ISOLATED p-MINIMAL SUBGROUPS

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1. Introduction

Suppose that p is a prime, P is a finite group and $S \in \mathrm{Syl}_p(P)$. Then P is *p-minimal* if S is not normal in P and S is contained in a unique maximal subgroup of P. Now suppose that G is a finite group and $S \in \mathrm{Syl}_p(G)$, if $S \le H \le G$ and S is not normal in H, then we call P a *p-parabolic subgroup* of G. In most cases the prime p will be evident from the context in which we are working and in these cases we often simply call P a parabolic subgroup of G. The set of maximal parabolic subgroups of G (containing S) is denoted by $\mathcal{M}_G(S)$ and the set of p-minimal parabolic subgroups of G (containing S) is denoted by $\mathcal{P}_G(S)$. Suppose that G is a Lie type group defined in characteristic p. Then the parabolic subgroups of G in the traditional sense are also p-parabolic subgroups in our context, though it should be noted that we do not require that our p-parabolic subgroups contain the Borel subgroup of G. If R is a rank 1 parabolic subgroup in G, then $P = O^{p'}(R)$ is a p-minimal parabolic subgroup of G. Moreover it is easy to see that in the Lie type groups we have $G = \langle \mathcal{P}_G(S)\rangle N_G(S)$ and it turns out that this is a property of p-minimal parabolic subgroups in general (see 2.1). Suppose that $P \in \mathcal{P}_G(S)$ and set

$$B = B_P = \langle \mathcal{P}_G(S) \setminus \{P\}, N_G(S)\rangle.$$

Notice that when P is Lie type group in characteristic p, $B_P \in \mathcal{M}_G(S)$ for all $P \in \mathcal{P}_G(S)$. In fact minimal p-parabolic subgroups in the Lie type groups have two further properties. The first is that $O_p(B_P) \not\le O_p(P)$ and the second is that $P/O_p(P)$ is a rank 1 Lie type group in characteristic p. We shall see that for p sufficiently large ($p \ge 11$) these two properties characterize Lie type groups in characteristic p among the finite simple groups.

We now make these notions more precise. We say that $P \in \mathcal{P}_G(S)$ is *isolated with respect to* A provided A is a normal p-subgroup of B_P and $A \not\le O_p(P)$. Notice that $A \le O_p(B_P)$ and so we also have $O_p(B_P) \not\le O_p(P)$. Suppose that P is a p-minimal group, $S \in \mathrm{Syl}_p(P)$ and M is the maximal subgroup of P containing S. Set $R = \bigcap M^P$, the core of M in P, and $E = O^p(P)R/R$. Then, loosely approximating the structure of a rank 1 Lie type group, we say that P is *narrow* if either E is a simple group or E is elementary abelian and M acts primitively on E. Our first result, proved in Section 2 is a basic structure theorem for groups which possess a narrow, p-minimal, isolated parabolic subgroup.

Theorem 1.1. [**Thm1**] *Suppose that $P \in \mathcal{P}_G(S)$ is narrow and isolated in G. Set $Y = \langle O^p(P)^G\rangle$. Then either $Y/O_p(Y)$ is quasisimple or $Y = O^p(P)$ and $G = B_P P$.*

Suppose that G is a p-minimal simple group, then so long at the unique maximal subgroup of G is a p-local subgroup, G satisfies the hypothesis of the theorem.

In keeping with the structure of the Lie type groups in characteristic p (perhaps extended by field automorphisms) we wish to restrict our further attentions to those candidates for LS which have P/Q_P a rank 1 Lie type group in characteristic p extended by automorphisms of order p. Thus we let $\mathcal{L}_1(p)$ consist of groups H with $O_p(H) = 1$ and $O^p(H)$ isomorphic to $O^p(L)$ for some (either adjoint or universal) rank 1 Lie type group in characteristic p. This has the effect of including some groups which are a little bit smaller than we might expect. For example, as $O^2({}^2\mathrm{B}_2(2))$ has order 5, $\mathrm{Dih}(10)$ and ${}^2\mathrm{B}_2(2)$ are both in H. A similar phenomena occurs with $\mathrm{PSU}_3(2) \cong 3^2 : \mathrm{Q}_8$. The extra condition that we will impose is contained in the following definition.

Definition 1.2. [**p-restricted**] *Suppose that p is a prime and G is a group and $S \in \mathrm{Syl}_p(G)$. Then $P \in \mathcal{P}_G(S)$ is* p-restricted *in G if P isolated in G and $P/Q_P \in \mathcal{L}_1(p)$. If G possesses a p-restricted p-minimal parabolic subgroup, then we say that G is p-restricted.*

Notice that if $P \in \mathcal{P}_G(S)$ and $P/Q_P \in \mathcal{L}_1(p)$, then P is narrow. Thus 1.1 tells us that when $O_p(G) = 1$, then $Y = \langle O^p(P)^G \rangle$ is either quasisimple or $Y = O^p(P)$. In this latter case we also have that Y is quasisimple or Y is soluble and $O^p(P)$ is isomorphic to one of $\mathbb{Z}_3$, Q_8, 2^2 or 3_+^{1+2} or 3^2.

Theorem 1.3. [**Thm2**] *Suppose that p is a prime, G is a finite group, $X = F^*(G)$ is a non-abelian simple group and G/X is a p-group. If G is p-restricted, then either X is a Lie type group defined in characteristic p or*

(a) [**a**] $p = 2$ *and* $X \cong \mathrm{Alt}(12)$ *(see 4.2).*

(b) [**b**] *X is a Lie type group in characteristic r with $r \neq p$, $p \in \{2, 3\}$ and the possibilities for p, X, P and B_P are as listed in Table 1.*

(c) [**c**] *X is a sporadic simple group and the possibilities for (X, p) are as follows:* $(\mathrm{Mat}_{12}, 2)$, $(\mathrm{Mat}_{12}, 3)$, $(\mathrm{Mat}_{22}, 2)$, $(\mathrm{J}_2, 2)$, $(\mathrm{J}_2, 3)$, $(\mathrm{Mat}_{23}, 2)$, $(\mathrm{HS}, 2)$, $(\mathrm{J}_3, 2)$, $(\mathrm{Mat}_{24}, 2)$, $(\mathrm{McL}, 3)$, $(\mathrm{He}, 2)$, $(\mathrm{Ru}, 2)$, $(\mathrm{Suz}, 2)$, $(\mathrm{Suz}, 3)$, $(\mathrm{O'N}, 2)$, $(\mathrm{Co}_3, 2)$, $(\mathrm{Co}_3, 3)$, $(\mathrm{Co}_2, 2)$, $(\mathrm{Co}_2, 3)$, $(\mathrm{Fi}_{22}, 2)$, $(\mathrm{Fi}_{22}, 3)$, $(\mathrm{HN}, 2)$, $(\mathrm{HN}, 5)$, $(\mathrm{Ly}, 5)$, $(\mathrm{Th}, 2)$, $(\mathrm{Th}, 3)$, $(\mathrm{Fi}_{23}, 2)$, $(\mathrm{Fi}_{23}, 3)$, $(\mathrm{Co}_1, 2)$, $(\mathrm{Co}_1, 3)$, $(\mathrm{Co}_1, 5)$, $(\mathrm{Fi}_{24}, 2)$, $(\mathrm{Fi}_{24}, 3)$, $(\mathrm{BM}, 2)$, $(\mathrm{BM}, 3)$, $(\mathrm{BM}, 5)$, $(\mathrm{M}, 2)$, $(\mathrm{M}, 3)$, $(\mathrm{M}, 5)$, $(\mathrm{M}, 7)$.

In particular, if $p \geq 11$, X is a Lie type group in characteristic p and, if $p \geq 5$, then X is a Lie type group or one of the sporadic groups HN, Ly, Co_1, BM *or* M.

Group	Condition on r^a	P	B
$\mathrm{PGL}_2(r^a)$	$5 \pmod 8$	$\mathrm{Sym}(4)$	$\mathrm{Dih}(2(r^a-1))$
$\mathrm{PGL}_2(r^a)$	$3 \pmod 8$	$\mathrm{Sym}(4)$	$\mathrm{Dih}(2(r^a+1))$
$\mathrm{PSL}_2(7) \cong \mathrm{PSL}_3(2)$		$\mathrm{Sym}(4)$	$\mathrm{Sym}(4)$
$\mathrm{PSL}_2(9) \cong \mathrm{Alt}(6)$		$\mathrm{Sym}(4)$	$\mathrm{Sym}(4)$
$\mathrm{P\Sigma L}_2(9) \cong \mathrm{Sym}(6)$		$\mathrm{Sym}(4) \times 2$	$\mathrm{Sym}(4) \times 2$
$\mathrm{PSL}_2(5) \cong \mathrm{Alt}(5)$		$\mathrm{PSL}_2(5)$	$\mathrm{Alt}(4)$
$\mathrm{PGL}_2(5) \cong \mathrm{Sym}(5)$		$\mathrm{PGL}_2(5)$	$\mathrm{Sym}(4)$
$\mathrm{PGL}_2(19)$		$\mathrm{Dih}(40)$	$\mathrm{Sym}(4)$
$\mathrm{PSU}_3(r^a)$	$3 \pmod 8$	$2^{\cdot}\,\mathrm{Sym}(4) * 4$	$(r^a+1)^2 : \mathrm{Sym}(3)$
$\mathrm{PSU}_3(r^a) : 2$	$3 \pmod 8$	$2^{\cdot}\,\mathrm{Sym}(4) * \mathrm{Q}_8$	$(r^a+1)^2 : (2 \times \mathrm{Sym}(3))$
$\mathrm{PSU}_3(3) \cong \mathrm{G}_2(2)'$		$4^2 : \mathrm{Sym}(3)$	$2^{\cdot}\,\mathrm{Sym}(4) * 4$
$\mathrm{PSU}_3(3) : 2 \cong \mathrm{G}_2(2)$		$4^2 : (2 \times \mathrm{Sym}(3))$	$2^{\cdot}\,\mathrm{Sym}(4) * \mathrm{Q}_8$
$\mathrm{PSL}_3(r^a) : 2$	$5 \pmod 8$	$2^{\cdot}\,\mathrm{Sym}(4) * \mathrm{Q}_8$	$(r^a-1)^2 : (2 \times \mathrm{Sym}(3))$
$\mathrm{PSU}_4(3)$		$2^{2+2+2}.\,\mathrm{Sym}(3)$	$2^{1+4}.(\mathrm{Sym}(3) \times \mathrm{Sym}(3))$
$\mathrm{PSU}_4(3).2_1$		$2^{2+2+2}.\,\mathrm{Sym}(3).2$	$2^{1+4}.(\mathrm{Sym}(3) \times \mathrm{Sym}(3)).2$
$\mathrm{PSU}_4(3).2_2$		$2^{2+2+2}.\,\mathrm{Sym}(3).2$	$2^{1+4}.(\mathrm{Sym}(3) \times \mathrm{Sym}(3)).2$
$\mathrm{PSU}_4(3).2_2$		$2^{2+2+2}.\,\mathrm{Sym}(3).2$	$2^{1+4}.(\mathrm{Sym}(3) \times \mathrm{Sym}(3)).2$
$\mathrm{PSU}_4(3).4$		$2^{2+2+2}.\,\mathrm{Sym}(3).4$	$2^{1+4}.(\mathrm{Sym}(3) \times \mathrm{Sym}(3)).4$
$\mathrm{PSU}_4(3).2^2_{122}$		$2^{2+2+2}.\,\mathrm{Sym}(3).2^2$	$2^{1+4}.(\mathrm{Sym}(3) \times \mathrm{Sym}(3)).2^2$
$\mathrm{PSU}_4(3).2^2_{133}$		$2^{2+2+2}.\,\mathrm{Sym}(3).2^2$	$2^{1+4}.(\mathrm{Sym}(3) \times \mathrm{Sym}(3)).2^2$
$\mathrm{PSU}_4(3).\mathrm{Dih}(8)$		$2^{2+2+2}.\,\mathrm{Sym}(3).\mathrm{Dih}(8)$	$2^{1+4}.(\mathrm{Sym}(3) \times \mathrm{Sym}(3)).\mathrm{Dih}(8)$
$\mathrm{PSU}_4(3) : 2_1$		$2^{1+2+1+2}.\,\mathrm{Sym}(3).2$	$2^4.\,\mathrm{Alt}(6).2$
$\mathrm{PSU}_4(3) : 2_1$		$2^{1+2+1+2}.\,\mathrm{Sym}(3).2$	$2^4.\,\mathrm{Alt}(6).2$
$\mathrm{PSU}_4(3) : 2_{122}$		$2^{1+2+1+2}.\,\mathrm{Sym}(3).2^2$	$2^4.\,\mathrm{Alt}(6).2^2$
$\mathrm{PSU}_6(3)$		$\leq 4^5.\,\mathrm{Sym}(6)$	$\frac{1}{2}\,\mathrm{GU}_2(3) \wr \mathrm{Sym}(3) \cap \mathrm{PSU}_6(3)$
$\mathrm{PGU}_6(3)$		$\leq 4^5(\mathrm{Sym}(6) \times 2)$	$\frac{1}{2}\,\mathrm{GU}_2(3) \wr \mathrm{Sym}(3)$
$\mathrm{PSp}_4(3) \cong \mathrm{PSU}_4(2)$			
$\mathrm{PSp}_6(3)$			
$\mathrm{P\Omega}^+_7(3)$			$\frac{1}{2}\mathrm{O}_1(3) \wr \mathrm{Sym}(7)) \cap X$
$\mathrm{P\Omega}^+_7(3) : 2$			$\frac{1}{2}\mathrm{O}_1(3) \wr \mathrm{Sym}(7))$
$\mathrm{P\Omega}^+_8(3)$			$\frac{1}{2}\mathrm{O}^+_4(3) \wr \mathrm{Sym}(2) \cap X$
$\mathrm{P\Omega}^+_8(3) : 2$			$\frac{1}{2}\mathrm{O}^+_4(3) \wr \mathrm{Sym}(2)$
$\mathrm{P\Omega}^+_{12}(3)$			$\frac{1}{2}\mathrm{O}^+_4(3) \wr \mathrm{Sym}(3) \cap X$
$\mathrm{P\Omega}^+_{12}(3) : 2$			$\frac{1}{2}\mathrm{O}^+_4(3) \wr \mathrm{Sym}(3)$
${}^2\mathrm{G}_2(3) \cong \mathrm{SL}_2(8) : 3$		$\mathrm{SL}_2(8) : 3$	$7^3 : 21$
$\mathrm{G}_2(3)$		$4^2 : \mathrm{Dih}(12)$	$2^{1+4} : 3^2.2$
$\mathrm{G}_2(3)$		$4^2 : \mathrm{Dih}(12).2$	$2^{1+4} : \mathrm{Sym}(3) \times \mathrm{Sym}(3)$
${}^3\,\mathrm{D}_4(3)$		$4^2 : \mathrm{Dih}(12)$	$(\mathrm{SL}_2(3) * \mathrm{SL}_2(27)).2$
$\mathrm{E}_7(3)$			$2^3.(\mathrm{PSL}_2(3))^7.2^4.\mathrm{PSL}_3(2)$
$\mathrm{E}_7(3).2$			$2^3.(\mathrm{PSL}_2(3))^7.2^4.\mathrm{PSL}_3(2).2$

Table 1: Lie type group exceptions with $p = 2$

Group	Condition on r^a	P	B
$\mathrm{PGL}_3(r^a)$	$r^a \equiv 4,7 \pmod 9$	$3^2 : \mathrm{SL}_2(3)$	$(r^a-1)^2 : \mathrm{Sym}(3)$
$\mathrm{PGU}_3(r^a)$	$r^a \equiv 2,5 \pmod 9$	$3^2 : \mathrm{SL}_2(3)$	$(r^a+1)^2 : \mathrm{Sym}(3)$
$\mathrm{PGL}_3(7)$		$3^2 : \mathrm{SL}_2(3)$	$3^2 : \mathrm{Sym}(3)$
$\mathrm{PSU}_4(r^a)$	$r^a \equiv 2,5 \pmod 9$	$3^{1+2}_+ : \mathrm{SL}_2(3)$	$\frac{1}{(2,r^a+1)}(r^a+1)^3 : \mathrm{Sym}(4)$
$\mathrm{PSU}_4(2)$		$3^3 : \mathrm{Sym}(4)$	$3^{1+2}_+ : \mathrm{SL}_2(3)$
$\mathrm{PSU}_5(2)$		$3 \times 3^{1+2}_+.\mathrm{SL}_2(3)$	$3^4.\,\mathrm{Sym}(5)$
$\mathrm{PSU}_6(2)$		$3^5 : \mathrm{Alt}(6)$	$3^{1+4}_+(\mathrm{Q}_8 \times \mathrm{Q}_8).3$
$\mathrm{PSU}_6(2) : 3$		$3^6 : \mathrm{Alt}(6)$	$3^{1+4}_+(\mathrm{Q}_8 \times \mathrm{Q}_8).3^2$
$\mathrm{PSp}_4(2) \cong \mathrm{PSL}_2(9)$		$\mathrm{PSL}_2(9)$	$3^2 : 4$
$\mathrm{P}\Omega_8^+(r^a)$	$r^a \equiv 2,5 \pmod 9$	$3^{1+2}.\mathrm{SL}_2(3) \times 3$	$\mathrm{O}_2^-(r^a) \wr \mathrm{Sym}(4)$
$\mathrm{P}\Omega_8^+(r^a) : \langle \tau \rangle$	$r^a \equiv 2,5 \pmod 9$	$3^{1+4}_+.\mathrm{SL}_2(3)$	$\mathrm{O}_2^-(r^a) \wr \mathrm{Sym}(4).3$
${}^2\mathrm{E}_6(2)$			$\frac{1}{3}\mathrm{PSU}_3(2) \wr \mathrm{Sym}(3).3 \cap X$
${}^2\mathrm{E}_6(2).3$			$\frac{1}{3}\mathrm{PSU}_3(2) \wr \mathrm{Sym}(3).3^2 \cap X$
$\mathrm{E}_8(2)$			$3^2.(\mathrm{PSU}_3(2)^4).3^2.\,\mathrm{GL}_2(3)$

Table 1: Lie type group exceptions with $p = 3$

For our intended application of 1.3 we will know additional information about the over groups of the p-restricted minimal parabolic subgroup. Indeed we will know for such a maximal p-parabolic subgroup M that $M/O_p(M)$ is $\mathrm{SL}_n(p^b)$ or $\mathrm{PSL}_n(p^b)$ perhaps extended by a group of field automorphisms (for some $n \geq 2$ and natural number b). Our next result can is designed to make 1.3 more immediately applicable in the circumstances just described.

Corollary 1.4. [**L3-restricted**] *Suppose that p is a prime, G is a finite group, $X = F^*(G)$ is a non-abelian simple group and G/X is a p-group. If $P \in \mathcal{P}_G(S)$ is p-restricted and there is a $P_1 \in \mathcal{P}_G(S)$ such that $O^p(\langle P, P_1\rangle / O_p(\langle P, P_1\rangle) \cong \mathrm{SL}_3(p^b)$ or $\mathrm{PSL}_3(p^b)$ for some integer b, then is either X is a Lie type group defined in characteristic p or $p = 2$ and X is one of the following sporadic simple groups .....*

Suppose that p is a prime. Then we use $\mathcal{R}_p$ to denote the set of group G for which there is $P \in \mathcal{P}_p(G)$ which is narrow and p-restricted. One of our goals is to determine those simple groups which are in $\mathcal{R}_p$.

For a group H, we denote the preimage of $F(H/O_p(H))$ by $F_p(H)$ and the preimage of $\Phi(H/O_p(H))$ by $\Phi_p(H)$. The remainder of our group theoretic notation is standard as can be found in [1].

2. Groups with isolated p-minimal parabolic subgroups

Recall from the introduction that for a prime p, a group P is called *p-minimal* if for a Sylow p-subgroup S of P, S is not normal in P and S is contained in a unique maximal subgroup of P. For a group G and $S \in \mathrm{Syl}_p(G)$, we denote the set of p-minimal parabolic subgroups of G (which contain S) by $\mathcal{P}_G(S)$. For an arbitrary subgroup R of G, we set $Q_R = O_p(R)$.

The following result is elementary to prove.

Lemma 2.1. [**gen**] $G = \langle \mathcal{P}_G(S) \rangle N_G(S)$.

We shall also need the following general result.

Lemma 2.2. [**no subnormal**] *Let M be a maximal subgroup of the finite group H and let N be a subnormal subgroup of H with $N \leq M$. Then $N \leq \bigcap M^H$.*

Proof. Suppose that H is a counterexample to the statement and select N subnormal in H with $N \leq M$ of maximal order. Let $N = N_0 \trianglelefteq N_1 \trianglelefteq \ldots \trianglelefteq N_k \trianglelefteq N_{k+1} = H$ be a subnormal chain from

N to H. By Wielandt's subnormal lemma, $\langle N^M\rangle \trianglelefteq\trianglelefteq H$. Because of the maximal choice of N, we have $\langle N^M\rangle = N \trianglelefteq M$. Also by the maximal choice of N, $N_1 \not\leq M$. Therefore, $N \trianglelefteq \langle N_1, M\rangle$. Since M is a maximal subgroup of H, we have $N \trianglelefteq H$ as claimed. □

Lemma 2.3. [**basic p-minimal**] *Assume that P is p-minimal, $S \in \mathrm{Syl}_p(P)$ and M is the maximal subgroup of P containing S.*

(a) [**a**] *$\bigcap M^P$ is p-closed, that is $S \cap \bigcap M^P = Q_P$.*
(b) [**d**] *If $O_p(O^p(P)) = 1$, then $\bigcap M^P$ is nilpotent.*
(c) [**b**] *If $O^p(P)$ is p-closed, then P is a $\{t,p\}$-group for some prime $t \neq p$.*
(d) [**c**] *If N is a subnormal subgroup of P with $N \leq M$, then $N \cap S \leq Q_P$.*

Proof. Let $F = \bigcap M^P$ and set $T = S \cap F$. Plainly $Q_P \leq T$ and, by the Frattini Argument, $P = FN_P(T)$. Since $N_P(T) \geq S$ and P is p-minimal, we have $P = N_P(T)$. So $T \leq Q_P$ and (a) holds.

Assume that $O_p(O^p(P)) = 1$. Then (a) implies that $F \cap O^p(P)$ is a p'-group. Let $T \in \mathrm{Syl}_t(F)$ for some prime $t \neq p$. Then $T \leq O^p(P)$ and so $T \leq F \cap O^p(P)$. The Frattini Argument gives $P = N_P(T)(F \cap O^p(P))$. Since p does not divide $|F \cap O^p(P)|$, we infer that $N_P(T)$ contains a Sylow p-subgroup of P. Thus T is normal in P and F is nilpotent.

Without loss of generality, we now assume that $Q_P = 1$.

For (c) let t be a prime such that t divides $|P/M|$. Note that $t \neq p$. Since $O^p(P/Q_P)$ is a p'-group, there exists an S-invariant Sylow t-subgroup T of $O^p(P)$. Then, as $T \not\leq M$, $P = ST$ as p is p-minimal. Thus (c) holds. Finally (d) follows from (a) and 2.2. □

Definition 2.4. [**def:restricted**] *Suppose that G is a group, $S \in \mathrm{Syl}_p(G)$, $P \in \mathcal{P}_G(S)$, $B = B_P = \langle \mathcal{P}_G(S) \setminus \{P\}, N_G(S)\rangle$ and A is a normal subgroup of B. If $A \not\leq Q_P$, then we say that P is* isolated *in H with respect to A.*

Note that if P is isolated in H with respect to A, then P is also isolated in H with respect to Q_B. Furthermore, we note that if P is isolated in H with respect to A, then certainly $A > 1$ and so $Q_B > 1$.

The next lemma is the primary structural result about groups which possess an isolated p-minimal subgroup.

Lemma 2.5. [**quasi**] *Suppose that $P \in \mathcal{P}_H(S)$ is isolated in H with respect to A. Set $Y = \langle O^p(P)^H\rangle$ and $F = \bigcap B^H$. Then*

(a) [**a-1**] *Suppose that $S \leq M \leq H$ and $M \not\leq B$, then $P \leq M$, $B \cap M$ is a maximal subgroup of M; furthermore, P is isolated in M with respect to A;*
(b) [**a-2**] *$P \cap B$ is the unique maximal subgroup of P containing S;*
(c) [**a**] *B is a maximal subgroup of H;*
(d) [**b**] *$N_H(T) \leq B$ for all $A \leq T \trianglelefteq S$;*
(e) [**c**] *if $O^p(H)$ is p-closed, then $H = BP$ and $\langle A^H\rangle = \langle A^P\rangle$;*
(f) [**d**] *if R is a normal subgroup of H and $R \not\leq B$, then $Y \leq R$;*
(g) [**e**] *if R is a proper characteristic subgroup of Y, then $Y \leq F$;*
(h) [**e+1**] *$[F, Y] \leq Q_H$;*
(i) [**f**] *either $Y = O^p(P)$ or YQ_H/Q_H is semisimple; and*
(j) [**g**] *if KQ_H/Q_H is a component in YQ_H/Q_H and $Y \neq O^p(P)$, then $Y = \langle K^S\rangle$ and $K \cap P \not\leq B$.*

Proof. Suppose that $S \leq M \leq H$ and $M \not\leq B$. Then, by 2.1, $M = \langle \mathcal{P}_M(S)\rangle N_M(S)$. Since $N_M(S) \leq B$ and P is the unique member of $\mathcal{P}_H(S)$ which is not contained in B, we have $P \in \mathcal{P}_M(S)$.

Since $\mathcal{P}_M(S) \setminus \{P\} \subseteq \mathcal{P}_B(S)$, we have $M \cap B = \langle \mathcal{P}_M(S) \setminus \{P\}\rangle N_M(S)$. Now let $M \cap B < D \le M$. Then also $\mathcal{P}_D(S) \le \mathcal{P}_M(S)$ and so as $D > M \cap B$, we must have $P \in \mathcal{P}_D(S)$ and we conclude that $D = M$. Thus $M \cap B$ is a maximal subgroup of M. Finally, as $A \trianglelefteq B$, $A \trianglelefteq B \cap M$ and $A \not\le Q_P$ and so P is isolated in M with respect to A. This proves (a). Parts (b) and (c) follow immediately from (a).

Suppose that $A \le T \trianglelefteq S$. Since $A \not\le Q_P$, P is not in $N_H(T)$. Taking $M = N_H(T)$, (a) gives $N_H(T) \le B$. So (d) holds.

For the proof of the remaining statements assume that $Q_H = 1$.

Suppose that $X := O^p(H)$ is p-closed. Then, as $Q_H = 1$, X is a p'-group. Since $O^p(P) \le X$, $O^p(P)$ is also a p'-group. Therefore, by 2.3(c), P is a $\{t, p\}$-group for some prime $t \ne p$. Since p and $|X|$ are coprime, for each prime divisor r of $|X|$ there is an S-invariant Sylow r-subgroup S_r of X. If $r \ne t$, then $P \not\le S_rS$ and so $S_r \le B$ by (a). Hence, by considering $|X|$, we have $H = (X \cap B)S_tS$. We now consider S_tS, we have that $(S_t \cap B)S$ is a maximal subgroup of S_tS by (a) and so $S_t \cap B$ is a maximal S-invariant subgroup of S_t. Since S_t is nilpotent, $N_{S_t}(S_t \cap B) > S_t \cap B$ and of course $N_{S_t}(S_t \cap B)$ is also normalized by S. Hence $S_t \cap B$ is normal in S_t. In particular, we have that $(S_t \cap B)O^p(P)$ is a subgroup of S_t and, as $S_t \cap B$ is a maximal with respect to being S-invariant, we infer that $S_t = (S_t \cap B)O^p(P)$. Hence

$$H = (B \cap X)S_tS = (B \cap X)O^p(P)S = (B \cap X)P = BP.$$

Finally, we have $\langle A^H\rangle = \langle A^{BP}\rangle = \langle A^P\rangle$ and this completes the proof of (e).

Set $Y = \langle O^p(P)^H\rangle$. Suppose R is a normal subgroup of H which is not contained in B. Then, by (a), $RS \ge P$ and so $O^p(P) \le R$. Thus (f) holds. Plainly (g) is a direct corollary of (f).

Set $F = \bigcap B^H$. Then as $F \le B$, F normalizes A. Therefore, $A \cap F \le Q_F \le Q_H = 1$. Therefore, as $[F, A] \le A \cap F$, $1 = [F, A] = [F, \langle A^H\rangle]$. Since $\langle A^P\rangle \ge O^p(P)$, we have $\langle A^H\rangle \ge \langle O^p(P)^H\rangle = Y$. Therefore, $[F, Y] = 1$ and (h) is true.

Suppose that Y is not perfect. Then $Y' < Y$ hence $Y' \le Z(Y)$ by (g) and (h). In particular, Y is nilpotent and so p-closed. Applying part (e) to YS, we have that $\langle A^H\rangle = \langle A^{BY}\rangle = \langle A^Y\rangle = \langle A^P\rangle = AO^p(P)$ and so $Y = O^p(P)$. If on the other hand, Y is perfect, then Y is semisimple by (g) and (h). So (i) holds.

Let K be a component of Y. Then Y is not nilpotent, so Y is semisimple by (j) and, in particular, K is normal in Y. Hence, as $H = YB$, $\langle K^B\rangle = \langle K^{YB}\rangle = \langle K^H\rangle$ and so $K \not\le B$. Therefore, $\langle K^S\rangle \ge O^p(P)$ by (a). If L is a component of Y not contained in $\langle K^S\rangle$, then $O^p(P) \le \langle K^S\rangle \cap \langle L^S\rangle \le Z(Y) \le B$, a contradiction. Therefore, $K^H = K^B = K^S$ and the first part of (j) holds. Assume that $K \cap P \le B \cap P$. Since $O^p(P) \le Y$, $K \ne Y$. Furthermore, as $(K \cap P) \trianglelefteq\trianglelefteq P$, 2.3(d) gives $K \cap S \le Q_P$. Hence $Y \cap S = Y \cap Q_P \trianglelefteq P$ and, in particular, $O^p(P)$ is p-closed. Set $R = N_Y(Y \cap S)S$. The $O^p(R)$ is p-closed and $R \ge P$. Therefore, $R = (R \cap B)P$ and $\langle A^R\rangle = \langle A^P\rangle \le P$ by (e). Suppose that $A \le N_H(K)$. Then $[A, K \cap R] \le K \cap \langle A^R\rangle \le K \cap P \le B$ which normalizes A. It follows that $[A, K \cap R]$ is a p-group and so, as $K \cap R$ is p-closed, $[A, K \cap R] \le Q_P \cap K$. Therefore, $[A, R] = \langle [A, K \cap R]^S\rangle \le Q_P$. But then $\langle A^R\rangle = \langle A^P\rangle$ is a p-group, a contradiction. Hence $A \not\le N_H(K)$. Therefore, Q_B properly permutes the components of Y. Since $[K \cap B, Q_B]$ is a p-group, we get $K \cap B \le (K \cap S)F$ and then that $Y \cap S = O_p(Y \cap B)$. But then $Y \cap S \trianglelefteq \langle O^p(P), B \cap Y\rangle = Y$ and we conclude that H is p-closed. Now a final application of (e) indicates that $Y \le \langle A^H\rangle = \langle A^P\rangle \le P$. So $Y = O^p(Y) = O^p(P)$ and thus (g) holds. $\square$

To control the structure of Y in (**??**) further we have to impose further conditions on the isolated p-minimal subgroup P.

Definition 2.6. [**def:narrow**] *Let P be a p-minimal group and $S \in \mathrm{Syl}_p(P)$. Let M be the maximal subgroup of P containing S, $R = \bigcap M^P$ and $E := O^p(P)R/R$. Then P is called* narrow *provided that either*

(a) [**a**] *E is non-abelian and simple; or*
(b) [**b**] *E is elementary abelian and B acts primitively on E.*

Lemma 2.7. [**qs**] *Suppose that $P \in \mathcal{P}_G(S)$ is narrow and isolated in G. Then either P/Q_P is soluble or $O^p(P)Q_P/Q_P$ is quasisimple.*

Proof. Let $M = B \cap P$, then by 2.5(b), M is the unique maximal subgroup of P containing S. Put $R = \bigcap M^P$. Then R is normal in P and contained in B, therefore, $[Q_B, R] \le Q_B \cap R \le S \cap R = Q_P$. Thus R/Q_P is centralized by $\langle Q_B^P \rangle \ge O^p(P)$. Thus $RQ_P/Q_P \le Z(O^p(P)Q_P/Q_P$. Since P is narrow, we have either P is soluble or $O^p(P)Q_P/Q_P$ is quasisimple. □

Theorem 2.8. [**simple**] *Suppose that $P \in \mathcal{P}_H(S)$ is narrow and isolated in H with respect to A. Assume that $Q_H = 1$ and set $Y = \langle O^p(P)^H \rangle$. Then either $Y = O^p(P)$ or Y is quasisimple.*

Proof. We may assume that $Y \neq O^p(P)$. Set $M = P \cap B$. The by 2.5(b) M is the unique maximal subgroup of P containing S. Put $R = \bigcap M^P$. By 2.5(i) and 2.5(j), Y is semisimple and for any component K of Y, $Y = \langle K^S \rangle = \langle K^M \rangle$ and $K \cap P \not\le B$. If K is normalized by S, then $Y = \langle K^S \rangle = K$ is quasisimple and we are done. Hence we assume that K is not normalized by S and look for a contradiction. Suppose first that $O^p(P)R/R$ is a non-abelian simple group. Then, as $(K \cap P)R/R$ is normalized by $O^p(P)R/R$, we have $(K \cap P)R \ge O^p(P)R$. Now selecting $s \in S$ such that $K^s \neq K$, we have $(O^p(P)R/R)' \le [K, K^s]R/R = (K \cap K^s)R/R$ which is abelian, a contradiction. Therefore, $O^p(P)R/R$ is an elementary abelian t-group for some prime $t \neq p$ and $O^p(P)$ is p-closed. Hence P is a $\{t,p\}$-group. We have that $O^p(K \cap P) \le X$ and so $X = \langle O^p(K \cap P)^S \rangle$. Put $R^* = Z(Y) \cap X$ and $D = O^p(K \cap P)$. Then $X \cap Z(Y)$ is normal in P and contained in B. Therefore $R^* \le R$. Note that, as Y is semisimple,

$$X/R^* \cong XZ(Y)/Z(Y) \cong \prod_{T \in D^M} T.$$

For a group L, let $\Phi_p(L)$ denote the full preimage of $\Phi(L/O_p(L))$. Then, as P is narrow and soluble, $\Phi_p(X) \le R \cap X$ and, as P is p-minimal Maschke's Theorem implies that $\Phi_p(X) = R \cap X$. On the other hand

$$\Phi_p(X/R^*) = \prod_{T \in D^M} \Phi_p(T)$$

and so we conclude that

$$XR/R \cong X/X \cap R = \prod_{T \in D^M} T/\Phi_p(T).$$

Since P is narrow, it must be that $D^M = D$ and so K is normalized by M, a contradiction. □

¿From Section **??** onwards we will be investigation specific simple groups with an eye to showing that they have or do not have an isolated narrow p-minimal parabolic subgroup. The next few results in this section will be applied to proper subgroups of such groups. We continue the notation from the previous lemmas. In particular, if $P \in P_G(S)$ is narrow and isolated, then we set $Y = \langle O^p(P)^G \rangle$.

Lemma 2.9. [**comps**] *Suppose that G is a group, $O_p(G) = 1$, $F(G) = C_G(E(G))$ and G operates transitively on the components of G. If $P \in \mathcal{P}_G(S)$ is narrow and isolated, then $E(G) = Y$ and, in particular, there is exactly one component in G.*

Proof. Assume that P is narrow and isolated in G. Then, if $Y \neq O^p(P)$ or $Y = O^p(P)$ and $O^p(P)$ is not soluble, 2.8 implies that Y is a component of G. Hence, as by hypothesis G acts transitively on its components, we get $Y = E(G)$. Thus $Y = O^p(P)$ is soluble. Since Y is a normal subgroup of G, Y centralizes $E(G)$. Since $G = BY$, we have that $Q_BY \trianglelefteq G$ and Q_BY centralizes $E(G)$. But then $Q_BY \leq C_G(E(G)) = F(G)$ by assumption. Thus Q_BY is nilpotent and this contradicts $Q_B \not\leq Q_P$. □

Lemma 2.10. [**quot**] *Suppose that $P \in \mathcal{P}_G(S)$ is isolated in G. Then $C_B(Y)$ is normal in G and if $X \leq C_B(Y)$ is normal in G, then*

(a) [**a**] *$PX/X \in \mathcal{P}_G(SX/X)$ is isolated in G/X; and*
(b) [**b**] *if P is narrow, then PX/X is narrow.*

Proof. We first of all note that $G = BY$. So $C_B(Y) = B \cap C_G(Y)$ is normalized by $G = BY$. Now suppose that $X \leq C_B(Y)$ is normal in G. We claim that PX/X is a p-minimal parabolic subgroup which is narrow and isolated in G/X. Suppose that $\bar{R} \in \mathcal{P}_{G/X}(SX/X)$ is not contained in B/X. Let R be the full preimage of R in G. Then $R \geq S$ and $R \not\leq B$ so $P \leq R$ by 2.5(a). Furthermore, as $\bar{R} \in \mathcal{P}_{G/X}(SX/X)$ and $B \geq X$, $B \cap R$ is the maximal subgroup R containing SX. Since $P \not\leq B$, we infer that $R = PX$ and consequently $\bar{R} = PX/X$ is the unique p-minimal parabolic subgroup of G/X not contained in B/X. Let U be the full preimage of $O_p(\overline{R})$ and let $U_p = U \cap S$. We have $R = N_R(U_p)U$ by the Frattini lemma. In particular, as $R \not\leq B$ and $U \leq XS \leq B$, $N_R(U_p) \not\leq B$. Since $S \leq N_R(U_p)$, it follows that $P \leq N_R(U_p)$ and so $U_p \leq Q_P$. Since $Q_B \not\leq Q_P$, it follows that $Q_B \not\leq U_p = U \cap S$. Therefore, $Q_BX/X not \leq UX/X$ and PX/X is isolated in G/X. This proves (a).

Suppose that P is narrow. Then $PX/X \in \mathcal{P}_G(S)$ by (a). We have so $PX/X \cong P/P \cap X$ and, putting $F = \bigcap(B \cap P)^P$, we have $P \cap X \leq F$. It follows that $O^p(P)F/F \cong \Omega^p(P/X)(FX/X)/(FX/X)$ and so PX/X is narrow. □

Corollary 2.11. [**quot2**]*Assume that $P \in \mathcal{P}_G(S)$ is narrow and p-restricted. Let $Y = \langle O^p(P)^G \rangle$ and assume that $Y = O^p(P)$ is soluble. If the holomorph of Y is soluble but not abelian, then G' is not perfect.*

Proof. We have $G/C_B(Y)$ is isomorphic to a subgroup of the holomorph of Y which is soluble but not abelian. □

Lemma 2.12. [**Opnormal**] *Suppose that $P \in \mathcal{P}_G(S)$ is narrow and isolated, $L \trianglelefteq H$ with L soluble and $C_H(L) \leq L$. Assume also that $O_p(H) = 1$. Then $O^p(P) \trianglelefteq H$ and P is soluble.*

Proof. Set $Y = \langle O^p(P)^H \rangle$. Then $Y = O^p(P)$ or Y is quasisimple. Assume that $O^p(P)$ is not soluble. Then, because $Y \trianglelefteq H$, $O_p(Y) = 1$ and so when $Y = O^p(P)$ we also have that Y is quasisimple by 2.7. Thus in any event Y is quasisimple. Therefore, as L is soluble, $L \cap Y$ is soluble and so $L \cap Y \leq Z(Y)$. Hence $[L, Y, Y] = 1$ and the three subgroup lemma gives $Y = [Y, Y] \leq C_H(L) \leq L$, a contradiction. □

Lemma 2.13. [**sbnrml2**] *Suppose that H is a group and $A \leq H$. Set $L = \langle A^H \rangle$. Assume that $AF(L) \trianglelefteq L$, $C_{H/F(L)}(L/F(L)) = 1$ and that there exists $h \in H$ such that $[A^h, A] \leq F(L)$. If $Y \trianglelefteq H$, then $Y/F(Y)$ is not a non-abelian simple group.*

Proof. Suppose that $Y/F(Y)$ is a non-abelian simple group. Since $Y \trianglelefteq H$, $F(Y) \le F(H)$ and since $[F(H), L] \le F(L)$, $C_{H/F(L)}(L/F(L)) = 1$ implies that $F(H) = F(L)$. Now $L \cap Y \trianglelefteq Y$ and so either $Y \le L$ or $L \cap Y = F(Y)$. In the latter case we have $[L, Y] \le F(L)$ which means that $Y \le F(L)$, a contradiction. Therefore, $Y \le L$. Since $F(L)A$ is normal in L, we either have $AF(L) \cap Y = F(Y)$ or $AF(L) \ge Y = Y$. In the latter case we have $Y \le AF(L) \cap A^h F(L)$ where h is as in the statement of the lemma. This gives

$$[Y, Y] \le [AF(L), A^h F(L)] \le F(L),$$

which is a contradiction. Therefore, $AF(L) \cap Y = F(Y) \le F(L)$. But $Y \le L$ and so Y normalizes $AF(L)$ and this means that $[AF(L), Y] \le AF(L) \cap Y \le F(L)$. Thus $[\langle A^H \rangle, O^p(P)] \le F(L)$ and, since $C_{H/F(L)}(L/F(L)) = 1$, we conclude that $Y \le F(L)$ and once again have a contradiction. □

Corollary 2.14. [**P soluble**] *Suppose that H is a group, $O_p(H) = 1$ and $A \le H$. Set $L = \langle A^H \rangle$. Assume that $AF(L) \trianglelefteq L$, $C_{H/F(L)}(L/F(L)) = 1$ and that there exists $h \in H$ such that $[A^h, A] \le F(L)$. If $P \in \mathcal{P}_H(S)$ is narrow and P is restricted in H, then $O^p(P) \trianglelefteq H$ and $O^p(P) \le F(L)$.*

Proof. Let M be a maximal subgroup of P containing a Sylow p-subgroup S of P and set $R = \bigcap M^P$ Set $Y = \langle O^p(P)^H \rangle$. Then by (**??**) either Y is quasisimple or $Y = O^p(P)$. However (**??**) implies that $Y/F(Y)$ is not simple. So $Y = O^p(P)$. Hence $O_p(P) = 1$ and R is nilpotent by (**??**)d. Since P is narrow (**??**) implies that $O^p(P)/F(O^p(P))$ is abelian and hence p-closed. But then $O^p(P)$ is a t-group for some prime t and $O^p(P) \le F(L)$ as claimed. □

Corollary 2.15. [**P soluble2**] *Suppose that H is a group, $O_p(H) = 1$ and $A \le H$. Set $L = \langle A^H \rangle$. Assume that $AF(L) \trianglelefteq L$, $C_{H/F(L)}(L/F(L)) = 1$ and that there exists $h \in H$ such that $[A^h, A] \le F(L)$. If L is perfect and $F(L) = Z(L)$, then $H \notin \mathcal{R}_p$.*

Proof. Assume that $H \in \mathcal{R}_p$. By the previous corollary we have that $O^p(P) \le F(L)$. Let $B = \langle \mathcal{P}_H(S) \setminus \{P\} \rangle$. Then, as $H \in \mathcal{R}_p$, $BO^p(P) = H$ and $B < H$. As $O^p(P) \le Z(L) \le L$, $L = L \cap BO^p(P) = (L \cap B)O^p(P)$ and $L \cap B < L$. Therefore $L' \le ((L \cap B)Z(L))' \le L \cap B < L$, a contradiction. □

Lemma 2.16. [**non-local p-minimal**] *Suppose $G \ge H \ge S$ and $Q_H = 1$. Then $H \ge P$.*

Proof. Suppose $H \le B$. Then $Q_B \le Q_H = 1$, a contradiction. Thus $H \not\le B$ and $H \ge P$ by 2.5(a). □

I don't think that this next one is used.

Lemma 2.17. [**no cubic**] *Suppose that Q_B is abelian and that no non-trivial normal subgroup of S acts non-trivially and cubically on Q_B. Then P is not of characteristic p.*

Proof. Suppose not and let D be normal subgroup of P in Q_P minimal such that $[D, O^p(P)] \ne 1$. Then $D = [D, O^p(P)] \le O^p(P)$ and so, by the minimality of D, $[D, D, D] \le [D, D, O^p(P)] = 1$. In particular, as Q_B normalizes D, $[Q_B, D, D, D] = 1$ and so by assumption $[Q_B, D] = 1$. Since $Q_B \not\le Q_P$, we get $[D, O^p(P)] = 1$, which is a contradiction. □

We finish this section with one final general lemma. It exploits the fact that isolated p-minimal parabolic subgroups are normalized by $N_G(S)$.

Lemma 2.18. [**ngs maximal**] *Suppose that $P \in \mathcal{P}_G(S)$ is isolated in G. Suppose that one of the following holds:*

1. [**1**] *$N_G(S)$ acts irreducibly on S.*

2. [**2**] $N_G(S) = B$.
3. [**3**] $N_G(S)$ *is contained in a unique maximal subgroup of* G.

Then $P \trianglelefteq G$

Proof. Suppose (1) holds. Since $N_G(S) \leq B$ we get that $Q_B = S$ and so (2) holds.

Suppose (2) holds. Then, by 2.5(c), $N_G(S)$ is a maximal subgroup of G and so (3) holds.

So we may assume that (3) holds. Since $N_G(S) \leq B$, we get that B is the unique maximal subgroup of G containing $N_G(S)$. Since $N_G(S)P \leq N_G(P)$ and $P \not\leq B$, we have $G = N_G(P)$ and so $P \trianglelefteq G$. □

3. General observations

¿From here on we wish to determine the possibilities for Y in 2.8(**??**)iven that we know that $O^p(P/Q_P)$ is a rank 1 Lie type group. So we shall assume the following hypothesis:

Hypothesis 3.1. [**hyp**] *p is a prime, G is a group with $Q_G = 1$ and*

(a) [**a**] *$X = O^p(G)$ is a non-abelian simple $\mathcal{K}$-group; and*
(b) [**b**] *there is a $P \in \mathcal{P}_G(S)$ such that P is narrow and isolated in G (with respect to A) and $O^p(P/Q_P)$ is a rank 1 Lie type group.*

We next establish some elementary consequences of 3.1

Lemma 3.2. [**s cyclic**] *If S is cyclic, then $\langle A^P \rangle \trianglelefteq G$.*

Proof. Suppose that $|SQ_P/Q_P| = p$, then, as $A \not\leq Q_P$, $S = A = Q_B$ and the result follows from 2.18(**??**) So assume that $|SQ_P/Q_P| > p$, then as S is cyclic, we have $P/Q_P \cong \mathrm{Suz}(2)$ or $3^{1+2}_+.4$. In particular, $p = 2$ and so transfer (see [**?**, 37.7]) implies that $O^2(G)$ is group of odd order. Therefore the lemma follows from 2.5(e). □

Proposition 3.3. [**GL**]*[Gorenstein Lyons] Suppose that X is a finite simple $\mathcal{K}$-group and that S is a Sylow p-subgroup of X. If S is abelian, then $N_G(S)$ acts irreducibly on $\Omega_1(S)$.*

Proof. See [3, 12-1,pg 158]. □

This allows us easily to establish the following lemma.

Lemma 3.4. [**SinXnotabl**] *If G satisfies 3.1 and X has abelian Sylow p-subgroups, then $G \neq X$.*

Proof. Suppose that $G = X$ and X has abelian Sylow p-subgroups. Then, as $Q_B > 1$, (**??**) and $N_G(S) \leq B$ implies that $\Omega_1(S) \leq Q_B$ and $B = N_G(\Omega_1(S))$. If $Q_P > 1$, then as Q_P is normalized by $N_G(S)$, we have $\Omega_1 \mathrm{Z}(Q_P) = \Omega_1(S)$. But then $P \leq B$, a contradiction. Thus $Q_P = 1$. Since $O^p(P/Q_P)$ is a rank 1 Lie type group, we have that S is elementary abelian. Hence 2.18 applies and we have a $P \trianglelefteq G$, a contradiction. □

Lemma 3.5. [**not char**] *Suppose that P is not of characteristic p and Q_B is abelian. Then Q_B is elementary abelian.(except maybe for $Sz(2)$ or $Ree(3)'$).*

Proof. Since Q_BQ_P/Q_P is an abelian normal subgroups of $(P \cap B)/Q_P$ we see (except for Sz(2) that $\Phi(Q_B) \leq Q_P$ and so $\Phi(Q_B)$ is normal in P and in B. Hence Q_B is elementary abelian □

Lemma 3.6. [**irr on z(s)**] *Suppose that P is not of characteristic p, $N_G(S)$ is irreducible on $\Omega_1 \mathrm{Z}(S)$ and $q > 3$. Then $Q_P = 1$ and $|Q_B| = q$.*

Proof. By 3.5 Q_B is elementary abelian. Suppose $q > 3$. Then $O^p(P) \cap Q_P = 1$. By irreducibility of $N_G(S)$ on $\Omega_1 \mathrm{Z}(S)$ we get $Z(S) \le O^p(P) \cap S$ and so $\Omega_1 \mathrm{Z}(S) \cap Q_P = 1$. Hence $Q_P = 1$. □

Lemma 3.7. [**norm**] *If P is not soluble, then $N_{O^p(P)}(S \cap O^p(P)) \le B$ and $Q_B Q_P \trianglelefteq N_{O^p(P)}(S \cap O^p(P))S$.*

Proof. Let $R = N_{O^p(P)}(S \cap O^p(P))$. Since R is normalized by S and, as P is not soluble, $RS \not\geq P$, $R \le B$ by 2.5(a). This of course then gives $Q_B Q_P \trianglelefteq RS$. □

Lemma 3.8. [**QBnX=1**] *Suppose $Q_B \cap O^p(P) = 1$. Then P is soluble and, in particular, $p \le 3$.*

Proof. Assume that $Q_B \cap X = 1$ and put $R = N_{O^p(P)}(S \cap O^p(P))$. Suppose that P is not soluble. Then by 3.7, R normalizes Q_B. Now $[Q_B, R] \le O^p(P) \cap Q_B \le X \cap Q_B = 1$. Therefore, $Q_B Q_P/Q_P \le Z(O^p(RS/Q_P))$. Since, from the structure of P/Q_P, $C_{P/Q_P}(RQ_P/Q_P) = 1$, we infer that $Q_B \le Q_P$, which is a contradiction. □

Lemma 3.9. [**order8**] *If $Q_B \le Z(S)$ and $Z(S)$ is cyclic, then $|S| \le p^3$.*

Proof. As $Q_B \le Z(S)$, $[Q_P, Q_B] = 1$ and so P is not p-constrained. Therefore $\Omega_1(Q_B) = \Omega_1(Z(S)) \cap Q_P = 1$ and so $Q_P = 1$. Since $Q_B Q_P$ is normalized by $N_{O^2(P)}(S \cap O^2(P))$, the structure of the rank 1 Lie type groups gives $O^p(P)Q_P/Q_P$ is defined over $\mathrm{GF}(p)$. But then $|S| = |S/Q_P| \le p^3$, as claimed. □

4. Alternating groups

It looks like we can do this for narrow p-isolated P and not insist on Lie type. I guess it lets in $\mathrm{Sym}(p)$ and $Sym(9)$.

In this section we determine those groups G are p-restricted and have $O^p(G) = F^*(G)$ an alternating group of degree at least 5. Among the small alternating and symmetric groups there are a number of isomorphisms with the classical groups and these examples always lead to G being p-restricted for the appropriate prime. Before proving our main result on the symmetric groups we recall these isomorphisms.

Lemma 4.1. [**smallalts**] *We have the following isomorphisms*

(a) [**a**] $\mathrm{Sym}(3) \cong \mathrm{SL}_2(2)$;
(b) [**b**] $\mathrm{Alt}(4) \cong \mathrm{PSL}_2(3)$;
(c) [**c**] $\mathrm{Alt}(5) \cong \mathrm{SL}_2(4) \cong \mathrm{PSL}_2(5)$;
(d) [**d**] $\mathrm{Alt}(6) \cong \mathrm{PSL}_2(9)$;
(e) [**e**] $\mathrm{Sym}(6) \cong \mathrm{Sp}_4(2)$;
(f) [**f**] $\mathrm{Alt}(8) \cong \mathrm{PSL}_4(2)$;
(g) [**g**] $\mathrm{Sym}(8) \cong \mathrm{O}_6^+(2)$.

Proof. These facts are well known. But see for example [6, 2.9.1]. □

Lemma 4.2. [**altcase**] *Suppose that p is a prime, G is a group such that $F^*(G) = O^p(G) \cong \mathrm{Alt}(n)$ with $n \ge 5$ and that G is not isomorphic to a Lie type group defined in characteristic p. If G is p-restricted, then $p = 2$ and $n = 12$. Furthermore, in this case if P is p-restricted in G, then B is the stabilizer of a system of imprimitivity with blocks $\{\Omega_1, \Omega_2, \Omega_3\}$ of size 4 and P, which is isomorphic to a subgroup of index at most 2 in $(\mathrm{Sym}(2) \wr \mathrm{Sym}(4)) \times (\mathrm{Sym}(2) \wr \mathrm{Sym}(2))$, is contained in $\mathrm{Stab}(\Omega_1 \cup \Omega_2) \times \mathrm{Stab}(\Omega_3)$.*

Proof. Let $X = F^*(G) = O^p(G) \cong \mathrm{Alt}(n)$ with $n \geq 5$, $S \in \mathrm{Syl}_p(G)$ and assume that $P \in \mathcal{P}_G(S)$ is p-restricted in G. Because of 3.2 we may assume that S is not cyclic and so in particular, $n > 2p-1$. Since, for $p \leq 3$, Alt(6) is a Lie type group in characteristic p, we may assume that $n \geq 7$. Suppose that $n = 7$. Then $p \in \{2,3\}$. When $p = 3$, then $N_G(S)$ operates irreducibly on S so 2.18 delivers $G = P$, a contradiction. So assume that $p = 2$. If $G = \mathrm{Alt}(7)$, then 2.16 implies $P \leq \mathrm{Alt}(6) \in \mathrm{M}_G(S)$ and $P \leq \mathrm{Sym}(5) \in \mathrm{M}_G(S)$, which is impossible as Sym(5) is p-minimal. For $G = \mathrm{Sym}(7)$, the same argument shows that $P \leq \mathrm{Sym}(6)$ with $P = \mathrm{Sym}(2) \wr \mathrm{Sym}(3)$ and $B = \mathrm{Sym}(5) \times \mathrm{Sym}(2)$. But P must also be contained in $\mathrm{Sym}(3) \times \mathrm{Sym}(4)$ and we have a contradiction.

Now we assume that $n \geq 8$ and that if $n = 8$, then $p \neq 2$ (as $\mathrm{Alt}(8) \cong \mathrm{SL}_4(2)$). We consider G acting on the set $\Omega = \{1, \ldots, n\}$. Set $Z = \Omega_1(Z(Q_B))$. Suppose first that B operates primitively on Ω. Then Z operates transitively on Ω and, as Z is abelian, Z acts regularly on Ω and, in particular, $|\Omega| = |Z| = p^a$ for some $a \in \mathbb{N}$. As $n > p$, we have $a \geq 2$. Select a p-cycle $x \in S$ (or when $p = 2$ a product of two transpositions). Then

$$|\operatorname{Fix}_\Omega(x)| = p^a - p = |C_Z(x)| \leq p^{a-1}$$

($2^a - 4 \leq 2^{a-1}$ when $p = 2$) which has no solution in for our values of $n = p^a$. Therefore, B does not act primitively on Ω.

Assume that B is not transitive on Ω and let Ω_1, Ω_2 be proper subsets of Ω fixed by B with $|\Omega_2| = k \leq \frac{n}{2}$. So $B \leq \mathrm{Stab}_G(\Omega_1) \times \mathrm{Stab}_G(\Omega_2)$. Since B is a maximal subgroup of G we must have $B = \mathrm{Stab}_G(\Omega_1) \times \mathrm{Stab}_G(\Omega_2)$ and $k \neq \frac{n}{2}$ (otherwise B would not be maximal in G). Since B is a p-local subgroup, we infer that $\mathrm{Sym}(\Omega_1)$ or $\mathrm{Sym}(\Omega_2)$ must be a p-local subgroup. Thus, as $n \geq 8$ and $k \neq \frac{n}{2}$, we have that $k = 2, 3, 4$ and $Q_B = O_p(\mathrm{Sym}(\Omega_2))$. Now let $T = \langle Q_B^g \mid g \in G, Q_B^g \leq S \rangle$. Then $Q_B \leq T \trianglelefteq S$. By 2.5(d), we have that $N_G(T) \leq B$, but we have $N_G(T) \geq O_p(\mathrm{Sym}(\Omega_2)) \wr \mathrm{Sym}([n/|\Omega_2|])$ which plainly does not normalize Q_B. Thus B operates transitively but not primitively on Ω. Let $\mathcal{B} = \{\Theta_1, \ldots, \Theta_r\}$ be a system of imprimitivity for B on Ω with $|\Theta_1| = k$. Then $B \leq \mathrm{Stab}_G(\mathcal{B})$ and, since B is a maximal subgroup of G and B is a p-local subgroup of G, we have $B = \mathrm{Stab}_G(\mathcal{B})$ and $k \in \{2,3,4\}$. Suppose that $k = 2$ or 4. Then $p = 2$. Write the 2-adic decomposition of n as $n = 2^{a_1} + 2^{a_2} + \ldots 2^{a_l}$ and notice that as $n \geq 8$, $a_1 \geq 3$. For $i = 1, \ldots, l$, let S_{a_i} represent a Sylow 2-subgroup of $Sym(2^{a_i})$. Then take $S = S_{a_1} \times S_{a_2} \times \cdots S_{a_l} \cap G$ as our"standard" Sylow 2-subgroup. If $k = 2$, we note that the subgroup $H := \mathrm{Sym}(4) \wr \underbrace{\mathrm{Sym}(2) \wr \cdots \wr \mathrm{Sym}(2)}_{a_1 - 2} \times S_{a_2} \times \ldots S_{a_l} \cap G$ is not contained in B and consequently must contain P. Suppose that $n \geq 9$ or $G = \mathrm{Sym}(n)$. Then H is actually p-minimal and we have that $H = P$, but H is not narrow. Thus we have that $G = \mathrm{Alt}(n)$ and $n = 9$. So we have that $S \leq \mathrm{Alt}(8)$ and $G = P$ is 2-minimal, a contradiction to the uniqueness of P (as $P \leq H$ also). Next assume that $k = 4$. Then set $H := \mathrm{Sym}(2) \wr \mathrm{Sym}(4) \wr \underbrace{\mathrm{Sym}(2) \wr \cdots \wr \mathrm{Sym}(2)}_{a_1 - 3} \times S_{a_2} \times \ldots S_{a_l}$. Again we have that $H \not\leq B$ and so $P \leq H$. Furthermore, if $a_1 \geq 4$, H is 2-minimal and so $H = P$ contradicting the structure of P. Thus $a_1 = 3$ and since $k = 4$ divides n we infer that $n = 12$, and we have the example listed in the lemma.

Finally suppose that $k = 3$. Then let $n = b_1 3^{a_1} + \cdots + b_l 3^{a_l}$ be the 3-adic decomposition of n. Let S_{a_i} be a Sylow 3-subgroup of $\mathrm{Sym}(3^{a_i})$ and $S_{a_l}^{b_l}$ be a Sylow 3-subgroup of $\mathrm{Sym}(b_i 3^{a_i})$. Then we may suppose that $S = S_{a_1}^{b_1} \times \cdots \times S_{a_l}^{b_l}$. We first note that Alt(9) itself is 3-minimal and so it is impossible that $G = \mathrm{Alt}(9)$. Then taking $H := \mathrm{Sym}(9) \wr \underbrace{\mathrm{Sym}(3) \wr \cdots \wr \mathrm{Sym}(3)}_{a_1 - 2} \wr \mathrm{Sym}(b_1) \times S_{a_2}^{b_2} \times \cdots \times S_{a_l}^{b_l} \cap \mathrm{Alt}(n)$ is not contained in B. Thus H contains P and we have a contradiction via 2.9 if either $a_1 > 3$ or $b_1 > 1$ and otherwise we use the fact that Sym(9) is not 3-restricted. □

5. Lie type groups in characteristic not p

In this section we begin our investigation of the groups which satisfy 3.1 and have X a Lie type group defined in characteristic $r \neq p$. Our objective here is to prove two general results.

Lemma 5.1. [**para-arg**] *Suppose that X is a simple group of Lie type defined over* $\mathrm{GF}(r^a)$ *and assume that for every parabolic subgroup R of X, $G = N_G(R)X$. If $S \cap X$ is contained in a proper parabolic subgroup R of G, then G is not p-restricted.*

Proof. Suppose that R is a parabolic subgroup of X and assume that R contains $S \cap X$. Then as $N_G(R)X = G$, we have that $N_G(R)$ contains S. It follows that S normalizes a standard Levi-complement L of R and thus $S \leq LO_r(R)$ and $S \leq LO_r(R)^x$ for $x \in G$ with $O_r(R) \cap O_r(R)^x = 1$.

Setting $H_1 = SO_r(R)$ and $H_2 = SO_r(R)^x$ we have that $O_p(H_1) = O_p(H_2) = 1$ and so 2.16 implies that $P \leq H_1 \cap H_2$. But then $O^p(P) \leq O^p(H_1) \cap O^p(H_2) = O_r(H_1) \cap O_r(H_2) = 1$ and we have a contradiction. Thus the lemma holds. □

Lemma 5.2. [**abelian**] *Suppose that $X = O^p(G)$ is a Lie type group $G(r^a)$ with $p \neq r$. If X has abelian Sylow p-subgroups and $P < G$, then $p \leq 3$ and $G > X$.*

Proof. If $X = G$, we simply cite 3.4 to obtain $G = P$. So assume that $G > X$ and $p \geq 5$. Since $O^p(P) \leq X$, $O^p(P)$ has abelian Sylow p-subgroups and so, as $p \geq 5$, $O^p(P/Q_P) \cong \mathrm{SL}_2(p^b)$ or $\mathrm{PSL}_2(p^b)$ for some $b \geq 1$ and P/Q_P is the same group extended by a cyclic group of field automorphisms. Since the Sylow p-subgroups of $O^p(P)$ are abelian, we infer further that $O^p(P) \cong \mathrm{SL}_2(p^b)$ or $\mathrm{PSL}_2(p^b)$ and that $[Q_P, O^p(P)] = 1$. Since $N_G(S \cap O^p(P)Q_P)S \leq B$, $Q_BQ_P \in \mathrm{Syl}_p(O^p(P)Q_P)$ and $\Phi(Q_B) \leq Q_P$. Therefore $\Phi(Q_B) \trianglelefteq \langle B, P\rangle$, so Q_B is elementary abelian. Since Q_B is normalized by $N_G(S \cap O^p(P))$ and $N_G(S \cap O^p(P))$ acts irreducibly on $S \cap O^p(P)$ and centralizes Q_P, we infer that $Q_B \cap O^p(P) = S \cap O^p(P) > 1$. In particular, $Q_B \cap X > 1$ so $Q_B \cap \Omega_1(S \cap X) > 1$ and 3.3 implies that $Q_B \cap X = \Omega_1(S)$.

Set $S_0 = S \cap X$ and consider $S_0O^p(P)$. Since S_0 is abelian and $O^p(P) \cong \mathrm{SL}_2(p^b)$ or $\mathrm{PSL}_2(p^b)$, we infer that $\Phi(S_0) \leq Q_P$. But then $\Phi(S_0) \cap \Omega_1(S_0) \trianglelefteq N_X(S_0)$, so 3.3 implies that $\Omega_1(S_0) \leq Q_P$. But then $\Omega_1(S_0) \trianglelefteq \langle B, P\rangle = G$ so S_0 is elementary abelian.

Since $p > 3$, G/X does not contain a graph automorphism. If X has a diagonal automorphism of order p, then using $p > 3$ again, we have that p divides $(n, \Phi_1(r^a))$ and $X \cong \mathrm{PSL}_n(r^a)$ or p divides $(n, \Phi_2(r^a))$ and $X \cong \mathrm{PSU}_n(r^a)$. In the first case we have that X contains a monomial subgroup $\frac{1}{(n,r^a-1)^2}(r^a-1)^n.\mathrm{Sym}(n)$ and in the second case $\frac{1}{(n,r^a+1)^2}(r^a+1)^n.\mathrm{Sym}(n)$ and, as $p > 3$, these subgroups exhibit that fact that S_0 is non-abelian. Thus X has no diagonal automorphisms of order p. So G/X must consist of field automorphisms. As $p > 3$, this means that $r^a = r^{pa_0}$ for some integer a_0. Letting m be the order of r^a mod p, [3, pg. 112] presents the relationship

$$\frac{\Phi_m(r^{pa_0})}{\Phi_m(r^{a_0})} \equiv \Phi_m(r^{a_0})^{\phi(p)} \pmod{p}.$$

Finally, [3, 10-1 (2)] shows that the exponent of S_0 is at least p^2 (using here that X has no diagonal automorphisms of order p) and so the Sylow p-subgroups of X are not elementary abelian, which is a contradiction. This final contradiction finishes the proof of the lemma. □

We remark here that there are examples of 3-restricted groups G with $X = O^p(G)$ a simple Lie type group with abelian Sylow 3-subgroups and $|G/X| = 3$ (see 6.11).

6. Linear and unitary groups

In this section we investigate the linear and unitary groups. To simplify our notation we use $\mathrm{GL}_n^+(r^a)$ and $\mathrm{GL}_n^-(r^a)$ to represent the general linear and unitary groups respectively. The notation $\mathrm{GL}_n^\epsilon(r^a)$ denotes either of the groups. So here $\epsilon = \pm$. However we will also write $r^a - \epsilon$ and in this case we regard ϵ as ± 1 according to $\epsilon = \pm$. Throughout this section $\overline{X} \cong \mathrm{SL}_n^\epsilon(r^a)$ and p is a prime with $p \neq r$. Recall that $\Gamma\mathrm{L}_n^\epsilon(r^a)$ has $\overline{X}$ as a normal subgroup and includes all the diagonal and field automorphisms of $\overline{X}$. In the event that $\overline{X} \cong \mathrm{SL}_n(r^a)$, we denote the inverse transpose automorphism by ι and, for ease of notation, when $\overline{X} \cong \mathrm{SU}_n(r^a)$ or $\mathrm{SL}_2(r^a)$, we take ι to be the trivial automorphism. Let V represent the natural linear space when $\overline{X} \cong \mathrm{SL}_n(r^a)$ and the natural unitary space when $\overline{X} \cong \mathrm{SU}_n(q)$.

For the remainder of this section we take

$$\overline{X} \leq \overline{G} \leq \Gamma\mathrm{L}_n^\epsilon(r^a) : \langle \iota \rangle$$

with $\overline{G}/\overline{X}$ a p-group and $\overline{G}$ non-soluble. Finally set $G = \overline{G}/F(\overline{G})$, $X = \overline{X}F(\overline{G})/F(\overline{G})$ and for $\overline{S} \in \mathrm{Syl}_p(\overline{G})$, $S = \overline{S}F(\overline{G})/F(\overline{G})$. Our objective is to determine for which values of n and r^a, G is p-restricted. The first few lemmas form the base of our final induction arguments.

Our first lemma helps us apply induction.

Lemma 6.1. [**subquot**] *Suppose that $\overline{R} \leq F(\overline{G})$ is normal in G and $P \in \mathcal{P}_{\overline{G}}(\overline{S})$ is p-restricted in $\overline{G}$. Then either $\overline{G}/ovR$ is p-restricted or $O^p(\overline{P}) \leq \overline{R}$.*

Proof. If $O^p(\overline{P}) \not\leq \overline{R}$, then $\overline{RS} \leq B$ and $\overline{R} \leq C_{\overline{B}}(\overline{Y})$. Thus the result follows from 2.10. □

Lemma 6.2. [**sl2**] *If $X \cong \mathrm{PSL}_2(r^a)$ and P is p-restricted in G, then one of the following holds:*

(a) [**a**] $p = 2$, $r^a \equiv 3, 5 \pmod 8$, $G \cong \mathrm{PGL}_2(r^a)$, $B = C_G(\Omega_1(Z(S)))$ *and* $P \cong \mathrm{Sym}(4)$;
(b) [**a+1**] $p = 2$, $G \cong \mathrm{PGL}_2(19)$, $B \cong \mathrm{Sym}(4)$ *and* $P \cong \mathrm{Dih}(40)$ *with* $P/O_2(P) \cong \mathrm{Dih}(10)$;
(c) [**b**] $p = 2$, $G \cong \mathrm{PSL}_2(7)$, $\mathrm{PSL}_2(9)$ *and* $B \cong P \cong \mathrm{Sym}(4)$ *or* $G \cong \mathrm{P\Sigma L}_2(9) \cong \mathrm{Sym}(6)$ *and* $B \cong P \cong \mathrm{Sym}(4) \times 2$;
(d) [**c**] $G = P \cong \mathrm{PSL}_2(5) \cong \mathrm{Alt}(5)$ *or* $\mathrm{PGL}_2(5) \cong \mathrm{Sym}(5)$;
(e) [**d**] $p = 3$, $X \cong {}^2\mathrm{G}_2(3) \cong \mathrm{PSL}_2(8).3 = \mathrm{P\Gamma L}_2(8)$.

Proof. We first consider the case when $p = 2$. Set $S_0 = S \cap X$. Let t be a central involution in S_0 and put $D = C_G(t)$. We shall often use the following straight forward observation.

1°. [**1**] *If $P \leq D$, then $P/O_2(P) \cong \mathrm{Sym}(3)$, $\mathrm{Dih}(10)$ or $\mathrm{Frob}(20)$ and $D \cap B$ is a maximal subgroup of D of index 3 in the first case and 5 in the last two cases.*

Suppose first that $|S_0| = 4$. Then $r^a \equiv 3, 5 \pmod 8$ and consequently a is odd. Therefore X admits no field automorphisms of order 2 and G is isomorphic to a subgroup of $\mathrm{PGL}_2(r^a)$. If $G = X$, then 5.2 implies that $G = P$ and in this case (d) holds. Hence we have $G \cong \mathrm{PGL}_2(r^a)$. So $S \cong \mathrm{Dih}(8)$ and $N_G(S_0) \cong \mathrm{Sym}(4)$. Assume that $P \leq D$. Then, as B is a 2-local subgroup of G, we must have $B \leq N_G(S_0)$. Hence (1°) implies that $|D| = 24$ if $P/O_2(P) \cong \mathrm{Sym}(3)$ and $|D| = 40$ if $P/O_2(P) \cong \mathrm{Dih}(10)$. Since $r^a \equiv 3, 5 \pmod 8$ and $|D| = r^a \pm 1$, we infer that the second possibility occurs and that $G \cong \mathrm{PGL}_2(19)$ which is possibility (b). Next suppose that have $P \not\leq D$. Then $B \geq D$ and, as $N_G(S_0) \cong \mathrm{Sym}(4)$ is 2-minimal and is not contained in B, we have $P = N_G(S_0)$. This delivers the examples listed in part (a). (Here we may mention that if $a > 1$, then P is contained in the subfield subgroups $\mathrm{PGL}_2(r^{a/b})$ whenever b divides a.)

Assume that $|S_0| \geq 8$ and let F_1 and F_2 be non-conjugate fours subgroups of S_0.

2°. [**2**] $Q_P \cap X > 1$.

Suppose that $Q_P \cap X = 1$. Then $P \not\leq D$ and so $B \geq D$. Additionally, we have $O_2(O^2(P)) \leq Q_P \cap X = 1$ and so, as $O^2(P/Q_P) \in \mathcal{L}_1(p)$, $O^2(P) \cong \mathrm{SU}_3(2^n)$, $\mathrm{PSU}_3(2^n)$, ${}^2\mathrm{B}_2(2^n)$, $\mathrm{SL}_2(2^n)$ for some $n \in \mathbb{N}$ or we have one of our unusual cases $O^2(P) \cong 3$, 5, 3_+^{1+2}, $3^2.2$, $2{\cdot}\mathrm{SL}_2(4)$. Since S_0 is a dihedral group and the Sylow 3 and 5-subgroups of X are cyclic, either $O^2(P) \cong \mathrm{SL}_2(4)$ or P is soluble. In the latter case we have that $|S/Q_P| \leq 8$ $|S| = 2^3|Q_P|$ leaving only the extreme case with $P/Q_P \cong \mathrm{PSU}_3(2) \cong 3^2 : \mathrm{Q}_8$ as a possibility and this contradicts the fact that $S/Q_P \cong S_0$ is dihedral. So $O^2(P) \cong \mathrm{SL}_2(4)$ and again by considering orders we get $P/O_2(P) \cong \mathrm{SL}_2(4) : 2$ and $|S_0| = 8$. Since $\mathrm{SL}_2(4) : 2$ contains a subgroup $P_1 = \mathrm{Sym}(4)$ and since P_1 is 2-minimal, $B \geq \langle P_1, D\rangle$ and hence $Q_B \cap X = 1$. But then 3.8 implies that P is soluble, a contradiction.

3°. [**3**] *If $P \not\leq D$, then (c) holds.*

Assume that $P \not\leq D$. Then necessarily $B \geq D$. By (2°) we have that $Q_P \cap X > 1$. Therefore, as P does not centralize t, we have $Q_P \cap X$ is a fours group F of S. Consequently S also normalizes F and so S_0 is a dihedral group of order 8. Furthermore, as in this case the Sylow 2-subgroups of $\mathrm{PGL}_2(r^a)$ are dihedral of order 32, we infer that $G \leq \mathrm{P\Sigma L}_2(r^a)$. Letting $F_1 \neq F$ also be a fours group of S_0, we have $H = N_G(F_1)$ is also a minimal parabolic subgroup (because H and P are conjugate in $\mathrm{P\Gamma L}_2(\mathrm{r^a})$). Thus $H \leq B$ also. But then $D \cong \mathrm{Dih}(8)$ or $\mathrm{Dih}(8) \times 2$ and we infer that one of the possibilities in part (c) occurs.

We now assume that $P \leq D$. Suppose that $Q_B \cap X > 1$. If $|S_0| > 8$, then $\langle P, B\rangle \leq C_G(t)$ and we have a contradiction. So $|S_0| = 8$ and $Q_B \cap X$ is a fours group F_1 of S_0. Furthermore, F_1 is normalized by S. Let F_2 be a fours group of S_0 not equal to F_1. Then $N_2 = N_X(F_2)S$ is 2-minimal and so as $N_2 \not\leq D$ and $P \leq D$, $N_2 \leq B$. But then $1 = O_2(\langle N_1, N_2\rangle) = Q_B \cap X$, a contradiction. Therefore, $Q_B \cap X = 1$ and in particular, $S_0 \leq C_X(Q_B)$. It follows at once that G is not isomorphic to a subgroup of $\mathrm{PGL}_2(r^a)$ which has dihedral Sylow 2-subgroups. Hence X admits field automorphisms and consequently as $|S_0| = 8$, $r^a \equiv 7, 9 \pmod{16}$ and a is even. So in fact $r^a \equiv 9 \pmod{16}$ and hence $D \cap X$ has order $r^a - 1$. Notice that the centre of the Sylow 2-subgroup of $\mathrm{Aut}(X)$ has order 2. Let θ be an automorphism of X which normalizes S and maps Q_B to Q_B^θ. Then $D^\theta = D$ and $N_G(Q_B^\theta)$ is also a 2-local subgroup and which is unequal to $B = C_G(Q_B)$ (as $Q_BQ_B^\theta \geq \langle t\rangle$). Therefore, $P \leq N_G(Q_B^\theta)$, But DS is invariant under θ and so $O^2(P) = O^2(PS) \leq N_G(Q_B^\theta)^{\theta^{-1}} = B$, a contradiction.

Assume that $p \geq 3$. Since the Sylow p-subgroups of $\mathrm{PSL}_2(r^a)$ are abelian, $p = 3$ and $G > X$ by 5.2. Now $O^3(P) \leq X$ has cyclic Sylow 3-subgroups. Since $O^3(P) \in \mathcal{L}_1(3)$, have $O^3(P) \cong \mathrm{Q}_8$, 2^2 or ${}^2\mathrm{G}_2(3)'$. By considering the Sylow 2-subgroup structure of X we deduce that $O^3(P) \cong 2^2$ or r^a is a power of 2 and $O^3(P) \cong {}^2\mathrm{G}_2(3)'$. If, in the latter case, $r^a > 8$, then $O^3(P)$ is not normalized by $N_G(S)$ and we have a contradiction. Thus if $O^3(P) \cong {}^2\mathrm{G}_2(3)'$, we have $G = P$ and we are done. So $O^3(P) \cong 2^2$ and $P/Q_P \cong \mathrm{PSL}_2(3)$. Now $Q_P \cap X$ is cyclic and centralized by $O^3(P)$. Since the centralizer of an involution in X is dihedral, we infer that $Q_P \cap X = 1$. Hence $P \cap X \cong \mathrm{PSL}_2(3)$ and so X has a cyclic Sylow 3-subgroup of order 3. Since X admits field automorphisms of order 3, we in fact have $|S \cap X|$ is divisible by 9, a contradiction. □

If p is odd then set $d = \mathrm{ord}(p, \epsilon r)$ and if $p = 2$ put $d = 1$ if $r^a \equiv \epsilon \pmod 4$ and otherwise put $d = 2$. Define $\overline{L^*} = \mathrm{GL}^\epsilon_d(r^a) \wr \mathrm{Sym}(s) \times \mathrm{GL}^\epsilon_{n-sd}(r^a)$ where $s = \left[\frac{n}{d}\right]$, put $\overline{L} = \overline{L^*} \cap \overline{G}$ and finally $L = \overline{L}/F(\overline{G})$. Then we may suppose that $S \cap X \leq L$ and that L is normalized by S (notice that the natural realization of L is closed under inverse-transpose operation). We shall use similar notation for other imprimitive subgroups of $\overline{G}$. Let $\overline{K^*}$ be the base group of $\overline{L^*}$ and $\overline{K} = \overline{K^*} \cap G$.

In the next lemma we use the well-known fact that the irreducible section of the natural $\mathrm{GF}(p)\,\mathrm{Sym}(s)$-permutation module has dimension $s - 1$ if $(s, p) = 1$ and $s - 2$ if p divides s. In the latter instance the permutation module is uniserial.

Lemma 6.3. [**sln1**] *Suppose that $G \le \mathrm{GL}_n^\epsilon(r^a)$, $n = ds > 2$ and assume that $L \le H$. If $O_p(H) > 1$, then either*

(a) [**a**] *$H > L$, $n = s = 4$, $d = 1$, $r^a = 4 + \epsilon$ and $\overline{H} \sim 4 * 2_+^{1+4}.(2 \times \mathrm{Sym}(6)) \cap \overline{G}$; or*

(b) [**b**] *$H = LS$ and either $d = 1$ or $d = 2$, $r^a \in \{2, 3\}$ and $\overline{L^*} \cong \mathrm{GL}_2^+(r^a) \wr \mathrm{Sym}(s)$.*

Proof. Assume that $d = 1$ and let R be a normal p-subgroup of L. Put $R_0 = \Omega_1(R)$. If $\overline{R_0} \le \overline{K}$, then as $R_0 \ne 1$, $|R_0| = p^{s-1}$ if $(p, s) = 1$ and $|R_0| \ge p^{s-2}$ if $(p, s) = p$. In either case, we have that the homogeneous components of $\overline{R_0}$ on V coincide with those of K and consequently $H = LS$. So we may assume that $R_0 \not\le K$. This means that K is a 2-group and $s = 4$ or 2. The case $s = 2$ has been ruled out by assumption and so we have that $s = 4$ and $X \cong \mathrm{PSL}_4(r^a)$. Now by the definition of d, we have the 4 divides $r^a - \epsilon$ and consequently, as $[K, R_0] \le K \cap R_0$ which is elementary abelian, we have that $r^a = 4+\epsilon$ as claimed in (a). So suppose that $d = 2$. If $r^a = 2$, then an argument similar to the one above shows that H normalizes $O_3(L)$ and we are done. So suppose that $r^a = 3$, let R be a normal 2-subgroup of H. Then $R \le O_2(L)$ and this is contained in the base group $\overline{K}$ of $\overline{L}$. Consider $\overline{R}_0 = \overline{R} \cap Z(\overline{K})$. Since $Z(\overline{K})$ is the natural $\mathrm{GF}(2)\,\mathrm{Sym}(s)$ permutation module for $\overline{L}$, we either have $|\overline{R}_0| = 2$ or $|\overline{R_0}| \ge 2^{n-1}$. If $|\overline{R_0}| > 2$, then the $\overline{K}$ and $\overline{R_0}$ have the same homogeneous components and we are done. So $\overline{R_0} \le Z(G)$. But then $R_0 \cap Z(K) = 1$, a contradiction as $Z(K) \ge Z(S)$. Finally assume that $r^a > 3$ and that $d \ge 2$. Then as L is a p-local subgroup and K is non-soluble, we have that $O_p(L) \le Z(K)$ and then, as $d \ge 2$, we have that $d = 2 = p$ and $r^a - \epsilon$ is not divisible by 4. We now argue as in the $r^a = 3$ case to obtain our contradiction. □

Lemma 6.4. [**QBX**] *Suppose that $n = ds$ and $B \ge LS$, then $Q_B \cap X > 1$.*

Proof. Suppose that $Q_B \cap X = 1$. Then $[Q_B, L \cap X] = 1$. Notice that $\overline{Q_B}$ induces the same type of automorphism on each $\mathrm{SL}_d^\epsilon(r^a)$ contained in $\overline{K}$ as it does on X. Suppose that $d > 2$ or p is odd and $d \ge 2$. Then the Sylow p-subgroup structure of $\mathrm{Out}(X)$ is identical to that of $\mathrm{SL}_d^\epsilon(r^a)$ and so in these cases it is impossible for $Q_B \cap X = 1$. Assume that $p = 2$ and $d = 2$. Then the inverse transpose automorphism of X induces an inner automorphism of $\mathrm{SL}_2(r^a)$ but not of $\mathrm{GL}_2(r^a)$ and thus in this case also it is impossible for Q_B to centralize K. Assume that $d = 1$. In this case, as inversion is not a field automorphism, we deduce that the only possibility is that Q_B induces a diagonal automorphism of X. But then $Q_B \le L$ and the result now follows. □

Lemma 6.5. [**monomial1**] *Suppose that t is a prime and $L \le G$ is as above with $d = 1$ so that $L \cap X \sim \frac{1}{(r^a-\epsilon,n)}(r^a - \epsilon)^{n-1}.\mathrm{Sym}(n)$ and $n \ge 4$. If Y is a non-trivial elementary abelian t-group of rank at most 2 and $Y \trianglelefteq L$, then $n = 4$ and $p = 2$.*

Proof. This is the result of an elementary calculation. □

Lemma 6.6. [**monomial2**] *Suppose that $L \cap X \sim \frac{1}{(r^a-\epsilon,n)}(r^a - \epsilon)^{n-1}.\mathrm{Sym}(n)$ with $n \ge 3$. Set $T = F(L)$. If $P \in \mathcal{P}_{LS}(S)$ is p-restricted in LS, then either*

(a) [**a**] *T is a p-group; or*

(b) [**b**] *P is soluble and $n = 3$.*

Proof. If $P \in \mathcal{P}LS(S)$ is p-restricted in LS and T is not a p-group. Then $O_p(LS) \cap X \le O_p(L)$ which is abelian. Furthermore, $E(LS/O_p(LS)) = 1$, and so $O^p(P/O_p(LS))$ is normal and soluble in $LS/O_p(LS)$ by **??**. Since $O^p(P) \le O^p(LS) \le X$, we have that $O^p(P/O_p(LS)) \le T/O_p(LS)$. Because T is abelian, we have $O^p(P) \trianglelefteq L$ and, as $O^p(P)$ is a t-group for some prime $t \in \{2, 3\}$ of rank at most 2, we either have $n \le 3$ or $n = 4$ and $t = 2$ by 6.5. The former case gives (b). In the latter case, we have $p = 3$ and $Q_B O^p(P)O_3(LS)/O_3(LS) \cong \mathrm{PSL}_2(3)$. Since also $[Q_B, C_B(YO_3(LS)/O_3(LS)] = 1$, we obtain a contradiction in this case. □

Lemma 6.7. [**monomial3**] *Suppose that $n = p \geq 5$ and $d = 1$, Then LS contains every proper p-local subgroup of G which contains S.*

Proof. We have $\overline{L^*} \cong (r^a - \epsilon) \wr \mathrm{Sym}(p)$. Let R be a proper p-local subgroup of G containing S. Set $Q_R = O_p(R)$ and $Q_L = O_p(L)$. Now $LS/F(L) \cong \mathrm{Sym}(p) \times T$ where T is a cyclic p-subgroup (which may be trivial) consisting of Frobenius automorphisms. In particular, we have $S/F(LS)$ is abelian and so $Q_R' \leq Q_L$. If $V|_{\overline{Q_R'}}$ is not homogeneous, then the homogeneous components of $\overline{Q_R'}$ coincide with the homogeneous components of $\overline{Q_L}$ and we conclude that $R \leq L$ as claimed. If on other other hand $V|_{\overline{Q_R'}}$ is homogeneous, then $\overline{Q_R}' \leq Z(\overline{L})$. It follows that $[\overline{Q_L}, \overline{Q_R}, \overline{Q_R}] \leq Z(\overline{L})$. In particular, we see that Q_R is not inducing field automorphisms on Q_L. Furthermore, since $\overline{Q_L}/\Phi(\overline{Q_L})$ is isomorphic to a section of the $\mathrm{GF}(p)$-permutation module for $\mathrm{Sym}(p)$ of order at least p^{p-1}, $p \geq 5$ and $[\overline{Q_L}, \overline{Q_R}, \overline{Q_R}] \leq Z(\overline{L})$ imply that $Q_R \leq Q_M$. If $V|_{\overline{Q_R}}$ is also homogeneous, then $Q_R \leq Z(X)$ and and $R = G$, a contradiction. Therefore, $V|_{Q_R}$ is not homogeneous and again we have that $R \leq M$. □

Lemma 6.8. [**L3andU3**] *Suppose that $p = 2$, $X \cong \mathrm{PSL}_3^\epsilon(r^a)$ and P is p-restricted in G. Then one of the following holds.*

(a) [**a**] $G \cong \mathrm{PSU}_3(r^a)$ *with* $s \equiv 3 \pmod 8$, $P \cong 2^{\cdot}\,\mathrm{Sym}(4) * 4$ *and* $B \cong (s+1)^2 : \mathrm{Sym}(3)$.
(b) [**b**] $G \cong \mathrm{PSU}_3(r^a) : 2$ *with* $s \equiv 3 \pmod 8$, $P \cong 2^{\cdot}\,\mathrm{Sym}(4) * \mathrm{Q}_8$ *and* $B \cong (s+1)^2 : (2 \times \mathrm{Sym}(3))$.
(c) [**c**] $G \cong \mathrm{PSU}_3(3) \cong \mathrm{G}_2(2)'$, $P \cong 4^2 : \mathrm{Sym}(3)$ *and* $B \cong 2^{\cdot}\,\mathrm{Sym}(4) * 4$.
(d) [**d**] $G \cong \mathrm{PSU}_3(3) : 2 \cong \mathrm{G}_2(2)$, $P \cong 4^2 : (2 \times \mathrm{Sym}(3))$ *and* $B \cong 2^{\cdot}\,\mathrm{Sym}(4) * \mathrm{Q}_8$.
(e) [**e**] $G \cong \mathrm{PSL}_3(r^a) : \langle \iota \rangle$ *with* $s \equiv 5 \pmod 8$, $P \cong 2^{\cdot}\,\mathrm{Sym}(4) * \mathrm{Q}_8$ *and* $\mathrm{B} \cong (s-1)^2 : (2 \times \mathrm{Sym}(3))$.

Proof. Let $S \in \mathrm{Syl}_2(G)$ and assume that $P \in \mathcal{P}_G(S)$ is p-restricted. If $r^a = 3$, we use the ATLAS [2] to see that statements (a), (b), (c) and (d) hold. So assume that $r^a > 3$.

Set $\overline{L_1^*} = \mathrm{GL}_2^\epsilon(r^a) \times \mathrm{GL}_1^\epsilon(r^a)$, $\overline{L} = \overline{L^*} \cap G$. Let K_1 be the component of L_1. Then K_1 is S-invariant. By 6.3, $O_2(\langle K_1, L, S\rangle) = 1$ and so $P \leq K_1 S$ or $P \leq L_1 S$ (or both). Assume that $B \geq K_1 S$, then $Q_B \leq O_2(K_1 S) \leq \Omega_2(LS) \leq Q_P$, which is a contradiction (note here it could be that $r^a - \epsilon$ is a power of 2 and that $Q_B \not\leq X$ induces ι). Thus $P \leq K_1 S$ and is p-restricted therein. Using (**??**) together with $s \equiv 1 \pmod 4$ implies that $r^a \equiv \epsilon(5) \pmod 8$ and that $P \not\leq LS$. Noting also that when $\epsilon = +$, $\overline{S \cap X}$ does not operate irreducibly on V, 5.1 implies that $G > X$. Thus we have the examples in (a), (b) and (e).

So suppose that $s \equiv -\epsilon \pmod 4$. Then $L \cong \mathrm{GL}_2^\epsilon(r^a)$ contains a Sylow 2-subgroup of X. Suppose that R is a 2-local subgroup of X which contains $S \cap X$. Then the decomposition of V restricted to $\overline{\Omega_1(Z(Q_R))}$ is preserved by R and we see that $P \cap X \leq N_X(\Omega_1(Z(Q_R))) \leq (r^a - \epsilon)^2 : \mathrm{Sym}(3)$ or $P \leq LS$. The former subgroup does not contain a Sylow 2-subgroup of X and so we infer that every 2-local subgroup of X which contains $S \cap X$ is contained in LS. Hence we cannot have both $Q_B \cap X > 1$ and $Q_P \cap X > 1$. Suppose that $Q_B \cap X = 1$. Then Lemma 3.7 implies that P is soluble and that P is not 2-constrained. Now $L'S < L$ and contains a Sylow 2-subgroup of G. Thus $P \leq L'S$. Then as P is not 2-constrained, Lemma 6.2 implies that $s = 19$ and $G \cong \mathrm{PGL}_3^+(19) : 2$ and that $(B \cap L)Z(L)/Z(L) \cong \mathrm{Sym}(4)$. But then Q_B normalizes $L/Z(L)$ and centralizes $(B \cap L)Z(L)/Z(L)$. It follows from the structure of $\mathrm{Aut}(\mathrm{PGL}_2(19))$ that $[Q_B, L] \leq Z(L)$, but then $[O^2(P), L/Z(L)] = 1$, a contradiction.

Next suppose that $Q_P \cap X = 1$. Then $B \geq N$ and so $|Q_B \cap X| = 2$. Since $|Q_B Q_P/Q_P|$ is also cyclic, we infer that P/Q_P has Sylow 2-subgroups of order at most 8. Therefore, $|S| \leq 8|Q_P|$ so that $|S \cap X| \leq 8$, which is of course a contradiction. □

Lemma 6.9. [**4dim**] *Suppose that $p = 2$, and X is isomorphic to $\mathrm{PSL}_4^\epsilon(r^a)$. If $P \in \mathcal{P}_G(S)$ is 2-restricted, then $X \cong \mathrm{PSU}_4(3)$ and the possibilities for P and B and G are as described in Table 1. In particular, if $G \geq \mathrm{PGU}_4(3)$, then $P \leq LS$.*

Proof. Let $X = \mathrm{PSL}_4^\epsilon(r^a)$. If $r^a = 3$, we inspect the Atlas [2] and see that in the case that $X = \mathrm{PSL}_4(3)$ there is no candidate for B and so we have that $X \cong \mathrm{PSU}_4(3)$ and the possibilities for B and P follow. So assume that $r^a > 3$. Since $p = 2$, $\overline{L_1} = \mathrm{GL}_2^\epsilon(r^a) \wr \mathrm{Sym}(2) \cap \overline{G}$ is normalized by S. Using 2.9 and $r^a > 3$ we infer that $B \geq L_1 S$. Assume that $s - \epsilon \equiv 0 \pmod 4$. Then $d = 1$ and $\overline{L_1} \neq \overline{L}$. By 6.3, B doesn't contain LS and so $P \leq LS$. Since $Q_B \leq O_2(L)$, we have $Q_B \leq O_2(L_1) \leq Q_P$, a contradiction. So assume that $r^a + \epsilon \equiv 0 \pmod 4$, then $Q_B \cap X = \Omega_1(Z(S))$ has order 2 and again Lemma 3.9 finishes this case. □

Lemma 6.10. [**su63**] *Suppose that $X \cong \mathrm{PSU}_n(3)$ with $n \in \{5, 6, 7\}$. If $P \in \mathcal{P}_G(S)$ is 2-restricted, then $X \cong \mathrm{PSU}_6(3)$ and $\overline{B} \geq \mathrm{GU}_2(3) \wr \mathrm{Sym}(3) \cap G$ and $P \leq LS \sim 4^5.\mathrm{Sym}(6)$.*

Proof. Suppose first that $X \cong \mathrm{PSU}_5(3)$. Then $L \cong 4^4.\mathrm{Sym}(5)$ is a 2-minimal minimal parabolic. Thus either $L \leq B$ or $P = L$. Let $\overline{L_1} = \mathrm{GU}_4(3) \times GU_1(3) \cap \overline{G}$. Then $L_1 \cong \mathrm{GU}_4(3)$. If $B \geq L_1$, then, using 6.3, $P \leq LS$ and $Q_B \leq O_2(L) \leq Q_P$, a contradiction. Therefore, $P \leq L_1 S$ and $B \geq L$. However, since $L_1/O_2(L_1) \cong \mathrm{PGU}_4(3)$, applying 6.9 we obtain $P \leq LS \leq B$, a contradiction .

Next consider $X \cong \mathrm{PSU}_6(3)$. Then $L \cap X \cong 4^5.\mathrm{Sym}(6)$. We set $\overline{L_1} = (\mathrm{GU}_4(3) \times \mathrm{GU}_2(3)) \cap \overline{G}$ and $\overline{L_2} = \mathrm{GU}_2(3) \wr \mathrm{Sym}(3) \cap \overline{G}$. If $P \leq L_2 S$ and we 2.12 implies that $O^p(P)O_2(L_2S)/O_2(L_2S)$ is normal in $L_2S/O_2(L_2S)$ and has order 3. Then $Q_B O^p(P) O_2(L_2S)/O_2(L_2S) \cong \mathrm{Sym}(3)$ and is a direct factor of $L_2S/O_2(L_sS)$, which is a contradiction. Hence $B \geq L_2 S$ and by 6.3 B does not contain L and so $P \leq LS$.

Suppose finally that $X \cong \mathrm{PSU}_7(3)$. Then $\overline{L} \cong 4^7 : \mathrm{Sym}(7)$. Since $\mathrm{Sym}(7)$ is not 2-restricted, we have $B \geq LS$. Setting $\overline{L_1} = \mathrm{GU}_6(3) \times \mathrm{GU}_1(3) \cap \overline{G}$, we have that $P \leq L_1 S$ we are in the $\mathrm{PSU}_6(3)$ configuration. In particular, $P \leq (L_1 \cap L)S \leq B$, a contradiction. □

We next consider the situation when $p = 3$ and $X \cong \mathrm{PSL}_3^\epsilon(r^a)$. However, before initiating the investigation we draw attention to the fact that when $r^a - \epsilon \equiv 3, 6 \pmod 9$, then $3_+^{1+2} : \mathrm{SL}_2(3)$ is contained in $\mathrm{GL}_3^\epsilon(r^a)$ and contains a Sylow 3-subgroup of $\mathrm{GL}_3(r^a)$. (See [4, 6.5.3].)

Lemma 6.11. [**3dimchr3**] *Suppose that $p = 3$ and $X \cong \mathrm{PSL}_3^\epsilon(r^a)$.If $P \in \mathcal{P}_G(S)$ is 3-restricted, then $G \cong \mathrm{PGL}_3^\epsilon(r^a)$, $r^a - \epsilon \equiv 3, 6 \pmod 9$ with $r^a \neq 4$ and either*
(a) [**a**] $P \cong 3^2 : \mathrm{SL}_2(3)$*; or*
(b) [**b**] $r^a = 7$ *and* $P \cong (3^2 \times 2^2) : \mathrm{Sym}(3)$.

Proof. Notice that 3 divides one of $r^a + 1$ and $r^a - 1$. If $r^a - \epsilon$ is not divisible by 3, then $d = 2$ and $L \cong \mathrm{GL}_2^\epsilon(r^a)$. Since $O_3(LS) = 1$, we infer that $P \leq LS$ and LS is 3-restricted. Then calling upon 6.2 delivers a contradiction. So we may assume that 3 divides $r^a - \epsilon$. In this case $\overline{L}^* \cong \mathrm{GL}_1^\epsilon(r^a) \wr \mathrm{Sym}(3)$. Let $\overline{K}$ be the base group of $\overline{L^*}$ and $\overline{K_X} = \overline{K} \cap \overline{X}$.

If $r^a = 7$ we examine the Atlas [2] and observe the configuration in (a) and (b). So assume that $r^a \neq 7$. Let t be a prime dividing $r^a - \epsilon$ and, if $l = 2$, assume that t^2 divides $r^a - 1$. Let $T \in \mathrm{Syl}_t(K_X)$. Then T normalized by S and, by the choice of t (and using 2.8), $P \not\leq ST$. It follows that $ST \leq B$ and $Q_B \leq O_3(ST) = O_3(K_X S)$. If $P \leq LS$, then $Q_P \geq O_3(K_X S)$ and we have a contradiction. Therefore $P \not\leq LS$ and, by 6.3 $B = LS$ and $Q_B = O_3(LS)$.

If P is of characteristic 3, then $O_3(O^3(P)) > 1$. Let D be a normal subgroup of $O^3(P)$ chosen minimal such that $[D, O^3(O_3(P))] \neq 1$. If $\overline{D}$ is abelian, then $D \leq Q_B$ and we have a contradiction as $[D, Q_B] \neq 1$. Therefore D is non-abelian and hence irreducible on V. In particular, the centralizer of D is cyclic and $P/C_P(D) \cong \mathrm{SL}_2(3)$. By considering the order of S we now obtain the configuration listed in part (a).

Suppose that P is not of characteristic 3. Then by 3.5 either Q_B is elementary abelian or $O^3(P) \cong \mathrm{Ree}(3)' \cong \mathrm{SL}_2(8)$. In the former case, it follows that G/X does not have field automorphisms, that $|S| = 3^3$ and that X has elementary abelian Sylow 3-subgroups. Therefore, the only possibilities for $O^3(P)$ in this case are $\mathrm{SL}_2(9)$, $\mathrm{PSL}_2(9)$, 2^2 or Q_8. Since 3 does not divides $|\operatorname{Out}(\mathrm{PSL}_2(9))|$, the first two cases deliver S elementary abelian of order 27 which is impossible. Therefore, $O^3(P)$ is a 2-group and $|Q_P| = 9$. But then Q_P is abelian and P is contained in a conjugate of L, a configuration we have already seen off. So assume that Q_B is not elementary abelian and that $O^3(P) \cong \mathrm{Ree}(3)' \cong \mathrm{SL}_2(8)$. By considering representations of the normalizer of a Sylow 2-subgroup of $O^3(P)$, we see that $r = 2$. Now $|S \cap X| > 27$ and so $Q_P \cap X > 1$. Since $[O^3(P), Q_P \cap X] = 1$, and $O^3(P) \not\leq LS$, $\overline{O^3(P)}$ is contained in $\mathrm{GL}_2^\epsilon(r^a) \times \mathrm{GL}_1^\epsilon(r^a)$ in a unique way. But then $\overline{S}$ has to normalize this configuration, a contradiction. □

We next consider the immediate repercussions if 6.11.

Lemma 6.12. [**4dim2**] *Suppose that* $p = 3$, $X \cong \mathrm{PSL}_n^\epsilon(r^a)$ *with* $4 \leq n \leq 5$ *and* $P \in \mathcal{P}_G(S)$ *is* 3-*restricted. Then either*

(a) [**a**] $G = X \cong \mathrm{PSU}_4(r^a)$ *with* $s \equiv 2, 5 \pmod 9$, $P \cong 3^{1+2} : \mathrm{GL}_2(3)$ *and* $B \cong \frac{1}{(2,s+1)}(s+1)^3 : \mathrm{Sym}(4)$;

(b) [**b**] $G = X \cong \mathrm{PSU}_4(2)$, $P \cong 3^3 : \mathrm{Sym}(4)$ *and* $B \cong 3_+^{1+2} : \mathrm{SL}_2(3)$; *or*

(c) [**c**] $G = X = \mathrm{PSU}_5(2)$, $P \cong 3 \times 3_+^{1+2}.\mathrm{SL}_2(3)$ *and* $B = 3^4.\,\mathrm{Sym}(5)$.

Proof. If $X \cong \mathrm{PSL}_n(r^a)$, then $\overline{S} \cap \overline{X}$ cannot act irreducibly on V and consequently $S \cap X$ is contained in a parabolic subgroup of X contrary to 5.1. So assume that $X \cong \mathrm{PSU}_n(r^a)$. We consider the case $n = 4$ first. Assume that $d = 2$. Then $\overline{L^*} \cong \mathrm{GU}_2(r^a) \wr 2$, since $r^a > 3$, 2.9 implies that $L \leq B$. But then $Q_B \leq F(L)$ and we have that 3 divides $r^a + 1$, which is a contradiction. So $d = 1$ and we have that $\overline{L^*} = (s+1) \wr \mathrm{Sym}(4)$. Set $\overline{L_1} = (\mathrm{GU}_3(r^a) \times \mathrm{GU}_1(r^a)) \cap \overline{G}$. Assume that $B \geq L_1$. Then, by 6.3, $P \leq LS$. If $r^a > 2$, then $Q_B \leq O_3(L_1) \leq O_3(L) \leq Q_P$, a contradiction. Thus if $B \geq L_1$, then $s = 2$ and (b) holds. So assume that $P \leq L_1$. Then 6.11 gives $s \equiv 2, 5 \pmod 9$ as claimed in (a).

Assume now that $n = 5$. Arguing as in the $n = 4$ case we have that 3 divides $d = 1$. Set $\overline{L_1} = (\mathrm{GU}_3(r^a) \times \mathrm{GU}_2(r^a)) \cap \overline{G}$. Since $\mathrm{Sym}(5)$ is not 3-restricted, we have that $B \geq L$ from 6.6. Assume that $s > 2$, then B contains at least one of the components from L_1 and then $B \geq \langle LS, L_1S \rangle$, a contradiction to(**??**). So $s = 2$ and we have $X \cong \mathrm{SU}_5(2)$. Finally, we inspect the subgroup structure in the Atlas [2] to obtain the result as stated in (c). □

The situation in $X = \mathrm{SU}_4(r^a)$ with $r^a \equiv 2, 5 \pmod 9$ is more exotic than a first look suggests. In the case that $s = 2$, we have that $U_4(2) \cong \mathrm{Sp}_4(3)$. So suppose that $s > 2$. Then in the monomial group $(s+1)^3.\,\mathrm{Sym}(4)$, there are three subgroups isomorphic to $3^3 : \mathrm{Sym}(4)$ each containing $N_G(S)$ call them P_1, P_2, P_3 with $P_i \cap P_j = N_G(S)$ whenever $i \neq j$. Let $P = 3_+^{1+2}\mathrm{SL}_2(3)$. Then, up to change of notation we have, we have $\langle P, P_1 \rangle \cong \langle P, P_2 \rangle \cong \mathrm{PSp}_4(3)$ and $\langle P, P_3 \rangle \cong \mathrm{GU}_3(r^a)$. The embedding of $\mathrm{PSp}_4(3)$ into $\mathrm{SU}_4(r^a)$ stems from the fact that $\mathrm{SU}_4(r^a) \cong \Omega_6^-(r^a)$ and $\mathrm{PSp}_4(3)$ has index 2 in the Weyl group of type E_6. Finally we note that the two subgroups $\langle P, P_1 \rangle$ and $\langle P, P_2 \rangle$ are conjugate in $\mathrm{GU}_4(r^a)$.

Because of the situation for $\mathrm{SU}_5(2)$ we need to individually inspect $\mathrm{SU}_6(2)$ and $\mathrm{SU}_7(2)$.

Lemma 6.13. [**u62andu72**] *Suppose that* $X \cong \mathrm{SU}_6(2)$ *or* $\mathrm{SU}_7(2)$ *and* P *is* 3*-restricted in* G*. Then* $X \cong \mathrm{SU}_6(2)$, $|G/X| \leq 3$, $P \cap X \cong 3^5 : \mathrm{PSL}_2(9) \cong 3^5 : \mathrm{Alt}(6)$ *and* $B \cap X \sim 3_+^{1+4}.(\mathrm{Q}_8 \times \mathrm{Q}_8) : 3$.

Proof. For $G \cong \mathrm{SU}_6(2)$, we just inspect the Atlas[2]. So suppose that $X \cong \mathrm{SU}_7(2)$. Then $\underline{L} \cong 3^6 : \mathrm{Sym}(7)$. Since Sym(7) is not 3-restricted, we have $B \geq LS$. Setting $\overline{L_1} = \mathrm{GU}_6(2) \times \mathrm{GU}_1(2) \cap \overline{G}$. We have that $P \leq L_1 S$. From the $\mathrm{PSU}_6(2)$ example we now read that $P \sim 3^6 : \mathrm{Alt}(6) \leq B$, which is a contradiction. □

Lemma 6.14. [**dimp2**] *Suppose that* $p \geq 5$ *and* $n \leq p$*. If* $X \not\cong \mathrm{PSL}_n^\epsilon(r^a)$*, then* G *is not* p*-restricted.*

Proof. Suppose for a contradiction that P is p-restricted in G. If $n < p$ or $d > 1$, then X has abelian Sylow p-subgroups and 5.2 delivers a contradiction. Therefore, $n = p$ and $d = 1$. So $\overline{L^*} \cong (r^a - \epsilon) \wr \mathrm{Sym}(p)$. Using $p \geq 5$, 6.7 and 3.8 we have that $B = LS$ and $Q_P = 1 \cap X$. Since Q_B is normalized by $N_P(O^p(P) \cap S)$ and $Q_B \cap X = O_p(LS) \cap X$ is abelian, we have that $O^p(P) \cong \mathrm{SL}_2(p^b)$, $\mathrm{PSL}_2(p^b)$ for some a and $Q_B \cap O^p(P) \in \mathrm{Syl}_p(O^p(P))$ is elementary abelian. Since $Q_P \cap X = 1$, we then have $b = p - 2$ and $P \cap X = O^p(P)$ because $\mathrm{SL}_2(p^{p-2})$ has no field automorphisms. This contradicts the fact that X has non-abelian Sylow p-subgroups. □

We at last come to the induction argument.

Lemma 6.15. [**LU-generic**] *Suppose that* $X \cong \mathrm{PSL}_n^\epsilon(r^a)$ *and assume one of the following conditions hold:*

(a) [**a**] *if* $p = 2$ *and* $r^a > 3$ *or* $X \cong \mathrm{PSL}_n(r^a)$*, then* $n \geq 5$*;*
(b) [**a1**] *if* $p = 2$ *and* $X \cong \mathrm{PSU}_n(3)$*, then* $n \geq 8$*;*
(c) [**b**] *if* $p = 3$ *and* $r^a > 2$ *or* $X \cong \mathrm{PSL}_n(r^a)$*, then* $n \geq 6$*;*
(d) [**b2**] *if* $p = 3$ *and* $X \cong \mathrm{PSU}_n(2)$*, then* $n \geq 8$*;*
(e) [**c**] *if* $p \geq 5$*, then* $n > p$*.*

Then G *is not* p*-restricted.*

Proof. Furthermore, assume that n as defined in the statement of the lemma and then is chosen minimally so that G is p-restricted. Recall that $\overline{L^*} = \mathrm{GL}_d^\epsilon(r^a) \wr \mathrm{Sym}(s) \times \mathrm{GL}_{n-ds}^\epsilon(r^a)$ where $s = \left[\frac{n}{d}\right]$ and L is normalized by S.

1°. [**1**] $ds = n$.

Assume that $ds < n$. Plainly $d > 1$. Then $S \cap X$ centralizes an $n - ds$ dimensional subspace of V and so, in particular, it centralizes a 1-dimensional subspace W of V. If $\epsilon = +$ and G/X consist of diagonal or graph automorphisms or $\epsilon = -$ and W is singular, then $S \cap X$ is contained in a parabolic subgroup of X and we have a contradiction via 5.1. In the other cases we have W is non-degenerate and, setting $\bar{H} = \mathrm{GL}_{n-1}^\epsilon(r^a)$ (fixing W), $S \cap X \leq H$ and H is normalized by S. Because of the choice of n, 6.9, 6.10, 6.12, 6.13 and 6.14 imply that H is not p-restricted. Hence $B \geq H$. Since $Q_B > 1$, we infer that p divides $r^a - \epsilon$ and so $d = 1$ when p is odd, a contradiction. So $p = 2$. In this case, since $d > 1$, $|Q_B| = 2$, and $n - 1$ is even. 3.7 implies that $P/Q_P \cong \mathrm{SL}_2(2)$, (a subgroup of) $\mathrm{PSU}_3(2)$, (a subgroup of) $\mathrm{SU}_3(2)$ or (a subgroup of) ${}^2\mathrm{B}_2(2)$. Since $d = 2$, $Q_B \leq [S \cap X, S \cap X, S \cap X] \leq Q_P$ (from the structure of a Sylow 2-subgroup in $\mathrm{GL}_2(r^a)$), which is a contradiction.

2°. [**2**] $s \geq p$.

If $s < p$, then from the structure of L we see that $S \cap X$ is abelian. Hence 5.2 implies that $p \leq 3$. But then $d \leq 2$ and $n = ds < dp \leq 6$ so the restrictions on n deliver a contradiction.

3°. [**3**] *If* $d > 1$*, then* $B \geq LS$ *and either*

(a) [**i**] $p = 3$, $d = r^a = 2$ *and* $\overline{L^*} \cong \mathrm{GL}_2^+(2) \wr \mathrm{Sym}(n/2)$;
(b) [**ii**] $p = 2$, $d = 2$, $r^a = 3$ *and* $\overline{L^*} \cong \mathrm{GL}_2^+(3) \wr \mathrm{Sym}(n/2)$.

If $B \geq LS$, then by 6.4 $Q_B \cap X > 1$ and 6.3 implies that one of (a) or (b) holds. So for a contradiction assume that B does not contain LS. Then $P \leq LS$. If $\mathrm{GL}_d^\epsilon(r^a)$, is not soluble, we apply 2.9 to obtain $s = 1$ and this contradicts (2°). Therefore $\mathrm{GL}_d(r^a)$ is soluble. Using (**??**) we have that $O^p(P) \leq F_p(LS)$. Suppose that $d \geq 2$ and $r^a = 2$. If $O_p(LS) = 1$, then $F_p(LS)$ is a 3-group and we have $O^p(P)$ is a 3-group from which we infer that $p = 2 = r^a$, a contradiction. Therefore, $p = 3$, and $\overline{L^*}/O_3(\overline{L^*}) \sim 2 \wr \mathrm{Sym}(n/2)$ and $n \geq 6$ by (c) or $\mathrm{Q}_8 : 3 \wr \mathrm{Sym}(n/3)$ and $n \geq 8$ by (d). Furthermore, from 2.5(e) we have that $LS = (B \cap LS)O^p(P)$, $|Q_B O_3(LS)/O_3(LS)| = 3$ and $O^p(P)O_3(LS)/O_3(LS)$ is either elementary abelian of order 4 or isomorphic or Q_8 and is normal in $LS/O_3(LS)$. The only possibility is that $n/2 = 3$ and that $LS/O_3(LS) \cong 2 \wr \mathrm{Sym}(3) \cong 2 \times \mathrm{Sym}(4)$. So $L \cong \mathrm{SL}_2(2) \wr \mathrm{Sym}(3) \leq X = G = \mathrm{SL}_6(2)$. Now in this case L can be embedded in $H \cong \mathrm{Sp}_6(2)$ and so we must have that $\mathrm{Sp}_6(2)$ is 3-restricted. However, a look in the Atlas [2] confirms that a Sylow 3-subgroup of $\mathrm{Sp}_6(2)$ is contained in a unique maximal subgroup and consequently $\mathrm{Sp}_6(2)$ is a 2-minimal parabolic group, a contradiction. Suppose that next that $r^a = 3$. Then just as above we argue that $p = 2$. Then $\overline{L^*}/O_2(\overline{L^*}) \sim \mathrm{Sym}(3) \wr \mathrm{Sym}(n/2)$ and $O^2(P)O_2(L)/O_2(L)$ has order 3. Considering again the centralizer of Q_B in $L/O_2(L)$, we obtain a contradiction. This proves (3°).

4°. [**4**] *If $d = 1$, then $LS \leq B$.*

Suppose that $B \not\geq LS$. Then 6.5 gives either $O^p(P)$ is soluble and $n \leq 3$, which contradicts $n \geq 4$, or $r^a - \epsilon$ is a power of p and $LS/O_p(LS)$ is p-restricted. So $LS/O_p(LS) \in \mathcal{R}_p$. If $p = 2$, then as $n \geq 6$, we have $LS/O_2(LS) \cong \mathrm{Sym}(6)$, $\mathrm{Sym}(8)$ or $\mathrm{Sym}(12)$ by 4.1 and 4.2. Suppose that $LS/O_2(LS) \cong \mathrm{Sym}(6)$ and define $\overline{L_1} = \mathrm{GL}_2^\epsilon(r^a) \wr \mathrm{Sym}(3) \cap G$. Then $S \cap G$ is contained in L_1. If $r^a > 3$, then 2.9 implies that L_1S is not 2-restricted. Thus, if $r^a > 3$, then $B \geq L_1S$ and we get $Q_B \leq O_p(L) \leq Q_P$, a contradiction. Therefore, $r^a = 3$. But then L_1 is soluble and so, if $P \leq L_1S$, $O^2(P)(L_1)O_2(L_1)/O_2(L_1)$ has order 3 and $L_1/O_2(L_1) = C_{B/O_2(L_1)}(O^2(P)O_2(L_1)/O_2(L_1) \times Q_B O^2(P)O_2(L_1)/O_2(L_1)$ has a direct factor isomorphic to $\mathrm{Sym}(3)$, a contradiction. Hence $B \geq L_1S$. Finally consider, $\overline{L_2} = (\mathrm{GL}_2^\epsilon(3) \times \mathrm{GL}_4^\epsilon(3)) \cap G$. Then as $B \geq L_1$, $P \leq L_2S$. And furthermore, letting K be the component in L_2, we have $P \leq KS$ is 3-restricted. It follows that $\epsilon = -$ but then we contradict our supposition on the size of n as given in (b).

Next assume that $n = 8$ and set $\overline{L_1} = \mathrm{GL}_4^\epsilon(r^a) \wr \mathrm{Sym}(2) \cap \overline{G}$. By 2.9, $B \geq L_1S$ and so $P \leq LS$ and $Q_B \leq Q_P$, a contradiction. So suppose that $n = 12$ and set $\overline{L_1} = \mathrm{GL}_4^\epsilon(r^a) \times \mathrm{GL}_8^\epsilon(r^a) \cap \overline{G}$. If $P \leq L_1S$, then we must have $P \leq \mathrm{GL}_8(r^a)$ and this contradicts the minimal choice of n. Thus $B \geq L_1S$ and again $Q_B \leq Q_P$, a contradiction. Thus $p > 2$ and consequently $n \leq 6$. Hence $p \geq 5$. This then forces $n = 5$ and $L/O_5(L) \cong \mathrm{Sym}(5)$ and this configuration has been considered in 6.14. This proves (3°).

5°. [**5**] *p does not divide s.*

Suppose that p divides s. Assume additionally that $d = 1$. Set $\overline{L_1} = \mathrm{GL}_{dp}^\epsilon(r^a) \wr \mathrm{Sym}(s/p) \cap \overline{G}$. Then, as $s > p$, 2.9 implies that $L_1S \leq B$. Therefore (4°) gives $B \geq \langle L, L_1 \rangle S$. Since $Q_B > 1$, this contradicts 6.3. Suppose that $d = 2$. Then restrictions (b) and (e) imply that $n \geq 8$. In particular, $n > dp$ and so $s/p > 1$. Now we apply the above argument to obtain a contradiction.

Let $t = n - d[s/p]$. Then $1 \leq t \leq p - 1$. Let $\overline{L_1} = \mathrm{GL}_{d[s/p]}^\epsilon(r^a) \times \mathrm{GL}_t^\epsilon(r^a)$. Then $P \leq L_1S$. If L_1 has two components, K_1 and K_2 say, then at least one of them is contained in B, a contradiction. Therefore, L_1 has at most one component. Since $n > 4$, we infer that n has exactly one component say K_1. Furthermore, we must have K_1S is p-restricted. If $t = 1$, we have a contradiction from our usual lemmas. Thus $t = 2$ and $r^a \in \{2, 3\}$. The restrictions on and the minimal choice of n then mean that $n = 8$. If $p = 2$, we obtain $s = 4$ or $s = 8$ and we contradict (5°). If $p = 3$, we have

$K_1 \cong \mathrm{SU}_6(3)$ and K_1S is 3-restricted. By 6.10 we then have $P \le L \cap K_1S \le B$, a contradiction. This completes the proof of the lemma. □

7. Symplectic groups

In this section we suppose that r is a prime with $r \neq p$ and that $X \cong \mathrm{PSp}_{2n}(r^a)$. As usual we have G/X is a r-group. Notice that, as $p \neq 3$, G/X never involves the exceptional graph automorphism which only appears in characteristic 2. We define $\mathrm{G\Gamma Sp_{2n}(r^a)}$ to be $\mathrm{Sp}_{2n}(r^a)$ decorated with its diagonal automorphism of order 2 and its field automorphisms. So $[\mathrm{G\Gamma Sp_{2n}(r^a)} : \mathrm{Sp_{2n}(r^a)}] = 2\mathrm{a}$. Now as in the unitary case we let $\overline{X}$ be $\mathrm{Sp}_{2n}(r^a)$ and $\overline{G} \le \mathrm{G\Gamma Sp_{2n}(r^a)}$ with $\overline{G}/\overline{X}$ a p-group. Then $G = G/Z(G)$ and $X = \overline{X}/Z(G)$. Define $d = \mathrm{lcm}(2, \mathrm{ord}(p,r))$ and set $s = \left[\frac{2n}{d}\right]$. Define $\overline{L} = \mathrm{Sp}_d(r^a) \wr \mathrm{Sym}(s) \times \mathrm{Sp}_{2n-ds)}(r^a) \cap \overline{G}$. Then $L \cap X$ contains a Sylow p-subgroup of $S \cap X$ and L is S-invariant.

Lemma 7.1. [**spcases**] *Assume that* $X \cong \mathrm{PSp}_{2n}(r^a)$ *with* $n \ge 2$. *If* $X \in \mathcal{R}_p$, *then one of the following holds*

(a) [**a**] $p = 3$, $X \cong \mathrm{PSp}_4(2)$ *and* $P = X$;
(b) [**b**] $p = 2$ *and* $X \cong \mathrm{PSp}_4(3) \cong \mathrm{PSU}_4(2)$; *or*
(c) [**c**] $p = 2$, $X \cong \mathrm{PSp}_6(3)$ *and* $B = \ldots$

Proof. Suppose first that $2n - ds \neq 0$. then $S \cap X$ centralizes an isotropic vector in V and consequently $S \cap X$ is contained in a parabolic subgroup of X and this contradicts 5.1. So $n - ds = 0$. If $d = 2n$, then X has abelian Sylow p-subgroups. Thus $p = 2, 3$ by 5.2. But then $2n \le d \le 2$, a contradiction. So we have

1°. [**1**] $\quad$ *$ds = n$ and $s > 1$.*

Assume that $P \le LS$. Then $LS \in \mathcal{R}_p$ and so 2.9 implies that $\mathrm{Sp}_d(r^a)$ is soluble. That is $d = 2$ and $r^a = 2$ or $r^a = 3$. Arguing exactly as in 6.15 3 delivers a contradiction unless $2n = 4$ and $\overline{L} \cong \mathrm{SL}_2(2) \wr 2$ or $\overline{L} \cong \mathrm{SL}_2(3) \wr 2$. Thus we have examples (a) and (b).

Henceforth we assume that $B \ge LS$. In particular, as $Q_B > 1$, we have that either $p = 3$ and $G \cong \mathrm{Sp}_{2n}(2)$ or $p = 2$. In any case we have that $d = 2$.

Assume that $2n = 4$. If $r^a \le 3$, then again possibilities (a) and (b) holds. So assume that $r^a > 3$. Then as $Q_B > 1$, we infer that $p = 2$ and that $|Q_B| = 2$ contrary to 3.9.

We now assume $2n > 4$ and when $p = 2$ and $r^a = 3$ that $2n > 6$. Furthermore, select n minimal so that the above conditions on n are satisfied and $G \in \mathcal{R}_p$. Write $s = 2k + l$ with $l \le 1$ and set $\overline{L_1} = \mathrm{Sp}_4(r^a) \wr Sym(k) \times \mathrm{Sp}_{2l}(r^a)$ and factor L_1 as $\overline{K_1} \times \overline{K_2}$ with $K_2 \cong \mathrm{Sp}_2(r^a)$. Then as L_1S and LS generate G, L_1S contains P. Since $P \not\le LS$, we infer that $PS \le K_2$. Then 2.9 implies that $k = 1$ and that $K_2 \cong \mathrm{Sp}_4(r^a) \in \mathcal{R}_p$. It follows that $p = 2$ and $r^a = 3$ or $p = 3$ and $r^a = 2$. The first case fails because of the choice of n, the second case indicates that $G \cong \mathrm{Sp}_6(2)$ and we have already seen that this group does not satisfy $\mathbb{R}_3$. □

8. The orthogonal groups

Assume first that $X \cong \mathrm{PO}^{\epsilon}_{2n}(r^a)$ with $n \ge 4$ and no triality automorphism.

Suppose that p and r are distinct primes and set $d = \frac{1}{2}\,\mathrm{lcm}(2, \mathrm{ord}(p, r^a))$.

$$\eta = \begin{cases} + & \mathrm{ord}(p,r) \text{ odd and } p \text{ odd} \\ + & p = 2 \text{ and } r^a \equiv 1 \pmod 4 \\ - & \mathrm{ord}(p,r) \text{ even and } p \text{ odd} \\ - & p = 2 \text{ and } r^a \equiv 3 \pmod 4. \end{cases}$$

Let

$$s = \begin{cases} \left[\frac{n}{d}\right] & d \text{ does not divide } n \\ \left[\frac{n}{d}\right] & d \text{ divides } n \text{ and } \eta^{\left[\frac{n}{d}\right]} = \epsilon \\ \left[\frac{n}{d}\right] - 1 & d \text{ divides } n \text{ and } \eta^{\left[\frac{n}{d}\right]} \neq \epsilon. \end{cases}$$

Finally put

$$\overline{L} = \mathrm{O}^{\eta}_{2d}(r^a) \wr \mathrm{Sym}(s) \times \mathrm{O}^{\theta}_{2(n-ds)}(r^a).$$

Then $\overline{L}$ contains a Sylow p-subgroup of $\mathrm{O}^{\epsilon}_{2n}(r^a)$.

Lemma 8.1. [**orth1**] *Suppose that $X \cong \mathrm{P}\Omega_{2p}(r^a)$, $d = 1$ and $\eta^p = \epsilon$. If $p \geq 3$, then every p-local subgroup of X which contains $S \cap X$ is contained in L.*

Proof. Suppose that R is a p-local subgroup of X containing $S \cap X$ and put $Q_R = O_p(R)$. Let $Z_R = Z(Q_R)$. If $Z_R \leq O_p(L)$, then the homogeneous components of $\overline{Z_R}$ on V coincide with those of $\overline{O_p(L)}$ and we have $R \leq L$. So assume that $Z_R \not\leq O_p(L)$. But then Z_R operates quadratically on $O_p(L)$ and, as $\Omega_1(O_p(L))$ is the permutation module for $\mathrm{Sym}(p)$ and $p \geq 3$, we have a contradiction. □

Lemma 8.2. [**Orth2**] *Suppose that $X \cong \mathrm{P}\Omega^{\epsilon}_{2n}(r^a)$ with $n \geq 4$ and $G \in \mathcal{R}_p$, then one of the following holds:*

(a) [**a**] $p = 2$, $X \cong \mathrm{P}\Omega^+_8(3)$ *and* $\overline{B} = \mathrm{O}^+_4(3) \wr \mathrm{Sym}(2)$;
(b) [**b**] $p = 2$, $X \cong \mathrm{P}\Omega^+_{12}(3)$ *and* $\overline{B} \cong \mathrm{O}^+_4(3) \wr \mathrm{Sym}(3)$.
(c) [**c**] $p = 3$, $X \cong \mathrm{P}\Omega^+_8(r^a)$ *with* $r^a \cong 2, 5 \pmod 9$, $\overline{B} \cong \Omega^-_2(r^a) \wr \mathrm{Sym}(4)$ *and* $P \sim 3^{1+2}.\mathrm{SL}_2(3) \times 3$.

Proof. Suppose that n is chosen minimally so that $\mathrm{P}\Omega^{\mu}_{2n-2}(r^a) \notin \mathcal{R}_p$.

1°. [**1**] *$n = sd$.*

Suppose first that p is odd. Then G does not involve the graph automorphism of G and $S \cap X$ centralizes a non-degenerate subspace of V of dimension $2(n - ds)$. In particular, $S \cap X$ centralizes a singular vector and hence $S \cap X$ is contained in a parabolic subgroup of G. Suppose then that $p = 2$. Then we have that $\overline{L} = \mathrm{O}^{\eta}_2(r^a) \wr \mathrm{Sym}(s) \times \mathrm{O}^{\theta}_2(r^a)$, $\eta \neq \theta$ and $s = \left[\frac{n}{d}\right] - 1$. Observe that a Sylow 2-subgroup of $\mathrm{O}^{\theta}_2(r^a)$ has order 4. Thus we also see that in this case S. Set $\overline{L}^* = \Omega 0^{\eta}_2 \wr \mathrm{Sym}(s) S$. Then L^* leaves invariant two distinct anisotropic 1-spaces. Thus $\overline{L}^* S$ is contained in two proper subgroups of G each isomorphic to $\mathrm{O}_{2n-1}(r^a)$. Since $|Q_B| > 2$, we have P is contained in the intersection of these two groups and this means that $P \leq L^* S$. But then $r^a = 3$ and we have $2n = 8$ and $\overline{L^*}$ involves $\mathrm{PSU}_4(3)$, $2n = 10$ and $\overline{L^*}$ involve $\mathrm{P}\Omega^+_8(3)$ or $2n = 14$ and $\overline{L^*}$ involves $\mathrm{P}\Omega^+_{12}(3)$. In the first case we note that $\overline{L_2}\Omega^+_6(3) \times \mathrm{O}^-_2(3)$ also contains a Sylow 2-subgroup of G and we obtain $B \geq LS$ for a contradiction from 6.9. In the second and third cases we use the fact that $\mathrm{O}_9(3)$ contains a subgroup $\mathrm{O}_1(3) \wr \mathrm{Sym}(9)$ case $\mathrm{O}_{13}(3)$ contains a subgroup $\mathrm{O}_1(3) \wr \mathrm{Sym}(13)$ and apply 4.2 to see that in each case B must contain this subgroup. This contradicts the structure of B as described in (a) and (b). Thus we have that $n = sd$.

2°. [**2**] *$s > 2$.*

For $p = 2$ or 3, this follows because of the requirements on the size of n. So $p \geq 5$ and $s \geq 2$ follows from 5.2. So we assume that $s = p$.

3°. [**3**] *$B \geq LS$.*

Suppose on the contrary that $P \le LS$. Assume for a moment that $p > 3$. Then $O^p(P)$ is not soluble. Thus (2°), 2.9 and **??** together imply that $O^{\epsilon}_{2d}(r^a)$ is a p-group. But then the only possibility is that $p = 2$, a contradiction. Thus $p \in \{2, 3\}$. Thus $d = 1$ and $\overline{L} \cong \mathrm{O}^{\eta}_2(r^a) \wr \mathrm{Sym}(s)$. Assume that $\mathrm{O}^{\eta}_2(r^a)$ is not a 2-group. Then, since the derived subgroup of L is perfect when $n \ge 5$, we have that $n = 4$ from 2.11. Easy TO SEE $n \ne 4$. Write a nice argument. Therefore, we have is a 2-group and $\mathrm{Sym}(n) \in \mathcal{R}_2$. Therefore 4.2 implies that $n \in \{4, 6, 8, 12\}$. Set $m = 2, 2, 4, 4$ according as $n = 4, 6, 8, 12$ and define $\overline{L_1} = \mathrm{O}^{\mu}_{2m}(r^a) \wr \mathrm{Sym}(n/m)$ where $\mu = \eta^m$. If $\mathrm{O}^{\mu}_{2m}(r^a)$ is not soluble, then, by 2.9, $B \ge L_1S$ and Q_B has order 2^{m-1} and is contained in Q_P, a contradiction. Therefore, $\mathrm{O}^{\mu}_{2m}(r^a)$ which means that $m = 2$, $r^a = 3$ and $\mu = +$. Hence $X \cong \mathrm{P}\Omega^+_8(3)$ or $\mathrm{P}\Omega^+_{12}(3)$ which is a contradiction to our choice of n. Hence (3°) holds.

Since $B \ge LS$, we must have that LS is a p-local subgroup. Thus

4°. [4] $L \cong \mathrm{O}^{\eta}_2(r^a) \wr \mathrm{Sym}(n)$.

Suppose for a moment that $p \ge 5$. Then 5.2 implies that the Sylow p-subgroups of X are not abelian. Thus $n \ge p$. If $n = p$, then **??** implies that $Q_P = 1$ and the argument in **??** works to give a contradiction. Therefore,

5°. [5] $n > p$.

Write $n = lp + k$ where $k \le p - 1$. Then put $\overline{L_1} = \mathrm{O}^{\tau}_{2rp}(r^a) \times \langle I_{2k} \rangle$. Then S normalizes L_1. Assume that $lp \ne n$. Since $L_1S \not\le B$, we have that $P \le L_1S$ and so $L_1S/O_p(L_1S) \in \mathcal{R}_p$. If $p \ge 5$, this immediately contradicts the minimal choice of n. Therefore, $p \le 3$. If $p = 3$, then we require $lp < 4$ which means that $n = 4$ or $n = 5$ and that $\mathrm{O}^{\mu}_6(r^a) \in \mathcal{R}_3$ and this again contradicts our supposition on n. If $n = 5$, then set $\overline{L_2} = \langle I_4 \rangle \times \mathrm{O}^-_4(r^a)$ and note that as $P \le L_1S$, $L_2S \le B$, a contradiction. So suppose that $p = 2$. Then $2n = 12$ and $r^a = 3$. But then from the example in $\mathrm{O}^+_{12}(3)$ we read that $P \le L \le B$, a contradiction. Hence $n = rp$. Now set $\overline{L_2} = \mathrm{O}^{\rho}_{2dp}(r^a) \wr \mathrm{Sym}(r)$ where $\rho = \eta^p$. Then L_2 is normalized by S. Plainly $L_1S \not\le B$ and so $P \le L_1S$. Since $r > 1$ by (5°), $2dp = 4$ and $r^a = 3$ as well as $p = 2$. Investigating the structure of $\overline{L_2}$ using 2.10 readily reveals a contradiction. This completes the proof of the lemma. □

Lemma 8.3. **[Orthodd]** *Suppose that r is odd and $X \cong \mathrm{P}\Omega_{2n+1}(r^a)$ with $n \ge 3$. If $G \in \mathcal{R}_p$, then $p = 2$, $X \cong \mathrm{P}\Omega_7(3)$ and $\overline{B} = \mathrm{O}_1(3) \wr \mathrm{Sym}(7)$.*

Proof. We have that $|\mathrm{O}_{2n+1}(r^a) : \mathrm{O}^{\epsilon}_{2n}(r^a)| = (r^{na} + \epsilon)r^b$ for some b. Thus a Sylow p-subgroup of $\mathrm{O}_{2n+1}(r^a)$ fixes either a plus point or a minus point when acting on V. Since these point stabilizers H have $O_p(H)$ of order at most 2, we $H \in Rp$ by 3.9. Thus the possibilities for p, r^a and n may be read from 6.9 and 8.2. Suppose that $n = 3$. Then $\overline{H} \cong 2 \times \mathrm{O}^{\epsilon}_6(r^a)$. Thus 6.9, 6.12 and 6.14 imply that p is either 2 or 3. If $p = 2$, then from 6.9 we get $\epsilon = -$ and $r^a = 3$ and (**??**) holds. If $p = 3$, then again $\epsilon = -$ and $r^a \equiv 2, 5 \pmod 9$. But then $r^{3n} + 1$ is divisible by 3, a contradiction.

Assume now that $n \ge 4$. Then **??** implies that $p = 2$ or 3. Assuming that $p = 2$, the subgroups $\mathrm{O}_1(3) \wr \mathrm{Sym}(9)$ and $\mathrm{O}_1(3) \wr \mathrm{Sym}(13)$ yield contradictions. So $p = 3$ and $\overline{H} \cong \mathrm{O}^+_8(r^a) \times 2$. Set $\overline{L} = \mathrm{O}^-_6(r^a) \times \mathrm{O}_3(r^a)$. Then L is S invariant. Since $r^a > 3$, $\mathrm{O}_3(r^a)$ is not soluble. Furthermore, writing $\overline{L} = \overline{K_1} \times \overline{K_2}$ with $K_2 \cong \mathrm{O}_3(r^a)$, we have that $K_2 \not\le H$ and so $K_2 \le B$, but then $B \ge \langle K_2, B \cap \overline{H} \rangle = G$. □

Finally in this section we come to the situation when $p = 3$, $X \cong \mathrm{P}\Omega^+_8(3)$ and S does not normalize all the parabolic subgroups of X. (So the triality automorphism of X is having an influence. From the examples in **??** we only need consider the case when $r^a \not\equiv 2, 5 \pmod 9$ (otherwise the example in X immediately lifts to an example in G with G/X a 3-group.

Lemma 8.4. [**triality**] *Suppose that* $X \cong \mathrm{P}\Omega_8^+(3)$ *and* G/X *is a* 3*-group. If* $G \in \mathcal{R}_3$*, then* $r^a \equiv 2, 5 \pmod 9$.

9. Exceptional groups of Lie type

In this section we suppose that $\overline{X}$ is a universal group of Lie type defined over a field over $\mathrm{GF}(r^a)$. We use the notation introduced in [4, page 237] writing

$$|X| = r^{NA} \prod_i \Phi_i(r^a)^{n_i}$$

where $\Phi_i(x)$ is the cyclotomic polynomial for for i-th roots of unity. The product $\prod_i \Phi_i(r^a)^{n_i}$ in the case when X is an exceptional group is conveniently presented in [3, Table 10:2]. We set $d = \mathrm{ord}(p, r^a)$ if p is odd and, if $p = 2$, $d = 1$ when $r^a \equiv 1 \pmod 4$ and otherwise $d = 2$.

Lemma 9.1. [**OrdSylow**] *Let* $\overline{X}$ *be a universal group of Lie type defined over* $\mathrm{GF}(r^a)$*,* $\overline{S}$ *be a Sylow* p*-subgroup of* $\overline{X}$ *with* p *odd. Then*

$$|\overline{S}| = p^b \Phi_d(r^a)_p^{n_d}$$

where $b = \sum_{pd|i} n_i$*. Furthermore, if* $b = 0$*, then the Sylow* p*-subgroups of* X *are abelian.*

Proof. Consult [3, Equation (*) page 113] and [4, Theorem 4.10.2 (c)].

Lemma 9.2. [**bigdexcep**] *Suppose* $G \in Rp$ *and that* $d = \mathrm{ord}(p, r^a) > 2$*. Then* $d = 4$*,* $p = 5$ *and* $X \cong \mathrm{E}_8(r^a)$.

Proof. Since $d.2$, $p \geq 5$. We show that other than in the $d = 4$, $p = 5$, $X \cong \mathrm{E}_8(r^a)$ cases the Sylow p-subgroups of X are abelian. The result then follows form 5.2. Since $d > 2$ and $p \geq 5$, $pd \geq 20$ (note $d = 3$ and $p = 7$ gives $pd = 21$). From [3, Table 10:2] the only exceptional group involving $\Phi_k(r^a)$ with $k \geq 20$ is $\mathrm{E}_8(r^a)$. So consider $\mathrm{E}_8(r^a)$. If $d = 3$, we see no $\Phi_{21}(r^a)$, if $d = 4$ we can only have $p = 5$ and if $d \geq 5$, there are also no possibilities. □

Lemma 9.3. [**E8p5**] *Suppose that* $d = 4$*,* $p = 5$ *and* $X \cong \mathrm{E}_8(r^a)$*. Then* $G \notin \mathcal{R}_5$.

Proof. In this case we have that 5 divides $\Phi_4(r^a) = r2a + 1$. Using [3, Table 4-1 (37)] we see that X contains a subgroup L isomorphic to $\mathrm{SU}_5(r^{2a})$ and by comparing orders we see that this subgroup contains a Sylow 5-subgroup of X and furthermore L is invariant under S. ¿From 6.15 we have that $LS \not\geq P$ and therefore, $B \geq LS$ and $|Q_B| = 5$. But $Z(S)$ is cyclic and so we have a contradiction with 3.9. □

Lemma 9.4. [**SuzandRee**] *If* $G \in \mathcal{R}_p$ *and* $X \cong {}^{\mathrm{B}}_2(2^a)$ *or* ${}^2\mathrm{G}_2(3^a)'$*, then* $p = 2$ *and* $X \cong {}^{\mathrm{G}}_2(3)'$.

Proof. If $X \cong {}^2\mathrm{B}_2(2^a)$, then $p > 3$ and the Sylow p-subgroups of X are abelian. Therefore, 5.2 shows that $G \notin Rp$. Let $X \cong {}^2\mathrm{G}_2(3^a)$ with $a \geq 3$. in this case also the Sylow 2-subgroups are abelian and so 5.2 delivers either a contradiction or $X \cong {}^2\mathrm{G}_2(3)' \cong \mathrm{SL}_2(8)$. □

Guess

Lemma 9.5. [**ReeF**]*If* $G \in \mathcal{R}_p$*, then* $X \not\cong {}^2\mathrm{F}_4(2^a)'$.

Proof. By 9.1 we only need to consider the situation when $p = 3$ and $d = 2$. we have $|S \cap X| = 3|\Phi_d(2^a)^2|_3$. If $r^a = 2$, we have $G = X = {}^{\mathrm{F}}_2(2)'$ and the Atlas [2] shows us that there are no 3-local subgroups which are maximal subgroups of G. So assume that $2^a \geq 8$. Using [7, Table 5.1 and 5.2] we see that $S \cap X$ is contained in maximal subgroups $L_1 = (2^a + 1)^2.\mathrm{GL}_2(3)$ and $L_2 \cong \mathrm{SU}_3(2^a) : 2$. According to 6.11 $L_2 \notin \mathcal{R}_3$, so $B \geq L_2$ and $P \leq L_1$. But then $Q_B \leq Q_P$, a contradiction. □

Lemma 9.6. [**F4**]*If $G \in \mathcal{R}_p$, then $X \not\cong \mathrm{F}_4(r^a)$.*

Proof. By 9.1 and 5.2 we have that $p \in \{2, 3\}$. Notice that as $p \neq r$, the graph automorphism of $\mathrm{F}_4(2^a)$ makes no appearance in this discussion. We first suppose that $p = 2$. Then, using **??**Table 5.1]LSS, there is a maximal subgroup $L \in X$ with $L \cong 2.\Omega_9(r^a)$. By 8.3, $L \notin \mathcal{R}_p$. Therefore, $B = L$. But then $|Q_B| = 2$ and **??** provides a contradiction. Next assume that $p = 3$. Then we set $L = (2, q-1)^2.\mathrm{P}\Omega_8^+(r^a).\mathrm{Sym}(3)$. By 8.2. Notice that L contains $S \cap X$ and that L is not a 3-local subgroup. Hence, if $G \in \mathcal{R}_p$ then L must also be in $\mathcal{R}_p$. But then 8.2 implies that $r^a \equiv 2, 5 \pmod 9$ and that $P \le C_G(Z(S))$. Furthermore, we note that $B \cap L_2 \sim (q+1)^4 : W(\mathrm{F}_4)$ where $W(\mathrm{F}_4)$ denotes the Weyl group of type F_4. (As a subsystem subgroup the $E(L_1)$ is generated by the root spanned by the D_4 subsystem {} details.) If $r^a = 2$, we consult the Atlas [2] and see that there is a unique maximal 3-local subgroup and so $B = N_G(Z(S))$. But then we have $P \le B$, a contradiction.

Suppose then that $r^a > 2$ and let $L_2 = 3.(\mathrm{PGU}_3(r^a) \times \mathrm{PGU}_3(r^a)).3.2$ be as in [7, Table 5.2]. Then L_2 can be chosen to contain $S \cap X$. Now $P \le L_2$ and so P is contained in exactly one of the components of L_2 and the other component must be contained in B. But then $B = G$ if r^a is odd and if $r = 2$, then B contains a subgroup isomorphic to L_1 (but this time generated by short root groups). In either case we have a contradiction as $Q_B = 1$. □

Get details for the above argument.

Lemma 9.7. [**G2 and 3D4**] *Suppose that $X = O^p(G)$ is isomorphic to either $\mathrm{G}_2(r^a)$ or ${}^3\mathrm{D}_4(r^a)$, then $X \cong \mathrm{G}_2(3)$ or ${}^3\mathrm{D}_4(3)$ and the possibilities for G, B, and P are listed in Table 1.*

Proof. By Lemma 5.2, we need only consider the cases with $p = 2$ or $p = 3$. Suppose that $X \cong \mathrm{G}_2(r^a)$. If $r^a = 3$, we refer the reader to the Atlas [2] to verify the details needed to confirm that $\mathrm{G}_2(3)$ is an example. Let Δ be the root system associated with X and let M be the monomial group $\Phi_n(s) : \mathrm{Dih}(12)$ where n is 1 or 2 as appropriate for M to contain a Sylow 2-subgroup, respectively 3, of X. Let $\{\alpha, \beta\}$ be a fundamental system for Δ with α a long root. Let $T = \langle h_\alpha, h_\beta \rangle$ be the Torus of order $\Phi_1(s)^2$. Then $N = \langle X_\alpha, X_{-\alpha}, X_{\alpha+2\beta}, X_{-\alpha-2\beta} \rangle$ Then $Z(N) = \langle h_\alpha(-1) \rangle (= \langle h_{\alpha+2\beta}(-1) \rangle)$. In particular, we notice that $[T : T \cap N] = 2$ and so $|TN/N| = 2$. Since $|X|_{s'} = \Phi_1(s)^2\Phi_2(s)^2\Phi_3(s)\Phi_6(s)$, and $|NT|_{s'} = \frac{1}{2}(\Phi_1(s)\Phi_2(s))^2.2$, we infer that NTS contains a Sylow 2-subgroup of G. Since $s \ge 3$, NTS contains two normal components F_1 and F_2 say, and so at least one of these component must be contained in B. So suppose that $F_1 \le B$. Then $O_2(F_1S)$ is isomorphic to a Sylow 2-subgroup of $\mathrm{SL}_2(s)$ and consequently $O_2(F_1S)$ contains a unique involution, namely $t = h_\alpha(-1)$. Since $Q_B \le O_2(F_1S)$, we see that $B \le C_G(t)$. But then $Q_B = Z(S)$ is cyclic of order 2 and we have a contradiction via Lemma 3.9.

If $X \cong {}^3\mathrm{D}_4(s)$ and $s \neq 3$, then an argument just as above works. CHECK THIS. Need to show that $|Z(S)| = |Q_B| = 2$. Now suppose that $p = 3$. Then the following groups are overgroups of the Sylow 2-subgroup of X are $\mathrm{G}_2(3)$, $(7 \times \mathrm{PSU}_(3)).2$ and $(\mathrm{SL}_2(3) * \mathrm{SL}_2(27)).2$ and taking $B = (\mathrm{SL}_2(3) * \mathrm{SL}_2(27)).2$ and $P = 4^2.\mathrm{Dih}(12)$ we satisfy the conditions required for X to be $\widetilde{P}$-restricted. So this group appears on Table 1. □

Lemma 9.8. [**E6**]*Assume that $X \cong \mathrm{E}_6(r^a)$ or ${}^2\mathrm{E}_6(p^a)$ and $G \in \mathcal{R}_p$, then $p = 3$ and $X \cong {}^2\mathrm{E}_6(2)$ and $B \cap X \cap 3.(\mathrm{PSU}_3(2) \times \mathrm{PSU}_3(2) \times \mathrm{PSU}_3(2)).3.\mathrm{Sym}(3)$.*

look at p=5

Proof. Suppose first that $d = 2$ when $X \cong \mathrm{E}_6(r^a)$ and that $d = 1$ when $X \cong {}^2\mathrm{E}_6(r^a)$ and as is standard write $X \cong \mathrm{E}_6^\epsilon(r^a)$ with $\epsilon = +$ when X is not twisted and otherwise $\epsilon = -$. Because of the choice of d we see that $p \le 3$. Suppose that $p = 3$, then $S \cap X$ is contained in the subgroup

$L \cong \mathrm{F}_4(q)$. Using 9.6 we have that $B \geq L$ and then $Q_B = 1$, a contradiction. So suppose that $p = 2$ and $\epsilon = -$. Then take $L = (4, r^a - \epsilon).(\mathrm{P}\Omega^{\epsilon}_{10}(r^a) \times (r^a - \epsilon)/(4, r^a - \epsilon)).(4, r^a - \epsilon)$ (use root groups $X_{\alpha_3}, X_{\alpha_4}, X_{\alpha_5}, X_{\alpha_2}X_{-\alpha_0}$ and their negatives so that L is normalized by the graph automorphism. It follows that Q_B is cyclic of order 2 and we have a contradiction. If $p = 2$ and $d = 1$ we take L as above and see that Q_B is cyclic and obtain a contradiction via?? So suppose that $p = 3$ and $d = 1$ when $\epsilon = 1$ and $d = 2$ when $\epsilon = -$. In this case we take $L = 3.(\mathrm{PSL}^{\epsilon}_3(r^a) \times \mathrm{PSL}^{\epsilon}_3(r^a) \times \mathrm{PSL}^{\epsilon}_3(r^a)).3^2.\,\mathrm{Sym}(3)$ and when $(r^a, \epsilon) \neq (2, -)$, the components of L are permuted transitively by L. Thus in this case $B \geq L$ by 2.9 and then $|Q_B| = 3$. Thus $(r^a, \epsilon) = (2, -)$ and the lemma follows. □

Mention the Fischer groups?

Lemma 9.9. [**E7**] *Assume that* $X \cong \mathrm{E}_7(r^a)$ *and* $G \in \mathcal{R}_p$*, then* $p = 2$*,* $X \cong \mathrm{E}_7(3)$ *and* $B = 2^3.(PSL_2(3)^7).2^3.\mathrm{SL}_3(2)$.

Proof. We have to consider $p \in \{2, 3, 5, 7\}$. According as $d = 1, 2$ set $\epsilon = +, -$ and note that $L_1 = (q-\epsilon)^7 : (2 \times \mathrm{Sp}_6(2)$ contains $S \cap X$. Set $L_2 = (3, r^a-\epsilon).(\mathrm{E}^{\epsilon}_6(r^a) \times (r^a-\epsilon)/(3, r^a-\epsilon)).(3, r^a-\epsilon).2$. Then so long as $p \neq 2, 7$, $S \cap X \leq L_2$ and we easily derive contradictions in these cases??

For $p = 7$ we set $L_3 = f.\mathrm{PSL}_8(r^a).g.(2 \times (2/f))$ where f and g are powers of 2. Then **??** shows that $L_2 \leq B$, a contradiction. So suppose that $p = 2$. Then set $L_3 = 2(\mathrm{PSL}_2(r^a) \times \mathrm{P}\Omega^{+}_{12}(r^a)).2$. Set $L_4 = 2^3.(\mathrm{PSL}_2(r^a)^7).d^3.\mathrm{PSL}_3(2)$. Assume that $r^a > 3$. Then 6.2 and 8.2 imply that $L_3 = B$ and that Q_B has order 2 contrary to 3.9. Thus $p = 3$. Suppose that $B = L_3$. Then $P \leq L_4$ and $Q_B \leq Q_2(L_4) \leq Q_P$ for a contradiction. Therefore, $B = L_4$ as claimed. Need to argue $P \not\leq L_4 \cap L_3$ (use **??**).

Lemma 9.10. [**E8**] *Assume that* $X \cong \mathrm{E}_8(r^a)$ *and* $G \in \mathcal{R}_p$*, then* $p = 3$*,* $X \cong \mathrm{E}_8(2)$ *and* $B \cap X = 3^2.(\mathrm{PSU}_3(2) \times \mathrm{PSU}_3(2) \times \mathrm{PSU}_3(2) \times \mathrm{PSU}_3(2)).3^2.\,\mathrm{GL}_2(3)$.

Proof. We have to consider the possibilities $p = 2$, 3, 5 and 7. Recall also that $d \in \{1, 2\}$. For $p = 2$ or 7 the group $L = (2, r^a - 1).\mathrm{P}\Omega^{+}_{16}(r^a).2$ contains a Sylow p subgroup of X. Applying 8.2 and 3.9 we obtain a contradiction. For $p = 5$, we consider the subgroup $5.(\mathrm{PSL}^{\epsilon}_4(r^a) \times \mathrm{PSL}^{\epsilon}_4(r^a)).5.4$ where ϵ is chosen so that 5 divides $r^a - \epsilon$. Then 3.9 and 2.9 delivers a contradiction. So assume that $p = 3$. Set $L_1 = 3.(\mathrm{PSL}^{\epsilon}_3(r^a) \times \mathrm{E}^{\epsilon}_6(r^a)).3.2$ $L_2 = 3^2.(\mathrm{PSL}^{\epsilon}(r^a) \times \mathrm{PSL}^{\epsilon}(r^a) \times \mathrm{PSL}^{\epsilon}(r^a) \times \mathrm{PSL}^{\epsilon}(r^a)).3^2.\,\mathrm{GL}_2(3)$ (with $\epsilon = +$ if $d = 1$ and $\epsilon = -$ if $d = 1$). Then 2.9 implies that $P \leq L_1$ and $B = L_2$. Using 9.8 we obtain $r^a = 2$ and this completes the lemma. □

10. Sporadic groups

Before we begin the case by case investigation of the sporadic simple groups we note that, by Lemma **??**, we may assume that S is not cyclic.

Lemma 10.1. [**m11**] $X \neq \mathrm{Mat}_{11}$.

Proof. Suppose that $G \in \mathcal{R}_p$. Then $p \in \{2, 3\}$. Suppose that $p = 2$. Let $H \leq G$ with $H \sim \mathrm{Mat}_{10}$. Then H is a 2-minimal subgroup of G and $Q_H = 1$. Hence by 2.16 $H = P$, a contradiction. Next we consider $p = 3$. This time we note that $N_G(S)$ is a maximal subgroup of G and apply 2.18 to get a contradiction. □

Lemma 10.2. [**m12**] *Suppose* $X = \mathrm{Mat}_{12}$*. Then* $p = 2$ *or* 3 *and* B *is either of the two maximal* p*-local parabolic subgroups of* G.

Proof. For $p > 3$, S is cyclic and for $p \leq 3$, $\mathcal{M}(S) = \mathcal{P}(S)$ has size two. □

Lemma 10.3. [**j1**] *$X \neq \mathrm{J}_1$.*

Proof. For $p > 2$, S is cyclic and for $p = 2$, $N_G(S)$ acts irreducibly on S. So the lemma follows from 2.18. □

Lemma 10.4. [**m22**] *Suppose $X = \mathrm{Mat}_{22}$. Then $p = 2$ and B is is either of the two maximal 2-local parabolic subgroups of G.*

Proof. For $p \geq 5$, the Sylow p-subgroups are cyclic. For $p = 3$, $N_G(S)$ acts irreducibly on S. Thus by 2.18, $p = 2$. Now B can be either of the maximal 2-local parabolics of G and (by definition) P is the unique minimal parabolic not in B. □

Lemma 10.5. [**j2**] *Suppose that $X = \mathrm{J}_2$. The one of the following holds:*

(a) [**1**] *$p = 3$, $B \sim 3.\mathrm{PGL}_2(9)$ and $P \sim \mathrm{PSU}_3(3)$.*

(b) [**2**] *$p = 2$ and B is either of the two maximal 2-local parabolic subgroups of G.*

Proof. For $p = 7$, S is cyclic and for $p = 5$, $N_G(S)$ acts irreducibly on S. So $p \leq 3$. For $p = 3$, G has a unique maximal 3-local parabolic subgroup M and $M \sim 3.\mathrm{PGL}_2(9)$. Thus $B = M$ and $B \sim 3.\mathrm{PGL}_2(9)$. Note that G has a parabolic subgroup $H \cong \mathrm{PSU}_3(3)$. Then by 2.16 $P = H$ and (a) holds.

For $p = 2$, $\mathcal{P}(S) = \mathcal{M}(S)$ has size 2 and (b) holds. □

Lemma 10.6. [**m23**] *Suppose $X = \mathrm{Mat}_{23}$. Then $p = 2$, $B \cong 2^4.\mathrm{Alt}(7)$ and $P \cong 2^4.\mathrm{Sym}(5)$.*

Proof. For $p \geq 5$, the Sylow p-subgroups of G are cyclic. For $p = 3$, $N_G(S)$ acts irreducibly on S. Thus by 2.18, $p = 2$. Since $L = 2^4.\mathrm{Alt}(7)$ is a subgroup of G and $L/Q_L \in \mathcal{R}_2$ by 4.2 $B = L$. So P is the unique minimal parabolic subgroup of G not in B and the lemma holds. □

Lemma 10.7. [**hs**] *Suppose $X = \mathrm{HS}$. Then $p = 2$ and B is any of the two maximal local parabolic subgroups of G.*

Proof. For $p \geq 7$, the Sylow p-subgroups of G are cyclic. For $p = 3$, $N_G(S)$ acts irreducibly on S.

Suppose $p = 5$. If $G = X$, then no 5-local subgroup is maximal in G, a contradiction. If $G \neq X$, then $N_G(S)$ is maximal in G, again a contradiction.

Thus $p = 2$. Now B is either one of the maximal local parabolic subgroups and the P is the unique minimal parabolic not contained in B. □

Lemma 10.8. [**j3**] *Suppose $X \cong \mathrm{J}_3$. The $p = 2$ and B is any of the two maximal local parabolic subgroups of G.*

For $p > 3$, the Sylow p-subgroups are cyclic. For $p = 3$, $N_G(S)$ is maximal. So $p = 2$, $\mathcal{P}(S) = \mathcal{M}(S)$ has size 2 and the lemma holds. □

Lemma 10.9. [**m24**] *Suppose $X \cong \mathrm{Mat}_{24}$. The $p = 2$ and B is any of the three maximal 2-local parabolic subgroups.*

Proof. For $p > 3$, the Sylow p-subgroups are cyclic. No 3-local subgroup is maximal in G and so $p = 2$. Now B is one of the three maximal 2-local subgroups containing S and P the unique 2-minimal parabolic subgroup not contained in B. □

Lemma 10.10. [**mcl**] *Suppose* $X = \mathrm{McL}$*. Then* $p = 3$*,* $B = 3^{1+4}.2.\,\mathrm{Sym}(5)$ *and* $P = 3^4.\mathrm{PSL}_2(9)$*.*

Proof. For $p > 5$, the Sylow p-subgroups are cyclic. For $p = 5$, $N_G(S)$ is maximal in G. For $p = 3$, P has a unique 3-minimal parabolic of rank 1 type. Thus the lemma holds in this case.

Suppose $p = 2$. If $G \neq X$, then G has a 2-minimal parabolic subgroup $\mathrm{PSL}_3(4).2^2$ and we get a contradiction to 2.16. Thus $G = X$. Let $S \leq H \leq G$ with $H \sim 2^4.\,\mathrm{Alt}(7)$. Then $H \notin \mathcal{R}_2$ by 4.2 and so $H = B$. But there are two different choices for B, a contradiction. □

Lemma 10.11. [**he**] *Suppose* $X = \mathrm{He}$*. Then* $G = X$*,*$p = 2$ *and* $B \cong 2^6.3.\,\mathrm{Sym}(6)$

Proof. For $p > 7$, the Sylow p-subgroups are cyclic, for $p = 7$, $N_G(S)$ is maximal in G and for $p = 5$, $N_G(S)$ is irreducible on S. Thus $p \leq 3$. For $p = 3$, there exists a minimal parabolic subgroup $2^6.3^3$ $[\leq 2^6 3.\,\mathrm{Sym}(6)]$, a contradiction to 2.16.

Thus $p = 2$. Suppose $G \neq X$. Let $H \in \mathcal{P}_G(S)$. Then $H/Q_H \cong \mathrm{Sym}(3) \wr \mathrm{Sym}(2)$ or $\mathrm{PSL}_3(2).2$ and so there is no candidate for P.

Thus $G = X$. Since $2^{1+3+3}.\mathrm{PSL}_3(2)$ only contains two of the four members of $\mathcal{P}_G(S)$, $B \cong 2^6.\,\mathrm{Sym}(6)$ and the lemma is proved. □

Lemma 10.12. [**ru**] *Suppose* $X = \mathrm{Ru}$*. Then* $p = 2$ *and* B *is any of the two maximal* 2*-local parabolic subgroups of* G *.*

Proof. For $p > 5$, the Sylow p-subgroups of G are cyclic. For $p = 5$, $N_G(S)$ is maximal in G. For $p = 3$, G contains a 3-minimal parabolic subgroup $2^6.3^2$ $[\leq 2^6.\mathrm{G}_2(2)]$, contradicting 2.16. So $p = 2$ and $\mathcal{M}(S)$ has size two. □

Lemma 10.13. [**suz**] *Suppose that* $X \cong \mathrm{Suz}$*. The one of the following holds.*

1. [**1**] $p = 3$ *and* $B \cong 3^5.\,\mathrm{Mat}_{11}$*.*
2. [**2**] $p = 2$ *and* B *is any of the three maximal* 2*-local parabolic subgroups of* G*.*

Proof. For $p > 5$, S is cyclic and for $p = 5$, $N_G(S)$ acts irreducibly on S. So $p \leq 3$. For $p = 3$, G has a maximal parabolic subgroup $H \sim 3^5.\,\mathrm{Mat}_{11}$. Since $H \notin \mathcal{R}_3$ by 10.1 $H = B$. So (1) holds.

For $p = 2$ $\mathcal{M}(S)$ has size three and (2) holds. □

Lemma 10.14. [**on**] *If* $X = \mathrm{O'N}$*, then* $p = 2$ *then* $B \cap X \sim 4.\mathrm{PSL}_3(4).2$*.*

Proof. For $p > 7$ and $p = 5$, S is cyclic and for$p = 3$, $N_G(S)$ acts irreducibly on S so these cases do not arise. For $p = 7$, G contains a parabolic subgroup $H = \mathrm{PSL}_3(7).2$. Since $Q_H = 1$, $H \neq B$ and so $H \in \mathcal{R}_7$. But $N_H(S)$ is maximal in H, a contradiction to 2.18 applied to H in place of G.

Thus $p = 2$. Let $H \leq G$ with $H \cap X \sim 4.\mathrm{PSL}_3(4).2_1$. Then H is 2-minimal and we conclude that $B = H$. □

Lemma 10.15. [**co3**] *If* $X = \mathrm{Co}_3$*, then one of the following holds:*

1. [**1**] $p = 3$ *and* $B \cong 3^5.2.\,\mathrm{Mat}_{11}$*.*
2. [**2**] $p = 2$ *and* B *is any of the three maximal local parabolic subgroups of* G*.*

Proof. For $p > 5$, S is cyclic, so $p \leq 5$. For $p = 5$, G contains a parabolic subgroup $\mathrm{McL} : 2$, a contradiction to 10.10.

For $p = 3$, G contains a maximal parabolic subgroup $H \sim 3^5.2.\,\mathrm{Mat}_{11}$. Since $H \notin \mathcal{R}_3$, we conclude that $H = B$. So (1) holds.

For $p = 2$, $\mathcal{M}(S)$ has size three and (2) holds. □

Lemma 10.16. [**co2**] *If $X \cong \mathrm{Co}_2$, then one of the following holds:*

1. [**1**] $p = 3$ *and* $B \sim 3_+^{1+4}.2_-^{1+4}.\mathrm{Sym}(5)$.
2. [**2**] $p = 2$ *and B is any of the three maximal 2-local parabolic subgroups of G.*

Proof. If $p > 5$, S is cyclic and for $p = 5$, $N_G(S)$ is a maximal subgroup in G and so $p \leq 3$.

Suppose $p = 3$ and let $S \leq H \leq G$ be a local subgroups $H \sim 3_+^{1+4}.2_-^{1+4}.\mathrm{Sym}(5)$. Then $H \notin \mathcal{R}_3$ and so $H = B$ and (1) holds.

For $p = 2$, $\mathcal{M}(S)$ has size two and (2) holds. □

Lemma 10.17. [**fi22**] *Suppose $X = \mathrm{Fi}_{22}$. Then one of the following holds:*

1. [**1**] $p = 3$ *and* $B \sim 3_+^{1+6}.2^{1+2+2+2}.3^{1+1}.2$.
2. [**2**] $p = 2$ *and B is an arbitrary maximal local parabolic subgroup.*

Proof. For $p > 5$, S is cyclic and for $p = 5$, $N_G(S)$ acts irreducibly on S.

Suppose $p = 3$. Then G has a maximal parabolic subgroup H with $H \sim 3_+^{1+6}.2^{1+2+2+2}.3^{1+1}.2$. If $H \neq B$ we get that $O^p(P)$ is normal in H, a contradiction (as can been seen by intersection H with a subgroup $\Omega_7(3)$. So $H = B$ and (1) holds.

If $p = 2$ then (2) holds. □

Lemma 10.18. [**HN**] *Suppose $X = \mathrm{HN}$. Then one of the following holds:*

1. [**1**] $p = 5$ *and* $B‘ \sim 5_+^{1+4}.2_-^{1+4}.5.4$.
2. [**2**] $p = 2$ *and* $B \cap X \sim 2_+^{1+8}.\mathrm{Alt}(5) \wr \mathrm{Sym}(2)$.

Proof. If $p > 5$, then S is cyclic so $p \leq 5$.

If $p = 5$ let H be the maximal 5-local parabolic subgroup with $H \sim 5^{1+4}2^{1+4}.5.4$. Then, as H is soluble and $p \neq 2, 3$, $P \not\leq H$ and consequently $H = B$. Thus (1) holds.

If $p = 3$, then X contains a subgroup contains a subgroup $L \sim 3^{1+4}.4.Alt(5)$. Since $L \notin \mathcal{R}_3$, we have that $B = X$. Let H be the parabolic with $H/Q_H \cong 2.(\mathrm{PSL}_2(3) \times \mathrm{PSL}_2(3)).4$ Since $N_H(S)$ is a maximal subgroup of H, we get that $H \leq B$ by 2.18. But then $G = B$, a contradiction.

Suppose $p = 2$. Then G has maximal parabolic subgroup H with $H \cong 2_+^{1+8}.\mathrm{Alt}(5) \wr \mathrm{Sym}(2)$. using 2.9 we have that $H = B$ and (2) holds. □

Lemma 10.19. [**ly**] *Suppose $X = \mathrm{Ly}$. Then $p = 5$ and $H \sim 5_+^{1+4}.4.\mathrm{Sym}(6)$.*

Proof. For $p \geq 7$, S is cyclic. So $p \leq 5$.

Suppose that $p = 5$. Let $S \leq H \leq G$ with $H \sim 5_+^{1+4}.4.\mathrm{Sym}(6)$. Since $\mathrm{Sym}(6) \notin \mathcal{R}_5$, we get $B = H$ and the lemma holds in this case.

Suppose $p = 3$, then G has two maximal parabolic subgroup F, H with $F/Q_F \sim 2 \times \mathrm{Mat}_{11}$ and $H/Q_H \sim 2.\mathrm{Alt}(5).\mathrm{Dih}_8$. So by 4.2 and 10.1 neither F nor H are in $\mathcal{R}_3$, a contradiction.

Suppose that $p = 2$. Then G has a parabolic subgroups $H \sim 3.\mathrm{McL}.2$. Since $Q_H \neq 1$, $H \not\leq B$, but this contradicts 10.10 □

Lemma 10.20. [**th**] *Suppose $X = \mathrm{Th}$. Then one of the following holds:*

1. [**1**] $p = 3$ *and* $\{B, P\} = \mathcal{P}(S)$.
2. [**2**] $p = 2$ *and* $B \sim 2_+^{1+8}.\mathrm{Alt}(9)$.

Proof. For $p > 7$, S is cyclic, for $p = 7$, $N_G(S)$ acts irreducibly on S and for $p = 5$, $N_G(S)$ is maximal in G. Hence $p \leq 3$.

For $p = 3$, (1) holds.

For $p = 2$, let $S \leq H \leq G$ with $H \sim 2_+^{1+8}.Alt(9)$. Then $H \notin \mathcal{R}_2$ so $B = H$ and (2) holds. □

Lemma 10.21. [**fi23**] *Suppose that $X \cong \mathrm{Fi}_{23}$. Then one of the following holds:*

1. [**1**] $p = 3$ *and* $B \sim 3_+^{1+8}2_-^{1+6}.3_+^{1+2}.2.\,\mathrm{Sym}(4)$.
2. [**2**] $p = 2$ *and* $B \sim 2^{11}.\,\mathrm{Mat}_{23}$ *or* $B \sim 2^{6+2\cdot 4}.(\mathrm{Alt}(7) \times \mathrm{Sym}(3))$.

Proof. For $p \geq 7$, S is cyclic and for $p = 5$, $N_G(S)$ acts irreducibly on S. So $p \leq 3$.

Suppose that $p = 3$ and let $S \leq H \leq G$ with $H = 3_+^{1+8}.2_-^{1+6}.3_+^{1+2}.2.Sym(4)$. Since H is soluble and $O^p(P)$ is not normal in H, we get that $H = B$ and (1) holds.

Suppose $p = 2$ and let $S \leq H \leq G$ with $H \sim 2^{11}.\,\mathrm{Mat}_{23}$. If $H = B$ (2) holds. So suppose that $H \not\leq B$, then by 10.6, $(H \cap B)/O_2(H \cap B) \cong \mathrm{Alt}(7)$. Since B is a maximal 2-local subgroup of G, we conclude $B \sim 2^{6+2\cdot 4}.(\mathrm{Alt}(7) \times \mathrm{Sym}(3))$ and again (2) holds. □

Lemma 10.22. [**co1**] *Suppose $X = \mathrm{Co}_1$. Then one of the following holds:*

1. [**1**] $p = 5$ *and B is one of the two maximal 5-local parabolic subgroups of G.*
2. [**2**] $p = 3$ *and B is one of the three maximal 3-local parabolic subgroups of G. (Note here that the two maximal subgroups $N(3C^2)$ in the Atlas* [2] *need to be deleted, see Modular Atlas*[5]*.)*
3. [**3**] $p = 2$ *and B is one of the four maximal 2-local parabolic subgroups of G.*

Proof. For $p > 7$, S is cyclic and for $p = 7$, $N_G(S)$ is irreducible on S. So $p \leq 5$ and one of (1), (2) and (3) holds. □

Lemma 10.23. [**j4**] *Suppose $X = \mathrm{J}_4$. The $p = 2$ and B is one of the three maximal 2-local parabolic subgroups of G.*

Proof. If $p \leq 5$ but $p \neq 11$, then S is cyclic and we are done. If $p = 11$ then $N_G(S)$ is maximal so this case falls. If $p = 3$, then S is contained in $H \sim 2^{11}.\,\mathrm{Mat}_{24}$ which is not in $\mathcal{R}_3$ by 10.9. Since $Q_H = 1$ we have a contradiction. So $p = 2$ and the lemma is proved. □

Lemma 10.24. [**fi24**] *Suppose $X = \mathrm{Fi}_{24}$. The one of the following holds:*

1. [**1**] $p = 3$ *and* $B \sim 3^{1+10}.\mathrm{PSU}_5(2).2$.
2. [**2**] $p = 2$ *and B is any of the four maximal 2-local parabolic subgroups of G.*

Proof. For $p > 7$, S is cyclic and so $p \leq 7$. For $p = 7$, S is contained in $He.2$ and we obtain a contradiction via 10.11. For $p = 5$, $N_G(S)$ acts irreducibly on S, so this case fails.

Suppose $p = 3$. If $B \sim 3^7.\Omega_7(3)$, then $P/Q_P \cong \mathrm{Alt}(5)$, a contradiction as $\mathrm{Alt}(5) \notin \mathcal{R}_3$. Hence (1) holds in this case.

If $p = 2$, then (2) holds. □

Lemma 10.25. [**bm**] *Suppose $X \cong \mathrm{BM}$. Then one of the following holds:*

1. [**1**] $p = 5$ *and* $B \sim 5_+^{1+4}.2_-^{1+4}.\,\mathrm{Alt}(5).4$.
2. [**2**] $p = 3$ *and* $B \cong 3_+^{1+8}.2_-^{1+6}\mathrm{O}_6^-(2)$.
3. [**3**] $p = 2$ *and B is any of the four maximal 2-local parabolic subgroups of G.*

Proof. If $p > 7$, then S is cyclic and, if $p = 7$, $N_G(S)$ is irreducible on S. So $p \leq 5$.

Suppose $p = 5$ and let $S \leq H \leq B$ with $H \sim 5_+^{1+4}.2_-^{1+4}.\mathrm{Alt}(5).4$. Then $H \notin \mathcal{R}_5$ and so $H = B$ and (1) holds.

Suppose that $p = 3$ and let $S \leq H \leq B$ with $H \sim 3_+^{1+8}.2_-^{1+6}.\mathrm{O}_6^-(2)$. Then $H \notin \mathcal{R}_3$ and so $H = B$ and (2) holds.

Suppose that $p = 2$. The (3) holds. □

Lemma 10.26. [**m**] *Suppose that* $X \cong \mathrm{M}$. *Then one of the following holds:*

1. [**1**] $p = 7$ *and* $B \sim 7_+^{1+4}.6.\mathrm{Sym}(7)$.
2. [**2**] $p = 5$ *and* $B \sim 5_+^{1+6}.4.\mathrm{J}_2.2$.
3. [**3**] $p = 3$ *and* $B \sim 3_+^{1+12}.2.\mathrm{Suz}.2$ *or* $B \sim 3^{2+5+5\cdot 2}.(\mathrm{Mat}_{11} \times \mathrm{GL}_2(3))$.
4. [**4**] $p = 2$ *and* B *is any of the five maximal* 2*-local parabolic subgroups.*

Proof. For $p > 13$, S is cyclic, for $p = 13$, $N_G(S)$ is maximal in G and for $p = 11$, $N_G(S)$ is irreducible on S. Thus $p \leq 7$.

For $p = 7$ choose $S \leq H \leq G$ with $H \sim 7_+^{1+4}.6.\mathrm{Sym}(7)$. Then $H \notin \mathcal{R}_7$ and so $B = H$. Hence (1) holds.

For $p = 5$ choose $S \leq H \leq G$ with $H \sim 5_+^{1+6}.4.\mathrm{J}_2.2$. By 10.5 $H \notin calR_5$ and so $B = H$. Hence (2) holds.

For $p = 3$ choose $S \leq H \leq G$ with $H \sim 3_+^{1+12}.2.\mathrm{Suz}.2$. If $H = B$, then (3) holds. If $H \neq B$ then 10.13 implies that $H \cap B/O_3(H \cap B) \cong 2.\mathrm{Mat}_{11}.2$. Thus $B \sim 3^{2+5+5\cdot 2}.(\mathrm{Mat}_{11} \times \mathrm{GL}_2(3))$ and (3) holds.

If $p = 2$, then (4) holds.

11. The smaller list

In this section we assume in addition that if $\widetilde{P} \leq M \leq S$ and M/Q_M is a classical group extended by field automorphism, then the classical groups is $(S)L_n(q)$.

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