Turk J Math
() , 1 – 8.

A non-linear locally finite simple group with a p-group as centralizer

U. Meierfrankenfeld
Michigan State University
meier@math.msu.edu

Abstract

We show that there exists a non-linear, locally finite, simple group such that the centralizer of every non-trivial element is (locally solvable)-by-finite.

In [1, Problem. 3.8] Brian Hartley asked the following question:

Does there exists a non-linear infinite simple locally finite group in which the centralizer of every non-trival element is almost soluble, that is, has a soluble subgroup of finite index?

In this note we will give a partial answer to Brian's question: We will show that the answer is affirmative if solvable is replaced by "locally solvable". More precisely we prove:

Theorem A *(a) There exists a non-linear, locally finite, simple group such that the centralizer of every non-trivial element is (locally solvable)-by-finite.*

(b) Let p be a prime. Then there exists a non-linear, locally finite, simple group with an element whose centralizer is a p-group.

This theorem and its proof first appeared in my lecture notes on locally finite simple groups [2].

If W and I are sets, then W^I denotes the set of functions from I to W. If X is group, Y a subgroup of X and W a Y-set we define

$$W\Uparrow_Y^X = \{f : X \to W \mid f(xy) = f(x)^y \text{ for all } x \in X, y \in Y\}.$$

Note that, if we view X as a Y-set by right multiplication, then $W\Uparrow_Y^X$ just consists of the Y-equivariant maps from X to W.

If W is a Y-module and X/Y is finite, then the following lemma shows that $W\Uparrow_Y^X$ is the induced module for X. And if W is a group with Y acting trivially on W, then $W\Uparrow_Y^X$ is the base group of the wreath product $W \wr_{X/Y} X$.

Lemma 1 *Let X be a group, Y a subgroup of X and W a Y-set. Put $V = W{\Uparrow}_Y^X$. Then*

(a) X acts on V by $f^h(x) = f(hx)$ for all $x, h \in X$.

(b) Let I be a left transversal to Y in X. Then the restriction map

$$\rho_I : V \to W^I, f \to f\mid_I$$

is a bijection. In particular, V and $W^{X/Y}$ are isomorphic as sets.

(c) Define

$$\pi : V \to W, \pi(f) = f(1).$$

Then π is an onto Y-equivariant map.

(d) Suppose that t is a fixed-point for Y on W. Let $w \in W$ and define

$$\kappa_t(w) : X \to W, x \to \begin{cases} w^x & \text{if } x \in Y \\ t & \text{if } x \notin Y \end{cases}.$$

Then

(a) $\kappa_t(w) \in V$ and $\kappa_t : W \to V, w \to \kappa_t(w)$ is a 1-1 Y-equivariant map.

(b) $\pi(\kappa_t(w)) = w$.

(c) $\pi(\kappa_t(w)^x)) = t$ for all $x \in X \setminus Y$.

(e) Suppose in addition that W is a Y-group, that is, W is a group and for each $y \in Y$ the map $W \to W, w \to w^y$ is a homomorphism of groups. Then the maps ρ_I, π and κ_1 all are homomorphism of groups.

Proof. (a): We need to verify that $f^h \in V$ and $f^{hl} = (f^h)^l$ for all $f \in V, h, l \in X$. Let $x \in X$ and $y \in Y$. Since $f^h(xy) = f(h(xy)) = f((hx)y) = f(hx)^y = (f^h(x))^y$, $f^h \in V$. Also $f^{hl}(x) = f((hl)x) = f(h(lx)) = f^h(lx) = (f^h)^l(x)$ and so (a) holds.

(b): Let $x \in X$. Then $x = iy$ for some unique $i \in I$, $y \in Y$.

Let $f \in V$. Then $f(x) = f(iy) = f(i)^y$ and so f is uniquely determined by f_I. Thus ρ_I is 1-1.

Let $g \in W^I$. Define $f : X \to W$ by $f(x) = f(i)^y$. It is easy to verify that $f \in V$ and $f_I = g$. Hence ρ_I is onto.

(c): Let $f \in V$, $y \in Y$. Then $\pi(f^y) = f^y(1) = f(y\cdot 1) = f(1\cdot y) = f(1)^y = \pi(f)^y$. So π is Y-equivariant. Choose a left transversal containing 1. Then (b) implies that π is onto.

(d): Let $w \in W, y \in Y$ and $x \in X$. Then $x \in Y$ if and only if $xy \in Y$. Also by assumption $t = t^y$ and so

$$\kappa_t(w)(xy) = \begin{cases} w^{xy} & \text{if } xy \in Y \\ t & \text{if } xy \notin Y \end{cases} = \begin{cases} (w^x)^y & \text{if } x \in Y \\ t^y & \text{if } x \notin Y \end{cases} = (\kappa_t(w)(x))^y.$$

Thus $\kappa_t(w) \in V$.

Also $yx \in Y$ if and only if $x \in Y$. So

$$\kappa_t(w^y)(x) = \begin{cases} (w^y)^x & \text{if } x \in Y \\ t & \text{if } x \notin Y \end{cases} = \begin{cases} w^{yx} & \text{if } yx \in Y \\ t & \text{if } yx \notin Y \end{cases} = \kappa_t(w)(yx) = \kappa_t(w)^y(x).$$

Thus $\kappa_t(w^y) = \kappa_t(w)^y$ and (d:a) holds.

Since $1 \in Y$, $\pi(\kappa_t(w)) = \kappa_t(w)(1) = w^1 = w$ and so (d:b) holds.

Let $x \in X \setminus Y$. Then $\pi(\kappa_t(w)^x) = \kappa_t(w)^x(1) = \kappa_t(w)(x \cdot 1) = \kappa_t(w)(x) = t$. So also (d:c) holds.

It remains to prove (e). So suppose that W is a Y-group. Clearly W^X is a group via $(fg)(x) = f(x)g(x)$. Moreover, for $f, g \in W{\Uparrow}_Y^X$, $x \in X$ and $y \in Y$ we have

$$(fg)(xy) = f(xy)g(xy) = f(x)^y g(x)^y = (f(x)g(x))^y = (fg)(x)^y.$$

So $fg \in W{\Uparrow}_Y^X$. Similarly $f^{-1} \in W{\Uparrow}_Y^X$ and clearly $1 \in W{\Uparrow}_Y^X$. Hence $W{\Uparrow}_Y^X$ is a subgroup of W^X.

For any $J \subseteq X$, the restriction map $W^X \to W^J, f \to f\mid_J$ is a homomorphism. Thus ρ_I and π are homomorphism. $W \to W, w \to w^y$ and $W \to W, w \to 1$ are homomorphisms and so also κ_1 is a homomorphism. $\square$

Statement (c:a) in the following lemma is crucial for this paper. It allows us to enlarge a group Y to a group H while controlling some of the centralizers.

Lemma 2 *Let X, Y, W, G be groups with $Y \leq X$, $G \leq Y$, $W \trianglelefteq Y$, $Y = WG$ and $W \cap G = 1$. Then there exists a semidirect product $H = V \rtimes X$ and an embedding $\beta : Y \to H$ such that*

(a) $V \cong W^{Y/X}$ *as groups.*

(b) $V\beta(y) = Vy$ *for all* $y \in Y$.

(c) *Let* $y \in Y$, $w \in W$ *and* $g \in G$ *with* $y = wg$ *and* $y \notin g^W$. *Then*

  (a) $VC_H(\beta(y)) = VC_Y(y)$.

  (b) $\beta(y) \notin y^V$.

Proof. Let $y \in Y$ and $y = wg$ with $w \in W, g \in G$. Let ρ be the projection of Y onto G, that is $\rho(wg) = g$. Note that W is a Y-group via $w \to w^{\rho(y)}$. We denote this Y-group by W_ρ. Put $V = W_\rho \Uparrow_Y^X$. Then by Lemma 1 X acts on V and we can form the semidirect product, $H = V \rtimes X = \{(v,x) | v \in V, x \in X\}$. We view V and X as subgroups of H. So $H = VX$ and $V \cap X = 1$. Let $\pi : V \to W$ and $\kappa = \kappa_1 : W \to V$ be as in Lemma 1. Let $v \in V$.

1° $\quad$ $\pi(v^y) = \pi(v)^{\rho(y)} = \pi(v)^g$ *and* $\pi(v^{y^{-1}}) = \pi(v)^{g^{-1}}$.

The first statement follows from 1(c) and the definition of action of Y on W_ρ. Since $\rho(y^{-1}) = \rho(y)^{-1} = g^{-1}$, the second statement follows from the first.

2° $\quad$ *Define*

$$\beta : Y \to H, y \to (\kappa(w), y).$$

Then β is a monomorphism and $V\beta(y) = Vy$.

Clearly β is 1-1 and $V\beta(y) = Vy$. For $i = 1, 2$ let $y_i \in Y$ and $y_i = w_i g_i$ with $w_i \in W$, $g_i \in G$. Then

$$y_1 y_2 = w_1 g_1 w_2 g_2 = w_1 w_2^{g_1^{-1}} g_1 g_2,$$

and so

$$\beta(y_1 y_2) = (\kappa(w_1 w_2^{g_1^{-1}}), y_1 y_2).$$

On the other hand,

$$\beta(y_1)\beta(y_2) = (\kappa(w_1), y_1)(\kappa(w_2), y_2) = (\kappa(w_1)\kappa(w_2)^{y_1^{-1}}, y_1 y_2).$$

As κ is a Y-equivariant homomorphism,

$$\beta(y_1)\beta(y_2) = (\kappa(w_1 w_2^{g_1^{-1}}), y_1 y_2)$$

and so β is a homomorphism.

3° $\quad$ *Let $(v,x) \in C_H(\beta(y))$. Then $\kappa(w)v^{y^{-1}} = v\kappa(w)^{x^{-1}}$ and $xy = yx$.*

We compute

$$\beta(y)(v,x) = (\kappa(w), y)(v,x) = (\kappa(w)v^{y^{-1}}, yx)$$

and

$$(v,x)\beta(y) = (v,x)(\kappa(w), y) = (v\kappa(w)^{x^{-1}}, xy)$$

Thus (3°) holds.

4° *Suppose that $VC_H(\beta(y)) \neq VC_Y(y)$. Then $y \in g^W$.*

Since $\beta(C_Y(y)) \leq C_H(\beta(y))$ we have $VC_Y(y) \overset{(2^\circ)}{=} V\beta(C_Y(y)) \leq VC_H(\beta(y))$ and so there exists $(v, x) \in C_H(\beta(y))$ with $x \notin Y$.

By Lemma 1(d:b), $\pi(\kappa(w)) = w$. By (1°), $\pi(v^{y^{-1}}) = \pi(v)^{k^{-1}}$. Also since $x \notin Y$ and $\kappa = \kappa_1$, Lemma 1(d:c) implies $\pi(\kappa(w)^{x^{-1}}) = 1$. So applying π to both sides of the first equation in (3°) we obtain

$$w\pi(v)^{g^{-1}} = \pi(v).$$

Put $r = \pi(v)$. Then $wgrg^{-1} = r$, $wg = rgr^{-1}$ and $y = wg = g^{r^{-1}} \in g^W$.

5° *If $\beta(y) \in y^V$, then $y \in g^W$.*

Suppose that $\beta(y) = (1, y)^{(v,1)}$ for some $v \in V$. Then

$$(\kappa(w), y) = (v^{-1}, 1)(1, y)(v, 1) = (v^{-1}, y)(v, 1) = (v^{-1}v^{y^{-1}}, y)$$

and so $\kappa(w) = v^{-1}v^{y^{-1}}$. Applying π to both sides we conclude $w = \pi(v^{-1})\pi(v)^{g^{-1}}$. Put $r = \pi(v)$ then $w = r^{-1}grg^{-1}$ and $y = wg = r^{-1}gr = g^r$. Thus (5°) holds.

We are now in the position to prove the lemma: (a) follows from Lemma 1(b),(e); (b) from (2°); (c:a) from (4°); and (c:b) from (5°). □

The preceding Lemma allows to control centralizers under the condition $y \notin g^W$. The next lemma provides us with a tool to achieve this condition:

Lemma 3 *Let G be a finite group and Π a set of primes. Then there exist a finite abelian $\mathbb{Z}G$-module W and a monomorphism $\alpha : G \to W \rtimes G$ such that*

(a) W is a Π-group.

(b) $W\alpha(g) = Wg$ for all $g \in G$.

(c) $\alpha(g) \notin g^W$ for all non-trivial Π-elements g in G.

(d) If G is perfect, then $W = [W, G]$ and $W \rtimes G$ is perfect.

Proof. Let m be the Π-part of $|G|$. Put $B = (\mathbb{Z}/m\mathbb{Z})^G$ and $H = \mathbb{Z}/m\mathbb{Z} \wr G$, where the wreathed product is formed with respect to regular action of G on G. Then B is the base group of H and $H = BG$. For $f \in B$ put $||f|| = \sum_{g\in G} f(g)$. Put $W = [B, G]$ and note that $W = \{f \in B \mid ||f|| = 0\}$. Then W is a Π-group and (a) holds. Also if $G = G'$, the Three Subgroups Lemma implies $[B, G, G] = [B, G]$ and so $W = [W, G]$ and WG is perfect. Thus (d) holds.

Let $b \in B$ be defined by $b(1) = 1$ and $b(g) = 0$ for all $g \in G^{\#}$. Define $\alpha : G \to WG, g \to g^b = [b, g^{-1}]g$. Then α is monomorphism and (b) holds. It remains to prove (c). So let g be a non-trivial Π-element in G and suppose that $g^b = g^a$ for some $a \in W$. Put $n = |g|$ and $c = ba^{-1}$. Then $c \in C_B(g)$. Let I be a left transversal to $\langle g \rangle$. Then each element of G can be uniquely written as ig^k for some $i \in I$ and some $0 \leq k < n$. Since $c^g = c$, $c(i) = c(ig^k)$. Let $s = \sum_{i \in I} c(i)$. We conclude that $||c|| = ns$. Thus

$$1 = ||b|| = ||ac|| = ||a|| + ||c|| = 0 + ns = ns$$

in $\mathbb{Z}/m\mathbb{Z}$, a contradiction, as n divides m. $\square$

We now combine the embeddings from the two preceding lemmas into one:

Lemma 4 *Let G and F be finite groups and Π a set of primes. Then there exist a finite group G^* with $G \leq G^*$ and normal subgroup V of G^* such that:*

(a) V is an abelian Π-group and G^*/V is simple.

(b) $G \cap V = 1$.

(c) Let x be a nontrivial Π-element in G. Then $C_{G^*}(x)$ has a normal solvable Π-subgroup M_x with $C_{G^*}(x) = M_x C_G(x)$.

(d) G^* has a subgroup isomorphic to F.

(e) If G is perfect, G^* is perfect.

Proof. Let α and $Y = W \rtimes G$ be as in Lemma 3. Let X be any finite simple group containing Y as a subgroup and such that X has a subgroup isomorphic to F. Let β and V be as in Lemma 2. Put $G^* = V \rtimes X$. Let x be a Π-element in G and put $\gamma = \beta \circ \alpha$. Let $y \in C_Y(\alpha(x))$. Then $Wy = Wg$ for some $g \in G$. By Lemma 3(b), $W\alpha(x) = Wx$. From $[\alpha(x), y] = 1$ we conclude that $[x, g] \in W \cap G = 1$. Hence $g \in C_G(x)$ and so $C_Y(\alpha(x)) \leq WC_G(x) = W\alpha(C_G(x))$. By Lemma 3(c) $\alpha(x) \notin x^W$ and so by Lemma 2(c:a), $C_{G^*}(\beta(\alpha(x))) \leq VC_Y(\alpha(x))$. Thus

$$C_{G^*}(\gamma(x)) \leq VC_Y(\alpha(x)) \leq VW\alpha(C_G(x)).$$

By Lemma 2(b), $V\beta(y) = Vy$ for all $y \in Y$ and so $V\alpha(C_G(x)) = V\gamma(C_G(x))$. Thus

$$C_{G^*}(\gamma(x)) \leq VW\gamma(C_G(x)) = VWC_{\gamma(G)}(\gamma(x)).$$

Put $M_x = C_{VW}(\gamma(x))$. Then $C_{G^*}(\gamma(x)) = M_x C_{\gamma(G)}(\gamma(x))$. Identifying G with its image in G^* under γ we see that all parts of the lemma hold. $\square$

Let G be a locally finite group. Recall that a Kegel-cover for G is a set $\mathcal{K}$ of pairs subgroups of G such that

(i) If $(H, M) \in \mathcal{K}$, then H is a finite subgroup of G, $M \trianglelefteq H$ and H/M is simple.

(ii) For each finite subgroup F of G there exists $(H, M) \in \mathcal{K}$ with $F \leq H$ and $F \cap M$.

Otto Kegel proved that every locally finite, simple group has a Kegel cover. The following well-known Lemma is a partial converse:

Lemma 5 *Let G be a locally finite group with a Kegel cover $\mathcal{K}$. Suppose that for all $(H, M) \in \mathcal{K}$, H is perfect and M is solvable. Then G is simple.*

Proof. Let L be a non-trivial normal subgroup of G. Let $g \in G$. It suffices to show that $g \in L$. For this let $1 \neq l \in L$ and put $F = \langle l, g \rangle$. Then F is finite and so there exists $(H, M) \in \mathcal{K}$ with $F \leq H$ and $F \cap M = 1$. Since $l \in H \setminus M$ we have $L \cap H \nleq M$. Since $L \cap H$ is normal in H and H/M is simple this implies $H = (L \cap H)M$. Thus $H/L \cap H \cong M/M \cap L$ and since M is solvable, $H/L \cap H$ is solvable. As H is perfect we conclude that $H = L \cap H$. Thus $g \in F \leq H \leq L$. $\square$

Proposition 6 *Let G_0 be a finite, perfect group, Π a non-empty set of primes and for each positive integer n let F_n be a finite group. Then there exists a locally, finite simple group G with $G_0 \leq G$ and such that:*

(a) If x is a nontrivial Π-element of G, then $C_G(x)$ has a locally solvable, normal Π-subgroup M_x of finite index.

(b) If x is a nontrivial Π-element in G_0, then $C_G(x) = M_x C_{G_0}(x)$.

(c) F_n is isomorphic to a subgroup of G.

Proof. We will produce finite groups $G_n, n \in \mathbb{Z}^+$, and normal subgroups M_n of G_n such that for all $n \in \mathbb{Z}^+$

1°

(a) G_n is perfect.

(b) M_n is abelian and G_n/M_n is simple.

(c) $G_{n-1} \leq G_n$ and $G_{n-1} \cap M_n = 1$.

(d) If x is a nontrivial Π-element in G_{n-1}, then there exists a solvable normal Π-subgroup M_{nx} of $C_{G_n}(x)$ with $C_{G_n}(x) = M_{nx} C_{G_{n-1}}(x)$.

(e) G_n has a subgroup isomorphic to F_n.

Let $i \geq 0$ and suppose we already found $G_1, \ldots, G_i$ such that (a) -(e) hold for $1 \leq n \leq i$. Then G_i is perfect and so we can apply Lemma 4 to $G = G_i$ and $F = F_{i+1}$. Put $G_{i+1} = G^*$, $M_i = V$ and $M_{ix} = M_x$. Then by Lemma 4 we conclude that (a) to (e) hold for $n = i + 1$.

Put $G = \bigcup_{i=1}^{n} G_n$. Then $\{(G_n, M_n) \mid n \geq 1\}$ is a Kegel cover for G and by Lemma 5, G is simple. Let $x \in G$ be a nontrivial Π-element. Then $x \in G_n$ for some n. Put $M_x^n = 1$ and inductively, $M_x^{m+1} = M_x^m M_{(m+1)x}$. It follows from (1°)(d) and induction that:

2° *Let $m \geq n$. Then M_x^m is a solvable, normal Π-subgroup of $C_{G_m}(x)$ and $C_{G_m}(x) = M_x^m C_{G_n}(x)$.*

Put $M_x = \bigcup_{m=n}^{\infty} M_x^m$. Then by (2°), M_x is a locally solvable, normal Π-subgroup of $C_G(x)$ and $C_G(x) = M_x C_{G_n}(x)$. Thus the proposition is proved. □

Corollary 7 *Let Π be a non-empty set of primes. Then there exists a non-linear, locally finite, simple group G such that*

(a) The centralizer of every non-trivial Π-element has a locally solvable Π-subgroup of finite index.

(b) There exists an element whose centralizer is a locally solvable Π-group.

Proof. Fix $p \in \Pi$ and put $G_0 = \mathrm{Alt}(2p + 1)$. Let x the product of two disjoint p-cycle in $\mathrm{Sym}(2p+1)$. Then $x \in G_0$, G_0 is perfect and $C_{G_0}(x) \cong C_p \times C_p$. In particular, $C_{G_0}(x)$ is solvable Π-group. Apply Lemma 6 to this G_0 and with $F_n = \mathrm{Sym}(n)$. The resulting G is not linear and fulfills (a). Moreover, (b) holds for the element $x \in G_0 \leq G$. □

Proof of Theorem A:

Apply Corollary 7(a) with Π the set of all primes and Corollary 7(b) with $\Pi = \{p\}$.

References

[1] B. Hartley, Simple locally finite groups in *Finite and Locally Finite Groups* (B. Hartley et al, Eds.) Kluwer Academic, Dordrecht, **1995**, 1-44.

[2] U. Meierfrankenfeld, *Lecture Notes on Locally Finite Simple Groups*, Michigan State University **1997**, http://www.math.msu.edu/~meier/Classnotes/LFG/LFG_abstract.html