Maximal 2-local subgroups of the Monster and Baby Monster

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Abstract

The lists of the maximal 2-local subgroups of the Monster and Baby Monster simple groups in the Atlas are complete.

1 Introduction

The Monster and the Baby Monster are the two largest groups among the 26 sporadic finite simple groups. After the classification of the finite simple groups was announced in 1981 the focus of research in the area of finite simple groups moved toward the study of the properties of the known groups. One of the most important pieces of information about a simple group G is its list of maximal subgroups, taken up to conjugation in G.

Methods used to classify maximal subgroups H of G differ significantly depending on whether or not H is p-local. A subgroup H of G is *p-local*, where p is a prime number dividing the order of G, if H is the normalizer of a nontrivial p-subgroup. We say that H is a maximal p-local subgroup if H is maximal by inclusion among p-local subgroups of G. Notice that a maximal p-local subgroup H may or may not be maximal in G. Nevertheless, the classification of all maximal p-local subgroups of G for each p is an important step toward a complete determination of all maximal subgroups of G.

In the case where G is one of the 26 sporadic finite simple groups the lists of maximal p-locals subgroups–as well as lists of non p-local maximal subgroups–have been compiled and proven complete for almost all G by work of many people, but most notably, R.A. Wilson. One significant omission to date has been the lists of maximal 2-local subgroups of M and BM. The Atlas of Finite Groups [ATLAS] provides lists of the *known* maximal 2-local subgroups of M and BM. What was missing was a proof that these lists are in fact complete. In this paper and its sequel [M] we bridge this gap by supplying necessary proofs.

Let us now review the lists from [ATLAS]. Seven conjugacy classes of maximal 2-local subgroups were known for $G = M$. The corresponding structures are as follows (see [ATLAS] for the exact meaning of these structures; however, notice that [ATLAS] uses the notation B for the Baby Monster group; also we use the good old "Ω" and "Sp" where [ATLAS] uses "O" and "S"):

(1) $2 \cdot BM$;

(2) $2^2 \cdot ({}^2E_6(2)) : S_3$;

(3) $2_+^{1+24}.Co_1$;

(4) $2^2.2^{11}.2^{22}.(S_3 \times M_{24})$;

(5) $2^3.2^6.2^{12}.2^{18}.(L_3(2) \times 3 \cdot S_6)$;

(6) $2^5.2^{10}.2^{20}.(S_3 \times L_5(2))$;and

(7) $2^{10+16} \cdot \Omega_{10}^+(2)$.

Eight classes of maximal 2-local subgroups were known for $G = BM$. Their structures are shown in [ATLAS] as follows.

(1) $2 \cdot ({}^2E_6(2)) : S_3$;

(2) $(2^2 \times F_4(2)) : 2$;

(3) $S_4 \times {}^2F_4(2)$;

(4) $2_+^{1+22} \cdot Co_2$;

(5) $2^2.2^{10}.2^{20}.(M_{22} : 2 \times S_3)$;

(6) $2^3.[2^{32}].(S_5 \times L_3(2))$;

(7) $2^5.[2^{25}].L_5(2)$; and

(8) $2^9.2^{16}.Sp_8(2)$.

Recall that $O_2(H)$ denotes the largest normal 2-subgroup of H. We say that H is of *characteristic* 2 if $C_H(Q) \leq Q$, where $Q = O_2(H)$. We split the work as follows. In this papers we determine all maximal 2-local subgroups of M and BM that are of characteristic 2. The sequel [M] deals with the remaining classes. In the above lists the partition into the characteristic 2 type and non characteristic 2 type is as follows: For M, classes (1) and (2) are not of characteristic 2, while classes (3)–(7) are of characteristic 2. For BM, classes (1)–(3) are not of characteristic 2, while classes (4)–(8) are of characteristic 2. Notice that classifying the maximal 2-local subgroups that are not of characteristic 2 is rather more simple and some may even say that this part of the lists has been known to be complete. However, as we are unaware of any published proof, we include this subcase in our work. Needless to say, we believe that our result on the maximal 2-local subgroups of characteristic 2 is entirely new.

Our approach was in part motivated by the work on the geometries of the groups M and BM, by A.A. Ivanov and the second author (see [IS]). We noticed that the known maximal 2-local subgroups of characteristic 2 are either the normalizers of certain very special elementary abelian subgroups that we call *singular subgroups*, or the normalizers of yet another type of elementary abelian subgroups (of order 2^{10}) that we call *arks*. The above-mentioned geometries of M and BM consist of singular subgroups, while arks also have a geometrical meaning, namely, they correspond to certain natural subgeometries.

We introduce singular subgroups in Section 4, in which we also classify them up to conjugation. Arks are introduced in Section 5. They form a single conjugacy class of subgroups. Our choice of the word "ark" was motivated by the fact that an ark contains representatives of all "species" (*i.e.*, conjugacy classes) of singular subgroups.

We will now formally state the principal results of this paper. We start with the following definitions of the groups M and BM: The *Monster* M is a finite simple group with a large extraspecial 2-subgroup Q whose normalizer C has the following structure: $C \sim 2^{1+24}.Co_1$. (Here and in what follows we use $\sim$ as shorthand for "has structure", while $\cong$ as usual stands for "is isomorphic".) Recall that a group G is said to have a *large extraspecial subgroup* if for some involution $z \in G$ its centralizer $C = C_G(z)$ contains a normal extraspecial 2-subgroup Q such that $C_G(Q) \leq Q$. It easily follows from this that z is *2-central* in G, that is, the centralizer of Z contains a Sylow 2-subgroup of G. It also follows that G has a unique conjugacy class of 2-central involutions. Returning to the Monster group M, in addition to the above structure of C, we have that the Co_1-module arising in the action of C on $Q/\langle z\rangle$ is isomorphic to the module on $\Lambda/2\Lambda$, where Λ is the Leech lattice. This due to R. Griess who showed in [Gr2] that $\Lambda/2\Lambda$ is the only faithful Co_1-module in dimension 24.

For the purposes of this paper, the *Baby Monster* BM is simply the group

$H/\langle t\rangle$, where t is a non 2-central involution in the Monster M and $H = C_M(t)$. All the work in this papers takes place in M; singular subgroups, arks and the Baby Monster BM "live" in M.

Under these definitions, we prove the following.

Theorem 1 *The Monster M contains exactly 5 conjugacy classes of maximal 2-local subgroups of characteristic 2. They are:* (a) *the normalizers of singular subgroups of types 2^1, 2^2, 2^3 and 2^5_2; and* (b) *the normalizers of arks.*

Theorem 2 *The Baby Monster BM contains exactly 5 conjugacy classes of maximal 2-local subgroups of characteristic 2. Their preimages in H are:* (a) *the normalizers in H of special singular subgroups U of types 2^1, 2^2, 2^3 and 2^5_1; and* (b) *the normalizers of arks containing t.*

The exact meaning of the word "special" in this last theorem is as follows: With each singular subgroup U we associate in Section 4 a second subgroup Q_U. The special singular subgroups U are those for which $t \in Q_U$.

In simple words, Theorems 1 and 2 state that the lists of maximal 2-local subgroups of M and BM given in [ATLAS] are complete in their characteristic 2 part.

Finally, we need to explain our policy with respect to citing *vs.* proving. A lot of information is available about the two monsters. However, much of it exists as a sort of finite group theory lore, that is, with no proper published proof known. In particular, at least some of the information given in [ATLAS] should be considered as semi-lore because there is no proofs there and very little by way of citation. Of course, we cannot prove everything in a single paper, and so we decided to take an "inductive" approach. We use some "lore" information about the smaller simple groups involved in M (mostly, Co_1 and $\Omega^+_{10}(2)$). At the same time, we prove everything we need as far as the properties of M and BM themselves are concerned. Likely, some of the facts that we prove can be found in the available sources such as [As1], [AsSe], [Gr1], [Se] and many others. However, we believe that the bulk of the detailed information that we need cannot be covered by citation. Notice also that M. Aschbacher in [As1] determines all maximal subgroups of M containing a Sylow 2-subgroup.

2 Classes of involutions, I

In this section we classify conjugacy classes of involutions of a group H satisfying the following conditions:

(H1) $O_2(H) \sim 2^{24}$;

(H2) $H/O_2(H) \cong Co_1$; and

(H3) the action of $H/O_2(H)$ on $O_2(H)$ is equivalent to the action of Co_1 on $\hat{\Lambda} = \Lambda/2\Lambda$, the Leech lattice taken modulo 2.

If C is the centralizer of a 2-central involution z in the Monster group M then the group $H = C/Z$, where $Z = \langle z \rangle$, satisfies (H1)–(H3) and so the results of this section give us some insight into the structure of C.

We refer to [ATLAS], page 180, for a description of the Leech lattice Λ, terminology and notation related to Λ, and a summary of properties of Λ. A whole wealth of information about the Leech lattice can be found in [CS]. Let $(x, y) = \frac{1}{8}\sum_{i=1}^{24} x_i y_i$ be the integral inner product that exists on Λ. Let $\Lambda_n = \{x \in \Lambda | (x, x) = 2n\}$. One useful fact that is the following

Lemma 2.1 *The orbits of Co_1 on $W = \hat{\Lambda}^{\#}$ are the sets $\hat{\Lambda}_2$, $\hat{\Lambda}_3$ and $\hat{\Lambda}_4$.* □

For $i = 2$, 3 and 4, let $w_i \in \hat{\Lambda}_i$. The structure of the stabilizer of w_i in Co_1 is also well-known.

Lemma 2.2 *The following hold.*

(1) $C_{Co_1}(w_2) \cong Co_2$;

(2) $C_{Co_1}(w_3) \cong Co_3$;

(3) $C_{Co_1}(w_4) \cong 2^{11} : M_{24}$. □

Recall that in its action on $W = \hat{\Lambda}$ the group Co_1 preserves a nondegenerate quadratic form q defined as follows: if $u = \hat{x}$ for some $x \in \Lambda$ then $q(u) = \frac{1}{2}(x, x) \pmod 2$. Let Φ denote the symmetric bilinear form that corresponds to q: for $u = \hat{x}$ and $v = \hat{y}$, we have $\Phi(u, v) = q(u + v) + q(u) + q(v) = (x, y) \pmod 2$. It follows from the definition of q that w_2 and w_4 are singular, while w_3 is nonsingular.

Before we go on, let us record the following property that can be verified, say, using the description of Λ from [ATLAS].

Lemma 2.3 *There is no $\hat{\Lambda}_2$-pure subgroups 2^3 in $W = \hat{\Lambda}$.* □

According to [ATLAS], Co_1 contains three conjugacy classes of involutions. We will need to know how the involutions and their centralizers act on W.

Let t be an involution in Co_1 and let $C_t = C_{Co_1}(t)$. Define $U_t = C_W(t)$ and $V_t = [W, t]$. Since t is an involution, $V_t \subseteq U_t$. Furthermore, $\dim W/U_t = \dim V_t$. In fact, U_t is the orthogonal complement (with respect to Φ) of V_t, and so the C_t-modules W/U_t and V_t are dual to each other.

First, let t be an involution of type $2A$. Then C_t is an extension of an extraspecial group 2_+^{1+8} by $\Omega_8^+(2)$. The action on the 8-dimensional quotient of $O_2(C_t)$ provides an irreducible module for $C_t/O_2(C_t) \cong \Omega_8^+(2)$. We will refer to this module as to the *natural* module. Notice that $\Omega_8^+(2)$ has two more irreducible 8-dimensional modules, and we will refer to those as to the two *halfspin* modules. Notice also that the natural module and the halfspin modules are all self-dual.

Lemma 2.4 *If t is of type $2A$ then*

(1) *V_t has dimension 8 and U_t has dimension 16;*

(2) *C_t acts irreducibly on each of W/U_t, U_t/V_t, and V_t; furthermore, V_t and U_t/V_t are two non-isomorphic halfspin modules and $W/U_t \cong V_t$.*

Proof: Notice that t (or rather, its preimage in Co_0) can be chosen to act on the standard frame, inverting signs in an octad. This allows to establish (1) by direct computation. Let $x \in C_t$ be an element of order three, such that x has a 6-dimensional centralizer in the natural module. Then 2^{13} divides the order of the centralizer of x in C_t, and hence x is of type $3A$ (a Suzuki 3-element). According to [ATLAS], x acts fixed-point-freely on W, which implies that W/U_t, U_t/V_t, and V_t are halfspin modules for $C_t/O_2(C_t) \cong O_8^+(2)$. Since the halfspin modules are self-dual, we have $W/U_t \cong V_t$. Finally, if $U_t/V_t \cong V_t$ then C_t contains a 3-element with an 18-dimensional centralizer in W, which contradicts the information from [ATLAS]. □

Let t be an involution of type $2B$. Then $C_t \sim (2^2 \times G_2(4)).2$.

Lemma 2.5 *If t is of type $2B$ then*

(1) *$U_t = V_t$ and so they are of dimension 12;*

(2) *C_t acts transitively on $V_t^{\#}$; furthermore, $V_t^{\#} \subseteq \hat{\Lambda}_4$.*

Proof: Notice that 13 divides the order of $G_2(4)$, and it does not divide the orders of Co_2, Co_3 and $2^{11} : M_{24}$. Therefore, $G_2(4)$ fixes no non-zero vector in W. Again, since 13 divides the order of $G_2(4)$, the latter group has no nontrivial $GF(2)$-modules in dimensions less than 12. It follows that V_t has dimension at least 12. Hence, $U_t = V_t$ and they are both of dimension exactly 12. According to [MOD], V_t must be the natural module for $G_2(4)$. In particular, $G_2(4)$ is transitive on $V_t^{\#}$. Since $\hat{\Lambda}_4$ has odd length, t fixes a vector in $\hat{\Lambda}_4$. Now the transitivity implies that $V_t^{\#} \subseteq \hat{\Lambda}_4$. □

Finally, let t be of type $2C$. Then $C_t \sim 2^{11} : \mathrm{Aut}\, M_{12}$.

Lemma 2.6 *If t is of type $2C$ then*

(1) *$U_t = V_t$, and so they are of dimension 12;*

(2) *as a C_t-module, U_t is uniserial with submodules of dimension 1 and 11; furthermore, the non-zero vector fixed by C_t is from $\hat{\Lambda}_4$, and $\hat{\Lambda}_3 \cap V_t$ coincides with the setwise complement of the 11-dimensional submodule.*

Proof: In this case again t can be chosen inside the diagonal subgroup stabilizing the standard frame. Namely, t inverts signs in a dodecad. This allows to compute all vectors in W fixed by t and thus establish (1) and also that V_t contains some elements from $\hat{\Lambda}_3$. Since $U_t = V_t$, we have that V_t is totally isotropic (with regard to Φ). Since the vectors in $\hat{\Lambda}_3$ are non-singular, we obtain that $V_t \cap \hat{\Lambda}_3$ coincides with the complement of a hyperplane. So V_t contains an 11-dimensional subspace V_0 left invariant by C_t. Observe that C_t is fully contained in the stabilizer of the standard frame. So C_t stabilizes a vector $v \in \hat{\Lambda}_4$, the image of the standard frame. Clearly, $v \in V_0$. Since 11 divides the order of M_{12}, C_t acts irreducibly on $V_0/\langle v \rangle$. It remains to notice that v is the only vector in W fixed by C_t (indeed, already the diagonal group 2^{11} fixes no other vector in $W^{\#}$). The uniseriality now follows. □

We can now determine the conjugacy classes of involutions in a group H satisfying conditions (H1)–(H3). Let $E = O_2(H)$ and $\bar{H} = H/E$. First of all, Lemma 2.1 implies the following.

Lemma 2.7 *The group H has exactly three classes of involutions contained in E. If e_2, e_3 and e_4 are representatives of those classes then $C_H(e_2) \sim 2^{24}.Co_2$, $C_H(e_3) \sim 2^{24}.Co_3$, and $C_H(e_4) \sim 2^{24}.(2^{11} : M_{24})$.* □

We will classify the classes of involutions outside E case by case, depending on whether $\bar{x}$ is of type $2A$, $2B$ or $2C$. We start with a general lemma. Let $U = C_E(\bar{x})$ and $V = [E, \bar{x}]$. Let $X = \langle x, U \rangle$ and $\tilde{X} = X/V$. Let C be the full preimage in H of $C_{\bar{H}}(\bar{x})$.

Lemma 2.8 *Suppose $x \in H$ is an involution, and $\bar{x} \neq 1$. Then the following hold:*

(1) *An element $y \in xE$ is an involution if and only if $y \in xU$.*

(2) *The subgroups X and V are invariant under C; furthermore, E acts trivially on $\tilde{X}$.*

(3) *If $y, z \in xU$ then y and z are conjugate in H if and only if $\tilde{y}$ and $\tilde{z}$ are in the same $\bar{C}$-orbit.*

Remark. Part (2) contends that the action of $\bar{C} = C_{\bar{H}}(\bar{x})$ on $\tilde{X}$ is well defined, which allows us to view $\tilde{X}$ as a $\bar{C}$-module. Since $U = X \cap E$, $\tilde{U}$ is invariant under $\bar{C}$, and so $\bar{C}$ permutes the vectors in $\tilde{X} \setminus \tilde{U}$. The meaning of part (3) is that the orbits of $\bar{C}$ on $\tilde{X} \setminus \tilde{U}$ bijectively correspond to those conjugacy classes of involutions in H that map onto $\bar{x}^{\bar{H}}$.

Proof: If $e \in E$ then $(xe)^2 = [x, e]$ since both x and e are involutions. Part (1) follows. Clearly, V is invariant under C. Since X is generated by all the involutions from the coset $\bar{x} = xU$, X is C-invariant, too. Clearly, E acts trivially on $\tilde{U}$. Furthermore, E fixes $\tilde{x}$, because $V = [E, x]$. This proves (2). For (3), let $y, z \in xU$. If $z = y^h$ for some $h \in H$ then $\bar{h} \in \bar{C}$, since $\bar{y} = \bar{x} = \bar{z}$. So $\tilde{y}$ and $\tilde{z}$ are in the same $\bar{C}$-orbit. Reversely, suppose that $\tilde{y}^{\bar{c}} = \tilde{z}$ for some $c \in C$. Then $y^c = zv$ for some $v \in V$. Since $V = [E, x]$ there exists an element $e \in E$ such that $v = [e, x]$. However, $[e, x] = [e, z]$, since $z \in xU$. Therefore, $z^e = vz = zv$, implying that y and z are conjugate. □

We will first classify those involutions x for which $\bar{x}$ is of type $2A$. We will need the following fact proved in [Po].

Lemma 2.9 *Suppose Y is a $GF(2)$-module for $\Omega_8^+(2)$ that is an extension of an irreducible 8-dimensional submodule Y_0 by a 1-dimensional module. Then Y splits.* □

Lemma 2.10 *The group H has exactly three classes of involutions whose images in $\bar{H}$ are of type $2A$. If a_1, a_2 and a_3 are representatives of these classes then $C_H(a_i)$ has the structure $2^{16}.2^{1+8}.\Omega_8^+(2)$, $2^{16}.2^{1+8}.Sp_6(2)$ and $2^{16}.2^{1+8}.(2^6 : L_4(2))$, for $i = 1$, 2 and 3, respectively.*

Proof: Let x be an element of H such that $\bar{x}$ is of type $2A$. We will first show that the coset $\bar{x}$ contains an involution, and so x can be chosen to be an involution. Let $R = \langle x, E\rangle$. Then R is a normal subgroup of C, where C is defined, as above, as the full preimage in H of $C_{\bar{H}}(\bar{x})$. Let $U = C_E(x)$ and $V = [E, x]$. Clearly, $U = Z(R)$. Consider $X = R/U$. According to Lemma 2.4, C has two chief factors within X, of dimensions 8 and 1. This implies that X is an elementary abelian group, which we can view as a module for $\bar{C}$. Furthermore, by the same Lemma 2.4, the 8-dimension chief factor in X is not a natural module for $\bar{C}/O_2(\bar{C}) \cong \Omega_8^+(2)$. Therefore, $O_2(\bar{C})$ acts trivially on X. Now Lemma 2.9 implies that X contains a 1-dimensional subspace T invariant under C. Let R_0 be the full preimage in R of T. Clearly, R_0 is normal in C. Next, define $X_0 = R_0/V$. Again X_0 is an extension of an 8-dimensional chief factor U/V by a 1-dimensional R_0/U. We conclude again that X_0 is elementary abelian. Clearly, R_0 acts trivially on X_0. Furthermore, by Lemma 2.4 the 8-dimensional chief factor in X_0 differs, as a module, from the chief factors in $O_2(C)/R_0$, which means that $O_2(C)$ acts trivially on X_0. Applying again Lemma 2.9 we obtain a C-invariant 1-dimensional subspace T_0 in X_0. Let R_1 be the full preimage of T_0 in R. Setting $X_1 = R_1$, we observe for the third time that X_1 is an extension of an 8-dimensional chief factor V by a 1-dimensional one, R_1/V. Hence R_1 is elementary abelian. Since $R_1 \not\leq E$, we finally conclude that the coset $\bar{x}$ contains some involutions. Without loss of generality we can now assume that x is itself an involution and so Lemma 2.8 applies. In the notation introduced before Lemma 2.8, $\tilde{X}$ (which has already appeared above as $X_0 = R_0/V$) is the direct sum of a halfspin module and a 1-dimensional module. Therefore, C has three orbits on $\tilde{X} \setminus \tilde{U}$, of sizes 1, 120 and 135, and this immediately leads to the conclusion as in the lemma. □

The classification of involutions x with $\bar{x}$ of type $2B$ or $2C$ is an easy corollary of Lemmas 2.5 and 2.6.

Lemma 2.11 *For $L = B$ or C, H has a unique conjugacy class of involutions whose images in $\bar{H}$ are of type $2L$. If b and c are representatives of those two classes then $C_H(b) \sim 2^{12}.(2^2 \times G_2(4)).2$ and $C_H(c) \sim 2^{12}.(2^{11} : \mathrm{Aut}\, M_{12})$.*

Proof: Let x be an element of H such that $\bar{x}$ is of type $2L$. Then according to Lemmas 2.5 and 2.6, we have that $C_E(x) = [E, x]$. In particular, $x^2 = [e, x]$ for some $e \in E$. It follows that $(xe)^2 = x^2[x, e] = 1$, that is, the coset xE contains involutions. Furthermore, in the notation of Lemma 2.8 we have that $\tilde{X}$ is 1-dimensional, and so the claim follows. □

This completes the classification of conjugacy classes of involutions in H. According to Lemmas 2.7, 2.10 and 2.11, the group H contains eight classes of involutions. We will refer to these classes as to the classes $2e_i$, $2 \leq i \leq 4$, $2a_i$, $1 \leq i \leq 3$, $2b$ and $2c$.

3 A fusion lemma and an application

In the first part of this section G is an arbitrary group having a large extraspecial subgroup. This means that for some involution $z \in G$ the centralizer $C = C_G(z)$ contains a normal extraspecial 2-subgroup Q and, furthermore, $C_G(Q) \leq Q$. This implies that G contains a unique class of 2-central involutions (recall that a 2-central involution is an involution in the center of some Sylow 2-subgroup of G) and that z is itself 2-central. We let $\mathcal{S}$ denote the class of 2-central involutions in G. For $x = z^g \in \mathcal{S}$ we denote $C_x = C_G(x) = C^g$ and $Q_x = Q^g$. Thus, $C = C_z$ and $Q = Q_z$.

We will assume throughout this section that

$$(*) \quad \mathcal{S} \cap C \neq \{z\}.$$

Indeed the principal case of interest for us is where G is simple. However in that case the Z^*-theorem of Glauberman makes $\mathcal{S} \cap C = \{z\}$ impossible. In this section we prove that, modulo some small configurations, $(*)$ implies the following stronger condition:

$$(**) \quad \mathcal{S} \cap Q \neq \{z\}.$$

Let, as above, $Q = Q_z$ and let $\bar{Q}$ denote Q/Z where $Z = \langle z \rangle$.

Lemma 3.1 *Suppose $\mathcal{S} \cap Q = \{z\}$ and let $x \in \mathcal{S} \cap C$, $x \neq z$. Denote $E = Q \cap Q_x$. Then one of the following holds:*

(1) $E = 1$*; or*

(2) $|E| = 2$ *and either*

  (a) $\bar{E} \not\leq [\bar{Q}, x]$*, or*

  (b) $z \neq [x, y]$ *for all* $y \in Q$*; or*

(3) $|E| = 4$ *and furthermore, for* $W = \langle E, z, x \rangle$*,*

  (a) $N_G(W)$ *induces on* $W \cong 2^4$ *either* $O_4^-(2)$*, or* $\Omega_4^-(2)$ *acting as on the natural module;*

  (b) $|W \cap \mathcal{S}| = 5$ *and, under the identification of* W *with the orthogonal module, the involutions in* $W \cap \mathcal{S}$ *are the singular vertors; moreover, for each* $w \in W \cap \mathcal{S}$*,* $W \cap Q_w$ *is the perp of* w*.*

Proof: Suppose that $E \neq 1$. (Otherwise, (1) holds.) If $e \in E$ then $e^2 \in Z \cap \langle x \rangle = 1$, since E is contained in both Q and Q_x. Hence E is elementary abelian. Let U and V be defined as the full preimages in Q of $C_{\bar{Q}}(x)$ and $[\bar{Q}, x]$, respectively. Observe that since Q is extraspecial we have that $V = C_Q(U) = Z(U)$ and that $C_Q(x)$ is either equal to U or $[U : C_Q(x)] = 2$. In the latter case, $[x, y] = z$ for all $y \in U \setminus C_Q(x)$.

Notice that $[C_Q(x), E] \leq [Q, Q] \cap [C_Q(x), Q_x] \leq Z \cap Q_x$. By assumption, $\mathcal{S} \cap Q = \{z\}$ and hence $\mathcal{S} \cap Q_x = \{x\}$. We conclude that $z \notin Q_x$ and hence $[C_Q(x), E] = 1$.

Let $e \in E^{\#}$ and suppose $e = [x, y]$ for some $y \in Q$. Then $ex = x^y$ is a conjugate of x contained in Q_x, *i.e.*, $\mathcal{S} \cap Q_x \neq \{x\}$. This contradiction shows that no nontrivial element from E is an elementary commutator $[x, y]$, for $y \in Q$.

If $C_Q(x) \neq U$ then $[U : C_Q(x)] = 2$, which implies that $[C_Q(C_Q(x)) : V] = 2$. Hence $[E : E \cap V] \leq 2$. If $E \cap V = 1$ then $|E| = 2$, implying (2a). So let us assume that $E \cap V \neq 1$ and let $e \in (E \cap V)^{\#}$. Since $e \in V$, we have that either $e = [x, y]$ or $ez = [x, y]$ for some $y \in Q$. However, by the preceding paragraph $e \neq [x, y]$. So $ez = [x, y]$. Furthermore, $z = [x, t]$ for $t \in U \setminus C_Q(x)$. Thus, $[x, ty] = [x, t]^y [x, y] = zez = e$; a contradiction.

Now assume that $C_Q(x) = U$ and so $z \neq [x, y]$ for all $y \in Q$. Also, $E \leq V$, since $[C_Q(x), E] = 1$. If $|E| = 2$ then we obtain (2b). Hence we may assume that $|E| \geq 4$. Set $W = \langle E, z, x \rangle$. Our next step is to determine which involutions from W are in $\mathcal{S}$. First of all, the involutions in W fall into the following types: z, x, e, ze, xe, zx, and zxe, where e denotes an arbitrary involution from $E^{\#}$. Clearly, $z, x \in \mathcal{S}$. By assumption, no other involution in $Q \cup Q_x$ is in $\mathcal{S}$. Hence the involutions e, ze, and xe are not in $\mathcal{S}$. Since $E \leq V$, we have that $\bar{e} = [\bar{y}, x]$ for some $y \in Q$. We have shown above that $e \neq [x, y]$. Hence $ze = [x, y]$. It follows that $zxe = xze = x[x, y] = x^y$. Thus, all elements zxe are in $\mathcal{S}$. The element zx may or may not be in $\mathcal{S}$.

Observe also that the element y above normalizes W. Indeed, $\langle E, z \rangle$ is normal in Q, so y leaves it invariant. Also $x^y = zxe \in W$. Thus, $W^y = W$. We conclude that x and all the elements zxe are conjugate under $N_G(W)$. Symmetrically, z is conjugate under $N_G(W)$ to the elements zxe and so also to x. Notice that $W_x = \langle E, x \rangle$ is an index two subgroup of W such that $|W_x \cap \mathcal{S}| = \{x\}$. Pick an element $e \in E^{\#}$. By transitivity, there exists a subgroup W_{zxe} of index 2 in E such that $W_{zxe} \cap \mathcal{S} = \{zxe\}$. Since z and x are not in W_{zxe}, we have that $zx \in W_{zxe}$. Thus, $zx \notin \mathcal{S}$, which completes the enumeration of the elements in $W \cap \mathcal{S}$.

If $|E| > 4$ then for every $e' \in E^{\#}$ there exist elements e_1 and $e_2 = e_1 e'$ such that $e_1 \neq e \neq e_2$. Since both zxe_1 and zxe_2 are not in W_{zxe}, we conclude that $e' = (zxe_1)(zxe_2)$ is in W_{zxe}, *i.e.*, $E \leq W_{zxe}$. However, in that case all elements zxe', $e' \in E^{\#}$, are in $W_{zxe} \cap \mathcal{S}$, a contradiction. This establishes that $|E| = 4$ and, consequently, $|W \cap \mathcal{S}| = 5$. Observe that the elements y from the preceeding paragraph stabilize z. This proves that $N_G(W)$ induces a 2-transitive group on $W \cap \mathcal{S}$. Also $y^2 \in Z$ for all those elements y, which rules out the Frobenius group F_5^4. Hence $N_G(W)$ induces on $W \cap \mathcal{S}$ one of the groups $S_5 \cong O_4^-(2)$ or $A_5 \cong \Omega_4^-(2)$ and the claim (2) follows. □

In the remainder of this section $G = M$, the Monster, z is a 2-central involution in G, $Z = \langle z \rangle$, $C = C_z = C_G(z)$ and $Q = Q_z = O_2(C_z)$. Since by assumption M is simple, Glauberman's Z^* theorem [Gl] shows that $\mathcal{S} \cap C \neq \{z\}$. Recall that $\mathcal{S}$ denotes the conjugacy class of 2-central involutions, z^M. Recall alsoy that for $x = z^g$ we set $C_x = C^g$ and $Q_x = Q^g$. Since M has a large extraspecial subgroup, Lemma 3.1 applies to it. We use that lemma to prove the following.

Proposition 3.2 $Q \cap \mathcal{S} \neq \{z\}$.

Proof: Suppose $Q \cap \mathcal{S} = \{z\}$. Since $C \cap \mathcal{S} \neq \{z\}$, we can choose $x \in C \cap \mathcal{S}$, $x \neq z$. According to Lemma 3.1, one of the exceptional cases (2a), (2b), or (3) must hold. In particular, $E = Q \cap Q_x$ has size at most four.

Observe now that the group $H = \bar{C} = C/Z$ satisfies the conditions (H1)–(H3) from Section 2. In particular, we can use the classification of conjugacy classes of involutions obtained in that section. Let $D = C \cap C_x$ and $R = Q_x \cap C$. Clearly, R is normal in D, and $\bar{D}$ is of index two or one in $C_{\bar{C}}(\bar{x})$ depending on whether or not x and xz are conjugate in C.

Suppose first that $\bar{x}$ is in the class $2a_i$ for some i. Then also $z\langle x \rangle$ is in $2a_i$ in $C_x/\langle x \rangle$. In particular, R is of order at least 2^{16} (cf., Lemma 2.4). Consider $\tilde{C} = C/Q$. Since $E = Q \cap R$ is of order at most four, we obtain that $\tilde{D}$ contains a normal 2-subgroup $\tilde{R}$ of order at least 2^{14}. Comparing with Lemma 2.10, we see that $i = 3$ must hold. However, $i = 3$ also leads to a contradiction. Indeed, let $\tilde{Y}$ be the normal extraspecial subgroup 2^{1+8} of $C_{\tilde{C}}(\tilde{x}) \sim 2^{1+8}.\Omega_8^+(2)$. We have that $[\tilde{Y} : \tilde{Y} \cap \tilde{R}] \leq 2$ and $[\tilde{R}, \tilde{R}] \leq \langle \tilde{x} \rangle = Z(\tilde{Y})$. Therefore, all elements of $\tilde{R}$ centralize a hyperplane in the 8-dimensional quotient of $\tilde{Y}$, which is impossible.

Suppose next that $\bar{x}$ is in $2b$. Then also $z\langle x \rangle$ is in the class $2b$ in $C_x/\langle x \rangle$. Hence $|R| \geq 2^{12}$. Considering again $\tilde{C} = C/Q$ and taking into account that $|E| \leq 4$, we see that $\tilde{D}$ contains a normal 2-group of size at least 2^{10}, clearly contradicting Lemma 2.11.

Finally, suppose $\bar{x}$ is in the class $2c$. In this case our argument must be slightly more subtle. Let U be the full preimage in Q of $\bar{U} = C_{\bar{Q}}(\bar{x})$. Then U is a subgroup of order 2^{13}. We claim that $C_Q(x)$ is a proper subgroup of U. Indeed, according to Lemma 2.4, $\bar{U}$ contains elements from the class $2e_3$. Observe that the mapping $\bar{e} \mapsto e^2$ defines a nondegenerate quadratic form g on $\bar{Q}$. Since, as a module for $C/Q \cong Co_1$, $\bar{Q}$ is absolutely irreducible, this quadratic form is unique, and hence g is equivalent to the form q (cf. Section 2). In particular, if $\bar{e}$ is in $2e_3$ then e is of order four. Since $\bar{U} = [\bar{Q}, \bar{x}]$, we have that $\bar{e} = [\bar{q}, \bar{x}]$ for some $q \in Q$. Therefore, $[q, x] = e$ or e^3. Since, clearly, q can be chosen to be an involution, we obtain that x inverts e, *i.e.*, $e \notin C_Q(x)$.

This has two consequences. First, $C_Q(x)$ is of size 2^{12}, and symmetrically, also $|R| = 2^{12}$. (Clearly, $z\langle x \rangle$ must also be in the class $2c$ in $C_x/\langle x \rangle$.) Secondly, we record for further use that $z = [q, x]$ for some $q \in Q$.

Since $|E| \leq 4$, we have that $\tilde{R}$ (where, as above, $\tilde{C} = C/Q$) has size 2^{10}, 2^{11}, or 2^{12}. Comparing with Lemma 2.11 and using that $\tilde{R}$ is normal in $\tilde{D}$, we obtain that $|\tilde{R}| = 2^{11}$, and hence $|E| = 2$. This means that either (2a) or (2b) of Lemma 3.1 must hold. Above we recorded that $z = [q, x]$ for some $q \in Q$. Hence, in fact, it must be the case (2a). To obtain a contradiction in this last case, it remains to see that $\bar{E} \leq [\bar{Q}, x]$. However, this is clear because $\bar{E} \leq C_{\bar{Q}}(\bar{x}) = [\bar{Q}, \bar{x}] = [\bar{Q}, x]$. □

4 Singular subgroups

First, let G be again a group with a large extraspecial subgroup, that is, let there be an involution $z \in G$ and an extraspecial 2-subgroup Q normal in $C = C_G(z)$ such that $C_G(Q) \leq Q$. Adopt the notation from Section 3, that is, let $\mathcal{S} = \{z^G\}$ be the class of 2-central involutions in G and, for $x = z^g \in \mathcal{S}$, let $C_x = C_G(x) = C^g$ and $Q_x = Q^g$.

Let x and y be two 2-central involutions. We will say that x *perpendicular* to y if and only if $y \in Q_x$. The following important lemma is a slight improvement on [As0], Lemma 8.7 (3).

Lemma 4.1 *The perpendicularity relation is symmetric.*

Proof: If $|Q| > 2^3$ then this is proven in [As0], Lemma 8.7 (3). So suppose $Q \sim 2^{1+2}$. Suppose that the relation is not symmetric so that for some $x \in \mathcal{S}$ we have that $x \in Q$, but $z \notin Q_x$. In particular, there is no $g \in G$ such that $z^g = x$ and $x^g = z$. Since $x \in Q$, Q must be isomorphic to D_8. Since $C_G(Q_x) = \langle x \rangle$ and since $\operatorname{Out} D_8$ is of order two, we have that

$C_x = Q_x\langle z\rangle \cong D_{16}$. It remains to notice that an element from $N_{C_x}(U)$, where $U = \langle z, x\rangle$, permutes z and zx and, likewise, an element from $N_C(U)$ permutes x and zx. Hence the normalizer of U induces on it the full group S_3. Thus, there exists an element $g \in G$ such that $z^g = x$ and $x^g = z$; a contradiction. $\square$

Let U be a purely 2-central (*i.e.*, all involutions in U are in $\mathcal{S}$) elementary abelian 2-subgroup of G. We will say that U is *singular* if $U \leq Q_u$ for every $u \in U^\#$. If U is singular define $Q_U = \cap_{u\in U^\#} Q_u$ and $L_U = \langle Q_u \mid u \in U^\#\rangle$. Clearly, $U \leq Q_U \leq L_U$.

Lemma 4.2 *Let U be singular. Then the following hold:*

(1) *U and Q_U are normal in L_U and L_U acts trivially on Q_U/U;*

(2) *if $W \leq Q_U$ and $W \cap U = 1$ then $C_{L_U}(W)$ induces on U the full group $L_n(2)$ (where n is the rank of U); in particular (for $W = 1$), $N_G(U) = L_U C_G(U)$;*

(3) *if $W_1, W_2 \leq Q_U$, $W_1 \cap U = W_2 \cap U = 1$ and $W_1U = W_2U$ then there is an element $x \in C_{L_U}(U)$ such that $W_1^x = W_2$; in particular, $C_{L_U}(U)$ acts transitively on every coset qU, $q \in Q \setminus U$;*

(4) *if $|U| > 2$ then Q_U is elementary abelian.*

Proof: Let $U \leq U' \leq Q_U$. If $u \in U^\#$ then $U' \leq Q_u$. Since Q_u is extraspecial, U' is normal in Q_u and hence U' is normal in L_U. This proves (1).

For W as in (2), take $U' = WU$. Clearly, U' is elementary abelian and so we can view it as a $GF(2)$-vector space. Notice that Q_u induces on U' all transvections with center $\langle u\rangle$. Since $W \cap U = 1$, $C_{Q_u}(W)$ induces on U all transvections with center $\langle u\rangle$. This implies (2), since the group generated by all transvections of U is $L_n(2)$.

Let W_1 and W_2 be as in (3). We will use induction on the rank of W_1. If $W_1 = 1$ then there is nothing to prove. Otherwise, choose $W_1' \leq W_1$ such that $|W_1/W_1'| = 2$, and let $W_2' = W_2 \cap W_1'U$. By induction, there is an element $x' \in C_{L_U}(U)$ such that $(W_1')^x = W_2'$. Let $w_1 \in W_1 \setminus W_1'$ and let $\{w_2\} = W_2 \cap w_1U$. Notice that, by (1), $w_1^x \in w_1U = w_2U$ and hence $w_1^x = w_2u$ for some $u \in U$. If $u = 1$ then take $x = x'$. Otherwise, let y be an element of Q_u that induces on W_1U the transvection with center $\langle u\rangle$ and axis $W_2'U$. Clearly, $y \in C_{L_U}(U)$ and $W_1^x = W_2$, where $x = x'y$. This proves (3).

Finally, if $q \in Q_U$ then $q^2 \in \Phi(Q_u) = \langle u \rangle$ for every $u \in U^{\#}$. This proves (4). $\square$

Lemma 4.3 *A subgroup U is singular if and only if it is generated by a set of pairwise perpendicular 2-central involutions. Furthermore, if $U = \langle u_1, \ldots, u_k \rangle$ ($u_i \neq 1$ for all i) is singular then $Q_U = \cap_{i=1}^k Q_{u_i}$.*

Proof: We only need to prove the 'if' part of the first claim. Suppose $U = \langle u_1, \ldots, u_k \rangle$, where $u_1, \ldots, u_k$ are 2-central and pairwise perpendicular. By induction, $U' = \langle u_2, \ldots, u_k \rangle$ is singular. Since u_1 is perpendicular to $u_2, \ldots, u_k$, we have that $U' \leq Q_{u_1}$ which by Lemma 4.1 implies that $u_1 \in Q_{U'}$. By Lemma 4.2 (2), all involutions in $U \setminus U' = u_1U'$ are 2-central, since u_1 is 2-central. Finally, let $u \in U^{\#}$. Then $u \in U \leq Q_{u_i}$ for every i, since the involutions u_i are pairwise perpendicular. By Lemma 4.1, $U = \langle u_1, \ldots, u_k \rangle \leq Q_u$.

Suppose now that $U = \langle u_1, \ldots, u_k \rangle$ is singular. Clearly, $Q_0 = \cap_{i=1}^k Q_{u_i}$ contains Q_U. So it remains to see that $Q_0 \leq Q_U$. Let us use induction on k. The claim is obviously true if $k = 1$. Consider now the case $k > 1$ and set $U' = \langle u_2, \ldots, u_k \rangle$. By induction, $Q_{U'} = \cap_{i=2}^k Q_{u_i}$ and hence $Q_0 \leq Q_u$ for all $u \in U^{\#}$. However, this means that Q_0 is normal in Q_u. In particular, Q_0 is invariant under an element $x \in Q_u$ which induces on U a transvection taking u_1 to u_1u. Hence $Q_0 = Q_0^x \leq Q_{u_1}^x = Q_{u_1u}$. Thus $Q_0 \leq Q_u$ for all $u \in U^{\#}$. $\square$

We now switch back to the case $G = M$. Our goal is to classify all singular subgroups in M up to conjugation. Notice that Proposition 3.2 means that the perpendicularity relation on 2-central involutions in M is nontrivial, that is, there exist singular subgroups of size more than two. We start by getting the details of the perpendicularity relation in M. For that we need to know the fusion of involutions in Q. Let $\bar{C} = C/Z$, where $Z = \langle z \rangle$. Recall that the classes of involutions in $\bar{C}$ were determined in Section 2.

Lemma 4.4 *The group C has exactly two classes of involutions $x \neq z$, contained in Q. If q_2 and q_4 are representatives of those classes then $C_C(q_2) \sim 2^{1+23}.Co_2$ and $C_C(q_4) \sim 2^{1+23}.(2^{11} : M_{24})$. Furthermore, q_4 is 2-central and q_2 is not.*

Proof: For $x \in Q \setminus Z$, the mapping $\bar{x} \mapsto x^2$ defines a nondegenerate quadratic form g on $\bar{Q}$. Since the action of $C/Q \cong Co_1$ on $\bar{Q}$ is absolutely irreducible, g is unique and hence g is equivalent to the form q existing on

$\hat{\Lambda} = \Lambda/2\Lambda$ (cf. Section 2). The form q is zero on $\hat{\Lambda}_2$ and $\hat{\Lambda}_4$, and it is non-zero on $\hat{\Lambda}_3$. This means that x is an involution if and only if $\bar{x}$ belongs to the class $2e_2$ or $2e_4$. The involutions x and xz are conjugate in Q, because Q is extraspecial. Combined with Lemma 2.7, this establishes the first two claims of the lemma.

According to Proposition 3.2, at least one of q_2 and q_4 is conjugate to z in G. So, to complete the proof of the lemma, it suffices to show that $x = q_2$ is not 2-central. Suppose that $x \in \mathcal{S}$. Let $D = C_C(x) = C \cap C_x$ and $R = Q \cap Q_x$. From the structure of D (see above), it is clear that $R = O_2(D) \sim 2^{1+23}$. Since $R \leq Q$, we have that $[R, R] = Z$. Symmetrically, since $R \leq Q_x$ we have that $[R, R] = \langle x \rangle$, implying that $z = x$, a contradiction. □

In particular, if a 2-central involution $y \neq z$ is perpendicular to z then y is conjugate in C to $x = q_4$. This lemma implies that every singular subgroup $U \sim 2^2$ in M is conjugate to $\langle z, q_4 \rangle$. So there is only one conjugacy class of such subgroups.

Lemma 4.5 *Let $U \sim 2^2$ be singular. Then $W = Q_U/U \sim 2^{11}$ and $C_M(U)$ induces on W a group M_{24} acting as on the Todd module. Under the identification of W with the Todd module, the images of 2-central involutions from $Q_U \setminus U$ correspond to sextets, while the images of non 2-central involutions correspond to pairs.*

Proof: Without loss of generality, $U = \langle z, x \rangle$, where $x = q_4$. Let $D = C \cap C_x = C_M(U)$ and $R = Q \cap Q_x$. Notice that by Lemma 4.3 we have $R = Q_U$. Recall that $\bar{Q} = Q/Z$ affords a quadratic form g defined by $\bar{y} \mapsto y^2$. By Lemma 4.2 (4), Q_U is elementary abelian. In particular, $\bar{R}$ is a totally singular subspace with respect to g. This implies that $|\bar{R}| \leq 2^{12}$, and hence $|R| \leq 2^{13}$. On the other hand, both $C \cap Q_x$ and $C_x \cap Q$ have order 2^{24} and they are normal in D. Since $(C \cap Q_x) \cap (C_x \cap Q) = R$, the order of $(C \cap Q_x)(C_x \cap Q)$ is at least $2^{24+24-13} = 2^{35} = |O_2(D)|$ (see Lemma 4.4 for the structure of D). Hence $|R| = 2^{13}$.

It follows that $\bar{R}$ is a 12-dimensional subspace in $\bar{Q}$ invariant under the monomial group $D/Q \sim 2^{11} : M_{24}$. Such a subspace is known to be unique. Identifying $\bar{Q}$ with $\hat{\Lambda}$ and assuming that $\bar{x}$ is the image of the standard frame, we get that $\bar{R}^{\#}$ consists of the images of the vectors of the shape $\pm 8^1 0^{23}$ ($\bar{x}$), $\pm 4^2 0^{22}$ ($\hat{\Lambda}_2$, non 2-central), and $\pm 4^4 0^{20}$ ($\hat{\Lambda}_4$, 2-central). Each pair of coordinates gives four vectors of the second kind, mapping onto two elements in $\bar{R} \cap \hat{\Lambda}_2$. These two elements of $\bar{R}$ sum up to $\bar{x}$. Similarly, every sextet produces 96 vectors of the third kind (two frames), mapping onto two elements in $\bar{R} \cap \hat{\Lambda}_4$. These two elements of $\bar{R}$ again sum up to

$\bar{x}$. Thus, in $R/U \cong \bar{R}/\langle\bar{x}\rangle$, the nonidentity elements correspond simply to pairs and sextets. By Lemma 4.4, the elements from $R/U^{\#}$ corresponding to pairs (respectively, sextets) are the images of non 2-central (respectively, 2-central) involutions from $R \setminus U$. □

For the record, the normalizer of a singular subgroup $U \sim 2^2$ is now known to be an extension of a normal 2-subgroup of order 2^{35} by $S_3 \times M_{24}$. (The latter being the action of $N_M(U)$ on $U \times Q_U/U$.)

In the above proof, if we do not assume that $\bar{x}$ is the image of the standard frame then the condition for $\bar{y}$ to be in $\bar{R}$ looks as follows: Let $\{v_i\}$ be the frame corresponding to $\bar{x}$ (*i.e.*, $\hat{v}_i = \bar{x}$ and $v_i \in \Lambda_4$ for all i) and let u be a short vector in Λ (*i.e.*, a vector from $\Lambda_2 \cup \Lambda_3 \cup \Lambda_4$) such that $\bar{v} = \hat{y}$. Then $\bar{y} \in \bar{R}$ if and only if $(v_i, y) \in \{0, \pm 4, \pm 8\}$ for all i. When $\{v_i\}$ is the standard frame, this corresponds to the statement in the above proof about the shapes of the vectors mapping into $\bar{R}$.

Combining Lemma 4.2 (3) with the fact that M_{24} acts transitively on pairs and on sextets, we obtain the following.

Corollary 4.6 *If $U \cong 2^2$ is singular then $C_M(U)$ has exactly two conjugacy classes in $Q_U \setminus U$, one consisting of non 2-central involutions, and one other consisting of 2-central involutions.* □

Since $Q_U \setminus U$ contains a unique class of 2-central involutions, M has exactly one conjugacy class of singular subgroups 2^3.

Before we proceed further we need to understand better the perpendicularity relation among the elements in $Q_U \setminus U$, where U is a singular subgroup 2^2. For a non 2-central (respectively, 2-central) involution $y \in Q_U \setminus U$, let $P(y)$ (respectively, $S(y)$) be the pair (respectively, sextet) corresponding to $yU \in Q_U/U$.

We say that two sextets S_1 and S_2 intersect *evenly* if $|T_1 \cap T_2|$ is even for all tetrads $T_1 \in S_1$ and $T_2 \in S_2$. Suppose T_1 and T_2 are two tetrads and suppose $|T_1 \cap T_2| = 2$. Then the sextets defined by T_1 and T_2 intersect evenly if and only if $T_1 \cup T_2$ is contained in an octad. This allows us to compute that every sextet evenly intersects exactly 90 other sextets.

Lemma 4.7 *Let $U \sim 2^2$ be singular. Suppose $y, t \in Q_U \setminus U$, and suppose y is 2-central. Then*

(1) *if t is non 2-central then $t \in Q_y$ if and only if $P(t)$ is contained in one of the tetrads from $S(y)$; and*

(2) *if t is 2-central then $t \in Q_y$ if and only if $S(t)$ and $S(y)$ intersect evenly.*

Proof: Assume again that $U = \langle z, x\rangle$, where $x = q_4$. Let $\{v_i\}$ be the frame in Λ that corresponds to $\bar{y}$ and let u be a short vector in Λ such that $\hat{u}$ corresponds to $\bar{t}$. Then the vectors v_i are of the shape $\pm 4^4 0^{20}$, where the nonzero coordinates appear in a tetrad from the sextet $S(y)$. Similarly, u is of shape $\pm 4^2 0^{22}$ (respectively, $\pm 4^4 0^{20}$) with the nonzero coordinates appearing in the pair $P(t)$ (respectively, sextet $S(t)$) if t is non 2-central (respectively, 2-central). According to the remark after the proof of Lemma 4.5 we have $t \in Q_y$ if and only if $(v_i, u) \in \{0, \pm 4, \pm 8\}$ for all i. The claim of the lemma follows. $\square$

One implication of Corollary 4.6 is that M contains exactly one conjugacy class of singular subgroups 2^3. Indeed, pick a 2-central involution $y \in Q_{\langle z,x\rangle} \setminus \langle z, x\rangle$. Then every singular subgroup 2^3 is conjugate to $\langle z, x, y\rangle$.

Lemma 4.8 *Let $U \sim 2^3$ be singular. Then $W = Q_U/U \sim 2^6$ and $C_M(U)$ induces on W a group $3 \cdot S_6$ that acts on W irreducibly. Furthermore, $N_M(U)$ has two orbits on $W^\#$: an orbit of length 18 (images of non 2-central involutions from $Q_U \setminus U$) and an orbit of length 45 (images of 2-central involutions).*

Proof: Without loss of generality, $U = \langle z, x, y\rangle$. We set $U_0 = \langle z, x\rangle$ and $V = \tilde{Q}_{U_0}/U_0$. According to Lemma 4.5, $C_M(U_0)$ induces on V a group M_{24} acting on V as on the Todd module. Let $S = S(y)$ be the sextet corresponding to $\tilde{y}$ under the identification of V with the Todd module. According to Lemma 4.7, $\tilde{Q}_U^\#$ consists of elements corresponding to pairs contained in the tetrads of S and to sextets evenly intersecting S. By counting, $\tilde{Q}_U^\#$ consists of 91 sextets (including S) and 36 pairs. Hence $|\tilde{Q}_U| = 2^7$. This means that $|Q_U/U| = 2^6$. Furthermore, $(Q_U/U)^\#$ contains 45 (respectively, 18) elements that are images of 2-central (respectively, non 2-central) involutions.

Recall that $C_M(U_0)$ induces on V a group M_{24}. The stabilizer of S in the latter group is a subgroup $2^6 : 3 \cdot S_6$. Let $D = N_M(U) \cap C_M(U_0)$ be the full preimage in $C_M(U_0)$ of the stabilizer of S. According to Lemma 4.2 (2), $N_M(U) \cap N_M(U_0)$ induces on $U \setminus U_0$ a group S_4. Since D is normal in $N_M(U) \cap N_M(U_0)$ and since $C_M(U)$ is the kernel of the action of D on $U \setminus U_0$, we conclude that $C_M(U)$ induces the whole sextet stabilizer $2^6 : 3 \cdot S_6$ in its action on V. Thus, it induces a quotient of $2^6 : 3 \cdot S_6$ on Q_U/U. Let $a \in C_M(U)$ be a 3-element mapping into the normal 3-subgroup of the quotient $3 \cdot S_6$. Consider the action of a on V. Clearly, a stabilizes every tetrad in the sextet S. Let S' be a sextet evenly intersecting S. Observe that every tetrad from S meets exactly two tetrads from S'. Being a 3-element, if a stabilizes S' then it must stabilize it tetradwise. However, in that case a stabilizes every part of a partition of $\{1, \ldots, 24\}$ into 12 pairs (intersections of tetrads from S with tetrads from S'), which makes a to act on $\{1, \ldots, 24\}$ trivially. This contradiction shows that a cannot stabilize S' and hence a acts nontrivially on Q_U/U. Therefore, $C_M(U)$ induces on Q_U/U either $2^6 : 3 \cdot S_6$ or $3 \cdot S_6$.

It is easy to see that the stabilizer of S in M_{24} acts transitively on pairs contained in tetrads from S and on sextets evenly intersecting S. Consequently, $C_M(U)$ has orbits of size 18 and 45 on Q_U/U. This makes the action on Q_U/U irreducible, implying that the group induced by $C_M(U)$ is in fact $3 \cdot S_6$. □

For the record, this lemma and Lemma 4.2 (2) imply that $N_M(U)$, where U is a singular subgroup 2^3, is an extension of a normal subgroup of order 2^{39} by $L_3(2) \times 3 \cdot S_6$.

Also, let us record what we proved about the classes of 2-central and non 2-central involutions in Q_U.

Corollary 4.9 *If $U \cong 2^3$ is singular then $C_M(U)$ has exactly two conjugacy classes in $Q_U \setminus U$, one consisting of non 2-central involutions, and one other consisting of 2-central involutions.*

Proof: Follows from Lemma 4.2 (3). □

In particular, M contains a unique conjugacy class of singular subgroups 2^4.

Lemma 4.10 *Let $U \sim 2^4$ be singular. Then $W = Q_U/U \sim 2^3$. Furthermore, $W^\#$ contains exactly three elements that are images of non 2-central involutions, and these three elements generate W. The group $C_M(U)$ induces on W a group S_3.*

Proof: Without loss of generality, $U \geq U_0 = \langle z, x \rangle$, say, $U = \langle z, x, y, t \rangle$. We will work with the Todd module $V = \tilde{Q}_{U_0} = Q_{U_0}/U_0$. Let $S = S(y)$ and $S' = S(t)$. If $s \in Q_U \setminus U$ is a non 2-central involution then $P(s)$ is contained in a tetrad from S and in a tetrad from S'. Hence $T(s)$ must be one the twelve pairs $P_1, \ldots, P_{12}$ (partitioning $\{1, \ldots, 24\}$) that are intersections of tetrads from S with tetrads from S'. This proves that Q_U/U contains exactly three involutions that are images of non 2-central involutions. (Indeed, the twelve involutions in $\tilde{Q}_U$ merge into three involutions in Q_U/U.)

Let us be more specific. Since S and S' intersect evenly, there is a unique trio $T := \{O_1, O_2, O_3\rangle$ of which both S and S' are refinements (we view trios and sextets as partitions of $\{1, \ldots, 24\}$). Then every O_i is a union of some four pairs P_j. It is easy to see that pairs P_j and P_k produce the same element in Q_U/U if and only if they are contained in the same octad O_i. Thus, the octads O_i correspond to the "non 2-central" elements $a_i \in Q_U/U$. Clearly, the stabilizer of S and S' in M_{24} induces an S_3 on the trio T. Hence also $N_M(U)$ induces an S_3 on the three involutions a_i. Furthermore, since $N_M(U)$ induces a simple group $L_4(2)$ on U, we also have that $C_M(U)$ induces an S_3 on the a_i's. It remains to see that they are linearly independent and generate Q_U/U.

Observe that if P_j and P_k belong to distinct octads O_i then the sum (we switch to the additive notation in V and Q_U/U) of the elements from V corresponding to P_j and P_k is of sextet type and, furthermore, that sextet is not a refinement of T. This means that the sum of two distinct involutions a_i is nontrivial and "2-central". This implies the linear independence. Let b be an arbitrary "2-central" element from $(Q_U/U)^{\#}$, say, it is the image of an element of V that corresponds to a sextet $S'' = \{R_1, \ldots, R_6\}$. Observe that S'' evenly intersects both S and S'. In particular, $|O_i \cap R_j|$ is even for all i and j. Suppose for some i and j we have $|O_i \cap R_j| = 2$. (We will say that such an S'' is of the *first kind*.) Observe that O_i is a union of some four pairs P_k. If R_j meets two of these pairs then R_j meets a tetrad from S or from S' in just one point, a contradiction. Hence, $O_i \cap R_j$ coincides with some P_k. Similarly, considering a nontrivial intersection of R_j with some other $O_{i'}$ we obtain that R_j contains a second pair $P_{k'}$ and hence b is the sum of two of the a_i's. It remains to consider the case where $|O_i \cap R_j| \in \{0, 4\}$ for all i and j, that is, every R_j is fully contained in some O_i. (Then we will say that S'' is of the *second kind*.) Fix O_i and R_j with $R_j \subset O_j$. If $P_k \subset R_j$ then $b + a_i$ is "non 2-central", which means that $R_j \setminus P_k = P_{k'}$. However, since P_k and $P_{k'}$ are both in O_i, we get $b = a_i + a_i = 0$, a contradiction. Therefore, R_i meets each of the four pairs P_k partitioning O_i in one point. Fix $P_k \subset O_i$ and consider $c = b + a_i$. Then one of the preimages of c in V will correspond to the sextet S''' containing the tetrad $R_j \triangle P_k$ ($\triangle$ denotes symmetric difference of sets). If S''' is of the second kind then S''' contains a tetrad contained in $O_{i'} \neq O_i$. That tetrad of S''' will meet some tetrad of S'' in at least two points. This gives us two octads meeting in five points, a contradiction. Therefore, S''' is of first kind. By the above, c is in the span of a_i's and hence so is also b. □

We will continue using the notation a_i for the three "non 2-central" elements from Q_U/U. According to Lemma 4.10, $C_M(U)$ has three orbits on $(Q_U/U)^\#$: $\{a_1, a_2, a_3\}$ ("non 2-central"), $\{a_1+a_2, a_1+a_3, a_2+a_3\}$ ("2-central", sextets of the first kind), and $\{a_1+a_2+a_3\}$ ("2-central", sextets of the second kind).

We record this as the following

Corollary 4.11 *If $U \cong 2^4$ is singular then $C_M(U)$ has exactly three conjugacy classes in $Q_U \setminus U$, two consisting of 2-central involutions, and one other consisting of non 2-central involutions.* □

For the record, the normalizer of a singular subgroup $U \sim 2^4$ is an extension of a normal subgroup of order 2^{39} by $L_4(2) \times S_3$.

It follows from Corollary 4.11 that M contains two conjugacy classes of singular subgroups 2^5. One of these two classes is represented by $\langle z, x, y, t, s\rangle$ with the image of s in $Q_{\langle z,x,y,t\rangle}/\langle z, x, y, t\rangle$ being $a_1 + a_2$, while for the other the image of s can be chosen as $a_1 + a_2 + a_3$. We will write "a singular subgroup 2^5_1" (respectively, 2^5_2) for the two types of singular subgroups 2^5.

We will need the following corollary of Lemma 4.11.

Corollary 4.12 *Every singular subgroup 2^4 is contained in exactly three singular subgroups 2^5_1 and a unique singular subgroup 2^5_2.* □

To complete the classification of singular subgroups of M we need to discuss perpendicularity between the elements of $Q_U \setminus U$.

Lemma 4.13 *Suppose $U \sim 2^4$ is singular. Let s and r be two elements from $Q_U \setminus U$, whose images in Q_U/U are distinct. If s is 2-central and $r \in Q_s$ then the image of s is $a_i + a_j$ for some i and j. Furthermore, the image of r is either a_i or a_j.*

Proof: Without loss of generality, $U = \langle z, x, y, t\rangle$ as in lemma 4.10. Since $r \in Q_s$, we have that r and rs are both 2-central or both non 2-central. This implies that the image of s cannot be $a_1 + a_2 + a_3$. Hence the image of s coincides with some $a_i + a_j$. Next, it is easy to see that if r' maps onto a_i or a_j then $r' \in Q_s$. Since no element mapping onto $a_1 + a_2 + a_3$ can be in Q_s, we conclude that the image of $Q_s \cap Q_U$ in Q_U/U coincides with $\langle a_i, a_j\rangle$. □

The information in Lemma 4.13 allows us to determine Q_U for singular subgroups $U \sim 2^5$.

Lemma 4.14 *The following hold.*

(1) *If U is singular 2^5_1 then Q_U/U is of order two. Furthermore, all involutions in $Q_U \setminus U$ are non 2-central.*

(2) *If U is singular 2^5_2 then $Q_U = U$.*

Proof: Follows from Lemma 4.13. □

For the record, the normalizer of a singular 2^5_1 is an extension of a subgroup of order 2^{36} by $L_5(2)$, while the normalizer of a singular 2^5_2 is an extension of a subgroup of order $2^{36}3$ by $L_5(2)$.

Lemma 4.14 means that M contains no singular subgroups of order more than 2^5 and so we have completed the classification of the singular subgroups in M.

Proposition 4.15 *The Monster group M contains exactly 6 classes of nontrivial singular subgroups. The corresponding orders are 2, 2^2, 2^3, 2^4, 2^5 and 2^5.* □

Let $\mathcal{S}_i$, $1 \leq i \leq 4$, denote the conjugacy class of all singular subgroups 2^i of M. For $i = 5$, we will use the notation $\mathcal{S}_{5,1}$ and $\mathcal{S}_{5,2}$ for the conjugacy classes of singular subgroups 2^5_1 and 2^5_2, respectively.

Notice that in this section we only indicated the order of the normalizers of singular subgroups and their action on $U \times Q_U/U$. A more detailed information about the structure of these 2-local subgroups can be found in the appendix.

5 Arks

From this section on, $G = M$, the Monster simple group. In this section we construct and study a class of subgroups 2^{10} of M, associated with singular subgroups.

Let U be a singular subgroup 2^5_1. According to Lemma 4.12, every index two subgroup of U is contained in a unique singular 2^5_2. Let $\mathcal{A} = \{U' \in \mathcal{S}_{5,2} | [U : U \cap U'] | = 2\}$ and let $A(U)$, the *ark* defined by U, be the subgroup of M generated by all $U' \in \mathcal{A}$. Clearly, $A(U)$ is invariant under $N_M(U)$.

Lemma 5.1 *The ark $A(U)$ is elementary abelian of order 2^{10}. Furthermore, U and $A(U)/U$ are dual to each other as modules for $N_M(U)$.*

Proof: Suppose $U', U'' \in \mathcal{A}$ with $U' \neq U''$. Then $W = U' \cap U'' \cap U$ is a singular subgroup 2^3. Since $U', U'' \leq Q_W$ and since Q_W is elementary abelian by Lemma 4.2 (4), we have that U' and U'' commute elementwise and, therefore, $A = A(U)$ is elementary abelian.

Consider $\bar{A} = A/U$. If $U' \in \mathcal{A}$ then $\bar{U}'$ is of order two. This yields a mapping $V \mapsto \bar{a}_V$ from the set of index 2 subgroups $V < U$ to $\bar{A}^{\#}$. Namely, $\langle \bar{a}_V \rangle = \bar{U}'$, where $U' \in \mathcal{A}$ is the only singular 2^5_1 containing V. Clearly, the elements $\bar{a}_V$ generate $\bar{A}$. Furthermore, the subgroups V correspond to the elements in $(U^*)^{\#}$, where U^* is the dual of U. Therefore, in order to complete the proof of this lemma it suffices to establish the three-term relations: $\bar{a}_{V_1}\bar{a}_{V_2}\bar{a}_{V_3} = 1$ whenever V_1, V_2 and V_3 are three index two subgroups of U, containing a given index four subgroup $W < U$.

Consider $\hat{Q}_W = Q_W/W$. According to Lemma 4.8, $\hat{Q}_W$ is 6-dimensional (as a vector space over $GF(2)$) and $C_M(W)$ induces on $\hat{Q}_W$ a group $3 \cdot S_6$. Let $x \in C_M(W)$ be a 3-element that maps onto a nontrivial element in the center of that action. Let V be a singular subgroup 2^4 containing W. Then $\hat{V}$ is of order two, and we claim that if U' is the unique singular 2^5_2 containing V then $\hat{U}' = \hat{V}\hat{V}^x$. Indeed, on the one hand, each of the 45 (cf. Lemma 4.8) subgroups V is contained in a unique U'. On the other hand, each U' contains three subgroups V. Therefore, W is contained in exactly 15 singular subgroups 2^5_2. It follows that each of them is invariant under x, since S_6 cannot nontrivially act on $15/3 = 5$ points. This proves our claim.

We can now finish the proof of the lemma. Suppose V_1, V_2 and V_3 are the three index two subgroups of U, containing W. Let U'_i, $i = 1, 2, 3$, be the unique singular 2^5_2 containing V_i. Working again in $\hat{Q}_W = Q_W/W$, we obtain that the image of $\langle U'_1, U'_2, U'_3 \rangle$ in $\hat{Q}_W$ coincides with $\hat{U}\hat{U}^x$, since $\langle V_1, V_2, V_3 \rangle = U$. Thus, $\langle \bar{a}_{V_1}, \bar{a}_{V_2}, \bar{a}_{V_3} \rangle = \langle U'_1, U'_2, U'_3 \rangle / U$ is of order four and hence $\bar{a}_{V_1}\bar{a}_{V_2}\bar{a}_{V_3} = 1$ holds. □

Let $U \in \mathcal{S}_{5,1}$ and let $\mathcal{A}$, $A = A(U)$ and $\bar{A} = A/U$ be as above.

Lemma 5.2 *The following hold.*

(1) *If $a \in A \setminus U$ then $\langle U, a \rangle = \langle U, U' \rangle$ for some $U' \in \mathcal{A}$. In particular, every coset of U in A contains a 2-central involution.*

(2) *If $a \in A \setminus U$ is 2-central then $U \cap Q_a$ is of index two in U. Furthermore, au (where $u \in U$) is 2-central if and only if $u \in U \cap Q_a$.*

Proof: Part (1) follows directly from Lemma 5.1. Let $U' \in \mathcal{A}$ be such that $\langle U, a \rangle = \langle U, U' \rangle$. Then $\langle U, a \rangle \leq Q_W$, where $W = U \cap U'$. Since perpendicularity is symmetric, we have that $W \leq Q_a$. On the other hand, $U \not\leq Q_a$, because otherwise $\langle U, a \rangle$ must be singular in view of Lemma 4.3.

Thus, $U \cap Q_a = W$ is of index two in U. Clearly, $\langle W, a\rangle$ is a singular subgroup; in particular, au is singular if $u \in W$. Comparing now with Lemma 4.10, we see that all elements in $\langle U, a\rangle \setminus (U \cup U')$ are non 2-central. Consequently, $a \in U'$ (and hence $U' = \langle W, a\rangle$) and au is non 2-central for all $u \in U \setminus W$. □

It follows from this lemma that A contains exactly $31 \cdot 16 = 496$ non 2-central and $31 + 31 * 16 = 527$ 2-central involutions. Moreover, all non 2-central involutions in A are conjugate to q_2. Also notice that both non 2-central and 2-central involutions generate A.

Next, we analize the embedding of the ark $A = A(U)$ in C_u for $u \in U^{\#}$. First of all, we claim that Lemma 5.2 implies that $A \cap Q_u$ has index two in A. Indeed, u is contained in 15 subgroups $W = U \cap U'$, $U' \in \mathcal{A}$, and hence $\overline{A \cap Q_u}$ is of order 16. Let $a \in A \setminus (A \cap Q_u)$. Since A is generated by 2-central involutions, we can choose a to be 2-central.

Lemma 5.3 *We have $A \cap Q_u = [Q_u, a]$ and the image of a in $C_u/Q_u \cong Co_1$ is a $2A$-involution.*

Proof: Notice that Q_u normalizes U and hence it also normalizes A. Therefore, $[Q_u, A] \leq A \cap Q_u$. If the image of a in Co_1 is of type $2B$ or $2C$ then Lemmas 2.5 and 2.6 imply that $|[Q_u, a]| \geq 2^{12}$, a contradiction. Hence, the image of a is of type $2A$. Furthermore, it follows from Lemma 2.4 that $[Q_u, a]\langle u\rangle/\langle u\rangle$ is of order 2^8, implying that $[Q_u, a]\langle u\rangle = A \cap Q_u$. Since $[Q_u, a]$ is normal in Q_u, it contains u and hence $[Q_u, a] = A \cap Q_u$. □

Let $D = N_{C_u}(Q_u\langle a\rangle) \sim 2^{1+24}.2^{1+8}.\Omega_8^+(2)$. Let $\bar{C}_u = C_u/\langle u\rangle$.

Lemma 5.4 *A is normal in D.*

Proof: Let $R = \langle Q_u, a\rangle$. Then $\bar{R}$ is normal in $\bar{D}$. Observe that $\bar{R}$ has exactly two maximal elementary abelian subgroups: $\bar{Q}_u$ and $\bar{R}_0 = \langle C_{\bar{Q}_u}(\bar{a}), \bar{a}\rangle$. Since Q_u is normal in D, we conclude that R_0 (defined as the full preimage of $\bar{R}_0$ in D) is also normal in D. We claim that $A = Z(R_0)$. Indeed, clearly, $A \cap Q_u = [Q_u, a]$ is the center of $R_0 \cap Q_u$, because Q_u is extraspecial and because $\overline{R_0 \cap Q_u} = C_{\bar{Q}_u}(\bar{a})$. Hence, it remains to see that $[R_0 \cap Q_u, a] = 1$. However, this is clear: since $\bar{R}_0$ is abelian, we have that $[R_0 \cap Q_u, a] \leq \langle u\rangle$; on the other hand, by Lemma 5.2 (2), the involution au is not 2-central. Hence u cannot be written as a commutator $[r, a]$ for $r \in R$. Since R_0 is normal in D and $A = Z(R_0)$, we finally obtain that A is normal in D. □

Let $N = N_M(A)$.

Corollary 5.5 *The action of N on A is irreducible. In particular, $A = \langle u^N \rangle$.*

Proof: According to Lemma 5.1, A has two 5-dimensional composition factors as a module for $N_M(U) \leq N$. On the other hand, it follows from Lemmas 5.4, 5.3 and 2.4 that A has composition factors of dimensions 1, 8 and 1 as a module for $D \leq N$. $\Box$

In view of this lemma we can assume that a is conjugate to u in N. Let $A_0 = A \cap Q_a \cap Q_u$. Notice that $A_0 \sim 2^8$.

Lemma 5.6 *We have $C_D(\langle a, u \rangle) \sim 2^{10}.2^{16}.\Omega_8^+(2)$. In particular, $\bar{a}$ is of type $2a_1$ in $\bar{C}_u$ (cf. Section 2) and $C_D(\langle a, u \rangle) = C_a \cap C_u$. Furthermore, $C_a \cap C_u$ induces on A_0 a group $\Omega_8^+(2)$ acting as on a halfspin module.*

Proof: Since $A \cap Q_u = [Q_u, a]$, the orbit of a under Q_u consists of at least 2^8 elements. On the other hand, if a' is a 2-central involution in $A \setminus Q_u$ then $a'u$ is non 2-central by Lemma 5.2. Therefore, $A \setminus (A \cap Q_u)$ consists of exactly 2^8 2-central and 2^8 non 2-central involutions. Furthermore, all 2-central (respectively, non 2-central) involutions in $A \setminus (A \cap Q_u)$ are conjugate by Q_u.

This shows that $Q_u C_D(\langle a, u \rangle) = D$. Comparing with Lemma 2.4 we obtain that $C_D(\langle a, u \rangle) \sim 2^{10}.2^8.2^8.\Omega_8^+(2)$. Notice that the two 8-dimensional chief factors (again, see Lemma 2.4) provide nonisomorphic modules for the quotient $\Omega_8^+(2)$. Therefore, $O_2(C_D(\langle a, u \rangle))/A$ is elementary abelian and so we can record the structure of $C_D(\langle a, u \rangle)$ as $2^{10}.2^{16}.\Omega_8^+(2)$.

Comparing with Lemmas 2.10 and 2.11, we see that $\bar{a}$ must be of type $2a_1$ and that $C_D(\langle a, u \rangle) = C_a \cap C_u$. Clearly, A_0 is invariant under $C_a \cap C_u$. Since $x \mapsto \bar{x}$ establishes an isomorphism between A_0 and $[\bar{Q}_u, a]$, the last claim follows from Lemma 2.4 (2). $\Box$

Define a mapping $f : A \longrightarrow GF(2)$ as follows: for $x \in A$, $f(x) = 0$ if and only if x is the identity or a 2-central involution.

Lemma 5.7 *The mapping f is a nondegenerate quadratic form of plus type.*

Proof: We will switch to the additive notation in A. Decompose A as $A = \langle a, u \rangle \oplus A_0$. Then the restriction of f on $\langle a, u \rangle$ is a plus type form, because au is non 2-central. It was shown in the preceding lemma that $C_a \cap C_u$ induces on A_0 a group $\Omega_8^+(2)$ acting as on a halfspin module (which is a triality conjugate of the natural module). In particular, $C_a \cap C_u$ has two orbits on $A_0^{\#}$, of length 120 and 135. Thus, in order to show that the restriction of f to A_0 is a quadratic form of plus type it suffices to show that A_0 contains exactly 120 non 2-central involutions. However, this is clear. Indeed, by Lemma 5.2, each of the 15 cosets $a' + (U \cap A_0) = a' + (U \cap Q_u)$, with $a' \in A_0 \setminus (U \cap Q_u)$, contains exactly eight 2-central and eight non 2-central involutions. We have shown that the restriction of f on A_0 is also a plus type form.

It remains to verify the values of f on the elements $x + y$, $x \in \langle a, u\rangle^\#$ and $y \in A_0$. If $x = a$ or u then $f(x + y) = f(y)$ because y and $x + y$ are conjugate in Q_x. In view of Lemma 5.6, $C_a \cap C_u$ has orbits of length 120 and 135 on the set $a + u + A_0^\#$. Since the total number of non 2-central involutions in A is known to be 496, we compute that among the elements in $a+u+A_0^\#$ there are exactly 135 non 2-central involutions and 120 2-central involutions. Hence $f(a + u + y) = 1 + f(y)$ for all $y \in A_0^\#$. □

We can now pin down the structure of $N = N_M(A)$. Let $P_A = O_2(N)$.

Lemma 5.8 *We have $N \sim 2^{10+16}.\Omega^+_{10}(2)$. In particular, $P_A = C_M(A) \sim 2^{10+16}$.*

Proof: First of all, Lemma 5.6 yields that $C_M(A)$ is an extension of A by a group 2^{16}, *i.e.*, $C_M(A) \sim 2^{10+16}$. Consider now the action of N on A. Clearly, N leaves the form f invariant. So $N/C_M(A)$ is isomorphic to a subgroup of $O^+_{10}(2)$. We claim that it is isomorphic to $\Omega^+_{10}(2)$. Indeed, observe that D and $N_M(U)$ share a Sylow 2-subgroup T (indeed, the 2-parts of the orders of D and $N_M(U)$ coincide and hence as T we can take a Sylow 2-subgroup of $N_M(U)$ centralizing u). Consider an index two subgroup in U invariant under T and the unique singular 2^5_2, say U', containing that subgroup. Both U and U' are maximal totally singular with respect to f and T leaves invariant both U and U'. This yields that the image of T lies in $\Omega^+_{10}(2)$ and, moreover, the images of D and $N_M(U)$ lie in $\Omega^+_{10}(2)$, too. Comparing the orders we obtain that they are two maximal parabolics in $\Omega^+_{10}(2)$. Therefore, $N/C_M(A)$ is either $\Omega^+_{10}(2)$ or $O^+_{10}(2)$. It remains to notice that $N/C_M(A) \cong O^+_{10}(2)$ is impossible, because U and U' are not conjugate. □

For $x \in A^\#$, let $x^\perp$ be the orthogonal complement of $\langle x\rangle$ with respect to the symplectic form $(x_1, x_2) = f(x_1 + x_2) - f(x_1) - f(x_2)$ on A. (We continue using the additive notation in A.)

Corollary 5.9 *If $x \in A^\#$ is 2-central then $x^\perp = A \cap Q_x$.*

Proof: First of all, by the preceding lemma, N is transitive on 2-central involutions in A. Hence $A \cap Q_x$ has index two in A. Furthermore, since y and $x+y$ have the same type whenever $y \in Q_x$, we have that $f(y) = f(x+y)$ for all $y \in A \cap Q_x$. This proves that $A \cap Q_x \leq x^{\perp}$. □

This shows that a subgroup of A is singular if and only if it is totally singular with respect to f. Notice that A contains both a singular 2^5_1 and a singular 2^5_2 and so, indeed, an ark contains all species of singular subgroups. Furthermore, all singular subgroups of A of the same kind are conjugate in N. This implies, in particular, the following

Lemma 5.10 *If $U \in \mathcal{S}_{5,1}$ then $A(U)$ is the only ark containing U. If $U \in \mathcal{S}_{5,2}$ then U is contained in exactly three arks. Furthermore, those three arks are conjugate under $N_M(U)$.*

Proof: The first claim follows since $N_M(U) \leq N$ if $U \in \mathcal{S}_{5,1}$. If $U \in \mathcal{S}_{5,2}$ and $U \leq A$, we compute that $N_N(U)$ has index three in $N_M(U)$. □

6 Elementary abelian subgroups in P_A

Let A be an ark and $N = N_M(A)$. We first produce an inventory of the elements from $P_A \setminus A$. Since $A \leq Z(P_A)$, every coset xA with $x \in P_A \setminus A$ consists entirely of involutions or entirely of elements of order four.

Let $\tilde{N} = N/P_A \cong \Omega^+_{10}(2)$.

Lemma 6.1 *If u is a 2-central involution from A then $R = P_A \cap Q_u$ is of order 2^{17}. In particular, P_A is nonabelian.*

Proof: Notice that Q_u normalizes any singular 2^5_1 subgroup U such that $u \in U \leq A$, and hence Q_u normalizes $A = A(U)$. Thus, $Q_u \leq N$. Notice further that Q_u cannot be fully contained in P_A. Indeed, if $Q_u \leq P_A$ then Q_u has index two in P_A, which implies that Q_u must have a center of size at least 2^5; clearly a contradiction. Thus, $Q_u \not\leq P_A$, which means that $\tilde{Q}_u$ is a nontrivial normal subgroup of $\tilde{D}$, where $D = N \cap C_u$. Since $\tilde{D}$ is a maximal parabolic (the stabilizer of a singular vector from the natural module), we get that $\tilde{Q}_u \sim 2^8$. Hence $|R| = 2^{25-8} = 2^{17}$. Being a subgroup of an extraspecial group 2^{1+24}, R must be nonabelian. Hence, P_A is also nonabelian. □

We will now classify the cosets xA with $x \in P_A \setminus A$. It turns out that the cosets consisting of involutions correspond to singular subgroups 2^5_1 from A.

Lemma 6.2 *Suppose $U \le A$, $U \in \mathcal{S}_{5,1}$. Then*

(1) *$Q_U \le P_A$ and $Q_U \not\le A$; hence, $X = Q_U A \setminus A$ is a coset from $P_A \setminus A$, consisting of involutions; if $x \in X$ then $U = [P_A, x]$; and*

(2) *$K = N_M(U)$ has exactly two orbits on X; one of the orbits is $Q_U \setminus U$, and it consists of non 2-central involutions (conjugate to q_2); the other orbit is $X \setminus Q_U$, and it consists of 2-central involutions.*

Proof: Let $K = N_M(U)$. Notice that $\tilde{K}$ is a maximal parabolic in $\tilde{N} \cong \Omega_{10}^+(2)$; namely, it is the stabilizer of a maximal totally singular subspace U from the natural module A. Clearly, Q_U is invariant under K. Since K has two 5-dimensional chief factors in A and since Q_U/U has order two, we conclude that $Q_U \not\le A$. If $y \in A \setminus U$ is 2-central then $W = U \cap y^\perp$ is a singular 2^4. Since $\langle y, Q_U \rangle \le Q_W$ (which is abelian), the subgroup Q_U centralizes every y and hence $Q_U \le P_A$.

Recall that $K = N_M(U)$ is contained in N, because $A = A(U)$. Clearly, K acts on $X = Q_U A \setminus A$. Notice that $[P_A, Q_U] \le Q_U \cap A = U$ and hence $[P_A, Q_U] = U$, because K acts on U irreducibly. This implies that for $x \in Q_U \setminus U$ we have $[P_A, x] = U$. Since $A = Z(P_A)$, the same must be true for all $x \in X$. In particular, for all $x \in X$, all elements in xU are conjugate under P_A. Let $T = Q_U A$ and let $\hat{T} = T/U$. Clearly, $\hat{T}$ is the product of $\hat{A} \sim 2^5$ and $\hat{Q}_U \sim 2$. Furthermore, K stabilizes both $\hat{A}$ and $\hat{Q}_U$, and it acts transitively on $\hat{A}^\#$. Thus, K indeed has exactly two orbits on $xA = T \setminus A$.

We already know from Lemma 4.14 (1) that the involutions from $Q_U \setminus U$ are non 2-central. Since those involutions are contained in Q_u for $u \in U^\#$, they are conjugate to q_2. To see that the involutions in $X \setminus Q_U$ are 2-central, consider $W \le U$, $W \sim 2^4$. Let U' be the unique singular subgroup 2_2^5 containing W. Then, by definition of $A = A(U)$, we have $U' \le A$. Let $x \in Q_U \setminus U$ and $s \in U' \setminus W$. Comparing with Corollary 4.11 and with the definition of singular subgroups 2_2^5 (following Corollary 4.11), we see that xs is 2-central. Clearly, $xs \in X \setminus Q_U$, and so the claim follows. $\square$

In particular, this lemma shows that Q_U and hence also U can be recognized from the coset $X = Q_U A \setminus A$. Therefore, such cosets are in a natural bijection with the set of all singular subgroups 2_1^5 from A. The latter set has size 2295, which means that at least 2295 cosets of A in $P_A \setminus A$ consist of involutions. We claim that the remaining $(2^{16} - 1) - 2295$ cosets of A in $P_A \setminus A$ consist of elements of order four.

Suppose $x \in P_A$ is of order four. Then $s = x^2$ is an element of A and furthermore $s = y^2$ for each $y \in xA$.

Lemma 6.3 *If u is a 2-central involution from A then $(P_A \cap Q_u)A \setminus A$ contains exactly 120 cosets xA such that $u = x^2$. The group $K = N \cap C_u$ transitively permutes those cosets.*

Proof: Consider $R = P_A \cap Q_u$. By Lemma 6.1, $|R| = 2^{17}$. Let a be a 2-central involution from $A \backslash u^{\perp}$. Then $R \leq C_a$, the image of a in $C_u/Q_u \cong Co_1$ is of type $2A$, and, comparing with Lemma 2.4, we see that $R = Q_u \cap C_a$. Since $Q_u \leq N$, we have that $[Q_u, a] \leq Q_u \cap A = u^{\perp}$. Since $F = N \cap C_a \cap C_u$ involves $\Omega_8^+(2)$ (indeed, if we view A as the natural module for $\tilde{N} = N/P_A \cong \Omega_{10}^+(2)$ then a and u span in A a nondegenerate subspace of plus type), Lemma 2.4 gives us that $\bar{R} \sim 2^8$ and F induces on $\bar{R}$ a group $\Omega_8^+(2)$ acting as on a halfspin module. Here the bar indicates the image in $\bar{P}_A = P_A/A$.

Define a mapping $q : \bar{R} \longrightarrow GF(2)$ by $q(xA) = 0$ if $x^2 = 1$, and $q(xA) = 1$ if $x^2 = u$. Then q is a quadratic form on $\bar{R}$ and this form is invariant under K. Since the halfspin module for $\Omega_8^*(2)$ is triality conjugate to the natural module and since the latter admits a unique invariant quadratic form, the claims of the lemma follow. □

Since A contains 527 2-central involutions u, Lemma 6.3 accounts for $527 \cdot 120$ cosets xA consisting of elements of order four. Since $527 \cdot 120 = (2^{16} - 1) - 2295$, all the cosets of A in $P_A \setminus A$ have been accounted for. Thus, we obtain the following.

Lemma 6.4 *The group N has exactly two orbits on the nonidentity elements of $\bar{P}_A = P_A/A$. The smaller orbit has length 2295 and it consists of cosets containing involutions. The longer orbit has length $527 \cdot 120$ and it consists of cosets containing elements of order four.* □

In particular, $\bar{P}_A$ is irreducible as a module for $\tilde{N} = N/P_A \cong \Omega_{10}^+(2)$. We remark that this module is isomorphic to the halfspin module. Indeed, this follows from the fact that the stabilizer of maximal totally singular subspace $U \leq A$, $U \in \mathcal{S}_{5,1}$, fixes a vector in $\bar{P}_A$.

Additionally, Lemma 6.2 gives us the following.

Corollary 6.5 *The group N has exactly two conjugacy classes of involutions in $P_A \backslash A$. One class has length $2295 \cdot 32$, and it consists of non 2-central involutions conjugate to q_2. The other class has length $2295 \cdot (1024 - 32)$ and it consists of 2-central involutions.* □

We will not classify the classes of elements of order four in P_A. However, we will need the following fact.

Lemma 6.6 *If $x \in P_A$ is of order four then $x \in (P_A \cap Q_u)A$ where $u = x^2$.*

Proof: We have seen above that $P_A \cap Q_u$ is nonabelian and hence it contains an element y of order four. Clearly, $y^2 = u$. Since N has just one orbit on cosets from P_A/A that consist of elements of order four, there is a conjugate y^n, $n \in N$, of y which lies in the coset xA. Then $(y^n)^2 = x^2 = u$ and hence, without loss of generality, we may assume that $y = y^n$ lies in xA. Now since $y \in (P_A \cap Q_u)A$, we have that $x \in yA \leq (P_A \cap Q_u)A$. $\square$

Next, we need to know when two involutions from $P_A \setminus A$ commute. For an involution $x \in P_A \setminus A$ let $U(x, A)$ (or simply $U(x)$, if A is clear from the context) be the singular subgroup 2^5_1 from A, that corresponds to the coset xA. Recall that $U(x) = [x, P_A]$ (cf. Lemma 6.2 (1)).

Lemma 6.7 *Involutions x and y from $P_A \setminus A$ commute if and only if $U(x) \cap U(y)$ has size at least eight.*

Proof: Notice first of all that $[x, y] = [x', y']$ for arbitrary $x' \in xA$ and $y' \in yA$. Hence, commutation of x and y depends solely on $U' = U(x)$ and $U'' = U(y)$. In particular, we may assume that $x \in Q_{U'}$ and $y \in Q_{U''}$. Secondly, observe that U' and U'' meet in a subgroup of order 2, 2^3, or 2^5. Since $N = N_M(A)$ acts transitively on pairs (U', U'') with $U' \cap U''$ of a given size and since P_A is nonabelian, it suffices to show that x and y commute if $W = U' \cap U''$ has size eight. However, this is clear: both x and y are contained in Q_W, which is abelian. $\square$

This lemma allows us to determine now all maximal elementary abelian subgroups of P_A.

Lemma 6.8 *With respect to conjugation by $N = N_M(A)$, the group P_A has exactly two classes of maximal elementary abelian subgroups Y:*

(1) *for $W \leq A$, $W \in \mathcal{S}_2$, Y consists of A and all involutions $y \in P_A \setminus A$ such that $W < U(y)$; and*

(2) *for $V \leq A$, $V \in \mathcal{S}_{5,2}$, Y consists of A and all involutions $y \in P_A \setminus A$ such that $V \cap U(y)$ is of order 2^4.*

Proof: First of all, it follows from Lemma 6.7 that the subgroups Y from (1) and (2) are elementary abelian. (Indeed, in (1) if $U(y_1)$ and $U(y_2)$ both contain W then $U(y_1) \cap U(y_2)$ is of order at least 2^3; the other case is even easier.)

Let E now be an elementary abelian subgroup of P_A. Let $\mathcal{E} = \{U(x) | x \in E \setminus A\}$. It follows from Lemma 6.7 that if U and U' are distinct elements of $\mathcal{E}$ then $U \cap U'$ has order 2^3. Let us now show that if $U, U', U'' \in \mathcal{E}$ are pairwise distinct then $U \cap U' \cap U''$ is of order at least 2^2. Suppose not. Then $U \cap U' \cap U''$ has order two. Observe that by Lemma 4.3 the subgroup $W = \langle U \cap U', U \cap U'', U' \cap U'' \rangle$ is singular. Furthermore, since $(U \cap U') \cap (U \cap U'')$ has order two, we have $U = \langle U \cap U', U \cap U'' \rangle$, which means that $U \leq W$. Similarly, $U', U'' \leq W$; clearly, a contradiction, since U, U' and U'' are maximal singular. Thus, indeed, $U \cap U' \cap U''$ has order at least 2^2.

Fix $U \in \mathcal{E}$ and let $\mathcal{T} = \{U \cap U' | U' \in \mathcal{E}, U' \neq U\}$. This is a set of subgroups 2^3 from U, such that any two of them meet in a subgroup 2^2. We claim that one of the following two possibilities holds: (a) there is a subgroup $W \leq U$ of order four such that every $T \in \mathcal{T}$ contains W; or (b) there is a subgroup $W \leq U$ of order 16, such that every $T \in \mathcal{T}$ is contained in W. Let T_1, T_2 be distinct elements from $\mathcal{T}$, and let $W_1 = T_1 \cap T_2$ and $W_2 = \langle T_1, T_2 \rangle$. Then clearly W_1 is of order four and W_2 is of order 16. If every $T \in \mathcal{T}$ is contained in W_2 then we have case (b) with $W = W_2$. So suppose $T_3 \in T$ and $T_3 \not\leq W_2$. Observe that $T_3 \cap W_2$ has order four and hence $T_3 \cap W_2 = T_3 \cap T_1 = T_3 \cap T_2 = W_1$. Finally, consider an arbitrary $T \in \mathcal{T}$. If $T \not\geq W_1$ then $T \cap T_1 \neq T \cap T_2$ and hence $T = \langle T \cap T_1, T \cap T_2 \rangle \leq W_2$. However, this means that $T \cap T_3 = T_3 \cap W_2 = W_1$, that is, $T \geq W_1$, a contradiction. We proved that case (a) holds with $W = W_1$.

We can now complete the proof of the lemma. If $\mathcal{T}$ satisfies the condition in (a) then E is contained in the subgroup from (1) defined by W. If, on the other hand, $\mathcal{T}$ satisfies the condition from (b) then E is contained in the subgroup from (2), where V is defined as the unique singular 2^5_2 containing W. Indeed, every U' from $\mathcal{E}$ meets V in a subgroup of size at least eight. Since U' and V are nonconjugate maximal totally singular subgroups from A, we must have that $V \cap U'$ has size 16. □

We will use the following notation. For an ark A and a singular $W \leq A$, $W \in \mathcal{S}_2$, let $\mathrm{Ab}_2(A, W)$ (or simply, $\mathrm{Ab}_2(W)$) be the maximal elementary abelian subgroup of P_A defined by W as in (1). Similarly, if $V \leq A$ and $V \in \mathcal{S}_{5,2}$ then let $\mathrm{Ab}_5(A, V)$ (or just $\mathrm{Ab}_5(V)$) denote the maximal elementary abelian subgroup defined by V as in (2). Notice that $|\mathrm{Ab}_2(W)| = 2^{14}$ and $|\mathrm{Ab}_5(V)| = 2^{15}$.

Recall that $N = N_M(A)$ and let $\bar{P}_A = P_A/A$.

Lemma 6.9 *Suppose $W, V \leq A$ with $W \in \mathcal{S}_2$ and $V \in \mathcal{S}_{5,2}$. Let $Y = \mathrm{Ab}_2(W)$ and $Y' = \mathrm{Ab}_5(V)$. The following hold:*

(1) *$N_N(W)$ induces on $\bar{Y} \sim 2^4$ the group $L_4(2)$; and*

(2) *$N_N(V)$ induces on $\bar{Y}' \sim 2^5$ the group $L_5(2)$.*

Proof: The involutions $\bar{y} \in \bar{Y}$ bijectively correspond to the 2^5_1 subgroups U in A, that contain W. Since $W^\perp/W$ is a 6-dimensional orthogonal space of plus type, $N_N(W)$ induces on $\bar{Y}$ the group $\Omega^+_6(2) \cong L_4(2)$. Similarly, the involutions in $\bar{y}' \in \bar{Y}'$ bijectively correspond to index two subgroups in V. So $N_N(V)$ induces on $\bar{Y}'$ the group $L_5(2)$. □

Lemma 6.10 *Suppose $B \leq P_A$ is elementary abelian and $\bar{B} \sim 2^3$. Then there exist unique $W, V \leq A$ with $W \in \mathcal{S}_2$ and $V \in \mathcal{S}_{5,2}$, such that $\mathrm{Ab}_2(W)$ and $\mathrm{Ab}_5(V)$ contain B.*

Proof: Let $b_1, b_2, b_3 \in B$ and $\bar{B} = \langle \bar{b}_1, \bar{b}_2, \bar{b}_3 \rangle$. Then $W = U(b_1) \cap U(b_2) \cap U(b_3)$ and V is the unique singular 2^5_2 in A containing $(U(b_1) \cap U(b_2))(U(b_1) \cap U(b_3)) \sim 2^4$. □

We complete this section with a different construction of the maximal elementary abelian subgroups $\mathrm{Ab}_5(A, V)$. Suppose V is a singular subgroup 2^5_2 and let A_1, A_2 and A_3 be the three arks containing V (cf. Lemma 5.10). Furthermore, let $P_i = P_{A_i}$ for all i.

Lemma 6.11 *Let $\{i, j, k\} = \{1, 2, 3\}$ and $R = A_i A_j$. Then the following hold:*

(1) *the subgroups A_i and A_j commute elementwise and $A_i \cap A_j = V$; in particular, $R \sim 2^{15}$; furthermore, $R = \mathrm{Ab}_5(A_i, V) = \mathrm{Ab}_5(A_j, V)$;*

(2) *$P_i \cap P_j = R$; in particular, R is maximal abelian;*

(3) *R contains A_k; in particular, $R = A_i A_k = A_j A_k$.*

Proof: If $a \in A_i$ and $b \in A_j$ then a and b lie in Q_W for some subgroup $W \cong 2^3$ of V. Since Q_W is abelian, we have that a and b commute. In particular, $A_j \leq P_i$ and hence also $R \leq P_i$. The stabilizer K of V in $N_M(A_i)$ involves $L_5(2)$ acting irreducibly on both V and A_i/V (see Lemma 5.1). Since the stabilizer of A_j in K is of index at most two, we obtain that $A_i \cap A_j = V$. If $y \in A_j \setminus V$ then $U(y)$ contains an index two subgroup from V. Hence $A_j \leq \mathrm{Ab}_5(A_i, V)$, proving (1).

For (2), let $N = N_M(A_j)$, $K = N_N(V)$ and $S = P_i \cap N$. Since K acts on $\{A_i, A_k\}$, the index of $K_0 = N_M(A_i) \cap N$ in K is at most two. Notice that the image of K in $N/P_j \cong \Omega^+_{10}(2)$ is a maximal parabolic $2^{10}.L_5(2)$

with an irreducible action of $L_5(2)$ on the normal 2^{10}. This means that S, being normal in K_0, is either contained in P_j and so $S = P_i \cap P_j$, or R has index 2^{10} in S. In its turn, S has index at most two in P_i. Thus, in the first case $P_i \cap P_j$ has index two in P_i, which is clearly impossible. In the second case, $P_i \cap P_j$ has order 2^{15} or 2^{16}, implying that R has index one or two in $P_i \cap P_j$. Suppose $R \neq P_i \cap P_j$. Since R is in the center of $P_i \cap P_j$ and since R is a maximal elementary abelian subgroup by (1), all elements in $(P_i \cap P_j) \setminus R$ are of order four and they all square to the same involution in $A_i \cap A_j = V$. Since K_0 involves $L_5(2)$ acting transitively on the involutions from V, we obtain a contradiction, proving that $R = P_i \cap P_j$. This shows that R is self-centralized, which means that R is maximal abelian. So (2) is proven.

Since A_k commutes with both A_i and A_j, it is contained in $C_M(R) = R$. □

In particular, this lemma shows that the subgroup $\mathrm{Ab}_5(A, V)$ depends, in fact, only on V. So we will use the notation $\mathrm{Ab}_5(V)$.

7 Classes of involutions, II

In this section z is a 2-central involution in M, the Monster, $Z = \langle z \rangle$, $C = C_z$, and $Q = Q_z$. We classify conjugacy classes of involution in C and determine the fusion of these classes in M.

Let $\bar{C} = C/Z$. If $x \neq z$ is an involution from C then $\bar{c}$ is again an involution. Notice that the group $\bar{C}$ satisfies the conditions (H1)–(H3) from Section 2. The results from that section tell us that $\bar{C}$ has exactly eight classes of involutions. If $\bar{x}$ is an involution then x is either an involution or an element of order four, having the property that $x^2 = z$. Suppose x is an involution. Then $y = xz$ is also an involution, and either x and y are conjugate in C, or they are not. As a result, each conjugacy class of involutions from $\bar{C}$ leads to zero, one, or two conjugacy classes of involutions in C. We now have to decide which case takes place for each of the eight classes of involutions from $\bar{C}$.

The three classes contained in $\bar{Q}$, namely, $2e_2$, $2e_3$ and $2e_4$, were discussed in Lemma 4.4. There we proved that $2e_3$ produces a class of elements of order four, while each of $2e_2$ and $2e_4$ leads to a class of involutions. We denoted by q_2 and q_4 representatives of those classes and noted that q_4 is 2-central (*i.e.*, conjugate to z) and q_2 is non 2-central. Thus, in this section we only need to discuss the classes $2a_i$, $2b$ and $2c$.

Let $q = q_4$, $D = C \cap C_q$ and $R = Q \cap Q_q \sim 2^{1+23}.(2^{11} : M_{24})$. According to Lemma 4.5, $Q \cap Q_q$ has order 2^{13} and hence $(C \cap Q_q)(C_q \cap Q) = O_2(D)$. That is, the subgroup $C \cap Q_q$ maps in $\tilde{C} = C/Q \cong Co_1$ onto the diagonal subgroup 2^{11}. Since the latter contains representatives of the conjugacy classes $2A$ and $2C$ from Co_1, we may be able to find elements y with $\bar{y}$ in the classes $2a_1$, $2a_2$, $2a_3$ and $2c$ by looking at $C \cap Q_q$.

If x is an involution in $C \cap Q_q$ and $x \notin \langle z, q \rangle$ then $\langle z, x \rangle$ maps onto a size four subgroup of $Q_q/\langle q \rangle$. Under the identification of the latter with $\hat{\Lambda} = \Lambda/2\Lambda$, the Leech lattice modulo two, the images of z, x, and xz belong to $\hat{\Lambda}_2 \cup \hat{\Lambda}_4$, because these elements are involutions (cf. Lemma 4.4).

To proceed, we will need some information about the subgroups of order four in $\hat{\Lambda}$. According to [ATLAS], Co_1 has exactly fifteen orbits on such subgroups. Representatives of nine of those orbits contain elements from $\hat{\Lambda}_3$. The remaining six orbits are shown in Table 1. Let $\hat{U} \sim 2^2$ be a representative of one of these orbits. Then the second column contains the types of the three involutions from $\hat{U}$. For example, if the entry there is 244 then one involution lies in $\hat{\Lambda}_2$, while two involutions lie in $\hat{\Lambda}_4$. The third column shows the structure of the elementwise stabilizer (centralizer) of $\hat{U}$ in Co_1. The fourth column shows the group induced on $\hat{U}$ by the setwise stabilizer (normalizer) of $\hat{U}$ in Co_1. Notice that [ATLAS] shows a different structure (namely, $[2^{12}].L_3(2)$) of the elementwise stabilizer of $\hat{U}$ for $\hat{U}$ from orbit 6. That structure is incorrect.

Orbit	Type	Stabilizer	Induced group
1	222	$U_6(2)$	S_3
2	224	$2^{10} : \mathrm{Aut}\, M_{22}$	2
3	244	$2^{1+8} A_8$	2
4	444	$2^{4+12}.3 \cdot S_6$	S_3
5	444	$\mathrm{Aut}\, M_{12}$	S_3
6	444	$[2^{11}].L_3(2)$	S_3

Table 1: $(\hat{\Lambda}_2 \cup \hat{\Lambda}_4)$-pure subgroups 2^2 in $\hat{\Lambda}$

Lemma 7.1 *Suppose $x \in C$. Then the following hold:*

(1) *if $\bar{x}$ is of type $2a_1$ then x and xz are involutions; one of them is non 2-central (fused with q_2) and the other one is 2-central;*

(2) *if $\bar{x}$ is of type $2a_3$ then x and xz are 2-central involutions and they are conjugate in C;*

(3) *if $\bar{x}$ is of type 2c then x and xz are 2-central involutions and they are conjugate in C.*

Proof: Let x be an involution from $C \cap Q_q \setminus \langle z, q\rangle$. Let $U = \langle z, x\rangle$ and $\hat{U}$ be the image of U in $\hat{\Lambda}$. Notice that z maps onto an element from $\hat{\Lambda}_4$ (cf. Lemma 4.4). Thus, $\hat{U}$ cannot be in orbit 1 from Table 1. If $x \in Q$ and x is 2-central then $\langle z, q, x\rangle$ is singular. Comparing with with Lemma 4.8, we see that in this case $\hat{U}$ is in orbit 4. If $x \in Q$ and x is non 2-central, then xz is also non 2-central and hence $\hat{U}$ is of type 224, that is, $\hat{U}$ is in orbit 2. Thus, orbits 2 and 4 correspond to x's from Q.

Suppose next that we choose x so that $\hat{U}$ is in orbit 3. Then, according to Table 1 and Lemma 4.4, one of the elements x and xz is 2-central and the other one is non 2-central. In particular, $x \notin Q$. Let A be an ark containing z and q and let $a \in A$ be a 2-central involution contained in $q^{\perp}$, but not in $z^{\perp}$. Then az is non 2-central and hence $\langle z, a\rangle$ is conjugate to U in C_q. Furthermore, since the normalizer of $\hat{U}$ in Co_1 permutes the two elements from $\hat{\Lambda}_4 \cap \hat{U}$ and since U is contained in the extraspecial group Q_q, we get that $\langle z, a\rangle$ is conjugate to U in $C \cap C_q$. It follows from Lemma 5.8 that $C \cap C_a$ involves $\Omega_8^+(2)$. Comparing with Lemmas 2.10 and 2.11, we obtain that $\bar{a}$ (and hence also $\bar{x}$) is of type $2a_1$.

Next choose x so that $\hat{U}$ is in orbit 5. Then $C \cap C_x$ involves M_{12}, whose order is divisible by 11. Comparing with Lemmas 2.10 and 2.11, we immediately obtain that $\bar{x}$ is of type $2c$.

Finally, let x be such that $\hat{U}$ is in orbit 6. In this case both x and xz are 2-central, which rules out the possibility that $\bar{x}$ is of type $2a_1$. It follows from Table 1 that 2^{38} divides $|C \cap C_x|$. Comparing again with Lemmas 2.10 and 2.11, we see that $\bar{x}$ can only be of type $2a_3$. □

In fact, we have proved a bit more.

Lemma 7.2 *Suppose x is an involution in C. Then there exists a 2-central involution $q \notin \langle z, x\rangle$, such that $z, x \in Q_q$, if and only if either $x \in Q$ or $\bar{x}$ is of type $2a_1$, $2a_3$, or $2c$. Furthermore, if such q's exist then $C \cap C_x$ permutes them transitively.*

Proof: Only the transitivity claim requires proof. Let q and q' be 2-central involutions such that $z, x \in Q_q \cap Q_{q'}$. Since $q, q' \in Q$, there exists $c \in C$ such that $(q')^c = q$. Then $\overline{x^c}$ is of the same type as $\bar{x}$. It follows from the above that $\langle z, x^c\rangle$ and U are conjugate in C_q and moreover there is an element $s \in C \cap C_q$ such that $(x^c)^s = x$. Clearly, $cs \in C \cap C_x$ and $(q')^{cs} = q$. □

Our next goal is to show that the classes $2a_2$ and $2b$ do not lead to involutions. We start with $2a_2$.

Lemma 7.3 *If $x \in C$ and $\bar{x}$ is of type $2a_2$ then x is of order four.*

Proof: Let R be the full preimage in Q of $C_{\bar{Q}}(\bar{x})$ and let $F = \langle R, x \rangle$. Then $|R| = 2^{17}$ and $|F| = 2^{18}$. The coset $F \backslash R$ consists entirely of elements y with $\bar{y}$ of type $2a_i$ for some i. According to Lemma 7.1, y is an involution if $i = 1$ or 3. Suppose x is also an involution. Then all elements in $F \backslash R$ are involutions. This implies that $C_R(x)$ contains no elements of order four, implying that $C_R(x)$ is elementary abelian. Since $C_R(x) \leq Q$ and $|C_R(x)| \geq 2^{16}$, we get a contradiction. □

It remains to consider the class $2b$.

Lemma 7.4 *If $x \in C$ and $\bar{x}$ is of type $2b$ then x is of order four.*

Proof: Suppose by contradiction that x is an involution. We first show that x and xz are not conjugate in C. Indeed, according to Lemma 2.11, $C \cap C_x$ involves $G_2(4)$. Let D be the Sylow 13-subgroup from $C \cap C_x$. Clearly, if x and xz are conjugate in C then they are also conjugate in $N_C(D)$. Let $\tilde{C} = C/Q \cong Co_1$. According to [ATLAS], $C_{\tilde{C}}(\tilde{D}) \cong 13 \times A_4$. Notice also that $C_Q(D) = Z$. This means that $F = C_C(D)$ is an extension of Z by a group $13 \times A_4$. Now the fact that x is an involution yields that $F \cong 2 \times 13 \times A_4$. (Here Z is the direct factor of order two.) Consequently, one of x and xz is contained in the commutator subgroup of F and the other is not. Thus, x and xz cannot be conjugate in $N_C(D)$, and hence they are not conjugate in C.

Choose an involution $a \in Q$ such that $\bar{a} \notin C_{\bar{Q}}(\bar{x})$. Then for $b = [a, x]$ we have that $\bar{b} \in C_{\bar{Q}}(\bar{x})$. It follows from Lemma 2.5 that b is a 2-central involution. In particular, x commutes with b and, furthermore, x and xb are conjugate in $\langle a, x \rangle \leq C_b$. Shifting now our attention to C_b we see that the image of x in $C_b/\langle b \rangle$ cannot be of type $2a_1$ or $2b$ because x and xb are conjugate in C_b (cf. Lemma 7.1 for $2a_1$). Also, by Lemma 7.3, it cannot be of type $2a_2$. Thus, the image of x in $C_b/\langle b \rangle$ must be of type $2a_3$ or $2c$. Lemma 7.1 forces now that x is 2-central.

Clearly, the image of z in $C_x/\langle x \rangle$ must also be of type $2b$. Observe that $Q \cap C_x$ is of index at most four in $O_2(C \cap C_x)$. Symmetrically, $Q_x \cap C$ is of index at most four in $O_2(C \cap C_x)$. This shows that $Q \cap Q_x \neq 1$, that is, there exists a 2-central involution q such that $z, x \in Q_q$. Now Lemma 7.2 provides a contradiction. □

We summarize all the above as follows.

Proposition 7.5 *The group C has exactly seven conjugacy classes of involutions, three in Q and four in $C \setminus Q$.* $\square$

One important corollary of Lemmas 7.1, 7.3 and 7.4 is that the group M contains exactly two classes of involutions. Indeed, we have shown that every involution is either 2-central and hence fused with z, or non 2-central, fused with q_2. We record this as the following

Proposition 7.6 *Every involution in M is either conjugate with z or with q_2.* $\square$

A second important corollary is that we now know all pairs of commuting 2-central involutions in M.

Lemma 7.7 *Let a and b be two commuting 2-central involutions, $a \neq b$, and let $R = Q_a \cap Q_b$. Let bar indicate the image in $C_a/\langle a\rangle$. Then one of the following is true:*

(1) *$\bar{b}$ is of type $2e_4$, i.e., $b \in Q_a$ and $\langle a, b\rangle$ is singular;*

(2) *$\bar{b}$ is of type $2a_1$; moreover, $\langle a, b\rangle$ is contained in a unique ark A and $R = a^{\perp} \cap b^{\perp}$; in particular, $A = \langle a, b\rangle R$;*

(3) *$\bar{b}$ is of type $2a_3$ and $R \sim 2^5$; R contains a singular 2^4 subgroup W and every involution in $R \setminus W$ is non 2-central; subgroups $W_a = W\langle a\rangle$ and $W_b = W\langle b\rangle$ are singular 2^5_1; we have that $R\langle a\rangle = Q_{W_a}$ and $R\langle b\rangle = Q_{W_b}$; finally, $Q_W = R\langle a, b\rangle$; or*

(4) *$\bar{b}$ is of type $2c$, $R \cong 2$, and the involution in R is 2-central.*

Proof: According to Lemmas 4.4, 7.1, 7.3 and 7.4, $\bar{b}$ is of type $2e_4$, $2a_1$, $2a_3$ or $2c$ in $\bar{C}_a$. By Lemma 7.2 $C_a \cap C_b$ is transitive on the (nonempty) set of 2-central involutions $q \notin \langle a, b\rangle$ such that a and b are in Q_q. Dividing $|C_a \cap C_b|$ by $|C_a \cap C_b \cap C_q|$ (the latter can be found using Table 1) we obtain that the number of involutions q is equal to 7084, 135, 15 and 1 depending on whether the type of $\bar{b}$ is $2e_4$, $2a_1$, $2a_3$ or $2c$. We now turn to the concrete cases.

If $\bar{b}$ is of type $2e_4$ then, clearly, (1) holds. Suppose $\bar{b}$ is of type $2a_1$. Then a and b are contained in an ark A. Clearly, $a^{\perp} \cap b^{\perp} \sim 2^8$ lies in R. On the other hand, R is abelian and invariant under the action of $C_a \cap C_b$. It follows from Lemma 2.4 that R cannot have size more than 2^8 and hence it coincides with $a^{\perp} \cap b^{\perp}$. Now all claims of (2) follow.

Consider next the case where $\bar{b}$ is of type $2a_3$. Choose a singular 2^4 subgroup W and let A and B be two singular 2^5_1 containing W. Let $a' \in A \setminus W$ and $b' \in B \setminus W$. If the image of b' is of type $2a_1$ in $C_{a'}/\langle a' \rangle$ then both A and B are contained in the unique ark containing a' and b'. This is impossible because in an ark every singular 2^4 is contained in a unique singular 2^5_1. Also the image of b' cannot be of type $2c$ because all the involutions $q \in W$ have the property that $a', b' \in Q_q$. Hence the image of b' is of type $2a_3$ and without loss of generality we may assume that $a' = a$ and $b' = b$. Clearly, $W \leq R$ and W contains all 15 2-central involutions from R. Hence all involutions in $R \setminus W$ are non 2-central. In particular, if $c \in R \setminus W$ and $w \in W$ then cw is non 2-central. According to Lemmas 7.1, 7.3 and 7.4, this means that $c \in Q_w$. That is, $R \leq Q_W$. Since also $a, b \in Q_W$, the claim (3) follows from Lemmas 4.10 and 4.13.

Finally, suppose that $\bar{b}$ is of type $2c$. Let q be the only 2-central involution in R. Then it follows from Lemma 2.3 that $[R : \langle q \rangle] \leq 4$. Comparing with Lemmas 2.6 and 2.11 we see that $R = \langle q \rangle$. □

8 More on P_A

In this section A is an arc. For an involution $x \in P_A \setminus A$ let $W(x, A)$ (or simply $W(x)$) be the subgroup of A generated by all 2-central involutions in $a \in A$ such that $x \in Q_a$. Recall from Section 6 that $U(x, A)$ (or simply $U(x)$) is the singular subgroup 2^5_1 in A that corresponds to the coset xA.

Lemma 8.1 *We have $W(x) \leq U(x)$. If x is non 2-central then $W(x) = U(x)$; otherwise, $W(x)$ is an index two subgroup in $U(x)$.*

Proof: We first notice that $W(x)$ is a singular subgroup. Indeed, suppose $a, b \in A$ are 2-central involutions such that $x \in Q_a \cap Q_b$. If a and b are not perpendicular then (a, b) is as in Lemma 7.7 (2). However, in this case $Q_a \cap Q_b \leq A$, which means that $x \in A$, a contradiction. Hence $W(x)$ is singular.

If x is non 2-central then $x \in Q_U$, where $U = U(x)$. Clearly, this means that $U \leq W(x)$. Since U is maximal singular, we obtain that $W(x) = U$. Now suppose x is 2-central. Then for some $a \in A$, we have that $t = xa$ is non 2-central. Let $U = U(x) = U(t)$ and $W = U \cap a^{\perp}$. Clearly, the index of W in U is at most two. If $w \in W$ then $a \in Q_w$ and $t \in Q_w$. Hence also $x \in Q_w$, *i.e.*, $W \leq W(x)$. On the other hand, $W(x)\langle x \rangle$ is singular, and hence $|W(x)| \leq 2^4$. Thus, $W(x) = W$. □

The next result adds to Lemma 6.1.

Lemma 8.2 *Suppose u is a 2-central involution in A and x is an involution in $P_A \setminus A$. Then $x \in (P_A \cap Q_u)A$ if and only if $u \in U(x)$.*

Proof: Suppose first that $x \in (P_A \cap Q_u)A$, that is, for some $a \in A$ the element $t = xa$ is in Q_u. By Lemma 8.1, this means that $u \in U(t) = U(x)$. Reversely, suppose $u \in U(x)$. Let $a \in A$ be such that $t = xa$ is non 2-central. Since $u \in U(x) = U(t)$, Lemma 8.1 implies that $t \in Q_u$. Therefore, $x \in (P_A \cap Q_u)A$. □

9 Heart of the proof

Suppose $t \in M$, where either $t = 1$, or t is a non 2-central involution. We call a 2-central involution u *marked* if $t \in Q_u$. In this section we are going to prove the following result.

Proposition 9.1 *Let Q be a 2-subgroup of M such that $C_M(Q) = Z(Q)$ and $t \in Z(Q)$. Let $E = \Omega_1 Z(Q)$. Let $\mathcal{J}$ be the set of all those marked involutions $u \in E$ for which $|E \cap Q_u|$ reaches maximum. Then either $\langle \mathcal{J} \rangle$ is singular, or E is an ark.*

The proposition will be proven in a sequence of lemmas. We first show that $\mathcal{J}$ is nonempty.

Lemma 9.2 *There exists a marked 2-central involution in E. In particular, $\mathcal{J}$ is nonempty.*

Proof: If $t = 1$ then E contains a 2-central involution, since if T is a Sylow 2-subgroup of M containing Q then $Z(T) \leq Z(Q)$. So suppose now that $t \neq 1$. Suppose T_0 is a Sylow 2-subgroup of $C_M(t)$ containing Q and T is a Sylow 2-subgroup of M containing T_0. Suppose z is the 2-central involution in the center of T. Then $z \in C_M(Q)$ and hence $z \in E$. Also, since C_z contains a full Sylow 2-subgroup of $C_M(t)$, it follows that $\bar{t}$ cannot be of type $2a_1$ in $\bar{C}_z = C_z/\langle z \rangle$. Hence $t \in Q_z$, *i.e.*, z is marked. □

Thus, $\mathcal{J}$ is nonempty. We need to show that either $\langle \mathcal{J} \rangle$ is singular, or E is an ark. Notice that in both cases $\mathcal{J}$ is fully contained in an ark. Inside an ark, every pair of 2-central involutions (in particular, two involutions from $\mathcal{J}$) is either as in case (1), or as in the case (2) of Lemma 7.7. Therefore, we first show that the other two cases are impossible. Fix $a, b \in \mathcal{J}$, $a \neq b$. Let $U = \langle a, b \rangle$ and $R = Q_a \cap Q_b$. Notice that $t \in R$ because a and b are marked. Let $\bar{Q}_a = Q_a/\langle a \rangle$.

Lemma 9.3 *The pair* (a, b) *is not in case* (4) *of Lemma* 7.7*.*

Proof: Indeed, suppose the pair (a, b) is in case (4). Since the only nontrivial element q from R is 2-central, we have $t = 1$. In particular, q is marked. According to Lemma 7.7, $\bar{b}$ is of type $2c$, which means that, with respect to the identification of $Q_q/\langle q\rangle$ with $\hat{\Lambda}$, we have that $\hat{U}$ is in orbit 5 from Table 1. In particular, the image of $C_a \cap C_b$ in $C_q/Q_q \cong Co_1$ is isomorphic to M_{12} or $\mathrm{Aut}\, M_{12}$. We obtain that $C_a \cap Q_b$ (which is normal in $C_a \cap C_b$!) is contained in Q_q. Similarly, $C_b \cap Q_a \leq Q_q$.

We claim that $q \in E$. Indeed, Q centralizes a and b, hence also q, as it is the only nontrivial element in $R = Q_a \cap Q_b$. Therefore, $q \in C_M(Q) = Z(Q)$, that is, $q \in E$, as claimed. On the other hand, we have $E \cap Q_b \leq C_a \cap Q_b \leq Q_q$. This means that $E \cap Q_b \leq E \cap Q_q$, hence, by maximality of $|E \cap Q_b|$, we have $q \in \mathcal{J}$ and $E \cap Q_b = E \cap Q_q$. Symmetrically, $E \cap Q_a = E \cap Q_q$. However, as $b \notin Q_a$, we have $E \cap Q_b \neq E \cap Q_a$; a contradiction. □

The second case, where (a, b) is as in Lemma 7.7 (3), is harder and requires several lemmas. Let us start with some additional notation. By Lemma 7.7, $R = Q_a \cap Q_b \cong 2^5$ and it contains a singular 2^4 subgroup W. All elements in $R \setminus W$ are non 2-central. Furthermore, $W_a = \langle a, W\rangle$ and $W_b = \langle b, W\rangle$ are singular subgroups 2^5_1. By Lemma 5.10, W_a is contained in a unique ark $A_a = A(W_a)$. Similarly, let $A_b = A(W_b)$ be the only ark containing W_b. We have $A_a \neq A_b$, since (a, b) is not as in case (1) or (2) of Lemma 7.7. Let also $P_a = P_{A_a}$ and $P_b = P_{A_b}$.

By Lemma 4.12, W is contained in a unique singular 2^5_2 subgroup T. By the definition of an ark, $T \leq A_a$ and $T \leq A_b$. It follows from Lemma 6.11 that $A_a \cap A_b = T$ and that $S = P_a \cap P_b$ coincides with $A_a A_b$ and is elementary abelian. (It coincides with $\mathrm{Ab}_5(T)$.) In particular, $a \in P_b$ and $b \in P_a$. Notice that $W = T \cap W_a = T \cap W_b$. Also $W = W(b, A_a) < U(b, A_a)$ (cf. Lemma 8.1) and similarly $W = W(a, A_b) < U(a, A_b)$.

First of all, we note the following.

Lemma 9.4 *For* $x, y \in \{a, b\}$*,* $x \neq y$*, we have that* $W_x = U(y, A_x)$*. Also,* $1 \neq C_{W_x}(Q) \leq E$ *and all involutions in* W_x *are marked.*

Proof: By Lemma 8.1, $W = W(y, A_x) < U(y, A_x)$. Since W is contained in A_x in a unique singular 2^5_1, it follows that $W_x = U(y, A_x)$. The next claim follows from the fact that $E = C_M(Q)$. Since $t \in R$ (because a and b are marked) and since $R \leq Q_W$ by Lemma 7.7 (3), we conclude that $t \in Q_W$. Taking now in the account that also $t \in Q_x$ and using Lemma 4.3, we obtain that $t \in Q_{W_x}$, because $W_x = \langle x, W \rangle$. Thus, every involution in W_x is marked. $\square$

Let $E_a = E \cap P_a$ and $E_b = E \cap P_b$. Since $a \in P_b$ and $b \in P_a$, we have that $a \in E_b$ and $b \in E_a$.

Lemma 9.5 *We have $E \cap Q_a \leq P_a$ and hence $E \cap Q_a \leq E_a$. Symmetrically, $E \cap Q_b \leq E_b$.*

Proof: Since $E \leq C_b$ it suffices to show that $C_b \cap Q_a \leq P_a$. Let $c \in T \setminus W$. Clearly, A_a is the only ark containing a and c. Since $b, c \in Q_W$ and since $Q_a \cap Q_W = Q_{W_a}$ has index two in Q_W we have that $bQ_a = cQ_a$. It follows from Lemma 7.1 that $C_c \cap Q_a$ has order 2^{17}, namely, it is the full preimage in Q_a of $C_{\bar{Q}_a}(c)$. Since $C_{\bar{Q}_a}(b) = C_{\bar{Q}_a}(c)$, we obtain that $C_b \cap Q_a \leq C_c \cap Q_a$. On the other hand, $P_a \cap Q_a \leq C_c$ and according to Lemma 6.1 the size of $P_a \cap Q_a$ is exactly 2^{17}. Therefore, $Q_a \cap C_c = P_a \cap Q_a \leq Q_a$. $\square$

Since $E \cap Q_a$ is fully contained in E_a, the maximality property of $a \in \mathcal{J}$ implies that for every marked 2-central involution $s \in E_a$ we have $|E_a \cap Q_s| \leq |E_a \cap Q_a|$. Symmetrically, for every marked 2-central involution $s \in E_b$ we have that $|E_b \cap Q_s| \leq |E_b \cap Q_b|$. Since $b \notin Q_a$, it follows that $E_a \cap Q_a \neq E_a$ and similarly $E_b \cap Q_b \neq E_b$.

Our argument depends on how E_a and E_b embed into P_a and P_b, respectively. Since E_a and E_b are elementary abelian we can make use of the classification from Section 6.

Lemma 9.6 *For $x \in \{a, b\}$, if $E_x \cap Q_x$ has index more than two in E_x then $E_x A_x = \mathrm{Ab}_5(V)$ for some singular 2^5_2 subgroup $V_x \leq A_x$, and the index of $E_x \cap Q_x$ in E_x is four.*

Proof: We may assume that $x = a$. Suppose $[E_a : E_a \cap Q_a] > 2$. Notice that $E_a A_a$ is elementary abelian. It follows from Lemma 6.8 that either $E_a A_a = \mathrm{Ab}_5(V_a)$ for some singular 2^5_2 subgroup $V_a \leq A_a$, or the intersection F of all $U(s)$, $s \in E_a \setminus A_a$, is nontrivial. Suppose the latter. Since $b \in E_a \setminus A_a$, we get $F \leq W_a$. Therefore Q centralizes some $1 \neq e \in F$, as Q clearly normalizes F. It follows from Lemma 9.4 that $e \in E$ and e is marked. If $s \in E_a$ then $e \in U(s)$, which by Lemma 8.2 means that $E_a \leq (P_a \cap Q_a)A_a$. The latter group contains $P_a \cap Q_a$ as an index two subgroup. It follows that $[E_a : E_a \cap Q_e] \leq 2$. By the maximality property of a we now have that $[E_a : E_a \cap Q_a] = 2$, since $E_a \not\leq Q_a$. This is a contradiction. Thus, if $[E_a : E_a \cap Q_a] > 2$ then $E_a A_a = \mathrm{Ab}_5(V_a)$ for some singular 2^5_2 subgroup $V_a \leq A_a$. Let $K = \mathrm{Ab}_5(V_a)$. Observe that Q acts trivially on V_a, since it acts trivially on K/A_a. Hence $V_a \leq E$. Since $b \in E_a$, we have that $V_a \cap W_a \sim 2^4$. Let $e \in V_a \cap W_a$. Then e is a marked 2-central involution in E. Using Lemma 8.2, we see that $[K : K \cap Q_e] = 4$ and hence $[E_a : E_a \cap Q_e] \leq 4$. By the maximality of a, we now must conclude that $[E_a : E_a \cap Q_a] \leq 4$. □

Out of the two options given by this lemma, we will first dispose of the possibility that $E_x A_x = \mathrm{Ab}_5(V_x)$ for some singular 2^5_2 subgroup $V_x \leq A_x$.

Lemma 9.7 *For $x \in \{a, b\}$, we have that $E_x A_x / A_x \not\sim 2^5$. In particular, $[E_x : E_x \cap Q_x] = 2$.*

Proof: Suppose by contradiction that $K = E_x A_x = \mathrm{Ab}_5(V_x)$ for some singular 2^5_2 subgroup $V_x \leq A_x$. Then again Q acts trivially on V_x because it acts trivially on K/A_x. Suppose $x \notin V_x$. If $s \in E_x \cap Q_x$ then $U(s)$ contains x (cf. Lemma 8.2) and also $U(s)$ meets V_x in a subgroup of index two (because $s \in K = \mathrm{Ab}_5(V_x)$). Since $U(s)$ is singular, we have that $U(s) \cap V_x = V_x \cap Q_x$. This shows that $U(s) = \langle x, V_x \cap Q_x \rangle$ is unique, which means that $[E_x \cap Q_x : E_x \cap Q_x \cap A_x] = 2$. However, in that case the index of $E_x \cap Q_x$ in E_x is at least 16, a contradiction with Lemma 9.6. Thus, $x \in V_x$. In particular, $V_x \neq T$. Now let $y \in \{a, b\}$, $y \neq x$. Observe that $V_x \cap Q_y \leq W(y) = W \leq T$. Consequently, $V_x \cap Q_y \leq V_x \cap T$, which gives us that $[V_x : V_x \cap Q_y] \geq 4$. (Here we use that both V_x and T are singular subgroups 2^5_2 and hence $|V_x \cap T| \leq 2^3$.) As $V_x \leq E \cap P_y = E_y$, it follows that $[E_y : E_y \cap Q_y] \geq 4$. By Lemma 9.6, $E_y A_y = \mathrm{Ab}_5(V_y)$ for some singular 2^5_2 subgroup $V_y \leq A_y$. Repeating the above argument with y in place of x, we obtain that $y \in V_y$ and hence $V_y \neq T$, and also that $[E_x : E_x \cap Q_x] = 4$.

Since $U(y, A_x)$ meets both V_x and T in a subgroup of order 16, we conclude that that $|V_x \cap T| \geq 8$. Symmetrically, $|V_y \cap T| \geq 8$. As a result, $F = V_x \cap V_y \cap T \neq 1$. Clearly, Q normalizes F and so we can choose $e \in C_F(Q)$. Then e is a marked 2-central involution and $e \in E$. Since $e \in V_x$, we have that $[K : K \cap Q_e] = 4$ and therefore $[E_x : E_x \cap Q_e] \geq 4$. Because of the maximality of x, we must have that $E \cap Q_e = E_x \cap Q_e$ and that this subgroup has index four in E_x. In particular, $E \cap Q_x \leq E_x$ and, symmetrically, $E \cap Q_e \leq E_y$. Hence $E \cap Q_e \leq E_x \cap E_y] \leq P_x \cap P_y = S = \mathrm{Ab}_5(T)$. This shows that $E \cap Q_e$ is contained in $K \cap S$. By Lemma 6.10, $[E \cap Q_e : E \cap Q_e \cap A_x] \leq 4$. However, this means that $[E_x : E \cap Q_e] \geq 8$, a contradiction. □

Thus, we now know that $|E_x A_x / A_x| \leq 2^4$ and $[E_x : E_x \cap Q_x] = 2$ for $x = a$ and b. We will obtain the final contradiction by showing that $[E_b : E_b \cap Q_b]$ must at the same time be equal to four. However, the proof of this claim will be different for the following two cases: (a) $|E_aA_a/A_a| \leq 4$, and (b) $|E_aA_a/A_a| \geq 8$. We first consider the case (a).

Lemma 9.8 *If $|E_aA_a/A_a| \leq 4$ then the index of $E_b \cap Q_b$ in E_b is at least four.*

Proof: By assumption, $E_a = \langle E_a \cap A_a, b, c\rangle$ for some $c \in E_a$. Notice that since $b \notin Q_a$ and since $[E_a : E_a \cap Q_a] = 2$, we can choose $c \in Q_a$. Since b and c commute, Lemma 6.7 tells us that $U(b) \cap U(c)$ has size at least eight. Since b is 2-central, $W(b) = W$ is of index two in $U(b)$. Depending on the type of c, the subgroup $W(c)$ either coincides with $U(c)$ or is an index two subgroup in it. In any case, $F = W(b) \cap W(c)$ is nontrivial. In particular, we can select $1 \neq e \in C_F(Q)$. This e is a marked 2-central involution and $e \in E$. Since $b, c \in Q_e$, we have $[\langle A_a, b, c\rangle : \langle A_a, b, c\rangle \cap Q_e] \leq 2$, which gives us that $[E_a : E_a \cap Q_e] \leq 2$. By the maximality of a, we have that $e \in \mathcal{J}$ and, furthermore, that $E \cap Q_e$ is an index two subgroup in E_x.

Since $b, c \in Q_e$ and $E_a = \langle E_a \cap A_a, b, c\rangle$, we can choose $s \in E_a \cap A_a$ such that $s \notin Q_e$. Consider the subgroup $X = \langle a, s\rangle = \{1, a, s, as\}$. Clearly, $X \leq E_b$, because $X \leq E$ and $X \leq A_a \leq P_b$. We have $a \notin Q_b$. Also, both s and as are not contained in Q_e, because $a \in Q_e$ (the latter holds since both a and e are in $U(b)$). We claim that s and as do not belong to Q_b. For that, view $A_a \cap Q_b$ as a subspace of the orthogonal space A_a (cf. Lemma 5.7). All singular vectors in $A_a \cap Q_b$ form the subspace W. This implies that W is in the radical of $A_a \cap Q_b$. So if s or as is in Q_b then it must be also perpendicular to e, because $e \in W$.

We have shown that X trivially intersects Q_b. Since $X \leq E_b$, this means that $[E_b : E_b \cap Q_b] \geq 4$. □

This lemma and Lemma 9.7 rule out case (a). So it remains to deal with case (b), *i.e.*, we now assume that $|E_aA_a/A_a| \geq 8$. We borrow ideas for our argument from the proof of Lemma 9.7.

First a preparatory lemma.

Lemma 9.9 *There is a subgroup $D \leq E_a$ such that*

(1) *$|DA_a/A_a| \geq 2^3$ and $b \in D$;*

(2) *D is contained in $\mathrm{Ab}_5(V)$ for some singular 2^5_2 subgroup $V \leq A_a$;*

(3) *for every 2-central involution $e \in A_a$ we have that $D \not\leq Q_e$.*

Proof: We consider two cases: Either (a) $E_aA_a = \mathrm{Ab}_2(F)$ for some singular 2^2 subgroup $F \leq A_a$, or (b) $E_aA_a \neq \mathrm{Ab}_2(F)$ for all such F. Suppose we are in case (a), that is, $E_aA_a = \mathrm{Ab}_2(F)$ for some F. Let $\bar{P}_a = P_a/A_a$. Notice that since $b \in E_a$ we have that $F \leq U(b) = W_a$. Let $F = \{1, f_1, f_2, f_3\}$. Lemma 9.4 yields that each f_i is marked. Suppose $E_a \leq Q_{f_i}$ for some i. The set of all such f_i together with 1 forms a Q-invariant subspace F_0 of F. Hence Q centralizes some f_i with $E_a \leq Q_{f_i}$; a contradiction to the maximality of a. Thus, $E_a \not\leq Q_{f_i}$ for all i. In particular, there exist x_1, x_2 and $x_3 \in E_a$ such that $x_i \notin Q_{f_i}$. Moreover, we can choose x_1 and x_2 equal, because no group can be fully covered by two proper subgroups. Let D be the full preimage in E_a of a subgroup 2^3 from $\bar{E}_a$ that contains the three elements $\bar{x}_1 = \bar{x}_2$, $\bar{x}_3$, and $\bar{b}$. Clearly, (1) is satisfied for this D. Also, (2) follows from Lemma 6.10. In case (b) we simply take $D = E_a$. Then (1) is trivially satisfied, while (2) follows from Lemmas 6.8 and 6.10.

It remains to show that (3) holds in both cases. If $e \in A_a$ is a 2-central involution such that $D \leq Q_e$ then e lies in the intersection X of all $W(s)$, $s \in D \setminus A_a$. Reversely, if $e \in X$ then e is 2-central and $D \leq Q_e$. Suppose $X \neq 1$. In case (a) we have that F is the intersection of all $U(s)$, $s \in D \setminus A_a$. Therefore, $X \leq F$. However, no f_i can be in X because the corresponding x_i is in D, a contradiction. Suppose now we are in case (b). Since $b \in D$, we have that $X \leq W(b) = W$, and Lemma 9.4 implies that all $e \in X$ are marked. Finally, since Q centralizes D, it normalizes X, and hence it centralizes some $1 \neq e \in X$. That e is a marked 2-central involution in E_a with the property that $D = E_a \leq Q_e$. However, we know that no such e exists. □

Choose D as in this lemma and let $V \leq A_a$ be the singular subgroup 2^5_2 such that $D \leq \mathrm{Ab}_5(V)$. In view of Lemma 6.10 this V is unique. Since V is unique, Q normalizes V.

Lemma 9.10 *The group Q centralizes V and, in particular, $V \leq E$.*

Proof: If $s \in D \setminus A_a$ then we set $B(s) = V \cap U(s)$. If $s \in (D \cap A_a) \setminus V$ then we set $B(s) = V \cap s^{\perp}$. In both cases $B(s)$ is a hyperplane of V. Let F be the intersection of subgroups $B(s)$ for all $s \in D \setminus V$. Since Q normalizes every $B(s)$, it normalizes F and acts trivially on V/F. That is, $[V, Q] \leq F$. If $F = 1$ then there is nothing else to prove, so we assume that $F \neq 1$. Notice that since $|DA_a/A_a| \geq 2^3$ and since $F \leq U(s)$ for all $s \in D \setminus A_a$, the size of F is at most four. Notice also that every $e \in F$ is perpendicular to all of $D \cap A_a$. Finally, since $F \leq U(s)$ for all $s \in D \setminus A_a$, we have that $D \leq A_aQ_e$ for every $e \in F$, $e \neq 1$. Therefore, $[D : D \cap Q_e] = 2$ for every such e.

Set $X = D \cap Q_F$ and for a hyperplane H of F set $Y = Y(H)$ to be $D \cap Q_H$. (Here if $H = 1$ then Q_H is the entire group M and $Y = D$.) Since $[D : D \cap Q_e] = 2$ for every $e \in F$, we have $|D/Y| \leq 2$, $|Y/X| \leq 2$, and so $|XA_a/A_a| \geq 2$.

We claim that Y is never equal to X. Indeed, suppose $Y \leq Q_F$. If $|F| = 2$, we have $Y = D$ and so $D \leq Q_F$, a contradiction with the definition of D. Thus $|F| = 4$. Pick $x \in D \setminus Y$. Since $W(x)$ is of index at most two in $U(x)$ and since $F \leq U(x)$, there exists $1 \neq e \in F$ with $e \in W(x)$. For that e, we have $D = \langle Y, x\rangle \leq Q_e$, since $Y = X \leq Q_e$. This is a contradiction, proving that $Y \neq X$.

Thus, $D \cap A_a < X < Y$, and there exist $s_1, s_2 \in Y \setminus X$ with $s_1 A_a \neq s_2 A_a$. Then $U(s_1) \neq U(s_2)$, and since every singular subgroup 2^4 in A_a lies in a unique singular 2^5_2, we also get $B(s_1) \neq B(s_2)$. Since $s_i \notin Q_F$ but $F \leq U(s_i)$, the involution s_i is 2-central and $U(s_i) = FW(s_i)$. Notice that $F \cap W(s_i) = H$ for $i = 1, 2$. As Q centralies s_i, Q normalizes $W(s_i)$, F and H. Observe that $[F, Q] \leq H$ and also $[W(s_i), Q] \leq F \cap W(s_i) = H$. Thus, $[U(s_i), Q] \leq H$. Finally, since $V = U(s_1)U(s_2)$, we have $[V, Q] \leq H$. As H was an arbitrary hyperplane of F we conclude $[V, Q] = 1$ and $V \leq E$. □

The next question is whether $V = T$ or not.

Lemma 9.11 *We have $V \neq T$. In particular, $[E_b : E_b \cap Q_b] \geq 4$.*

Proof: Since $a \notin T$, it suffices to show that $a \in V$. Suppose $a \notin V$. If $s \in \mathrm{Ab}_5(V) \cap Q_a$ then $U(s) = \langle a, V \cap U(s)\rangle$. Since $U(s)$ is singular, $V \cap U(s)$ must coincide with $V \cap a^{\perp}$. Hence $U(s)$ is unique. This shows that $\mathrm{Ab}_5(V) \cap Q_a \cap A_a$ has index at most two in $\mathrm{Ab}_5(V) \cap Q_a$. Therefore, also $|(D \cap Q_a)A_a/A_a| \leq 2$. Since DA_a/A_a has size at least eight, we must have that $[D : D \cap Q_a] \geq 4$, yielding $[E_a : E_a \cap Q_a] \geq 4$, which is a contradition with Lemma 9.7. Hence $V \neq T$. Finally, notice that $V \leq E_b$ and $V \cap Q_b \leq V \cap W(b) \leq V \cap T$. Since $[V : V \cap T] \geq 4$, we conclude that $[E_b : E_b \cap Q_b] \geq 4$. □

Manifestly, the conclusion of this lemma contradicts Lemma 9.7, ruling out case (2) and thus showing that $\bar{b}$ cannot be of type $2a_3$ in $\bar{C}_a = C_a/\langle a\rangle$.

Corollary 9.12 *The pair (a, b) is not in case* (3) *of Lemma* 7.7. □

According to Lemma 9.3 and Corollary 9.12, any two involutions $a, b \in \mathcal{J}$ are either perpendicular, or (a, b) is as in case (2) of Lemma 7.7. If any two involutions in $\mathcal{J}$ are perpendicular then the subgroup generated by $\mathcal{J}$ is singular. Thus, it order to complete the proof of Proposition 9.1 all we need is to prove the following lemma.

Lemma 9.13 *If the pair* (a, b) *is as in case* (2) *of Lemma* 7.7 *then* E *is an ark and* $\langle \mathcal{J} \rangle = E \cap t^{\perp}$.

Proof: In this case $\bar{b}$ is of type $2a_1$ in $\bar{C}_a = C_a/\langle a \rangle$. By Lemma 7.7, $A = \langle a, b\rangle(Q_a \cap Q_b)$ is the unique ark containing a and b. Let $P = P_A$. Notice that $Q_a \cap C_b$ has order 2^{17} by Lemma 7.1 and that $P \cap Q_a$ also has size 2^{17} by Lemma 6.1. Since $P \leq C_b$, we conclude that $Q_a \cap C_b = P \cap Q_a$. In particular, $E \cap Q_a \leq P$. Symmetrically, $E \cap Q_b \leq P$. Let $D = (E \cap Q_a)(E \cap Q_b)$. Then $D \leq P$ and $E \cap Q_x = D \cap Q_x$ for $x \in \{a, b\}$. The maximality of x now shows that $|D \cap Q_e| \leq |D \cap Q_x|$ for all marked 2-central involutions from E. Let $y \in \{a, b\}$, $y \neq x$. Since $y \notin Q_x$, we have that $E \cap A \cap Q_x$ is a hyperplane of $E \cap A$. This implies that $|E \cap A \cap Q_x| = |E \cap A \cap Q_y|$ and hence also $|(E \cap Q_x)A/A| = |(E \cap Q_y)A/A|$. We will denote this latter number by r. We intend to prove that $r = 1$, that is, $E \cap Q_x \leq A$.

If $s \in P \setminus A$ then $U(s) = U(s, A)$ is singular, and hence it cannot contain both x and y. This means that $(E \cap Q_x)A/A$ and $(E \cap Q_y)A/A$ meet trivially in P/A. Therefore, $|DA/A| = r^2$. Furthermore, this shows that $[D : D \cap Q_x] = [DA/A : (D \cap Q_x)A/A] \cdot [D \cap A : D \cap Q_x \cap A] = 2r$. Since D is abelian we have that $|DA/A| \leq 2^5$ and hence $r \leq 4$.

Suppose first that $r = 2$ and let F be the intersection of all $U(s)$, $s \in D \setminus A$. Then F is a singular subgroup 2^3. Note that $t \in Q_a \cap Q_b \leq A$. This means that $t^{\perp}$ is a hyperplane in A and hence $F_0 = F \cap t^{\perp}$ is nontrivial. Clearly, Q normalizes F_0 so we can select $1 \neq e \in C_{F_0}(Q)$. This e is a marked 2-central involution in E. Since $e \in F$ we have that $[D : D \cap Q_e] \leq 2$. In view of maximality of x we must have that $2 \geq 2r = 4$, a contradiction.

Suppose now that $r = 4$. Pick a hyperplane H in D such that $D \cap A \leq H$. Let F be the intersection of all $U(s)$, $s \in H \setminus A$. Then $F \sim 2^2$ and hence again $F_0 = F \cap t^{\perp} \neq 1$. Choosing $1 \neq e \in C_{F_0}(Q)$, we see that e is a marked 2-central involution from E and $[D : D \cap Q_e] \leq 2[H : H \cap Q_e] \leq 4 < 2r = 8$. So again we have a contradiction with the maximality of x.

Thus, $r = 1$, that is, $E \cap Q_x \leq A$ for $x \in \{a, b\}$. Let e be any 2-central involution in $E \cap A$ that is perpendicular to t. Since $[E \cap A : E \cap A \cap Q_a] \leq 2$ the maximality of x implies that $e \in \mathcal{J}$, $E \cap Q_e \leq A$ and $[E \cap A : E \cap A \cap Q_e] = 2$. Consider $F = (E \cap A)^{\perp}$. We would like to show that $A \leq E$, that is we need to prove that $F = 1$. Suppose the contrary. We first remark that $t \in E \cap A$. Indeed, $t \in A$ and $t \in E = \Omega_1 Z(Q)$ by assumption. Since $t \in E \cap A$, every 2-central involution in F is marked. Recall now that the

2-central involutions in A are simply the singular vectors with respect to the quadratic form f on A. Since Q normalizes F, if the number of singular vectors in F is odd then Q centralizes a 2-central involution $e \in F$. Then e is marked and 2-central, $e \in E$ and $E \cap A \leq Q_e$, a contradiction with the maximality of x. Hence the number of singular vectors in F is even. This means that that either F is 1-dimensional containing a nonsingular vector, or 2-dimensional nondegenerate. Suppose F is 2-dimensional. Then F contains an odd number of nonsingular vectors and hence Q centralizes one of them, say c. Since $E = C_M(Q)$, we have that $c \in E$. Hence c is in the radical of the symplectic form on F, a contradiction, since that form is nondegenerate. If F is 1-dimensional then Q centralizes a hyperplane in A. Since $N_M(A)$ induces on A the group $\Omega_{10}^+(2)$, no element of Q can act on A as a transvection. This means that Q centralizes A, that is, $A \leq E$ and $W = 1$, a contradiction. Thus, $F = 1$ and $A \leq E$.

Now, $A \leq E$ implies that $Q \leq C_M(A) = P$. In particular, $E \leq P$. Suppose $s \in E \setminus A$. Then $W(s) \cap t^\perp \neq 1$. By the above every involution $e \in W(s) \cap t^\perp$ is in $\mathcal{J}$ and $E \cap Q_e \leq A$. This contradicts the fact that $s \in Q_e$. □

The proof of Proposition 9.1 is now complete.

10 Proof of the theorems

In this section we derive Theorems 1 and 2. We start with the Monster group M.

Proof of Theorem 1. Suppose N is a maximal 2-local subgroup in M such that $C_N(Q) \leq Q$, where $Q = O_2(N)$. Let $E = \Omega_1 Z(Q)$. If E is an ark then N is the normalizer of an ark, which agrees with Theorem 1. So now suppose that E is not an ark. Set $t = 1$ and let $\mathcal{J}$ be the set of all 2-central involutions $e \in E$ for which $|E \cap Q_e|$ reaches maximum. According to Proposition 9.1, $U = \langle \mathcal{J} \rangle$ is singular, since E is not an ark. Thus, N coincides with the normalizer of a singular subgroup. The singular subgroups of M have been classified in Section 4 (cf. Proposition 4.15). The normalizer of a singular 2^4 is not maximal because it is contained in the larger normalizer of a singular subgroup 2^5_2 (cf. Lemma 4.12). Also the normalizer of a singular subgroup 2^5_1 is not maximal, because it is contained in the normalizer of an ark (cf. Lemma 5.10). The remaining possibilities agree with Theorem 1. □

We now turn to the case of the Baby Monster BM. Recall that for us BM is defined as the group $\bar{H}$, where $H = C_M(t)$, t is a non 2-central involution in M, and $\bar{H} = H/\langle t\rangle$.

Proof of Theorem 2. Let $\bar{N}$ be a maximal 2-local subgroup of $\bar{H}$ and suppose $C_{\bar{H}}(\bar{Q}) \leq \bar{Q}$, where $\bar{Q} = O_2(\bar{H})$. Let N and Q be the full preimages of $\bar{N}$ and $\bar{Q}$ in H. Then $C_H(Q) \leq Q$ and $t \in Z(Q)$. Thus Proposition 9.1 applies to Q. Let $E = \Omega_1 Z(Q)$ and let $\mathcal{J}$ be the set of all marked 2-central involutions $e \in E$, for which $|E \cap Q_e|$ reaches maximum. According to Proposition 9.1, either E is an ark, or $U = \langle \mathcal{J} \rangle$ is singular. Suppose first that E is an ark. Notice that $t \in E$ and that t is a nonsingular vector in the orthogonal space E. The group $\Omega^+_{10}(2)$ induced on E acts transitively on nonsingular vectors. Thus, H has a unique conjugacy class of arks containing t. This leads to one of the cases from Theorem 2. Next suppose that E is not an ark and hence U is singular. Observe that U is generated by marked 2-central involutions which means that $t \in Q_U$. It follows from the results of Section 4 (namely, Lemma 4.4, Corollaries 4.6, 4.9 and 4.11, and Lemma 4.14; see also Lemma 4.2 (3)) that U cannot be a singular 2^5_2 and that in every other case the normalizer of U has a unique conjugacy class of non 2-central involutions in Q_U. Thus, H contains exactly five classes of singular subgroups U' having the property that $t \in Q_{U'}$. For the case $U' \sim 2^4$ we claim that in fact $N_H(U')$ is not a maximal subgroup (so $U = \langle \mathcal{J} \rangle$ can never be a singular 2^4). Namely, we claim that U' and t are contained together in a unique ark A and hence $N_H(U') \leq N_H(A)$. First notice that U' and t are contained in some ark. Indeed, if A is an ark containing U' then $A \cap Q_{U'}$ does contain some non 2-central involutions. Since $N_M(U')$ is transitive on non 2-central involutions from $Q_{U'}$, there must be an ark containing U' and t. It follows from Lemmas 4.12 that U' is contained in exactly three singular 2^5_1 subgroups V, each of which is in turn contained in a unique ark A. By Lemma 4.13, t belongs to Q_V for two of these V. If $t \in Q_V$ then by Lemma 6.2 we have for the corresponding ark A that $t \in P_A \setminus A$. Thus, there is at most one ark containing U' and t. We have shown that $U \not\sim 2^4$. The remaining four classes of possible singular subgroups U appear in Theorem 2. □

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