F-stability in finite groups

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October 13, 2009

1 Introduction

Let G be a finite group and p a prime. A subgroup P containing a Sylow p-subgroup of G is a *p-parabolic* subgroup of G, and P is a *local p-parabolic* subgroup if in addition $O_p(P) \neq 1$.

Moreover, G has *characteristic p* if $C_G(O_p(G)) \leq O_p(G)$; and G has *parabolic characteristic p* if every local p-parabolic subgroup has characteristic p.

The standard examples for groups of parabolic characteristic p are the finite simple groups of Lie type in characteristic p. In these examples every proper parabolic subgroup is a local p-parabolic subgroup, and for maximal parabolic subgroups M the normal subgroup $\Omega_1 Z(O_p(M))$, considered as a $GF(p)M$-module, has a remarkably restricted structure. In this paper we try to understand this phenomena in arbitrary finite groups.

What kind of properties of the module $\Omega_1 Z(O_p(M))$ should one aim at in general? A possible answer arose during our detailed study of the p-local structure of groups of local characteristic p in [MSS], where a group has local characteristic p if each of its p-local subgroup has characteristic p.

Definition 1.1 *Let A be an elementary abelian p-group and V a finite dimensional $GF(p)A$-module. Then A is*

(a) quadratic on V if $[V, A, A] = 0$,

(b) nearly quadratic on V if $[V, A, A, A] = 0$ and

$$[V, A] + C_V(A) = [v, A] + C_V(A) \text{ for every } v \in V \setminus [V, A] + C_V(A),$$

(c) an offender on V if $|V/C_V(A)| \leq |A/C_A(V)|$,

(d) a $2F$-offender on V if $|V/C_V(A)| \leq |A/C_A(V)|^2$,

(e) non-trivial on V if $[V, A] \neq 0$.

A p-subgroup Y of G is called p-reduced (for G) if Y is elementary abelian and normal in G, and $O_p(G/C_G(Y)) = 1$. The largest p-reduced subgroup of G is denoted by Y_G; for the existence of Y_G see 2.2(a).

Let M be a subgroup of G. Then M is F-stable (in G) if none of the elementary abelian p-subgroups of $N_G(Y_M)/C_G(Y_M)$ are non-trivial offenders on Y_M. Similarly, M is $2F$-stable (in G) if none of the elementary abelian p-subgroups of $N_G(Y_M)/C_G(Y_M)$ are non-trivial nearly quadratic $2F$-offenders on Y_M.

Modules admitting non-trivial $2F$-offenders have been investigated by Guralnick, Lawther and Malle in [GLM],[GM1],[GM2], and [L]. They have classified all pairs (V, G), where V is an irreducible $GF(p)G$-module and G is a known finite almost quasisimple group containing a non-trivial $2F$-offender on V.

Their result is a major generalization of earlier results, where G was assumed to contain a non-trivial offender.

For stating our results we need some further definitions.

Definition 1.2 *By $\mathcal{S}(X)$ we denote the subgroups of G containing X. Let S be Sylow p-subgroup of G.*

$$B(S) := C_S(\Omega_1 ZJ(S)),$$

$$C^*(G, S) := \langle C_G(\Omega_1 Z(S))), N_G(C) \mid 1 \neq C \text{ char } B(S)\rangle,$$

and

$$C^{**}(G, S) = \langle N_G(J(S)), C_G(\Omega_1 Z(S)\rangle.$$

A factorization family *for $\mathcal{S}(S)$ is a subset $\mathcal{F}(\mathcal{S}) \subseteq \mathcal{S}(S)$ with the following two properties:*

(i) For every $H \in \mathcal{S}(S)$ there exists $M \in \mathcal{F}(\mathcal{S})$ with $H \subseteq C_G(Y_H)M$ and $Y_H \leq Y_M$.

(ii) If $H \in \mathcal{S}(S)$ and $M \in \mathcal{F}(S)$ with $M \subseteq C_G(Y_M)H$ and $Y_M \leq Y_H$, then $Y_M = Y_H$ and $H \leq M$.

Property (i) implies

$$H/C_H(Y_H) \cong HC_G(Y_H)/C_G(Y_H) \cong (HC_G(Y_H) \cap M)C_G(Y_H)/C_G(Y_H),$$

so the action of H on Y_H is isomorphic to the action of $HC_G(Y_H) \cap M$ on the submodule Y_H of Y_M. In particular, it suffices to identify $M/C_M(Y_M)$ and its action on Y_M to identify $H/C_H(Y_H)$ and Y_H.

Property (ii) is the crucial one for applications since it has strong consequences. For example, if G is of parabolic characteristic p and $S \leq H \leq M \in \mathcal{F}(S)$ such that $M = HC_M(Y_M)$, then M is the unique maximal p-local subgroup of G containing H (see 3.5).

Of course, it is not clear a priori that factorization families exist. The existence (and uniqueness) will be established in Theorem 3.4.

Theorem 1.3 *Let G be a finite group and $S \in Syl_p(G)$. There exists a unique factorization family $\mathcal{F}(\mathcal{S})$ for $\mathcal{S}(S)$ in G. Moreover, at most one member of $\mathcal{F}(S)$ is F-stable, and*

$$\Omega_1 Z(S) \leq Y_M \text{ and } M = N_G(Y_M) \text{ for every } M \in \mathcal{F}(S);$$

in particular, the elements of $\mathcal{F}(S)$ are p-local subgroups of G if $S \neq 1$.

In the following results $\mathcal{F}(S)$ is always a factorization family for $\mathcal{S}(S)$. Recall that a finite group H is p-constrained if $H/O_{p'}(H)$ is of characteristic p.

Theorem 1.4 *Let G be a finite group and $S \in Syl_p(G)$, and let $1 \neq C \,\text{char}\, B(S)$ and $M := N_G(C)$. Suppose that there exists $N \in \mathcal{F}(S)$ that is F-stable.*

(a) If $C = B(S)$, then $Y_N = Y_M$ and $N = C_G(Y_M)M = N_G(Y_M)$.

(b) If $Y_N \leq O_p(M)$, then $Y_M = Y_N$ and $M \leq N$.

(c) If M is p-constrained, then $M = O_{p'}(M)(M \cap N)$.

Theorem 1.5 *Let G be a finite group and $S \in Syl_p(G)$, and let $M \in \mathcal{S}(S)$ such that $\Omega_1 Z(S) \trianglelefteq M$ or $M = N_G(C)$ for some $1 \neq C \operatorname{char} B(S)$. Suppose that there exists $N \in \mathcal{F}(S)$ that is $2F$-stable.*

(a) If $Y_N \leq O_p(M)$, then $M \leq N$.

(b) If M is p-constrained, then $M = O_{p'}(M)(M \cap N)$.

(c) The following hold for any p-constrained $H \in \mathcal{S}(\mathcal{B}(S))$ with $H \not\leq O_{p'}(H)N$ (where $\overline{H} = H/O_{p'}(H)$):

- *(a) $\overline{Y_N} \leq O_p(\overline{H})$.*
- *(b) $C_{O_p(\overline{H})}(\overline{Y_N}) \trianglelefteq \overline{H}$.*
- *(c) $Y_{\overline{H}}$ is not F-stable in $\overline{H}$.*
- *(d) $C^{**}(\overline{H}, \overline{T}) \leq \overline{H \cap N} < \overline{H}$, where $B(S) \leq T \in Syl_p(H)$.*

For groups of parabolic characteristic p more can be said about the members of the factorization family $\mathcal{F}(S)$.

Theorem 1.6 *Let G be a finite group of parabolic characteristic p and $1 \neq S \in Syl_p(G)$. Then the members of $\mathcal{F}(S)$ are maximal p-local subgroups of G. Moreover, if $N \in \mathcal{F}(S)$ is $2F$-stable and $H \in \mathcal{S}(B(S))$ with $B(S) \leq T \in Syl_p(H)$, then $C^*(H, T) \leq N$.*

Corollary 1.7 *Let G be a finite group of parabolic characteristic p and $S \in Syl_p(G)$. If S is contained in at least two maximal p-local subgroups of G, then there exists $M \in \mathcal{F}(S)$ such that M is not $2F$-stable.*

Let G and N be as in 1.6, and let H be a p-local subgroup containing S such that $H \not\leq N$. Then by 1.6 $C^*(H, S)$ is a proper subgroup of H. In this case the structure of H can be described precisely using the Local $C(G, T)$-Theorem proved in [BHS].

The proof of the above theorems relies heavily on two elementary results from [PPS] and [Ste], the L-Lemma and the qrc-Lemma. The authors found it remarkable that these results allow to study finite groups in this context without any $\mathcal{K}$-group assumption.

In fact, using the L-Lemma another result is proved, which is interesting in its own right and which can be used to improve the qrc-Lemma.

Theorem 1.8 *Let G be a finite group, $S \in Syl_p(G)$, and V be a finite dimensional faithful $GF(p)G$-module. Suppose that $O_p(G) = 1$ and S is contained in a unique maximal subgroup of G. Then $|A| = |V/C_V(A)|$ for every offender A of G on V.*

2 Elementary Properties

In this section G is a finite group, p is a prime, and $S \in Syl_p(G)$.

Notation 2.1 *Let X be a p-subgroup of G. A subgroup P of G is X-minimal if X is contained in a unique maximal subgroup of P and $X \not\leq O_p(P)$.*

Lemma 2.2 *Let L be a subgroup of G and P be a p-parabolic subgroup of L.*

(a) There exists a unique largest p-reduced subgroup Y_L of L.

(b) If Y is a p-reduced subgroup of P with $Y \leq O_p(L)$, then $\langle Y^L \rangle$ is p-reduced for L and so $Y \leq Y_L$.

(c) If L is of characteristic p, then $Y_P \leq Y_L$.

Proof: (a): Let A and B be p-reduced subgroups of L. It suffices to show that also AB is p-reduced. Then Y_L is the product of all p-reduced subgroups of L.

Since A is p-reduced, $B \leq O_p(L) \leq C_L(A)$ and so AB is elementary abelian. Let D be the inverse image of $O_p(L/C_L(AB))$. Since $C_L(AB) \leq C_L(A)$, $DC_L(A)/C_L(A) \leq O_p(L/C_L(A))$ and so $D \leq C_L(A)$. By symmetry, $D \leq C_L(B)$ and thus $D \leq C_L(A) \cap C_L(B) = C_L(AB)$.

(b): Since P is a p-parabolic subgroup of L, $O_p(L) \leq P$. Hence $[Y, O_p(L)] = 1$ since Y is p-reduced in P. By assumption $Y \leq O_p(L)$ and so $Y \leq \Omega_1 Z(O_p(L))$. In particular, $V := \langle Y_P^L \rangle$ is an elementary abelian normal subgroup of L.

Since P contains a Sylow p-subgroup of L, there exists $S_0 \leq P$ such that $S_0C_L(V)/C_L(V) = O_p(L/C_L(V))$ and $S_0 \in Syl_p(S_0C_L(V))$. As $S_0C_P(V) \trianglelefteq P$ and $C_P(V) \leq C_P(Y_P)$, we get that $S_0C_P(Y_P) \trianglelefteq P$. Hence $S_0C_P(V) \leq C_P(Y_P)$ since Y_P is p-reduced in P, and so $[V, S_0C_P(V)] = 1$ since $S_0C_P(V) \trianglelefteq P$. Thus V is p-reduced for L, and by (a) $V \leq Y_L$.

(c): As in (b), $[Y_P, O_p(L)] = 1$. Since L is characteristic p, $Y_P \leq O_p(L)$. So (b) implies $Y_P \leq Y_L$.
□

Lemma 2.3 *Let $X \leq S \leq P \leq G$. Suppose that P is X-minimal and $N \trianglelefteq P$. Then either $O^p(P) \leq N$ and $P = XN$, or $S \cap N \leq O_p(P)$. In particular, $P = XO^p(P) = \langle X^P \rangle$.*

Proof: Observe that $P = NN_P(S \cap N)$. As P is X-minimal, either $NX = P$ or $N_P(S \cap N) = P$, and in the second case $S \cap N \leq O_p(P)$.

Since $X \not\leq O_p(P)$, $S \cap XO^p(P) \not\leq O_p(P)$ and so $P = XO^p(P)$. A similar argument gives $P = \langle X^P \rangle$. □

Lemma 2.4 *Let A be an F-stable elementary abelian p-subgroup of G, and let Q be a p-subgroup of G with $A \trianglelefteq Q$. Then the following hold:*

(a) $A \leq Z(J(Q))$.

(b) $\langle A^{N_G(Q)} \rangle$ is elementary abelian.

Proof: (a): Let $B \in \mathcal{A}(Q)$. Then B acts on A, and $|B| \geq |C_B(A)A|$ by the maximality of B. Also $C_B(A) \cap A \leq A \cap B \leq C_B(A)$ and so $C_B(A) \cap A = A \cap B$. Hence

$$|C_B(A)||A||C_A(B)|^{-1} \leq |C_B(A)||A||A \cap B|^{-1} = |C_B(A)A| \leq |B|,$$

and $|A/C_A(B)| \leq |B/C_B(A)|$ follows. The F-stability of A gives $[A, B] = 1$ and (a) holds.

(b): This is a direct consequence of (a) since $Z(J(Q)) \trianglelefteq N_G(Q)$. □

Lemma 2.5 *Let Q be a normal p-subgroup of G with $C_G(Q) \leq Q$ and Y be an abelian p-subgroup of G. If $C_Q(Y) \trianglelefteq G$ and Q normalizes Y, then $Y \leq O_p(G)$.*

Proof: Observe that

$$[Q, Y] \leq Q \cap Y \leq C_Q(Y).$$

Since $C_Q(Y) \trianglelefteq G$ this shows that $\langle Y^G \rangle$ centralizes $Q/C_Q(Y)$ and $C_Q(Y)$. Hence $O^p(\langle Y^G \rangle)$ centralizes Q and since $C_G(Q) \leq Q$, $O^p(\langle Y^G \rangle) = 1$ and $\langle Y^G \rangle$ is a p-group. Thus $Y \leq O_p(G)$. □

Lemma 2.6 *Let A be a finite elementary abelian p-group and V a finite dimensional $GF(p)A$-module. Suppose that A is quadratic on V and $[v, A] = [V, A]$ for every $v \in V \setminus C_V(A)$. Then A is a quadratic offender on every A-submodule of V.*

Proof: Since every A-submodule of V satisfies the same hypothesis it suffices to show that A is an offender on V. Without loss, $[V, A] \neq 1$. Choose $W \leq [V, A]$ with $|[V, A]/W| = p$ and put $\overline{V} = V/W$. Let U be the inverse image of $C_{\overline{V}}(A)$ in V. Then $[U, A] \leq W$ and so $[V, A] \not\leq [U, A]$. Thus $U \leq C_V(A)$ and $C_{\overline{V}}(A) = \overline{C_V(A)}$; in particular, $|V/C_V(A)| = |\overline{V}/C_{\overline{V}}(A)|$. Note that $\overline{V}$ satisfies the hypothesis, so replacing V by $\overline{V}$ we may assume that $|[V, A]| = p$. Let $B < A$ with $|A/B| = p$. Since $[V, B]$ is at most 1-dimensional, B in place of A also satisfies the hypothesis of the lemma. Hence by induction on $|A|$, $|V/C_V(B)| \leq |B|$.

Let $a \in A \setminus B$. Since $|[V, a]| = p$, $|V/C_V(a)| \leq p$ and so also $|C_V(B)/C_V(B) \cap C_V(a)| \leq p$. But $C_V(A) = C_V(B) \cap C_V(a)$ and so

$$|V/C_V(A)| \leq |V/C_V(B)|p \leq |B|p = |A|.$$

□

3 A Partial Ordering

In this section G is a finite group, p is a prime, and $S \in Syl_p(G)$.

Notation 3.1 *Let A and B be subgroups of G. The relation $\ll$ on the subgroups of G is defined by*

$$A \ll B :\Longleftrightarrow A \subseteq C_G(Y_A)B \text{ and } Y_A \leq Y_B.$$

Furthermore, we define

$$A^\dagger := C_G(Y_A)A \text{ and } \mathcal{S}^\dagger := \{L \leq G \mid L = L^\dagger\}.$$

Lemma 3.2 *Let L and M be subgroups of G.*

(a) $Y_L \leq Y_{L^\dagger}$, $L \ll L^\dagger$, and $(L^\dagger)^\dagger = L^\dagger$.

(b) $\mathcal{S}^\dagger = \{L \leq G \mid C_G(Y_L) \leq L\}$.

(c) $\ll$ is reflexive and transitive.

(d) $L \subseteq C_G(Y_L)M$ if and only if $L \leq C_G(Y_L)N_M(Y_L)$.

(e) Suppose that $L \subseteq C_G(Y_L)M$ and $L \cap M$ is a p-parabolic subgroup of L and M. Then Y_L is p-reduced for $N_M(Y_L)$ and $L \ll N_M(Y_L)$.

(f) If $L = L^\dagger$, then $L \ll M$ if and only if $Y_L \le Y_M$ and $L = C_G(Y_L)(L \cap M)$.

(g) Restricted to $\mathcal{S}^\dagger$, $\ll$ is a partial ordering.

Proof: (a): Clearly Y_L is a p-reduced subgroup of $L^\dagger$, so $Y_L \le Y_{L^\dagger}$. Thus $C_G(Y_{L^\dagger}) \le C_G(Y_L) \le L^\dagger$ and $L^\dagger = (L^\dagger)^\dagger$.

(b): This is an immediate consequence of the definition of $L^\dagger$.

(c): Obviously $\ll$ is reflexive. If $A, B, C \le G$ with $A \ll B$ and $B \ll C$, then $Y_A \le Y_B \le Y_C$ and so $Y_A \le Y_C$. Also $C_G(Y_B) \le C_G(Y_A)$ and hence

$$A \subseteq C_G(Y_A)B \subseteq C_G(Y_A)C_G(Y_B)C = C_G(Y_A)C.$$

Thus $A \ll C$ and $\ll$ is transitive.

(d): If $L \subseteq C_G(Y_L)M$ then $L \le N_G(Y_L) \cap C_G(Y_L)M = C_G(Y_L)N_M(Y_L)$. The other direction is obvious.

(e): Since $L \cap M$ is a p-parabolic subgroup of L,

$$Y_L \le O_p(L) \le L \cap M \le N_M(Y_L),$$

so Y_L is an elementary abelian normal subgroup of $N_M(Y_L)$. Since $L \cap M$ is a p-parabolic subgroup of M, $C_G(Y_L)(L \cap M)$ and thus also $C_G(Y_L)L$ are p-parabolic subgroups of $C_G(Y_L)N_M(Y_L)$.

As Y_L is a p-reduced subgroup of $C_G(Y_L)L$, 2.2(b) shows that $Y_L = \langle Y_L^{C_G(Y_L)N_M(Y_L)} \rangle$ is p-reduced for $C_G(Y_L)N_M(Y_L)$. Hence Y_L is also a p-reduced subgroup of $N_M(Y_L)$. Thus $Y_L \le Y_{N_M(Y_L)}$ and so $L \ll N_M(Y_L)$.

(f): Since $L \in \mathcal{S}^\dagger$ we have $C_G(Y_L) \le L$ and so $L \subseteq C_G(Y_L)M$ implies $L = C_G(Y_L)(L \cap M)$. Now (f) is obvious.

(g): Let $L, M \in \mathcal{S}^\dagger$ with $L \ll M$ and $M \ll L$. Since $Y_L \le Y_M \le Y_L$, we have $Y_L = Y_M$. By (f) $L = C_G(Y_L)(L \cap M)$ and $M = C_G(Y_M)(M \cap L)$. Hence $Y_M = Y_L$ gives $L = M$. So the restriction of $\ll$ to $\mathcal{S}^\dagger$ is anti-symmetric. Now (g) follows (c). □

Notation 3.3 *Put $\mathcal{S}^\dagger(S) := \{L \in \mathcal{S}^\dagger \mid S \le L\}$. According to 3.2(g) $\ll$ restricted to $\mathcal{S}^\dagger(S)$ is a partial ordering on $\mathcal{S}^\dagger(S)$. We denote the set of maximal elements of $\mathcal{S}^\dagger(S)$ with respect to $\ll$ by $\mathcal{F}(S)$.*

Theorem 3.4 *$\mathcal{F}(S)$ is the unique factorization family for $\mathcal{S}(S)$.*

Proof: Let $\mathcal{G}$ be a factorization family for $\mathcal{S}(S)$ and let $M \in \mathcal{G}$. Clearly $M \le M^\dagger$ and by 3.2(a), $Y_M \le Y_{M^\dagger}$. So Condition (ii) of 1.2 gives $M = M^\dagger$. Thus $M \in \mathcal{S}^\dagger(S)$ and $\mathcal{G} \subseteq \mathcal{S}^\dagger(S)$.

Now let $\mathcal{G}$ be any subset of $\mathcal{S}^\dagger(S)$. Then Condition (i) of 1.2 is fulfilled for $\mathcal{G}$ if and only if for each $L \in \mathcal{S}(S)$ there exists $M \in \mathcal{G}$ with $L \ll M$. Since $L \ll L^\dagger$ and $\ll$ is transitive by 3.2, we conclude that $\mathcal{G}$ fulfills (i) if and only if $\mathcal{G}$ contains all the maximal elements of $\mathcal{S}^\dagger(S)$ with respect to $\ll$. And Condition (ii) holds if and only if all elements of $\mathcal{G}$ are maximal with respect to $\ll$ in $\mathcal{S}^\dagger(S)$. Thus $\mathcal{F}(S)$ is the unique factorization family for $\mathcal{S}(S)$. □

Lemma 3.5 *Let $M \in \mathcal{F}(S)$ and $H \in \mathcal{S}(S)$ with $M = C_M(Y_M)(M \cap H)$. If H is p-constrained, then $H = O_{p'}(H)(H \cap M)$. In particular, if G is of parabolic characteristic p and $S \le L \le M$ with $M = C_M(Y_M)L$, then M is the unique maximal p-local subgroup of G containing L.*

Proof: Put $\overline{H} = H/O_{p'}(H)$. Since $M = C_M(Y_M)(H \cap M)$, Y_M is p-reduced for $H \cap M$ and $\overline{Y_M}$ is a p-reduced subgroup of $\overline{H \cap M}$. So by 2.2(c) , $\overline{Y_M} \leq Y_{\overline{H}}$. Let $Y \leq S$ with $\overline{Y} = Y_{\overline{H}}$ and $K := N_H(Y)$. Then by the Frattini argument, $H = O_{p'}(H)K$. It follows that Y is a p-reduced subgroup of K, so $Y_M \leq Y \leq Y_K$.

As $YO_{p'}(H) \cap M = Y(O_{p'}(H) \cap M)$, we also get, using the Frattini argument one more time,

$$H \cap M = (O_{p'}(H) \cap M)(K \cap M) = O_{p'}(M \cap H)(K \cap M).$$

Thus $M = C_M(Y_M)(H \cap M) \leq C_G(Y_M)K$ since $O_{p'}(M \cap H)$ centralizes Y_M. Now 1.2(ii) implies that $K \leq M$ and so $H = O_{p'}(H)(H \cap M)$. Hence the first statement holds.

To prove the the second statement, let H be a p-local subgroup containing L. Then $M = C_M(Y_M)L$ implies $M = C_M(Y_M)(H \cap M)$. On the other hand H is of characteristic p since G has parabolic characteristic p, so H is p-constrained and $O_{p'}(H) = 1$. Hence by the first statement $H = O_{p'}(H)(H \cap M) \leq M$. □

Lemma 3.6 *Let $M \in \mathcal{F}(S)$, $S_0 := C_S(Y_M)$ and $M_0 := N_M(S_0)$.*

(a) $M = C_M(Y_M)M_0$, $S_0 = O_p(M_0)$ and $C_S(S_0) \leq S_0$.

(b) $\Omega_1 Z(S) \leq Y_M = Y_{M_0} = \Omega_1 Z(S_0)$.

Proof: (a): The Frattini argument gives $M = C_M(Y_M)M_0$. Hence $O_p(M_0) = S_0$, since Y_M is p-reduced. Clearly $Y_M \leq \Omega_1 Z(S_0)$, and so

$$C_S(S_0) \leq C_S(\Omega_1 Z(S_0)) \leq C_S(Y_M) \leq S_0.$$

(b): Let $S_0 \leq S_1 \leq S$ with

$$S_1 C_{M_0}(\Omega_1 Z(S_0))/C_{M_0}(\Omega_1 Z(S_0)) = O_p(M_0/C_{M_0}(\Omega_1 Z(S_0))).$$

Then $S_1 C_M(Y_M)/C_M(Y_M)$ is a normalized by $M_0 C_M(Y_M) = M$. Since $O_p(M/C_M(Y_M)) = 1$ we get $S_1 \leq C_M(Y_M)$, so $S_1 = S_0 = O_p(M_0)$ by (a), and $\Omega_1 Z(S_0)$ is p-reduced for M_0. Together with 3.2(a) this gives

$$Y_M \leq \Omega_1 Z(S_0) \leq Y_{M_0} \leq Y_{M_0^\dagger}.$$

In particular $M \ll M_0^\dagger$, and the maximality of M yields $Y_M = Y_{M_0^\dagger}$. Now (b) follows, since also $\Omega_1 Z(S) \leq \Omega_1 Z(S_0)$. □

Proof of Theorems 1.3 and 1.4:

By 3.4 $\mathcal{F}(\mathcal{S})$ is the unique factorization family for $\mathcal{S}(S)$. Let $M \in \mathcal{F}(S)$. By 3.6(b) $\Omega_1 Z(S) \leq Y_M$ and by 3.2(e) $M \ll N_G(Y_M)$. Hence the maximality of M gives $M = N_G(Y_M)$.

Assume that there exists $N \in \mathcal{F}(\mathcal{S})$ that is F-stable, i.e. $Y := Y_N$ is F-stable in $N_G(Y_N) = N$. Then by 2.4 $B(S) \leq C_G(Y)$ and

$$N = C_G(Y)N_N(B(S)) \subseteq C_G(Y)L, \quad \text{where } L := N_G(B(S);$$

in particular $Y \leq \Omega_1 Z(B(S)) \leq O_p(N_G(B(S)))$. Now 2.2(b) implies that $Y \leq Y_L$ and so by 1.2(ii) $Y = Y_L$. It follows that $N = N_G(Y_L)$. In particular, N is the unique F-stable member of $\mathcal{F}(S)$. This finishes the proof of 1.3 and also shows 1.4(a).

Now let $1 \neq C \operatorname{char} B(S)$ and put $M := N_G(C)$. Then $N_N(B(S)) \leq M$ and thus also $N = C_G(Y)(M \cap N)$. Suppose that $Y_N \leq O_p(M)$. Then as above 2.2(b) implies that $Y_N \leq Y_M$, and by 1.2(ii) $Y_N = Y_M$ and $M \leq N$. So 1.4(b) holds.

Suppose next that M is p-constrained. From $N = C_G(Y)(N \cap M)$ and 3.5 we get that $M = O_{p'}(M)(M \cap N)$. Hence 1.4(c) holds. □

4 The L-Lemma and the qrc-Lemma

In this chapter we will work with the following hypothesis.

Hypothesis 4.1 *Let P be a finite group of characteristic p, $T \in Syl_p(P)$, $Y \trianglelefteq T$, and $R := C_T(Y)$. Suppose that P is $RO_p(P)$-minimal with M being the unique maximal subgroup of P containing $RO_p(P)$.*

Notation 4.2 *Let X be a finite group and V a finite dimensional $GF(p)X$-module. By $c(V,X)$ we denote the number of non-central chief factors of X in V (in a given chief series). We define $q(V,X) := 0$ if every quadratically acting subgroup of X already centralizes V, and*

$$q(V,X) := \min\{\log_{|A/C_A(V)|} |V/C_V(A)| \big| A \leq X, [V,A,A] = 1 \neq [V,A]\}$$

otherwise. Moreover, $r(V,X) := 0$ if V does not possess non-central X-chief-factors, and

$$r(V,X) := min\{q(C,X) \mid C \text{ non-central } X\text{-chief-factor on } V\}$$

otherwise.

Lemma 4.3 (L-Lemma) *Assume Hypothesis 4.1. Let A be a subgroup of T such that $A \not\leq O_p(P)$. Then there exists a subgroup $L \leq P$ with $AO_p(P) \leq L$ satisfying:*

(a) $AO_p(L)$ is contained in a unique maximal subgroup L_0 of L, and $L_0 = L \cap M^g$ for some $g \in P$.

(b) $L = \langle A, A^x \rangle O_p(L)$ for every $x \in L \setminus L_0$.

(c) L is not contained in any P-conjugate of M.

Proof: See [PPS]. □

The next lemma is very similar to [Ste, 3.3].

Lemma 4.4 *Assume Hypothesis 4.1. Suppose $V := \langle Y^P \rangle$ is elementary abelian, $C_{O_p(P)}(Y) \ntrianglelefteq P$ and $c(V,P) = 1$. Then $[O_p(P), O^p(P)]$ is non-trivial quadratic offender on Y.*

Proof: Since P is $RO_p(P)$-minimal, we get from 2.3 that $P = RO^p(P)O_p(P)$. Put

$$Q := [O_p(P), O^p(P)], \quad W := [V, O^p(P)] \quad \text{and } D := C_V(O^p(P)).$$

Since $c(V,P) = 1$, $W/W \cap D$ is chief-factor for P on V. Hence $[W, O_p(P)] \leq D$. Note that $P = TO^p(P)$ normalizes YW and so $V = YW$. Thus $[Y,Q]D \trianglelefteq P$. Observe that R centralizes

$[Y,Q]D/D$. Since $O^p(P) \leq \langle R^P \rangle$ we conclude that $[Y,Q,O^p(P)] \leq D$. Hence $O^p(P)$ centralizes $[Y,Q]$. So $P = TO^p(P)$ normalizes $[Y,Q]$ and $[V,Q] = [Y,Q] \leq D$. It follows that

$$[V, O^p(P), O_p(P)] = [W, O_p(P)] \leq D \quad \text{and} \quad [O_p(P), O^p(P), V] = [Q, V] \leq D.$$

Hence the Three Subgroup Lemma implies $[V, O_p(P), O^p(P)] \leq D$ and so

$$(*) \qquad [V, O_p(P)] \leq D.$$

Pick $x \in Y \setminus D$. Since R centralizes x we conclude from $(*)$ that $P = RO^p(P)O_p(P)$ normalizes $\langle x^{O^p(P)} \rangle D$ and so $W \leq \langle x^{O^p(P)} \rangle D$. Put $X := [x, Q]$. Since $X \leq D$ it follows that

$$[W, Q] \leq [\langle x^{O^p(P)} \rangle D, Q] = [x, Q] = X.$$

As $[V, Q, O^p(P)] = 1 \leq X$ and $[V, O^p(P), Q] = [W, Q] \leq X$, the Three Subgroup Lemma implies $[Q, O^p(P), V] \leq X$. Since $[Q, O^p(V)] = Q$ we get $[V, Q] = X$. In particular,

$$[y, Q] = X \text{ for every } y \in Y \setminus C_Y(Q).$$

Now 2.6 shows that Q is a quadratic offender on Y.

If Q acts trivially on Y, then $Q \leq C_{O_p(P)}(Y)$ and so $C_{O_p(P)}(Y) \trianglelefteq TO^p(P) = P$, a contradiction. □

Lemma 4.5 *Let L be a finite group acting on a p-group E, and let A and B be p-subgroups of L and X and Z subgroups of E. Suppose that*

(i) $B \not\leq O_p(L)$,

(ii) $[E, A] \leq X \leq C_E(A)$ *and* $[E, B] \leq Z \leq C_E(B)$,

(iii) L *is* $AO_p(L)$*-minimal and* $[E, O^p(L)] \neq 1$,

(iv) X *is normalized by* E *and* $O_p(L)$, X *is abelian, and* $E = \langle X^L \rangle$.

Then

(a) $C_B(E) \leq B \cap O_p(L)$,

(b) $E = X^g Z = X^g C_E(B)$ *for some* $g \in L$,

(c) $Z = [E, B]C_Z(E) = [E, b]C_Z(E)$ *for all* $b \in B \setminus O_p(L)$,

(d) $[B, E, E, E] = 1$,

(e) $|Z/C_Z(E)| \leq |ZD/D| \leq |E/C_E(B)|$,

(f) $|BO_p(L)/O_p(L)| \leq |E/C_E(B)|$.

Proof: By (iii) there exists a unique maximal subgroup L_0 of L containing $AO_p(L)$, and by 2.3 $(\bigcap_{g\in L} L_0^g)/O_p(L)$ is a p'-group.

Pick $b \in B \setminus O_p(L)$. Then there exists $g \in L$ with $b \notin L_0^g$. Put $H := \langle A^g, b \rangle$. Then $L = HO_p(L)$ since $H \not\leq L_0^g$. Furthermore, we put $D := \bigcap_{g\in L} X^g$.

(a): Again by (iii) $O^p(L) \not\leq C_L(E)$, so 2.3 shows that $C_L(E)/C_{O_p(L)}(E)$ is a p'-group. Now (a) follows.

(b): By (iv) $O_p(L)$ normalizes X, so $X^L = X^H$. It follows that $D = \bigcap_{h\in H} X^h$ and $C_{X^g}(b) \leq C_{X^g}(H) \leq D$. From $E = \langle X^L \rangle = \langle X^H \rangle$ and (ii) we conclude that

$$(*) \qquad E = X^g[E,H] = X^g[E,A^g][E,b] = X^g[E,b] = X^g Z.$$

Since $Z \leq C_E(B)$ by (ii), (b) holds.

(c): Since $X^g \cap Z \leq C_{X^g}(H) \leq D$, we get $X^g \cap Z = D \cap Z$ and so by (b) and $(*)$

$$|E/X| = |E/X^g| = |ZX^g/X^g| = |Z/Z \cap X^g| = |Z/Z \cap D|.$$

Moreover, using $(*)$

$$Z = (X^g \cap Z)[E,b] \leq (Z \cap D)[E,b] \leq C_Z(E)[E,b] \leq C_Z(E)[E,B] \leq Z,$$

and so (c) holds.

(d): Since L is $AO_p(L)$-minimal, $A \not\leq O_p(L)$ and so (b) can be applied with A and A^g in place of A and B. Then $E = XX^t$ for some $t \in L$; in particular $[X^t, X] \leq X \cap X^t \leq D \leq Z(E)$. Thus $E' \leq D \leq Z(E)$ and $[B,E,E,E] \leq [E',E] = 1$. So (d) holds.

(e): By (ii) and (b)

$$C_E(B) \leq C_{X^g}(B)Z = C_{X^g}(H)Z = DZ.$$

Hence

$$|E/C_E(B)| \geq |E/ZD| = |X^g ZD/ZD| = |X^g/X^g \cap ZD| = |X^g/D|.$$

On the other hand, by (b) $|E/X| = |Z/Z \cap X^g| = |Z/Z \cap D|$, while the same result applied to A in place of B gives $|E/X| = |X/D| = |X^g/D|$. Since $D \leq Z(E)$ this gives

$$|E/C_E(B)| \geq |Z/Z \cap D| \geq |Z/C_Z(E)|.$$

(f): Let $x \in X^g \setminus D$ and suppose that $[x,b] \in D$. Then $\langle x \rangle D$ is normalized by $\langle X^g, b \rangle = H$ and so $x \in D$, a contradiction. This shows that $[x,c] \notin D$ for every $c \in B \setminus O_p(L)$. Since B acts quadratically on the abelian group E/D we conclude

$$|[x,B]D/D| = |\{[x,c]D \mid c \in B\}| \geq |BO_p(L)/O_p(L)|.$$

Note that by (ii), $[x,B]D \leq ZD$ and so (f) now follows from (e). □

Theorem 4.6 *Assume Hypothesis 4.1. Let V be a finite dimensional $GF(p)P$-module such that $[V, O_p(P)] = 0$ and $[V, O^p(P)] \neq 0$. Then $q(V,P) = 0$ or $q(V,P) \geq 1$.*

Proof: Let $A \leq T$ be a quadratic on V with $[V,A] \neq 0$. We need to show that that $|V/C_V(A)| \geq |A/C_A(V)$. The proof is by induction on $|A|$.

Let Y be a non-central P-chief factor in V. By 2.3 $C_T(Y) \leq O_p(P) \leq C_T(V)$. It follows that

$$|Y/C_Y(A)| \leq |V/C_V(A)| \text{ and } |A/C_A(Y)| = |A/C_A(V)|$$

for every $A \leq T$. Hence we may assume that

1° *V is a non-trivial simple P-module.*

We now apply 4.3. Then there exists $A \leq L$ such that L has the properties given in 4.3. In particular, there exists $g \in P$ such that $A \leq T^g \cap L \in Syl_p(L)$, and $L \cap M^g$ is the unique maximal subgroup of L containing $AO_p(L)$. Put $U := \langle C_V(T^g)^L \rangle$.

2° $[U, O^p(L)] \neq 0$ *and* $[U, A] \neq 0$.

By (1°) $C_V(T^g)$ is not P-invariant, so $N_P(C_V(T^g)) \leq M^g$. Since $L \not\leq M^g$, we get that $[U, O^p(L)] \neq 0$ and thus also $[U, A] \neq 0$.

3° *Put* $D := C_A(U)$. *Then* $|A/D| \leq |U/C_U(A)|$.

Observe that by the definition of U, $[U, O_p(L)] = 0$. Thus, for $E := U$, $B := A$, and $X := Z := C_U(A)$, L satisfies the hypothesis of 4.5. By 4.5(f)

$$|A/D| = |A/C_A(U)| \leq |A/A \cap O_p(L)| \leq |U/C_U(A)|.$$

So (3°) holds.

4° $|D/C_D(V)| \leq |V/C_V(D)|$.

Since $[U, A] \neq 0$, $D < A$ and (4°) follows by induction on $|A|$.

Using (3°) and (4°) we compute

$$\begin{aligned} |A/C_A(V)| &= |A/D||D/C_D(V)| \leq |U/C_U(A)||V/C_V(D)| \\ &\leq |C_V(D)/C_V(A)||V/C_V(D)| = |V/C_V(A)|. \end{aligned}$$

□

The next lemma is a variation of [Ste, 3.2].

Lemma 4.7 (*qrc-Lemma*) *Assume Hypothesis 4.1. Let* $V := \langle Y^P \rangle$. *Suppose that*

(i) $Y \leq \Omega_1 Z(J(O_p(P)))$,

(ii) $C_{O_p(P)}(Y) \not\trianglelefteq P$,

(iii) $J(R) \not\leq O_p(P)$.

Then $V \leq \Omega_1 Z(J(O_p(P)))$, $V \neq Y$, $N_L(Y) \leq M$, $[V, O^p(P)] \neq 1$, $C_T(V) \leq O_p(P)$ *and there exists* $A \in \mathcal{A}(R)$ *with*

$$[V, A, A] = 1 \neq [V, A] \text{ and } A \not\leq O_p(P).$$

Moreover, one of the following holds, where $q := q(Y, O_p(P))$, $r := r(V, P)$ *and* $c := c(V, P)$*:*

(a) $0 \neq q \leq 1$.

(b) $2 \leq c$, $1 \leq r$, *and* $(q-1)(rc-1) \leq 1$. *In particular,* $0 \neq q \leq 2$.

Proof: By (i) $V \leq ZJ(O_p(P))$. If $V = Y$, then $C_{O_p(P)}(Y) \trianglelefteq P$, a contradiction to (ii). Hence $V \neq Y$ and $RO_p(P) \leq T \leq N_P(Y) < P$. Since P is $RO_p(P)$-minimal we conclude that $N_P(Y) \leq M$. So $[V, O^p(P)] \neq 1$. Hence 2.3 gives $C_T(V) \leq O_p(P)$. In particular, (P, Y) satisfies Hypothesis III of [Ste].

As $C_T(V) \leq O_p(P)$, (iii) shows that there exists $A \in \mathcal{A}(R)$ such that $[V, A] \neq 1$. By the Timmesfeld Replacement Theorem [KS] we may assume that $[V, A, A] = 1$. Moreover, (i) implies that $A \not\leq O_p(P)$.

Suppose that $c = 1$. Then 4.4 shows that (a) holds. Thus, we may assume from now on that $c \geq 2$.

Suppose that $[A \cap O_p(P), V] = 1$. Again by 2.3 $O_p(P) \cap A = C_A(V) = C_A(U)$ for every non-central P-chief factor U of V. On the other and, by the maximality of A, $|V/C_V(A)| \leq |A/C_A(V)|$ and thus also $|U/C_U(A)| \leq |A/C_A(U)|$. Hence 4.6 implies that $c = 1$, which contradicts our assumption. We have shown that $[A \cap O_p(P), V] \neq 1$; in particular $q \neq 0$. Now [Ste, 3.2 (c)] and 4.6 yield (b). □

Lemma 4.8 *Assume hypothesis 4.1. Suppose that*

(i) $C_{O_p(P)}(Y) \ntrianglelefteq P$,

(ii) $Y \leq Z(J(O_p(P))) \cap Z(J(T))$,

(iii) $J(T) \not\leq O_p(P)$.

Then there exist subgroups $A \in \mathcal{A}(T)$ and L of P such that the following hold:

(a) L is $AO_p(L)$-minimal.

(b) $O_p(P)A \leq T \cap L \in SyL_p(L)$, and $M \cap L$ is the unique maximal subgroup of L containing $AO_p(L)$.

(c) $Y \ntrianglelefteq L$, and $V_0 := \langle Y^L \rangle$ is abelian.

(d) If $Y \leq Z(J(O_p(L)))$, then L and Y satisfies the hypothesis of 4.7 with L in place of P.

Proof: From (ii) $[Y, J(T)] = 1$ and so $J(T) \leq R$ and $J(R) = J(T)$. So the assumptions of 4.7 are fulfilled. In particular, V is elementary abelian and there exists $A \in \mathcal{A}(T)$ with $[V, A, A] = 1 \neq [V, A]$ and $A \not\leq O_p(P)$.

Hence we are allowed to apply the L-Lemma 4.3. This gives a subgroup L having the properties (a) – (c) given in 4.3. By 4.3(c) L is not a p-group, and so L is $AO_p(L)$-minimal. This is (a).

According to 4.3(a) there exists $g \in P$ such that $AO_p(L) \leq T^g \cap L \in Syl_p(L)$, and $L \cap M^g$ is the unique maximal subgroup containing $AO_p(L)$. Hence replacing A by $A^{g^{-1}}$ and L by $L^{g^{-1}}$ we may assume that (b) holds.

Clearly $Y \ntrianglelefteq L$ since $L \not\leq M$ but $N_P(Y) \leq M$. Since V is abelian, V_0 is abelian and (c) holds.

From $A \not\leq O_p(P)$ and (ii) we get that that $J(C_{T \cap L}(Y)) \not\leq O_p(L)$ and L is $C_{T \cap L}(Y)O_p(L)$-minimal. Hence 4.7(iii) holds for L and Y. Assume that $C_{O_p(L)}(Y) \trianglelefteq L$. Then also $C_{O_p(P)}(Y) \trianglelefteq L$ and thus $P = \langle T, L \rangle \leq N_P(C_{O_p(P)}(Y))$. This contradicts (i). Hence also 4.7(ii) holds for L and Y, and (d) follows. □

5 F-stability

In this section we explore the following hypothesis:

Hypothesis 5.1 *Let p be a prime and H a finite group. Suppose that Y is an elementary abelian p-subgroup of H such that for $T \in Syl_p(N_H(Y))$ and $R := C_T(Y)$ the following hold:*

(i) $Y \trianglelefteq N_H(J(R))$.

(ii) *Y is F-stable in H.*

(iii) *Either $Y \leq O_p(H)$ or H is of characteristic p.*

This hypothesis is motivated by the following observation:

Lemma 5.2 *Let G be a finite group, $S \in Syl_p(G)$ and $J(S) \leq H \leq G$, and let $\mathcal{F}(S)$ be a factorization family for $\mathcal{S}(S)$. Suppose that $N \in \mathcal{F}(S)$ is F-stable.*

(a) If $Y_N \leq O_p(H)$, then $Y := Y_H$ and H satisfy Hypothesis 5.1.

(b) If H is p-constrained and $\overline{H} := H/O_{p'}(H)$, then $\overline{Y_N}$ and $\overline{H}$ satisfy Hypothesis 5.1 in place of Y and H.

Proof: Let $T \in Syl_p(H)$ with $J(S) \leq T$. Put $Y := Y_N$ and $R := C_T(Y)$. Since $Y \trianglelefteq S$ and Y is F-stable, 2.4(a) implies that $Y \leq \Omega_1 Z(J(S)) \leq H$ and $J(S) = J(T) = J(R)$. Observe that $Y \leq O_p(N_G(J(S)))$ and so by 1.4(b), $N_G(J(S)) \leq N$. In particular, $T \leq N_G(Y)$ and so $T \in Syl_p(N_H(Y))$. Now (a) follows.

Assume that H is p-constrained. Then $\overline{H} = H/O_{p'}(H)$ is of characteristic p. By the Frattini-argument, $N_{\overline{H}}(\overline{Y})) = \overline{N_H(Y)}$ and $N_{\overline{H}}(J(\overline{R})) = \overline{N_H(J(R))}$. Moreover since Y is F-stable in G, $\overline{Y}$ is F-stable in $\overline{H}$. Thus Hypothesis 5.1 holds for $\overline{Y}$ and $\overline{H}$. □

Lemma 5.3 *Assume Hypothesis 5.1. Then $Y \trianglelefteq T$, $Y \leq Z(J(T))$, $J(R) = J(T)$, $N_H(T) \leq N_H(Y)$ and $T \in Syl_p(H)$.*

Proof: Clearly $Y \trianglelefteq T$. Thus by 2.4(a), $[Y, J(T)] = 1$. So $J(T) \leq R$ and $J(T) = J(R)$. Therefore $N_H(T) \leq N_H(J(R))$ and so by Hypothesis 5.1(i) $N_H(T) \leq N_H(Y)$. Hence $T \in Syl_p(H)$. □

Theorem 5.4 *Assume Hypothesis 5.1 and suppose $C_{O_p(H)}(Y) \ntrianglelefteq H$. Then Y is not $2F$-stable in H.*

Proof: If any subgroup of H satisfies the conclusion of 5.4 with respect to Y, then also H does. Thus we may assume:

1° *No proper subgroup of H satisfies the hypothesis of 5.4 with respect to Y.*

Put

$$H_0 = N_H(C_{O_p(H)}(Y)).$$

From 5.3 we conclude

2° *$N_H(T) \leq N_H(Y) \leq H_0$, $J(R) = J(T)$, $Y \leq ZJ(R))$ and $T \in Syl_p(H_0)$.*

Next we show:

3° *Let* $J(R)O_p(H) \leq \tilde{H} < H$. *Then* $C_{O_p(\tilde{H})}(Y) \trianglelefteq \tilde{H}$ *and* $\tilde{H} \leq H_0$.

By (2°) there exists $\tilde{R} \in Syl_p(C_{\tilde{H}}(Y))$ with $J(R) \leq \tilde{R}$ and $J(R) = J(T) = J(\widetilde{R})$, and so $Y \trianglelefteq N_{\tilde{H}}(J(\tilde{R}))$. Since $O_p(H) \leq O_p(\tilde{H})$, $Y \leq O_p(\tilde{H})$ or $\tilde{H}$ is of characteristic p. Moreover Y is F-stable in $\widetilde{H}$, and so Y and $\widetilde{H}$ satisfy Hypothesis 5.1. Now (1°) shows that $C_{O_p(\tilde{H})}(Y) \trianglelefteq \tilde{H}$. Hence $C_{O_p(H)}(Y) = O_p(H) \cap C_{O_p(\tilde{H})}(Y)$ is normal in $\tilde{H}$ and $\tilde{H} \leq H_0$.

4° H_0 *is the unique maximal subgroup of* H *containing* $J(R)O_p(H)$, *and* H *is* $J(R)O_p(H)$-*minimal.*

The first statement follows from (3°). If $J(R) \leq O_p(H)$, then by 5.3, $J(R) = J(O_p(H)) \trianglelefteq H$, and so by Hypothesis 5.1(i), $H \leq N_H(Y) \leq H_0$, a contradiction. Hence $J(R)O_p(H) \not\leq O_p(H)$, and H is $J(R)O_p(H)$-minimal.

5° *Put* $W := \langle Y^{H_0} \rangle$. *Then* W *is elementary abelian.*

We will first show that $Y \leq O_p(H_0)$. If $Y \leq O_p(H)$, this is obvious. Otherwise H is of characteristic p and by 5.3, $O_p(H)$ normalizes Y. So by 2.5 and (2°) $Y \leq O_p(H_0) \leq T$. Now (5°) follows from Hypothesis 5.1(ii) and 2.4(b).

Let $\mathcal{W}$ be the set of all p-subgroups D of H satisfying:

(a) $WO_p(H) \leq N_H(D) \not\leq H_0$.

(b) $D = J(D) \leq H_0$.

Clearly $1 \in \mathcal{W}$ and so $\mathcal{W} \neq \emptyset$. Pick $D \in \mathcal{W}$ such that first $|A|$ is maximal for $A \in \mathcal{A}(D)$ and then $|D|$ is maximal. Put $N := N_H(D)$ and $T_0 := DO_p(H)$ and let $T_1 \in Syl_p(N \cap H_0)$. Since $T_0 \leq O_p(N \cap H_0)$, $T_0 \leq T_1$. As W is H_0-invariant and by (2°) $T \in Syl_p(H_0)$, there exists $g \in H_0$ with $W^g = W$ and $T_1^g \leq T$; in particular $D^g \in \mathcal{W}$. Thus, after replacing D by D^g we may assume that $T_1 \leq T$.

6° $Y \leq Z(J(T_1))$, *and if* $Y \leq T_0$ *then* $Y \leq Z(J(T_0))$.

Since $T_1 \leq T$, T_1 normalizes Y. So (6°) follows from 2.4(a).

7° *Let* U *be a* p-*subgroup of* H_0 *containing* D. *Suppose that* $W \leq N_H(U)$ *and* $N_H(U) \not\leq H_0$. *Then* $J(U) = D$, *and if* $Y \leq U$ *then* $Y \leq Z(D)$.

Observe that $W \leq N_H(U) \leq N_H(UO_p(H))$. Hence $N_H(UO_p(H)) \not\leq H_0$ and $J(UO_p(H)) \in \mathcal{W}$. Since $D \leq U \leq UO_p(H)$, the maximal choice of D gives $D = J(UO_p(H)) = J(U)$.

Suppose that $Y \leq U$, then $J(U) = D \leq T_1 \leq N_H(Y)$ and so by 2.4(a), $Y \leq Z(J(U)) = Z(D)$.

8° $J(T) \neq D$ *and* $J(T_1) \neq D$.

Suppose $J(T) = D$. Then by 5.3 and Hypothesis 5.1(i), $N \leq N_G(J(T)) \leq N_G(Y)$ and so $N \leq N_H(C_{O_p(H)}(Y)) \leq H_0$, contrary to the choice of D.

Suppose $J(T_1) = D$. Then $N_T(T_1) \leq N_H(J(T_1)) = N$. So $N_T(T_1) \leq T_1$, $T = T_1$ and $J(T) = J(T_1) = D$, a contradiction.

9° *Let U be a p-subgroup of H_0 containing WD. Suppose that $J(U) \neq D$ or $Y \not\leq Z(D)$. Then U is not contained in any H-conjugate of H_0 other than H_0.*

Let $g \in H$ with $U \leq H_0^g$ and $U \leq T_2 \in Syl_p(H_0 \cap H_0^g)$. If $J(T_2) = D$, then also $D = J(U)$ and thus $Y \not\leq Z(D)$. Thus either $J(T_2) \neq D$ or $Y \not\leq Z(D)$. So (7°) gives $N_H(T_2) \leq H_0$. This implies $N_{H_0^g}(T_2) \leq H_0 \cap H_0^g$ and so $T_2 \in Syl_p(H_0^g)$. By (4°), H_0^g is the unique maximal subgroup of H containing T_2. Since $T_2 \leq H_0$ we get $H_0 = H_0^g$.

10° *$T_1 \in Syl_p(N)$, $J(T_1) \not\leq O_p(N)$, and $WJ(T_1)D$ is not contained in any other H-conjugate of H_0.*

By (8°) and (7°) $N_H(J(T_1)D) \leq H_0$, so $N_N(T_1) \leq N \cap H_0$ and $T_1 \in Syl_p(N)$. If $J(T_1) \leq O_p(N)$, then $J(T_1) = J(O_p(N))$ and $N \leq N_H(J(T_1)D) \leq H_0$, a contradiction.

Put $U := WJ(T_1)D$. By (8°), $J(U) \neq D$ and so the last statement in (10°) follows from (9°).

11° *There exists a $WJ(T_1)T_0$-minimal subgroup $H_1 \leq N$ such that $H_1 \cap H_0$ is a maximal subgroup of H_1 and $J(O_p(H_1)) = D$.*

By definition of $\mathcal{W}$, $N \not\leq H_0$. Choose $WJ(T_1)T_0 \leq H_1 \leq N$ such that H_1 is minimal with $H_1 \not\leq H_0$. Since $H_1 \not\leq H_0$, $N_H(O_p(H_1)) \not\leq H_0$. Also $WO_p(H)$ normalizes $O_p(H_1)$ and $D \leq O_p(H_1)$. So by (7°) $J(O_p(H_1)) = D$. Since $J(T_1) \neq D$ by (8°) we conclude $J(T_1) \not\leq O_p(H_1)$. Hence also $WJ(T_1)T_0 \not\leq O_p(H_1)$, and H_1 is $WJ(T_1)T_0$-minimal.

In the following let H_1 be as in (11°). Pick $WJ(T_1)T_0 \leq T_3 \in Syl_p(H_1 \cap H_0)$. Then H_1 is T_3-minimal and so $T_3 \in Syl_p(H_1)$. Since $T_3 \leq N \cap H_0$ and $T_1 \in Syl_p(H_0 \cap N)$, there exists $g \in N \cap H_0$ with $J(T_1) \leq T_3 \leq T_1^g$. Hence g normalizes $J(T_1)$, D and W, and thus also $WJ(T_1)T_0$. So replacing H_1 by H_1^g and T_3 by T_3^g we may assume that $T_3 \leq T_1 \leq T$.

Case 1 *The case $Y \leq O_p(H_1)$.*

12° *Y and H_1 satisfy the hypotheses of 4.8.*

Since $T_3 \leq T_1$, $Y \trianglelefteq T_3$. By (5°) and (6°) $Y \leq Z(WJ(T_1))$, so H_1 satisfies Hypothesis 4.1. Hence (8°) and (11°) give Hypothesis 4.8(iii), while 2.4(a) gives Hypothesis 4.8(ii).

Assume that $C_{O_p(H_1)}(Y) \trianglelefteq H_1$. As $O_p(H) \leq O_p(H_1)$, also $C_{O_p(H)}(Y) \trianglelefteq H_1$, which contradicts $H_1 \not\leq H_0$. Hence also Hypothesis 4.8(i) holds.

According to (12°) we are allowed to apply 4.8 to Y and H_1. Let L and V be with the properties given there. Since $Y \leq O_p(L) \leq T_1$, we get from 2.4(a) that $Y \leq Z(J(O_p(L)))$. Thus, by 4.8(d) L and Y satisfy the hypothesis of 4.7.

Since Y is $2F$-stable we are in case 4.7(b), so $0 \neq q(Y, O_p(H_1)) \leq 2$. Thus there exists non-trivial quadratic 2F-offender on Y and the lemma is proved in (Case 1).

Case 2 *The case $Y \not\leq O_p(H_1)$.*

By our assumption on H, in this case H has characteristic p. Hence also H_1 has characteristic p since $O_p(H) \leq H_1$. We now apply the L-Lemma 4.3 with W and H_1 in place of A and P. Then there exists $WO_p(H_1) \leq L$ such that

(i) L is $WO_p(L)$-minimal and

(ii) there exists $g \in H_1$ such that $L_0 := H_0^g \cap L$ is the unique maximal subgroup of L containing $WO_p(L)$.

13° $L_0 = L \cap H_0$, $Y \not\leq Z(D)$ *and* $J(O_p(L)) = D$.

Let g as in (ii). Then $WD \leq H_0 \cap H_0^g$ and $Y \not\leq Z(D)$ since $D \leq O_p(H_1)$. Hence (9°) implies $H_0 = H_0^g$; in particular $L_0 = L \cap H_0$ and $L \not\leq H_0$. Now (7°) also gives $J(O_p(L)) = D$.

According to (13°) we may assume, after conjugation by a suitable element of $H_0 \cap H_1$, that

14° $WO_p(L) \leq T \cap L \in Syl_p(L_0)$. *In particular* $O_p(L) \leq T$ *and* $O_p(L)$ *normalizes* Y.

By (13°), $Y \not\leq ZJ(O_p(L))$. Since $O_p(L)$ normalizes Y, we get from 2.4 that

15° $Y \not\leq O_p(L)$.

Put

$$A := W,\ B := Y,\ X := O_p(L) \cap A,\ E := \langle X^L \rangle,\ Z := O_p(L) \cap B.$$

By (5°) A is abelian, and by (14°) $O_p(L)$ normalizes A and B. Moreover, since $O_p(H_1) \leq O_p(L)$ and H_1 has characteristic p, $[E, O^p(L)] \neq 1$. It follows that the hypotheses of 4.5 are satisfied.

By 4.5(a), $C_Y(E) \leq Y \cap O_p(L)$ and so by 4.5(c), $[b, Y]C_Y(E) = [E, Y]C_Y(E) = Y \cap O_p(L)$ for all $b \in Y \setminus O_p(L)$. Moreover, by 4.5(d) $[Y, E, E, E] = 1$ and by 4.5(e),(f) we have $|Y/C_Y(E)| \leq |E/C_E(Y)|^2$. So E is a nearly quadratic $2F$-offender on Y. Hence the lemma also holds in (Case 2). □

Lemma 5.5 *Assume Hypothesis 5.1. Suppose that* $Y \ntrianglelefteq H$ *and* Y *is* $2F$*-stable. Then* $\Omega_1 Z(T) \ntrianglelefteq H$.

Proof: Let $T \leq P \leq H$ and P be minimal with $Y \ntrianglelefteq P$. By 5.3 $N_H(T) \leq N_H(Y)$, so $T \ntrianglelefteq P$ and P is T-minimal. Put

$$Q := C_{O_p(P)}(Y),\ V_0 := \Omega_1(Z(Q)),\ V := C_{V_0}(O_p(P)),\ \overline{P} := P/C_P(V).$$

If $Z(T) \not\leq O_p(P)$, $Z(T) \ntrianglelefteq P$. So we may assume $\Omega_1 Z(T) \leq O_p(P)$ and thus $\Omega_1 Z(T) = C_V(T)$. By 5.4 $Q \trianglelefteq P$. Since either $Y \leq O_p(H) \leq O_P(P)$ or P is of characteristic p, 2.5 implies $Y \leq O_p(P)$. Thus $Y \leq V_0$. Since $Y \ntrianglelefteq P$, we get that $[V_0, O^p(P)] \neq 1$. By 5.3, $J(R) = J(T)$ and so by Hypothesis 5.1(i), $J(R) \not\leq O_p(P)$. Hence 2.3 shows that $[O^p(P), J(R)] = O^p(P)$. Since $J(R)$ centralizes Y, $[O_p(P), J(R)] \leq O_p(P) \cap J(T) \leq Q$ and so $[O_p(P), O^p(P)] \leq Q$. The $P \times Q$-Lemma yields $[V, O^p(P)] \neq 1$.

Again 2.3 gives $C_T(V) = O_p(P)$ and $O_p(\overline{P}) = 1$. Moreover $\overline{J(T)} \neq 1$ since $J(R) \not\leq O_p(P)$. Hence $\overline{P}$ and V satisfy the hypothesis of [BHS, 5.6]. It follows that $[C_V(T), P] \neq 1$. Since $C_V(T) = \Omega_1 Z(T)$ and $P = \langle T^P \rangle$ we conclude that $\Omega_1 Z(T) \ntrianglelefteq P$ and so also $Z(T) \ntrianglelefteq H$. □

6 The Proof of Theorems 1.5 – 1.8

Recall that Theorems 1.3 and 1.4 have been proved in Section 3.

Proof of Theorem 1.5:

(a): Observe that $N = N_G(Y_N)$ by 1.3. Suppose $Y_N \leq O_p(M)$. If $M = N_G(C)$ for $1 \neq C \operatorname{char} B(S)$, then 1.5(a) follows from 1.4(b). If $\Omega_1 Z(S) \trianglelefteq M$, then 5.2(a) shows that Y_N and M satisfy Hypothesis 5.1. Hence 5.5 gives $Y_N \trianglelefteq M$ and so $M \leq N$.

(b): Put $\overline{M} := M/O_{p'}(M)$. Then 5.2(b) shows that $\overline{Y_N}$ and $\overline{M}$ satisfy Hypothesis 5.1. Thus 5.5 gives $\overline{Y_N} \trianglelefteq \overline{M}$. By the Frattini-argument $M = O_{p'}(M)N_M(Y_N) = O_{p'}(M)(M \cap N)$.

(c): Let $B(S) \leq H \leq G$ and H be p-constrained with $H \neq O_{p'}(H)(H \cap N)$, and let $B(S) \leq T \in Syl_p(N_H(Y_N))$. Put $\overline{H} := H/O_{p'}(H)$. Then again 5.2(b) shows that $\overline{Y_N}$ and $\overline{H}$ satisfy Hypothesis 5.1. Hence by 5.4, $C_{O_p(\overline{H})}(\overline{Y_N}) \trianglelefteq H$ and by 2.5, $\overline{Y_N} \leq O_p(\overline{H})$. From 5.5 applied to $N_{\overline{H}}(\Omega_1 Z(\overline{T}))$ we get $\overline{Y_N} \trianglelefteq N_{\overline{H}}(\Omega_1 Z(\overline{T}))$. Recall that $Y_N \leq \Omega_1 Z(J(S)) \leq O_p(N_G(J(S)))$. Thus by 1.4(b), $N_G(J(S)) \leq N$. Since $B(T) = B(S)$ we have $J(T) = J(S)$ and so $\overline{Y_N} \trianglelefteq C^{**}(\overline{H}, \overline{T})$. By the Frattini Argument $N_{\overline{H}}(\overline{Y_N}) = \overline{N_H(Y_N)} = \overline{H \cap N}$. Hence also $C^{**}(\overline{H}, \overline{T}) \leq \overline{H \cap N}$.

Proof of Theorem 1.6:

Let $P \in \mathcal{F}(S)$. By 1.3 P is a p-local subgroup of G. Let L be a maximal p-local subgroup containing P. By 2.2(c) $Y_P \leq Y_L$ and so $P \ll L$. Hence by 3.4 $P = L$.

Suppose that $N \in \mathcal{F}(S)$ is $2F$-stable. Let $M = N_G(C)$ for $1 \neq C \operatorname{char} B(S)$ or $M = N_G(\Omega_1 Z(S))$. Then $S \leq M$, so M has characteristic p since G is of parabolic characteristic p. Hence 1.5(a) implies $M \leq N$.

Let $H \in \mathcal{S}(B(S))$ and $B(S) \leq T \in Syl_p(H)$. Then $B(S) = B(T)$ and so $N_H(C) \leq N$ for $1 \neq C \operatorname{char} B(T)$. Also $T \leq S^g$ for some $g \in N_G(B(S)) \leq N$ and $\Omega_1 Z(S^g) \leq J(S) \leq T$, so $\Omega_1 Z(S^g) \leq Z(T)$ and $C_H(\Omega_1 Z(T)) \leq C_G(\Omega_1 Z(S^g)) \leq N^g = N$. Thus $C^*(H, T) \leq H \cap N$.

Proof of Corollary 1.7:

By 1.6 the members of $\mathcal{F}(S)$ are maximal p-local subgroups. We may assume that there exists a $2F$-stable $N \in \mathcal{F}(S)$.

Let L be a maximal p-local subgroup containing S with $L \not\leq N$ and choose $M \in \mathcal{F}(S)$ with $L \ll M$. Then $L \leq C_G(Y_L)M$. On the other hand, by 2.2(c) $\Omega_1 Z(S) \leq Y_L$ and so by 1.6 $C_G(Y_M) \leq C_G(\Omega_1 Z(S)) \leq N$. Since $L \not\leq N$ we conclude that $M \not\leq N$ and $M \neq N$. By 1.4 N is the only member of $\mathcal{F}(S)$ which is F-stable. Hence M is not F-stable.

Proof of Theorem 1.8:

Let P be the semi-direct product of G and V. Then $O_p(P) = V$ and $[V, O^p(P)] \neq 1$. Let A be an offender on V such that $|A||C_V(A)|$ is maximal. Because of [KS, 9.2.3] we may assume that A is quadratic on V Hence 4.6 implies $|A/C_A(V)| = |V|$, and 1.8 follows.

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