we'll drop the "k" from now on.

Lemma 4.4.2 *1. D is a subalgebra of $\mathcal{L}^k$.*

2. If P has nilpotent class m, then D has nilpotent class at most m.

3. If $[P[P[\underbrace{\ldots}_{n-times}[P[P,A]]\ldots]]] = 0$, then $D^n A = 0$.

4. $\dim(\mathrm{Ann}_A(D)) \geq \dim(C_A(P))$.

Proof: Notice that $[g_\alpha, g_\beta]U^-_{\alpha+\beta} = [x_\alpha(t_\alpha), x_\beta(t_\beta)]U^-_{\alpha+\beta} = x_{\alpha+\beta}(N_{\alpha\beta}t_\alpha t_\beta]U^-_{\alpha+\beta}$, where $[X_\alpha, X_\beta] = N_{\alpha+\beta}X_{\alpha+\beta}$ in $\mathcal{L}$. If $N_{\alpha+\beta} \neq 0$, then $[g_\alpha g_\beta] \in U^+_{\alpha+\beta} \setminus U^-_{\alpha+\beta}$. Hence, $\alpha+\beta \in \Phi$, so D is a subalgebra of $\mathcal{L}^k$, proving (1).

Now $[g_{\alpha_1}, g_{\alpha_2}, \dots, g_{\alpha_n}]U^-_{\alpha_1+\alpha_2+\dots+\alpha_n} = x_{\alpha_1+\alpha_2+\dots+\alpha_n}(rt_{\alpha_1}t_{\alpha_2}\dots t_{\alpha_n})U^-_{\alpha_1+\alpha_2+\dots+\alpha_n}$. So if $[g_{\alpha_1}, g_{\alpha_2}, \dots, g_{\alpha_n}] = 1$, then $r = 0$ and so $[X_{r_1}, X_{r_2}, \dots, X_{r_n}] = rX_{r_1+r_2+\dots+r_n} = 0$.

Now let $a \in A^+_\mu$ with $a = a_\mu + a^-_\mu$ where $a_\mu \in A_\mu$ and $a_\mu \in A^-_\mu$.
Then

$$[x_\alpha(t_\alpha), a] = \sum_{n=1}^{\infty} \frac{1}{n!} t^n_\alpha X^n_\alpha)a \in t_\alpha X_\alpha a_\mu + A^-_{\mu+\alpha}$$

So $[g_\alpha, a] \in t_\alpha X_\alpha a_\mu + A^-_{\mu+\alpha}$, and in particular,

$$[g_{\alpha_1}[g_{\alpha_2}[\dots[g_{\alpha_n}, a]\dots]] \in t_{\alpha_1}t_{\alpha_2}\dots t_{\alpha_n}X_{\alpha_1}X_{\alpha_2}\dots X_{\alpha_n}a_\mu + A^-_{\mu+\alpha_1+\alpha_2+\dots+\alpha_n}.$$

So, if $[P[P[\dots[P, A]\dots]] = 0$, then $X_{\alpha_1}X_{\alpha_2}\dots X_{\alpha_n}a_\mu \in A^-_{\mu+\alpha_1+\alpha_2+\dots+\alpha_n} \cap A_{\mu+\alpha_1+\alpha_2+\dots+\alpha_n} = 0$.

Hence $X_{\alpha_1}X_{\alpha_2}\dots X_{\alpha_n}A = 0$. That is, $D^nA = 0$, proving (2).

Choose $E_\mu \leq A_\mu$ so that $C_{A^+_\mu}(P) + A^-_\mu \geq E_\mu + A^-_\mu$ $(E_\mu = A_\mu \cap (C_{A^+_\mu}(P) + A^-_\mu))$. Let $E = \bigoplus_\mu E_\mu$. Then $\dim_k(E) = \dim_k(C_A(P))$.

Now, if $a \in C_{A^+_\mu}(P)$, then $a = a_\mu + a^-_\mu$, so $[g_\alpha, a] \in t_\alpha X_\alpha a_\mu + A^-_{\mu+\alpha}$ implies that $x_\alpha a_\mu = 0$.

Hence, $X_\alpha E = 0$ and so $DE = 0$, proving (3). □

Chapter 5

Quadratic Modules

5.1 Quadratic modules for $\mathfrak{g}$

For a root system Φ let $p_\Phi := \frac{(a,a)}{(b,b)}$ where a is a long and b is a short root in Φ. Note that if Φ is connected than $p_\Phi \in \{1,2,3\}$. If $\mathfrak{g} = \mathfrak{g}_\Phi(\mathbb{K})$ and $p_\phi = \operatorname{char}\mathbb{K}$, then $\mathfrak{g}_{\text{short}}$ (the subalgebra of $\mathfrak{g}$ generated by $\{\mathfrak{E}_\alpha \mid, \alpha \in \Phi_{\text{short}}\}$) is an ideal in $\mathfrak{g}$. Note that this happens for $p = 2$ and Φ of type B_n, C_n and F_4 and for $p = 3$ and Φ of type G_2. These cases will require special attention throughout this section.

Definition 5.1.1 *A module V for $\mathfrak{g}_\Phi(\mathbb{K})$ is called* quadratic *if $(\mathfrak{H}_\alpha - 1)\mathfrak{E}_\alpha V = 0$ for all long roots $\alpha \in \Phi$.*

The defintion of a quadratic module is motivited by the following lemma:

Lemma 5.1.2 [**quadratic in odd characteritic**] **Comment: A version of the follwing might be better at an earlier place** *With the notation from the previous proposition, let α be a long root. Then $(\frac{1}{2}\mathfrak{E}_\alpha^2 \in \mathcal{U}_\mathbb{Z}$ and so $(\frac{1}{2}\mathfrak{E}_\alpha^2$ acts on V. Note that $[(\frac{1}{2}\mathfrak{E}_\alpha^2, \mathfrak{E}_{-\alpha}] = (\mathfrak{H}_\alpha - 1)\mathfrak{E}_\alpha$ and so $(\frac{1}{2}\mathfrak{E}_\alpha^2$ annihilates V we get $(\mathfrak{H}_\alpha - 1)\mathfrak{E}_\alpha V = 0$. This indicated that maybe the correct definition for quadratic action for Lie algebras is $\frac{1}{2}\mathfrak{E}_\alpha^2 V = 0$. It works well in any characteritic. But we prefer to work with the slightly weaker condition $(\mathfrak{H}_\alpha - 1)\mathfrak{E}_\alpha V = 0$, since it can be phrased just in terms of the Lie algebra. And for* $\operatorname{char}\mathbb{K}$ *it turms out to be equivalent to $\mathfrak{E}_\alpha^2 e \equiv 0$: $\mathfrak{H}_\alpha\mathfrak{E}_\alpha \equiv \mathfrak{E}_\alpha$ implies $2\mathfrak{E}_\alpha \equiv \mathfrak{H}_\alpha\mathfrak{E}_\alpha - \mathfrak{E}_\alpha\mathfrak{H}_\alpha \equiv \mathfrak{E} - \mathfrak{E}_\alpha\mathfrak{H}_\alpha$ and so $\mathfrak{E}_\alpha\mathfrak{H}_\alpha = -\mathfrak{E}_\alpha$. So we can compute $\mathfrak{E}_\alpha\mathfrak{H}_\alpha\mathfrak{E}_\alpha$ in two different ways. First it equals $(\mathfrak{E}_\alpha\mathfrak{H}_\alpha)\mathfrak{E}_\alpha \equiv -\mathfrak{E}_\alpha\mathfrak{E}_\alpha \equiv \mathfrak{E}_\alpha^2$ and secondly $\mathfrak{E}_\alpha(\mathfrak{H}_\alpha\mathfrak{E}_\alpha) = \mathfrak{E}_\alpha\mathfrak{E}_\alpha = \mathfrak{E}_\alpha^2$. So if* $\operatorname{char}\mathbb{K} \neq 2$ *we can conclude $\mathfrak{E}_\alpha^2 \equiv 0$.*

The irreducible quadratic modules for $\mathfrak{g}_\Phi\mathbb{K}$ are fairly easily classified (see the next theorem). The remainder of the section will be devoted to show that some weaker conditions already imply that a module is quadratic. If V module for $\mathfrak{g}$ and $\mathfrak{E}_1, \mathfrak{E}_2 \in \mathfrak{g}$ we write $\mathfrak{E}_1 \equiv \mathfrak{E}_2$ if $(\mathfrak{E}_1 - \mathfrak{E}_2)V = 0$ (that is if the image of $\mathfrak{E}_1$ and $\mathfrak{E}_2$ in $\operatorname{End}(V)$ are equal.

Theorem 5.1.3 [**classification of quadratic modules for Lie algebras**] *Let* $\mathbb{K}$ *be an field,* Φ *a root system and* $\mathfrak{g} = \mathfrak{g}_\Phi(\mathbb{K})$ *the corresponding algebra. Let* $V = V(\lambda)$ *the irreducible restricted* $\mathfrak{g}$ *module of heighest weight* $\lambda \neq 0$*. Let* α *be the heighest long root. Then the following are equivalent:*

(a) V *is quadratic.*

(b) $(\mathfrak{H}_\alpha - 1)\mathfrak{G}_\alpha V = 0$

(c) $\mathfrak{G}_\beta \mathfrak{G}_\alpha V = 0$ *for all* $\beta \in \Phi$ *with* $(\beta, \alpha) > 0$.

(d) $(\lambda, \alpha^*) = 1$.

(e) $\lambda = \lambda_\beta$ *for some root* $\beta \in \Pi$ *with* $n^*_{\beta^*} = 1$*, where* n^*_γ *for* $\gamma \in \Pi^*$ *is defined by* $\alpha^* = \sum_{\gamma \in \Pi^*} n^*_\gamma \gamma$.

(f) $\lambda = \lambda_\beta$ *for some root* $\beta \in \Pi$ *with* $n_\beta = p_\beta$*, where* $p_\beta = \frac{(\alpha,\alpha)}{(\beta,\beta)}$*, and* n_β *is defined by* $\alpha = \sum_{\beta \in \Pi} n_\beta \beta$.

(g) *One of the follwing holds:* **Comment: labeling of roots needs to be introduced**

1. $\Phi = A_n$ *and* $\lambda = \lambda_i$ *for some* $1 \leq i \leq n$.
2. $\Phi = B_n$ *and* $\lambda = \lambda_1$ *or* λ_n.
3. $\Phi = C_n$ *and* $\lambda = \lambda_i$ *for some* $1 \leq i \leq n$.
4. $\Phi = D_n$ *and* $\lambda = \lambda_1, \lambda_{n-1}$ *or* λ_n.
5. $\Phi = E_6$ *and* $\lambda = \lambda_1$ *or* λ_6.
6. $\Phi = E_7$ *and* $\lambda = \lambda_1$.
7. $\Phi = E_8$*: No such module.*
8. $\Phi = G_2$ *and* $\lambda = \lambda_1$.
9. $\Phi = F_4$ *and* $\lambda = \lambda_1$

Proof: We assume without loss that $\mathbb{K}$ is algebraicly closed. (a)$\Longrightarrow$(b): Obvious from the definition of "quadratic"

(b)$\Longrightarrow$(c): Let $\beta \in \Phi$ with $(\beta, \alpha) > 0$. If $\beta =\neq \alpha$ then $(\alpha^*, \beta) = 1$ and so $[\mathfrak{H}_\alpha, \mathfrak{G}_\beta] = \mathfrak{G}_\beta$. Note that β is postive, so $\beta + \alpha \notin \Phi$ be the maximality of α. Thus $\mathfrak{G}_\alpha \mathfrak{G}_\beta = \mathfrak{G}_\beta \mathfrak{G}_\alpha$. Also by assumption $(\mathfrak{H}_\alpha - 1)\mathfrak{G}_\alpha =\equiv 0$ and so $\mathfrak{H}_\alpha \mathfrak{G}_\alpha = \mathfrak{G}_\alpha$. We compute:

$$\mathfrak{G}_\beta \mathfrak{G}_\alpha = [\mathfrak{H}_\alpha, \mathfrak{G}_\beta]\mathfrak{G}_\alpha = \mathfrak{H}_\alpha \mathfrak{G}_\beta \mathfrak{G}_\alpha - \mathfrak{G}_\beta \mathfrak{H}_\alpha \mathfrak{G}_\alpha = \mathfrak{H}_\alpha \mathfrak{G}_\alpha \mathfrak{G}_\beta - \mathfrak{G}_\beta \mathfrak{H}_\alpha \mathfrak{G}_\beta \equiv \mathfrak{G}_\alpha \mathfrak{G}_\beta - \mathfrak{G}_\beta \mathfrak{G}_\alpha = 0$$

It remains to show that $\mathfrak{G}_\alpha^2 \equiv 0$. If $p = 2$ this is obvious. So suppose $p \neq 2$. Then

$$0 = [\mathfrak{G}_\alpha, \mathfrak{G}_\alpha] =\equiv [\mathfrak{H}_\alpha \mathfrak{G}_\alpha, \mathfrak{G}_\alpha] = 2\mathfrak{G}_a^2$$

and (c) is proved.

(c)$\Longrightarrow$(d): Let $v_- = \omega_0(v_+)$ be a lowest weigth vector. Then v_- has weight $\omega_0(\lambda)$. Also no proper $\mathfrak{u}$ submodule of V contains v_- (**??**) and so $v^- \notin \mathrm{Ann}(\mathfrak{G}_\alpha)$. Hence $v := \mathfrak{G}_\alpha v_- \neq 0$ is a non zero weight vector with weight $\omega_0(\lambda) + \alpha$. Let

$$\mathfrak{q}_\alpha = \mathbb{K}\langle G_\beta \mid \beta \in \phi, (\alpha, \beta) > 0\rangle$$

and

$$\mathfrak{l}_\alpha = \mathbb{K}\langle G_\beta \mid \beta \in \phi, (\alpha, \beta) = 0\rangle$$

By Smith's Lemma4.2.4 $\mathrm{Ann}(\mathfrak{q}_\alpha)$ is an irreducible module for $\mathfrak{l}_\alpha$. Since v_+ is a highest weigth vector in $\mathrm{Ann}(\mathfrak{q}_\alpha)$ we concldue from **??** that all weights in $\mathrm{Ann}(\mathfrak{q}_\alpha)$ are of the form $\lambda + \mu$ for some $\mu \in \mathbb{N}(\Phi^- \cap \alpha^\perp)$.

Recall that with weights vectors we mean weight vectors for the cartan subgroup H of $G_\mathbb{K}(\Phi)$. In particular two weights in Λ which share a non-zero weight vector are equal. Thus

$$\omega_0(\lambda) + \alpha = \lambda + \mu$$

for some $\mu \in \Lambda$ with $(\alpha, \mu) = 0$. Note also that ω_0 has order two, peserves $(\cdot, \cdot)$ and $\omega_0(\alpha) = -\alpha$. So we compute

$$(\omega_0(\lambda) + \alpha, \alpha^*) = (\omega_0(\lambda), \alpha^*) + \ r\alpha\alpha^* = (\lambda, \omega_0(\alpha^*)) + 2 = -(\lambda, \alpha^*) + 2$$

On the other hand

$$(\omega_0(\lambda + \mu, \alpha)^* = (\lambda, \alpha^*) + (\mu, \alpha^*) = (\lambda, \alpha^*)$$

The last three displayed equations imply $2(\lambda, \alpha^*) = 2$. Since this is statement in $\mathbb{Z}$ we conclude $(\lambda, \alpha^*) = 1$.

(d)$\Longrightarrow$(a): Suppose that $(\lambda, \alpha^*) = 1$. Suppose that $\mathfrak{q}_a \mathfrak{G}_\alpha V \neq 0$. Then there exists a weight vector v of weight ρ and $\beta \in \Phi$ with $(\beta, \alpha) > 0$ such that $\mathfrak{G}_\beta \mathfrak{G}_\alpha v \neq 0$. Thus $\tilde{\rho} := \rho + \alpha + \beta$ is a weight on V. By **??** $-1 = -(\lambda, \alpha^*) \leq (\rho, \alpha^*)$ and $(\tilde{\rho}, \alpha^*) \leq (\lambda, \alpha^*) = 1$. Hence

$$1 \geq (\tilde{\rho}, \alpha^*) = (\rho, \alpha^*) + (\alpha, \alpha^*) + (\beta, \alpha^*) > -1 + 2 = 1$$

This contradiction shows that $\mathfrak{G}_\alpha V \leq \mathrm{Ann}(\mathfrak{q}_\alpha)$. Since $\mathfrak{l}_\alpha$ is irreducible on $\mathrm{Ann}(\mathfrak{q}_a$ and $\mathfrak{H}_\alpha$ commutes with $\mathfrak{l}_\alpha$, $\mathfrak{H}_\alpha$ acts as a scalar k on $\mathrm{Ann}(\mathfrak{q}_\alpha)$. Since $v_+ \in \mathrm{Ann}(\mathfrak{q}_\alpha$ this scalar is $(\lambda, \alpha^*) = 1$. Thus $(\mathfrak{H}_\alpha - 1$ annihilates $\mathfrak{G}_\alpha V \leq \mathrm{Ann}(\mathfrak{q}_a)$. Thus $(\mathfrak{H}_\alpha - 1)\mathfrak{G}_\alpha V = 0$. Since W acts transitively on the long roots, V is quadratic.

(d)$\Longleftrightarrow$ (e): Let $\lambda = \sum_{\beta \in \Pi} m_\beta \lambda_\beta$. Then each m_β is a non-negative integer and each n_γ^*

is a positive integer. Also $(\lambda, \alpha^*) = \sum_{\beta\in\Pi} m_\beta n^*_{\beta^*}$ and so (d) and (e) are equivalent.

(e)$\Longleftrightarrow$ (f): Follows from **??**

(e)$\Longleftrightarrow$ (g): Follows from a glance of at the highest short root of Φ^* (**??**). □

Definition 5.1.4 *A quadratic tuple is tuple $(\Phi, p, \lambda, \alpha, \beta)$ where Φ is a connected root system, λ is a non-zero dominant integral p-restricted weight, α and β are roots, and $V = V_{\mathbb{K}}(\lambda)$ for some field $\mathbb{K}$ with* $\operatorname{char}\mathbb{K} = p$ *such that*

(a) $\mathfrak{G}_\beta\mathfrak{G}_\alpha V = 0$.

(b) $\mathfrak{G}_\alpha V \neq 0 \neq \mathfrak{G}_\beta V$.

(c) *If $\alpha = \beta$ then $p \neq 2$.*

In the next few lemmas we will determine all the quadratic tuples. **Comment: We should once and for all introduce weight vectors for arbitrary fields: For the algebraicly closed case define it by the action of H, in general $v \in V(\lambda)$ is called a weight vector if $1 \otimes_{\mathbb{K}} v$ is a weight vector in $\bar{K} \otimes \mathbb{K}V$. Note that for p-restricted weights, V will be the direct sum of the weight spaces.(just start with the lowest weight vector and take images under the $\mathfrak{G}_\alpha$'s**

Lemma 5.1.5 [**quadratic tuple for a=b long**] *Let $(\Phi, p, \lambda, \alpha, \beta)$ be a quadratic tuple with $\alpha = \beta$ and α long. Then V is a quadratic module.*

Proof: By assumption $p \neq 2$. So the lemma follows from **??** □

Lemma 5.1.6 [**quadratic tuples for (a,b) positive and a long**] *Let $(\Phi, p, \lambda, \alpha, \beta)$ be a quadratic tuple with α long, $\alpha \neq \beta$ and $(\alpha, \beta) > 0$. Then V is a quadratic module.*

Proof: Without loss α is the highest long root. Then β is positive. Let $\Psi = \langle \alpha, \beta \rangle$, the root subsystem generated by α and β. Then Ψ is of type A_2, B_2 of G_2. In any case $\delta = \alpha - \beta$ is a root, $\alpha = \delta + \beta$, $\alpha + \beta$ is nor a root, $\mathfrak{G}_\alpha\mathfrak{G}_\beta = \mathfrak{G}_\beta\mathfrak{G}_\alpha \equiv 0$ and $r_{\delta\beta} + 1 = p_\Psi$.

Suppose first that $p \neq 2$ and $p \neq p_\Psi$.

Since $\mathfrak{G}_\beta\mathfrak{G}_\alpha \equiv 0$ taking the Lie bracket with $\mathfrak{G}_\delta$ gives $\pm p_\Psi \mathfrak{G}_\alpha^2 \equiv 0$. Thus $\mathfrak{G}_\alpha^2 = 0$ and we are done by 5.1.5.

Suppose next that $Pp = p_\Psi$. Then $p = p_\Phi$ and β is short. Let X be an irreducible $\mathfrak{g}_{\text{short}}$-submodule in V. If $\mathfrak{G}_\beta X = 0$ then also $\mathfrak{H}_\alpha = [\mathfrak{G}_\beta, \mathfrak{G}_{-\beta}]$ annihilates X. Thus by **??**(bb), $\mathfrak{H}_\alpha$ acys nilpotently on V. But $\mathfrak{H}_\alpha$ is semisimple on V and so $\mathfrak{H}_\alpha V = 0$. Hence by **??** $\mathfrak{G}_\beta V = 0$, a contrdiction to the definition of a quadratic tuple.

Thus $\mathfrak{G}_\beta X \neq 0$. Since $\mathfrak{G}_\alpha\mathfrak{G}_\beta X = 0$ we conclude $\operatorname{Ann}_X(\mathfrak{G}_\alpha) \neq 0$ and so by **??**(bc), $\mathfrak{G}_\alpha X \leq X$. By symmetry the same holds for any long root subalgebra of $\mathfrak{g}$ and so $\mathfrak{g}X \leq X$

and $V = X$. Thus $\mathfrak{g}_{\text{short}}$ acts irreducibly on V. Let $\mathfrak{q} = \mathbb{K}\langle \mathfrak{G}_\mu \mid \mu \in \Phi_{\text{short}}, (\mu, \alpha) > 0\rangle$ and $\mathfrak{l} = \mathbb{K}\langle \mathfrak{G}_\mu \mid \mu \in \Phi_{\text{short}} \cap \alpha^\perp\rangle$. Then $\mathfrak{q} + \mathfrak{l} + \mathfrak{h}_{\text{short}}$ is a parabolic subalgebra and so by 4.2.4 $\operatorname{Ann}_V(\mathfrak{q})$ is an irreducible $\mathfrak{l}$-module. Note that $\mathfrak{q}$ is an ideal in $\mathfrak{q}_\alpha + \mathfrak{l}_\alpha$ and so $\operatorname{Ann}_V(\mathfrak{q})$ is an irreducible module for $\mathfrak{q}_\alpha + \mathfrak{l}_\alpha$. It follows that $\mathfrak{q}_\alpha$ annihilates $\operatorname{Ann}_V(\mathfrak{q})$. On the other hand $W(\Phi \cap \alpha^\perp)$ acts transitively on $\{\mu \in \Phi_{\text{short}}, (\mu, \alpha) > 0\}$ and thus $\mathfrak{q}\mathfrak{G}_\alpha V = 0$ and so also $\mathfrak{q}_\alpha G_\alpha V = 0$. Thus V is quadratic by 5.1.3.

Suppose now that Ψ is of type A_2. We claim that $\mathfrak{G}_\mu \mathfrak{G}_\alpha \equiv 0$ for all $\mu \in \Phi$ with $(\mu, \alpha) > 0$. This is obvious if $\mu = \alpha$ or if (α, μ) is conjugate to (α, β) under $W(\Phi)$. If neither of this holds then Φ is of type A_n. Let V^* be $\mathfrak{g}$ module dual to V. Then $\mathfrak{G}_\alpha \mathfrak{G}_\beta V^* = 0$. Since $\mathfrak{G}_\alpha$ and $\mathfrak{G}_\beta$ commute, $\mathfrak{G}_\beta \mathfrak{G}_\alpha V^* = 0$. Now $V^* \cong V^\sigma$ where σ is the graph automorphism of $\mathfrak{g}$. Thus $\mathfrak{G}_{\sigma(\beta)} \mathfrak{G}_{\sigma(\alpha)} V = 0$. Now (α, μ) is conjugate under $W(\Phi)$ to $(\sigma(\alpha), \sigma(\beta)$ and we again conclude that $\mathfrak{G}_\mu \mathfrak{G}_\alpha \equiv 0$. Thus V is quadratic by 5.1.3.

Suppose finally that $p = 2$ and Ψ is of type G_2. Then β is short. Let $\gamma = \beta - \delta$. Then γ is a root, $r_{\delta\beta} = 3$, and $\alpha + \gamma$ is not a root.

$$0 \equiv [\mathfrak{G}_\beta \mathfrak{G}_\alpha, \mathfrak{G}_\gamma] = \pm 3 \mathfrak{G}_{\beta+\gamma} \mathfrak{G}_\alpha$$

Thus $\mathfrak{G}_{\beta+\gamma} \mathfrak{G}_\alpha \equiv 0$. Using the action of $W(\Phi \cap \alpha^\perp)$ we conclude that $\mathfrak{q}_\alpha \mathfrak{G}_\alpha \equiv 0$ and V is quadratic. □

Lemma 5.1.7 [**a long implies quadratic**] *Let $(\Phi, p, \lambda, \alpha, \beta)$ be a quadratic tuple with α long. Then V i quadratic.*

Proof: Without loss α is the highest long root. If $\beta = \alpha$ we are done by 5.1.5. So we may choose $\beta \in \Phi$ maximal with $\beta \neq a$, $\mathfrak{G}_\beta V \neq 0$ and $\mathfrak{G}_\beta \mathfrak{G}_\alpha V = 0$. If $(\beta, \alpha) > 0$ we are done by 5.1.6. So we may assume that $(\alpha, \beta) \leq 0$.

Suppose first that β is long. If Φ is of type A_1 then $\beta = -\alpha$ and so $2\mathfrak{G}_\alpha^2 = [\mathfrak{G}_\beta \mathfrak{G}_\alpha, \mathfrak{G}_\alpha, \mathfrak{G}_\alpha] = iv0$. Thus $\mathfrak{G}_\alpha^2 \equiv= 0$ and V is quadratic by 5.1.3 (Actually a moments thought even gives a contradiction).

So assume that $\Phi \neq A_1$. If Φ_{long} is connected there exists $\gamma \in \Pi(\Phi_{\text{long}})$ with $\beta + \gamma \in \Phi_{\text{long}}$. Then $N_{\beta\gamma} \neq 0$ and so $\mathfrak{G}_{\beta+\gamma} \mathfrak{G}_a = 0$. The maximal choice of β implies $\beta + \gamma = \alpha$. But then $(\alpha, \beta) > 0$.

So Φ_{lo} is disconnected, $\alpha \perp \beta, \Phi$ is of type C_n and $\gamma := \frac{1}{2}(\alpha - \beta) \in \Phi_{\text{short}}$. Then $N_{\beta\gamma} \neq 0$ and $\mathfrak{G}_{\gamma+\alpha} \mathfrak{G}_\alpha \equiv 0$. The maximal choice of γ implies $\mathfrak{G}_{\gamma+\alpha} V = 0$. In particular $p = 2$, $\mathfrak{g}_{\text{short}} V = 0$ and $[\mathfrak{H}_\beta, \mathfrak{g}]V = 0$. Thus $\mathfrak{H}_\beta$ acts as a scalar on V. Since $\alpha \perp b$, $\mathfrak{H}_\beta \mathfrak{G}_\alpha = [\mathfrak{G}_\beta \mathfrak{G}_\alpha, G_{-\beta} =\equiv 0$ and so $\mathfrak{H}_\beta V = 0$ But then $\mathfrak{g}$ acts nilpotent on V a contradiction.

Suppose next that β is not long. Note that the highest short root has positive inner product with α. So β is not the highest short root. Assume Φ_{short} is connect. Then we can choose $\gamma \in \Pi(\Phi_{\text{short}})$ with $\beta + \gamma \in \Phi_{\text{short}}$ and we get a contradiction to the maximal choice of β. Hence Φ_{short} is disconnected and Φ is of type B_n. If β is not perpendicular to α then $((b,a) < 0$, $N_{\beta\alpha} \neq 0$ and we get $G_{\alpha+\beta}\mathfrak{G}_\alpha = 0$, contradiction the maximality of β. So $\beta \perp \alpha$ and as above $\mathfrak{H}_\beta\mathfrak{G}_\alpha = 0$. Let $\gamma \in \Pi$ with $\beta + \gamma \in \Phi$. If $N_{\beta\gamma} \neq 0$, we get a contradiction to the maximality of β. Thus $p = 2$ and so $[Hb, \mathfrak{g}] = 0$ and $\mathfrak{H}_\beta$ centralizes V. But then $\mathfrak{g}_{\text{short}}V = 0$, a contradiction as β is short and $\mathfrak{G}_\beta V \neq 0$.

This settles the last case and the lemma is proved. □

Lemma 5.1.8 [**quadratic tuples with GaGb not 0**] *Let $(\Phi, p, \lambda, \alpha, \beta)$ be a quadratic tuple with $\mathfrak{G}_\alpha\mathfrak{G}_\beta V(\lambda) \neq 0$. The up to conjugacy under W $\Phi = A_n$, $\alpha = e_0 - e_n$ and either $\beta = -e_0 + e_1$ and $\lambda = \lambda_n$ or $\beta = -e_2 + e_n$ and $\lambda = \lambda_1$.*

Proof: Let V^* the dual of V. So $V^* = V(omega_0(\lambda))$. Then $\mathfrak{G}_\alpha\mathfrak{G}_\beta V^* = 0$ and we conclude that $\lambda \neq -\omega_0(\lambda)$. Thus $\Phi = A_n$, E_6, or $n \geq 5, n$ is odd and $\Phi = D_n$ Also $[G_\alpha, G_b] \neq 0$ and so $(\alpha, \beta) < 0$.

But in D_n for $n > 3$ and for E_6, W has a unique orbits on pairs of roots (γ, δ) with $(\gamma, \delta) < 0$. Namely for D_n all are conjugate to $(e_1 + e_2, -e_1 + e_3)$ and for E_6. Thus $(\alpha\, beta)$ is conjugate to (β, α) contradicting the assumptions.

Thus Φ is of type A_n. By 5.1.7 that V is quadratic and so by 5.1.3 $\lambda = \lambda_i$ for some $1 \leq i \leq n$.

Up to conjugation under W, we may assume $\alpha = e_0 - e_n$ and either $\beta = -e_0 + e_1$ or $\beta = -e_1 + e_n$. In view of the graph automorphismus it suffices to treat the case $\beta = -e_0 + e_n$. Let

$$\Sigma = \langle \beta, \Phi \cap \alpha^\perp \rangle = \{\pm(e_i - e_j) \mid 0 \leq i < j \leq n-1\}.$$

Then Σ is a closed root subsystem of type A_{n-1}. Also $\mathfrak{G}_\alpha V$ is invariant under $\mathfrak{l}_\alpha$ and $\mathfrak{G}_\beta$ and so under $\mathfrak{g}_\Sigma$. Since $\mathfrak{G}_b$ annihilates $\mathfrak{G}_\alpha V$ and $W(\Sigma)$ is tranisitive on Σ, g_σ annihilates $\mathfrak{G}_\alpha V$. As $v_+ \in \mathfrak{G}_\alpha V$ we conclude that $\lambda = \lambda_n$ and the lemma is proved. □

Lemma 5.1.9 [**quadratic tuples for (a,b) not postive and a long**] *Let $(\Phi, p, \lambda, \alpha, \beta)$ be a quadratic tuple with α long, $\alpha \neq \beta$ and $(\alpha, \beta) \leq 0$. Then one of the following holds:*

- (a) *$\Phi = A_n$, $\alpha = e_0 - e_n$ and either*
  - (aa) *$\lambda = \lambda_1$ and $\beta = e_1 - e_2$ or $-e_2 + e_n$ or*
  - (ab) *$\lambda = \lambda_n$ and $\beta = e_1 - e_2$ or $-e_0 + e_1$.*
- (b) *$\Phi = C_n$, $\lambda = \lambda_1$, $\alpha = 2e_1$ and either $\beta = 2e_2$ or $p \neq 2$, $n > 2$ and $\beta = e_2 - e_3$.*
- (c) *$\Phi = B_n, n \geq 3$, $\alpha = e_1 + e_2$ and either*
  - (ca) *$\lambda = \lambda_n$ and $\beta = e_1 - e_2$ or*
  - (cb) *$\lambda = \lambda_1$ and either $\beta = e_3 - e_4$ and $n \geq 4$, or $\beta = e_3$ and $p \neq 2$.*

(d) $\Phi = D_4$ $\alpha = e_1 + e_2$ *and one of the following holds:*

  (da) $\lambda = \lambda_1$ *and* $\beta = e_3 - e_4$ *or* $e_3 + e_4$.

  (db) $\lambda = \lambda_3$ *and* $\beta = e_1 - e_2$ *or* $e_3 + e_4$.

  (dc) $\lambda = \lambda_4$ *and* $\beta = e_1 - e_2$ *or* $e_3 - e_4$.

(e) $\Phi = D_n, n \geq 5, \alpha = e_1 + e_2$ *and either*

  (ea) $\beta = e_3 - e_4$ *and* $\lambda = \lambda_1$ *or*

  (eb) $\beta = e_1 - e_2$ *and* $\lambda = \lambda_{n-1}$ *or* λ_n.

Proof: Without loss α is the highest root. Let Ψ be the closed root subsystem generated by α and β. By 5.1.7 that V is quadratic and so by 5.1.3 $\lambda = \lambda_\mu$ for some $\delta \in \Pi$ with $n^*_{\mu^*} = 1$. Moreover, $\mathfrak{G}_\alpha V = \mathrm{Ann}(\mathfrak{q}_\alpha)$ and so $\mathfrak{G}_\beta \mathfrak{G}_\alpha V = 0$ just means that $\mathfrak{G}_\beta$ annihilates $V_\alpha := \mathrm{Ann}(\mathfrak{q}_\alpha)$.

Supose first that $(\beta, \alpha) = 0$. Then $\mathfrak{G}_b \leq \mathrm{Ann}_{\mathfrak{l}_\alpha}(V_\alpha)$. If $(\mu, \alpha) \neq 0$ then all of $\mathfrak{l}_\alpha$ annihilates V_α and (a) or (b) holds.

Suppose next that $(\beta, \alpha) < 0$. If $\mathfrak{G}_\alpha \mathfrak{G}_\beta V 0$, then (a) holds by 5.1.8 So we may assume that $\mathfrak{G}_\alpha \mathfrak{G}_\beta \equiv 0$. Then also $[\mathfrak{G}_\alpha, \mathfrak{G}_\beta] \equiv 0$. Since $(\beta, \alpha) < 0$, $\alpha + \beta$ is a root and since α is long $N_{\alpha\beta} \neq 0$. It follows that $G_{\alpha+\beta} \equiv 0$. Thus $p = p_\Phi$ and $\alpha + \beta$ is short. Since $\mathfrak{G}_\beta \not\equiv 0$, β is long. But the sum of two long roots always long , a contradiction to $\alpha + \beta$ short. □

Lemma 5.1.10 [**p=pphi and a and b short**] *Let* $(\Phi, p, \lambda, \alpha, \beta)$ *be a quadratic tuple and suppose that* $p = p_\Phi$ *and both* α *and* β *are short. Then* $\Phi = C_n$, $p = 2$ *and* $\lambda = \lambda_1$ *or* $\lambda_1 + \lambda_n$.

Proof: Note that Φ is B_n, C_n, G_2 or F_4 and Φ_{short} is of type A_1^n, D_n, A_2 and D_4 respectively. Moreover $W/W(\Phi_{\text{short}})$ induces the full group of graph automorphisms on Φ_{short}.

Let μ be the restriction of λ to Φ^*_{short}. Then all composition factors for $\mathfrak{g}_{\text{short}}$ on V are isomorphic to $V(\mu)$. Moreover $(\Phi_{\text{short}}, \mu, \alpha, \beta)$ is a quadraic tuple. This easily rules out the case $\Phi_{\text{short}} = A_1^n$. Hence Φ_{short} is connected and so by 5.1.7 $V(\mu)$ is quadratic for $\mathfrak{g}_{\text{short}}$. Since μ is invariant under all graph automorphism, 5.1.3 implies that $\Phi_{\text{short}} = D_n$ and $\mu =$ "μ_1". Then $\lambda = \lambda_1$ or $\lambda = \lambda_1 + \lambda_n$ and the lemma is proved. □

It remains to look at quadratic tuples where Φ has two root lengths, α and β are short and $p \neq p_\phi$,

Lemma 5.1.11 [**a=b short**] *Let* $(\Phi, p, \lambda, \alpha, \beta)$ *be a quadratic tuple with* $\alpha = \beta$ *short and* $p \neq \Phi_p \neq 1$.. *Then* V *is minuscule. That is one of the follwing holds*

(a) $\Phi = B_n$ *and* $\lambda = \lambda_n$.

(b) $\Phi = C_n$ *and* $\lambda = \lambda_1$

Proof: Without loss α is the highest short root. Since α is not the highest long, there exists $\gamma \in \Pi$ with $\alpha + \gamma \in \Phi$. Since α is the highest short root, $\alpha + \gamma$ is long, $N_{\alpha\gamma} = \pm p_\Phi$ and neither $\alpha + 2\gamma$ nor $2\alpha + \gamma$ are roots Thus

$$0 \equiv [\mathfrak{G}_\alpha^2, \mathfrak{G}_\gamma] = \pm 2 p_\Phi \mathfrak{G}_{\alpha+\gamma}\mathfrak{G}_\alpha$$

Since $\alpha = \beta$, $p \neq 2$. By assumtion $p \neq p_\Phi$ and so $\mathfrak{G}_{\alpha+\gamma}\mathfrak{G}_\alpha \equiv 0$. Thus by 5.1.7 V is quadratic. So $\lambda = \lambda_\delta$ for some $\delta \in \Pi$ so that δ^* appears once in the highest short root of Φ^*. A glance at the highest long root of Φ^* shows that δ appears once or twice in α^*. Thus $(\lambda, \alpha^*) \in \{1, 2\}$. Note that there exists a composition factor for $\mathbb{K}\langle G\alpha, \mathfrak{H}_\alpha \mathfrak{G}_{-\alpha}\rangle$ with heighest weight the restriction of λ. Since $\mathfrak{G}_a^2$ annihilates this composition factor $(\lambda, \alpha^*) = 1$. So λ is minuscule. □

Lemma 5.1.12 [**a,b short, (a,b) not negative**] *Let $(\Phi, p, \lambda, \alpha, \beta)$ be a quadratic tuple with both α and β short, $\alpha \neq \beta$, $(\alpha, \beta) \geq 0$ and $p \neq \Phi_p \neq 1$.. Then up to conjuagacy under W,*

$\Phi = C_n, \lambda = \lambda_1$, $\alpha = e_1 + e_2$ *and* $\beta = e_2 + e_3$ *or* $\beta = e_3 + e_4$.

Proof: Suppose that $\alpha + \beta$ is a long root. Then $N_{\alpha\beta} = p_\Phi \neq p$. By 5.1.8 $\mathfrak{G}_\beta\mathfrak{G}_\alpha \equiv 0$ and so $N_{\alpha\beta}G_{\alpha+\beta} \equiv 0$. Thus $G_{\alpha+\beta} \equiv 0$ a contradiction.

Thus $\alpha + \beta$ is not a long root. This rules out the case $\Phi = B_n$ and $\Phi = G_2$. It also shows that $(\alpha, \beta) > 0$ for F_4. Also $p \neq p_\phi = 2$ and in view of 5.1.11 we will be done if we can show that $\mathfrak{G}_\alpha^2 \equiv 0$.

Suppose that $(alpha, \beta) > 0$. Then $\langle \alpha, \beta \rangle$ is of type A_2. So $\gamma = \beta - \alpha$ is a short root, $\alpha + \gamma$ is not a root and $N_{\beta\gamma} = \pm 1 \neq 0$. Hence

$$0 \equiv [\mathfrak{G}_\beta\mathfrak{G}_\alpha, \mathfrak{G}_\gamma] = N_{\beta\gamma}G_\alpha^2$$

and so $\mathfrak{G}_\alpha^2 \equiv 0$.

Suppose next that $(\alpha, \beta) = 0$. Then $\Phi = C_n$, $n \geq 4$ and without loss $\alpha = e_1 + e_2$ and $\beta = e_3 + e_4$. Let $\gamma = e_2 - e_3$. Then $\beta + \gamma = e_2 + e_4$ is a root, $N_{\beta\gamma} = \pm 1 \neq 0$ and $\alpha + \gamma$ is not a root and so

$$0 \equiv [\mathfrak{G}_\beta\mathfrak{G}_\alpha, \mathfrak{G}_\gamma] = N_{\beta\gamma}\mathfrak{G}_{\beta+\gamma}\mathfrak{G}_\alpha$$

and so $\mathfrak{G}_{\beta+\gamma}\mathfrak{G}_\alpha \equiv 0$. Since $(\beta + \gamma, \alpha) > 0$, we are done by the previuos case. □

Lemma 5.1.13 [**a,b short, (a,b) negative**] *Let $(\Phi, p, \lambda, \alpha, \beta)$ be a quadratic tuple with both α and β short, $\alpha \neq \beta$, $(\alpha, \beta) < 0$ and $p \neq \Phi_p \neq 1$.. Then up to conjuagcay under W,*

Φ is of type G_2, $\lambda = \lambda_1$, $p = 2$, $\alpha = \alpha_1 + 2\alpha_2$, $\beta = \alpha_1 + \alpha_2$

Proof: By 5.1.8 $\mathfrak{G}_\alpha\mathfrak{G}_\beta \equiv 0$ and so $[\mathfrak{G}_\alpha, \mathfrak{G}_\beta] \equiv 0$.

Suppose that $\beta = -\alpha$ then $[\mathfrak{G}_\alpha, \mathfrak{G}_\beta] = \mathfrak{H}_\alpha$. By **??** $\mathfrak{H}_\alpha \equiv 0$ implies $\mathfrak{G}_\alpha \equiv 0$, a contradicion.

Thus $\beta \neq -\alpha$ and $(\alpha, \beta) \neq 0$ implies that $\alpha + \beta$ is a root. Hence $N_{\alpha\beta}G_{\alpha\beta} \equiv= 0$ and as $p \neq p_\Phi$ we conclude $N_{\alpha\beta} = 0$. $p \neq p_\phi$ implies $N_{\alpha\beta} = \pm 2$, $p_\phi \neq 2$ and so $\Phi = G_2$ and $p = 2$. Let $\Pi = \{\alpha_1, \alpha_2\}$ with α_1 short. Define

$$\Sigma_+ = \{\alpha_1, \alpha_1 + \alpha_2, -2\alpha_1 - \alpha_2\}$$

and

$$\Sigma^- = -\Sigma^+$$

Then $\Phi_{\text{short}} = \Sigma_+ \cup \Sigma_-$ and $W(\Phi_{\text{long}})$ acts transitively on $\Phi_{\text{long}}, \Sigma_+$ and Σ_-. Let $\epsilon \in \{+, -\}$ and $\delta, \mu \in \Sigma_\epsilon$ with $\delta \neq \mu$. Then (δ, μ) is conjugate under $W(\Phi)$ to (α, β) and so $\mathfrak{G}_\delta \mathfrak{G}_\mu \equiv 0$. Since $p = 2$ also $\mathfrak{G}_\delta^2 \equiv 0$. Moreover $[G_\delta, G_\mu] = \pm 2 G_{\delta+\mu} = 0$. Put

$$\mathfrak{q}_\epsilon = \mathbb{K}\langle G_\delta \mid \delta \in \Sigma^\epsilon \rangle$$

We conclude that $\mathfrak{q}_\epsilon$ is an commuative subalgebra of $\mathfrak{g}$ and that

$$q_\epsilon^2 \equiv 0$$

Also $\mathfrak{G}_{\alpha_2}$ commutes with $\mathfrak{G}_{\alpha_1+\alpha_2}$ and with $\mathfrak{G}_{-2\alpha_1-\alpha_2}$ and $[\mathfrak{G}_{\alpha_2}, \mathfrak{G}_{\alpha_1}] = \pm \mathfrak{G}_{\alpha_1+\alpha_2}$. Thus $[\mathfrak{G}_{\alpha_2}, \mathfrak{q}_+] \leq \mathfrak{q}_+$. Let $\mathfrak{l} = \mathfrak{g}_{\text{long}}$. The action of $W(\Phi_{\text{long}})$ implies $[\mathfrak{l}, \mathfrak{q}^+] \leq q_+$. Since $W(\Phi)$ interchanges Σ^+ and Σ^- we also have $[\mathfrak{l}, \mathfrak{q}^-] \leq \mathfrak{q}^-$. Thus we can apply **??** conclude that

$$V = V_+ \oplus V_-$$

where $V_\epsilon = \mathrm{Ann}_V(q_\epsilon)$.

Since V_ϵ is H invariant, $v_+ \in V_\epsilon$ for some $\epsilon \in \{+, -\}$. Hence v_+ is annihilated by q_ϵ and $\mathfrak{u} = \mathbb{K}\langle \mathfrak{G}_\delta \mid \delta \in \Phi^+ \rangle$. It is easy to see that $\mathfrak{g}$ is (as a Lie algebra) generated by $\mathfrak{q}_-$ and $\mathfrak{u}$. Thus $v_+ =\in V_+$ and $v+$ is annihilated by $\mathfrak{q}_+$ and $\mathfrak{u}$. In particular $\mathfrak{G}_{\pm(2\alpha_1+\alpha_2)} v_+ = 0$ and so $\mathfrak{H}_{2\alpha_1+\alpha_2} v_+ = 0$. Since $(2\alpha_1 + \alpha_2)^* = 2\alpha_1^* + 3\alpha_2^*$ and $p = 2$ we have $\mathfrak{H}_{2\alpha_1+\alpha_2} = \mathfrak{H}_{\alpha_2}$. Thus $\mathfrak{H}_{\alpha_2} v_+ = 0$ and so $\lambda = \lambda_1$. □

Comment: there probably exists more direct proof for the preceeding lemma, but I like the proof seens it treats G_2 for $p = 2$ like an A_3

5.2 Quadratic modules for Groups of Lie Type

Definition 5.2.1 *A quadratic system is a tuple* (M, V, A, D, p) *such that*

(a) M *is a finite group.*

(b) p *is a prime and* V *an irreducible faithful* $GF(p)M$*-module.*

(c) D *is a* p*-subgroup of* M *with* $A \leq Z(D)$ *and* $|D| > 2$.

(d) $M = \langle A^M \rangle D$.

(e) $[V, A, D] = 0$.

The purpose of this section is to study and (under some extra assumptions) classify quadratic system.

Lemma 5.2.2 [**[V,D,A]=0**] *Let (V, M, A, D, p) be a quadratic system. Then*

(a) $[V, D, A] = 0$.

(b) $M = O^p(M)D$.

Proof: (a) By the definition of a quadratic system $[V, A, D] = 0$ and $A \leq Z(D)$. Thus $[A, D, V] = 0$ and the Three Subgroup Lemma 2.0.1 implies $[D, V, A] = 0$. (b) Since $M = \langle A^M \rangle D$, $M = \langle D^M \rangle$. So (b) follows from 2.0.2 applied to $M/O^p(M)$. □

Lemma 5.2.3 [**imprimitive quadratic systems**] *Let (M, V, A, D, p) be a quadratic system and suppose that Δ is a system of primitivity for M on V. Then*

(a) *$p = 2$ and A acts non-trivially on Δ.*

(b) *$|D/\mathrm{C}_D(W)| = 2 = |W^Q|$ for all $W \in \Delta$ with $A \not\leq \mathrm{N}_M(W)$.*

(c) *$O^p(M)$ acts transitively on Δ.*

Proof:

Since V is faithful and $V = \sum \Delta$, there exists $W \in \Delta$ with $[W, A] \neq 0$). Suppose first that A acts trivially on Δ. Then $0 \neq [W, A] \leq C_W(D)$ and so D normalizes W. Since $M = \langle A^M \rangle D = C_G(\Delta)D$ we conclude that M normalizes W, a contradiction to the irreducibility of V.

So A acts non-trivially on Δ. Let W with $A \not\leq N_M(W)$. $[W, A, D] = 0$ implies $|W^A| = W^D| = p = 2$. Also $[W, \mathrm{N}_D(W)] \leq C_W(A)$ and so $[W, \mathrm{N}_D(W)] = 0$. Therefore $D/C_D(W) = 2$.

Suppose that $O^p(M)$ does not act transitively on Δ. Replacing Δ by $\{\sum W^{O^p(M)} \mid W \in \Delta\}$ we may assume that $O^p(M)$ acts trivially on Δ. Thus by 5.2.2(b) $M = C_M(\Delta)D$. Hence $\Delta = W^M = W^D$, $|\Delta| = 2$, $C_D(\Delta) = C_D(W) \leq C_M(V) = 1$ and so $|D| = 2$ a contradiction. □

Lemma 5.2.4 [**OpM irreducible in quadratic system**] *Let (M, V, A, D, p) be a quadratic system. Then $O^p(M)$ acts irreducible on V.*

Proof: By 5.2.3 V is homogenous on V. So the lemma follows from by **??**.

Definition 5.2.5 [**dtendec**] *Let K be a field, H a group and V a KH-module. Then a tensor decomposition of V for H is a tuple $(F, V_i, i \in I)$ such that*

(a) *$F \leq \mathrm{End}_K(V)$ is a field with $K \leq F$.*

(b) *H acts F-semilinear on V.*

(c) *Put $E = C_H(F)$ (the largest subgroup of H acting F-linear on V). Then V_i is an FE-promodule.*

(d) *As FE-modules, V and $\bigotimes_F\{V_i \in I\}$ are isomorphic.*

Lemma 5.2.6 [**qtp**] *Let Q be a group with $|Q| \geq 3$. $1 \neq Z \leq Z(Q)$, K a field with $\mathrm{char}\, K = p$, p a prime, V a faithful KQ-module with $[V, Z, Q] = 0$ and $(F, V_i, i \in I)$ a tensor decomposition of V for Q. Then Q acts F-linear and one of the follwing holds:*

1. *There exists $i \in I$ so that $[V_i, Z, Q] = 0$ and Q acts trivially on all other V_j's.*

2. *$p = 2$, Q is F-linear and there exist $i, j \in I$, $a_k \in \mathrm{End}_F(V_k)$ with $a_k^2 = 0$ (k=i,j)and a monomorphism $\lambda : Q \to (F, +)$ so for $q \in Q$,*

   (a) *For $k = i, j$, q acts on V_k as $1 + \lambda(q)a_i$.*

   (b) *Q centralizes all V_s's with $s \neq i, j$.*

Proof: Note first that as Z acts quadratically on V, Z is an elementary abelian p-group. Also $[V, Z, Q] = 0$ and $[Q, Z] = 1$. So the three subgroup lemma implies that $[V, Q, Z] = 1$.

Suppose that Q does not act F-linear. Note thet z induces some field automorphism σ on F. Let F_σ be the fixed field of σ in F. As z is quadratic on V, $f - f^\sigma \in F_\sigma$ for all $f \in F$. It easy to see that this implies $F = F_\sigma$ or $p = 2$ and F_σ has inded two in F. Moreover, $[V, z]$ is an F_σ-subspace centralized by Q. So Q is F_σ and $F_\sigma \neq F$. Since $[V, C_Q(F)]$ is an F-spave centralizes by z, $C_Q(F) = 1$. Thus $|Q| = 2$ in contradcition to the assumptions.

Suppose from now on the Q is F-linear. Since Z is a p-group, we mau assume that the V_i's are actually FZ-modules and not only promodules. If Q acts trivially on some V_k, V is a direct sum of copies of the FQ-module $\bigotimes_F\{V_i \mid i \in I - k\}$. So the latter has the same properties as V. Thus we may assume fom now on that Q acts non-trivially on each V_i. If $|I| = 1$, then 1. holds

Suppose next that $|I| = 2$ and say $I = \{1, 2\}$. Note that

$$[C_{V_1}(Z) \otimes V_2, Z] = C_{V_1} \otimes [V_2, Q].$$

Q acts as scalars on $[V_2, Z]$ and $[V_1, Z]$. Hence we may choose the promodules V_1 and V_2 so that $[V_i, Z, Q] = 0$ for $i = 1, 2$. For $q \in Q$ let q_i be the endomorposim $q - 1$ of V_i. Then $z_i q_i = 0$. Moreover, in $\mathrm{End}_F(V_1 \otimes V)$,

$$z - 1 = (1 + z_1) \otimes (1 + z_2) - 1\otimes = z_1 \otimes 1 + 1 \otimes z_2 + z_1 \otimes z_2.$$

Thus $[V, z, q] = 0$ implies

$$z_1 \otimes q_2 = -q_1 \otimes z_2$$

If $z_1 = 0$ then as V is faithful, $z_2 \neq 0$. Thus the previuos equation implies $q_2 = 0$ for q, a contradcition to the assumption that Q does not centalize V_2. Hence both z_1 and z_2 are not zero. Choosing $q = z$ we see that $p = 2$. Hence for arbitray q, $q_1 = \lambda(q)z_1$ and $q_2 = \lambda(q)z_2$ for some $\lambda(q) \in F$. Thus 2. holds in this case.

Suppose now that $|I| \geq 3$. Say $1, 2 \in I$ and but $W = \bigotimes_F \{V_i \mid i \in I \setminus \{1, 2\}$. Then $V \cong (V_1 \otimes V_2) \times W$. Then by the prviuos case Q acts faithfully on $V_1 \otimes V_2$ $z - 1$ and $q - 1$ are linear dependent on $V_1 \otimes V_2$. Let $\lambda = \lambda(q)$ be as above. Then on $v_1 \otimes v_2$

$q - 1 = (1 + \lambda z_1) \otimes (1 + \lambda z_2) - 1 \otimes 1 = \lambda(z_1 \otimes 1 + 1 \otimes z_2 + \lambda z_1 \otimes z_2)$.

On the otherhand $z - 1 = z_1 \otimes 1 + 1 \times z_2 + z_1 \otimes z_2$ and we conclude that $\lambda = 0, 1$ and so $|Q| = 2$, a contradiction. □

Theorem 5.2.7 [**same characteritic quadratic systems**] *Let (M, V, A, D, p) be a quadratic system. Suppose that*

(a) *M is a quotient of ${}^{\sigma}\mathrm{G}_{\Phi}(\mathbb{K})$ and* $\operatorname{char} K = p$.

(b) *$|D| > |K|$ or $|\Phi_D| \geq 2$.*

Then one of the following holds

(a)

Theorem 5.2.8 [**same characteritic quadratic systems with outer automorphism**] *Let (M, V, D, A, p) be a quadratic system and*

(a) *$\mathrm{F}^*(M)$ is a quotient of ${}^{\sigma}G_{\Phi}(\mathbb{K})$ and* $\operatorname{char} K = p$.

(b) *$D \not\leq F^*(M)$.*

Then $p = 2$, $M = O^{\epsilon}_{2n}(\mathbb{K}_{\sigma})$ and V is the corresponding natural module.

Proof:

5.3 Quadratic Pairs

Lemma 5.3.1 [**3 quadratic**] *Let* F *be a field with* $\operatorname{char} F \neq 2$, *$A$ an group and V an* F *A-module. Let $a, b \in A$ such that a, b and ab acts quadratically on V. Then $\langle a, \beta \rangle$ acts quadratically on V.*

Proof: Let $\alpha = a - 1 \in \operatorname{End}(V)$ and $\beta = b - 1$. Then $\alpha^2 = \beta^2 = 0$, $\alpha\beta = \beta\alpha$ and

$$ab - 1 = (1 + \alpha)(1 + \beta) = 1 = \alpha\beta + \alpha + \beta$$

Thus

$$0 = (ab - 1)^2 = \alpha^2\beta^2 + \alpha^2 + \beta^2 + 2\alpha\beta\alpha + 2\alpha\beta^2 + 2\alpha\beta = 2\alpha\beta$$

Since $\operatorname{char} \mathrm{F} \neq 2$ we get $\alpha\beta = 0$. □

Lemma 5.3.2 [**half quadratic**] *Let* F *be a field with* char $F = p > 0$ *and* $p \neq$*, let* A *be a finite abelian group,* F *an* F A*-module* $\mathcal{Q}$ *the set of non-trivial quadratically acting elements in* A*. Suppose that* $|\mathcal{Q}| \geq \frac{|A^{\#}|}{2}$*. The one of the following holds:*

1. A *acts quadratically on* V*.*
2. $p = 3$ *and* $|A/B| = 9$ *where* $B = C_A([V, A])$*.*

Let E be a maximal quadratic subgroup of A. If $E = A$ then (1) holds. So suppose $A \neq E$. Let $|A/E| = p^n$. For $a \in \mathcal{Q} \setminus E$ and put $E_a = \{e \in E \mid ea \in \mathcal{Q}\}$. Let $e \in E_a$. Then by 5.3.1 $\langle e, a\rangle$ is quadratic and we conclude that $E_a = C_E([V, a])$. In particular, E_a is a subgroup of E_a. Note also that $E_a\langle a\rangle$ is quadratic and contains all the quadratic elements in $E\langle a\rangle$ not contained in E. In particular, by maximality of E, $E_a \neq E$. Thus $E_a a$ contains at most $\frac{1}{p}|E|$ quadratic elements.

Hence

$$|\mathcal{Q}| \leq |E| - 1 + \frac{p^n - 1}{p}|E|$$

On the otherhand

$$|\mathcal{Q}| \geq \frac{1}{2}|A^{\#}| = \frac{1}{2}(p^n|E| - 1)$$

Hence

$$\frac{1}{2}(p^n|E| - 1) \leq |E| - 1 + \frac{p^n - 1}{p}|E|$$

$$(p^{n+1} - 2p^n - 2 - 2p) \leq -\frac{p}{|E|} \leq 0$$

$$(p - 2)(p^n - 2) \leq 6$$

Thus $p = 3$ and $n = 1$. So $A = E\langle a\rangle$ and E_a centralizes both $[V, E]$ and $[V, a]$. Thus $E_a \leq B$. If $E_a < B$, then $A = EB$ or $A = B\langle a\rangle$ and in both cases A acts quadratically, contradicting the maximal choice of E. Thus $B = E_a$ and (2) holds. $\square$

Bibliography

[LGCP] The Local Structure Of Finite Groups of Local Characteristic p. (2062)

[St] R. Steinberg, *Lectures On Chevalley Groups (1967) Yale University*