Let $F$ be a subfield of $\mathbb{R}$ and for $i = 1, 2$ let $\Phi_i$ be a root system in the Euclidean $F$-space $E_i$. Let $\Pi_i$ be a base for $\Phi_i$ and $C_i = (\langle \alpha, \beta \rangle)_{\alpha, \beta \in \Pi_i}$ the Cartan matrix for $\Phi_i$.

Define two matrices $(a_{ij})_{i, j \in I}$ and $(b_{kl})_{k, l \in K}$ to be isomorphic if there exists a bijection $g : I \rightarrow K$ with $a_{ij} = b_{g(i)g(j)}$ for all $i, j \in I$.

Let $\Phi = \Phi_1, E = E_1$ and $\Pi = \Pi_1$.

1. Show that the following are equivalent:
   
   (a) $\Phi_1$ and $\Phi_2$ are isomorphic root systems.
   
   (b) $\Gamma(\Pi_1)$ and $\Gamma(\Pi_2)$ are isomorphic graphs.
   
   (c) $C_1$ and $C_2$ are isomorphic matrices.

2. Suppose that $\Phi_1$ is connected and $\rho : E_1 \rightarrow E_2$ is an $F$-linear isomorphism with $\rho(\Phi_1) = \Phi_2$.
   Show that there exists $k \in F$ with $\langle \alpha, \beta \rangle = k\langle \rho(\alpha), \rho(\beta) \rangle$ for all $\alpha, \beta \in \Phi$.

3. Let $e \in E$ such that, for all $w \in W(\Phi)$, there exist non-negative $f_{\alpha} \in F$ with $w(e) = \pm \sum_{\alpha \in \Pi} f_{\alpha} \alpha$. Then $e = f_{\alpha} \alpha$ for some $f \in F$ and $\alpha \in \Phi$.
   (Hint: Suppose not. Show that there exists $d \in E$ with $d \perp e$ but $\delta \perp \alpha$ for all $\alpha \in \Phi$. Choose $w \in W$ with $w(d) \in C$. Consider $\langle w(d), w(e) \rangle$).

4. Let $K$ be a standard field, $L$ a perfect, semisimple Lie algebra over $K$ with Cartan subalgebra $H$ and root system $\Phi$. Let $\Psi$ be a root subsystem of $\Phi$ and let $A$ be the Lie subalgebra of $L$ generated by the $L_{\alpha}, \alpha \in \Psi$.
   Show that $A = (H \cap A) \oplus \bigoplus_{\alpha \in \Psi} L_{\alpha}$ and that $A$ is a perfect semisimple Lie algebra with Cartan subalgebra $H \cap A$ and root system isomorphic to $\Psi$. 