Group Theory I
Lecture Notes for MTH 913
S19

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Contents

Chapter 5

Groups with MIN

5.1 Basic properties of groups with MIN

Definition 5.1.1. [**def:locally d**] *Let $\mathcal{D}$ be a class of groups and G a group.*

(a) [**a**] *G is called a $\mathcal{D}$-group if $G \in \mathcal{D}$.*

(b) [**b**] *G is called a locally $\mathcal{D}$-groups if any finite subset of G is contained in a $\mathcal{D}$-subgroup of G.*

Definition 5.1.2. [**direct set**] *A partially ordered set $(I, \leq)$ is called directed if for all $i, j \in I$ there exists $k \in I$ with $i \leq k$ and $j \leq k$.*

Lemma 5.1.3. [**finite directed**] *Let $(I, \leq)$ a non-empty direct partially ordered set and J a finite subset of I. Then there exists $k \in I$ with $i \leq k$ for all $i \in I$.*

Proof. If $|J| = 0$ we can choose k to be any element of I. So suppose $J \neq \varnothing$ and let $a \in J$. By induction there exists $b \in I$ with $j \leq b$ with for all $j \in J \smallsetminus \{a\}$. Since I is direct there exists $k \in I$ with $a \leq k$ and $b \leq k$, and since I is partially ordered $j \leq b$ and $b \leq k$ implies $j \leq k$. □

Lemma 5.1.4. [**construct locally finite**] *Let $\mathcal{H}$ be a non-empty set of groups. Suppose that for all $A, B \in \mathcal{H}$ there exists $D \in \mathcal{H}$ such that $A \leq D$ and $B \leq D$. G. Put $G := \bigcup \mathcal{H}$ and for $H \in \mathcal{H}$ let $\cdot_H$ be the binary operation on H Define*

$$\mathcal{R} := \{(a, b, d) \mid D \in \mathcal{H}, a, b \in D, d = a \cdot_H b\}.$$

(a) [**a**] *$\mathcal{R}$ is binary operation on $G \times G$.*

(b) [**b**] *G is a group under $\mathcal{R}$ and each $H \in \mathcal{H}$ is a subgroup of G.*

(c) [**c**] *Any finite subset of G is contained in an member of $\mathcal{H}$. In particular, if each $H \in \mathcal{H}$ is finite, then G is locally finite.*

(d) [**d**] *If each $H \in \mathcal{H}$ is a subgroup of a group $\hat{G}$, then also G is a subgroup of $\hat{G}$.*

Proof. (a)Let $a, b \in G$ we need to show that there exists a unique $d \in G$ with $(a, b, d) \in \mathcal{R}$. Since $a, b \in G$ there exists $A, B \in \mathcal{H}$ with $a, \in A$ and $b \in \mathcal{B}$. By hypothesis there exists $D \in \mathcal{H}$ with $A, B \leq D$. Thus $a, b \in D$. Put $d = a \cdot_D b$. Then $(a, b, d) \in \mathbb{R}$. Suppose now that also $(a, b, e) \in \mathcal{R}$. Then there exists $E \in \mathcal{H}$ such that $a, b \in E$ and $e = a \cdot_E b$. By hypothesis there exists $F \in \mathcal{H}$ with $D, E \leq F$. Thus

$$d = a \cdot_D b = a \cdot_F b = a \cdot_E b = e$$

Thus d is unique.

(b)For $H \in \mathcal{H}$ let let 1_H the identity element of H. Let $a, b, c \in G$. Then there exists $A, B, C \in \mathcal{H}$ with $a \in A, b \in B$ and $c \in C$. Note that $(\mathcal{H}, \leq)$ is a direct partially ordered set so by 5.1.3 there exists $D \in \mathcal{H}$ with $A, B, C \leq D$. For $a = 1_A$ and $b = 1_B$ we conclude that $1_A = 1_D = 1_B$. Thus $1_A \cdot b = b = b \cdot 1_A$ and so 1_A is an identity for G. In particular, the inverse of a in A is also an inverse of a in G. As D is a group, $cdot_D$ is associative and since $a, b, c \in D$ the multiplication on G is associative.

(c)Let A be a finite subset of H. The for each $a \in A$ there exists $H_a \in \mathcal{H}$ with $a \in H_a$. By 5.1.3 there exists $K \in \mathcal{H}$ with $H_a \leq K$ for all $a \in A$. Then $A \subseteq K$.

(d)is obvious. □

Example 5.1.5. [**ex:locally finite**]

(a) [**a**] Every finite group is a locally finite group.

(b) [**b**] Let p be a prime and for $n \in \mathbb{N}$ let H_n be a cyclic group of order p chosen such that

$$H_0 \leq H_1 \leq H_2 \leq \ldots \leq H_n \leq \ldots$$

Define $\mathrm{C}_{p^\infty} = \bigcup_{n \in \mathbb{N}} H_n$. Then C_{p^∞} is an infinite locally finite group.

Example 5.1.6. [**fsym m**] Recall that for a set M, $\mathrm{Sym}(M)$ is symmetric group on M, that is the set of bijection from M to M with composition as binary operation. Let $\mathcal{F}$ be the set finite subsets of M and for $F \in \mathcal{F}$ define

$$H_F = \mathrm{C}_{\mathrm{Sym}(M)}(M \smallsetminus F) = \{g \in \mathrm{Sym}(M) \mid mg = m \text{for all} m \in M \smallsetminus F\}$$

Observe that $H_F \cong \mathrm{Sym}(F)$ and so H_F is a finite subgroups of $\mathrm{Sym}(M)$. Define

$$\mathrm{FSym}(M) = \bigcup_{F \in \mathcal{F}} H_F$$

If $E, F \in \mathcal{F}$, then $K := E \cup F \in \mathcal{F}$ and $H_E, H_F \leq H_K$. Thus (5.1.4)(c)shows that $\mathrm{FSym}(M)$ is a locally finite subgroup of $\mathrm{FSym}(M)$.

For $g \in \mathrm{FSym}(M)$ define $\mathrm{supp}(g) = M \smallsetminus \mathrm{C}_M(g)\} = \{m \in M \mid mg \neq m$. Observe that

$$\mathrm{FSym}(M) = \{g \in \mathrm{Sym}(M) \mid \mathrm{supp}(g) \text{ is finite}\}$$

and that $\mathrm{FSym}(M)$ is a normal subgroup of $\mathrm{Sym}(M)$.

Definition 5.1.7. [**def:min**] *Let G be a group. We say that G fulfills the minimum condition on subgroups if every non-empty set of subgroups of G has a minimal element. If this is the case we also say that G is a group with* MIN.

Definition 5.1.8. [**def:gcirc**] *Let G be a group. Then G° is the intersection of all the subgroups of finite index in G.*

Lemma 5.1.9. [**basic gcirc**] *Let G be a group and $A, B \leq G$ with G/B finite.*

(a) [**a**] *$|A/A \cap B|$ is finite.*

(b) [**b**] *If G/A is finite, then also then $G/A \cap B$ is finite*

(c) [**c**] $A^\circ \leq G^\circ$,

(d) [**d**] $B^\circ = G^\circ$.

Proof. (a)Just observe that $|A/A \cap B| = |AB/B| \leq |G/B$.

(b)Observe that $|G/A \cap B| = |G/A||A/A \cap B$ and use (a).

[**c**] Let $\mathcal{M}$ be the set of subgroups of finite index of G. Let $M \in \mathcal{M}$. By (a)$|A/A \cap M|$ is finite. The definition of A° shows that $A^\circ \leq A \cap M \leq M$. Thus $A^\circ \leq \bigcap \mathcal{M} = G^\circ$.

[**d**] Let $\mathcal{N}$ be the set of subgroups of finite index of B. Let $N \in \mathcal{B}$. Then both B/N and G/B are finite. Hence $G/N = |G/B||B/N|$ is finite. Thus $G^\circ \leq N$ and so $G^\circ \leq \bigcap \mathcal{N} = B^\circ$. By (c)$B^\circ \leq G^\circ$, so $B^\circ = G^\circ$. □

Lemma 5.1.10. [**gcirc and min**] *Let G be a group with* MIN. *Then $G^\circ = (G^\circ)^\circ$ and G° is smallest subgroup of finite index in G.*

Proof. Let $\mathcal{M}$ be the set of subgroups of finite index of G. Note that $G \in \mathcal{M}$, so $\mathcal{M}$ is not empty and thus has a minimal element B. Let $A \in \mathcal{M}$. By (5.1.9)(b)$A \cap B$ has finite index in G. As $A \cap B \leq B$ the minimal choice of B gives $A \cap B = B$. Thus $B \leq A$. It follows that B the smallest element of $\mathcal{M}$, so $B = \bigcap \mathcal{M} = G^\circ$. As $|G/B|$ is finite, (5.1.9)(d)shows that $B^\circ = G^\circ$, that is $(G^\circ)^\circ = G^\circ$. □

Lemma 5.1.11. [**basic min**] *Let G be a group with* MIN.

(a) [**a**] *Every section of G fulfills* MIN.

(b) [**b**] *G is periodic, that is every element in G has finite order.*

Proof. (a)Let $B \trianglelefteq A \leq G$ and $\mathcal{M}$ a non-empty set of subgroups of A/B. Let $D \leq G$ be minimal with $B \leq D \leq A$ and $D/B \in \mathcal{M}$. Then D/B is a minimal element of $\mathcal{M}$.

(b)Let $g \in G$. By (a)$\langle g \rangle$ fulfills MIN and so $\langle g \rangle \ncong \mathbb{Z}$. Thus $\langle g \rangle$ is finite.

□

5.2 Divisible groups

Definition 5.2.1. [**def:divisible**] *A group A is called divisible of it is abelian and for all $a \in A$ and $n \in \mathbb{Z}^\sharp$ where exists $b \in A$ with $b^n = a$.*

Example 5.2.2. [**ex:divisible**]

(a) [**a**] $(\mathbb{Q}, +)$ is divisible. Indeed if $a \in \mathbb{Q}$ and $n \in \mathbb{Z}^\sharp$, then $B := \frac{1}{n}a \in \mathbb{Q}$ and $na = b$.

(b) [**b**] Let p be a prime. Then C_{p^∞} is divisible. Indeed recall that $\mathrm{C}_{p^\infty} = \bigcup_{k=0}^\infty H_k$ where $H_i \cong \mathrm{C}_{p^k}$ and $H_k \leq H_{k+1}$. Let $a \in \mathrm{C}_{p^\infty}$ and $n \in \mathbb{Z}^\sharp$. Then there exists $k, l \in \mathbb{N}$ and $m \in \mathbb{Z}^\sharp$ with $a \in H_k$, $n = mp^l$ and $p \nmid m$. The function $\alpha : H_{k+l} \to H_{k+l}, d \to d^{p^l}$ is a homomorphisn with kernel of order p^l and image of order p^l. Thus $\operatorname{Im} \alpha = H_k$ and there exists $d \in H_{k+l}$ with $a = d^{p^l}$. Since $p \nmid m$, the function $\beta : H_{k+l} \to H_{k+l}, b \to b^m$ is a homomorphism with trivial kernel and so is an isomorphism. Thus there exists $b \in H_{k+l}$ with $b^n = d$. Then

$$b^n = b^{mp^l} = d^{p^l} = a$$

Remark 5.2.3. [**rm:divisible**] *There does not exists a non-trivial, divisible, finite group.*

Proof. Let G be a finite divisible group. Let $a \in G$ and $n := |G|$. Then $a = b^n$ for some $b \in G$. As $n = |G|$ we get $b^n = 1$, so $a = 1$ and $G = 1$. □

Lemma 5.2.4. [**basic divisible**] *Let A be an abelian group and D a divisible subgroup of A. Then $A = D \oplus K$ for some $K \leq A$.*

Proof. Let $\mathcal{M} = \{L \leq A \mid L \cap D\}$. If $\mathcal{L}$ is a totally ordered subset of $\mathcal{M}$, then $\bigcup \mathcal{L}$ is an upper bound for $\mathcal{L}$ in $\mathcal{M}$. Hence Zorn's lemma implies that exists a maximal element K of $\mathcal{M}$. We will show that $A = KD$. Since A is abelian we know that $KD \trianglelefteq A$ and we can consider the group $\overline{A} = A/KD$. Let $a \in A$ and let $n = |\overline{a}|$ be the order of $\overline{a}$ in $\overline{A}$. We will now prove:

(∗) [**1**] *Suppose $\langle a \rangle K \cap D \neq 1$. Then $n \in \mathbb{Z}^+$ and $a^n \in KD \smallsetminus K$.*

Let $d \in \langle a \rangle K \cap D$ with $d \neq 1$. Then $d = a^m k$ with $m \in \mathbb{Z}$ and $k \in K$. As $1 \neq d \in D$ and $D \cap K = 1$ we have $d \notin K$. Since $d = a^m k$ and $k \in K$ we get $a^m \notin K$. Note that $a^m = dk^{-1} \in KD$ and so $\overline{a}^m = 1$. In particular, $\overline{a}$ has finite order and $m = nl$ for some $l \in \mathbb{Z}^+$. Then $(a^n)^l = a^m \notin K$ and so $a^n \notin K$. As $n = |a|$ we have $\overline{a}^n = 1$, so $a^n \in KD$.

(∗∗) [**2**] *There exists $e \in aD$ with $\langle e \rangle K \cap D = 1$.*

If $\langle a \rangle K \cap D = 1$ we can choose $e = a$. So suppose $\langle a \rangle K \cap D \neq 1$. Then (∗)shows that $n \in \mathbb{Z}^+$ and $a^n = KD$. Hence $a^n = kd$ with $k \in K$ and $d \in D$. Since D is divisible, there exists $b \in D$ with $b^n = d$. Put $e = ab^{-1}$. Then $e \in aD$, $eD = aD$ and $|\overline{e}| = \overline{a}| = n$. Also

$$e^n = (ab^{-1})^n = a^n (b^n)^{-1} = (kd)d^{-1} = k \in K$$

The contrapositive of (∗)applied to e in place of a now shows that $\langle e \rangle \cap D = 1$. Thus (∗∗)holds.

We now can show that $a \in KD$. By (∗∗)there exists $e \in aD$ with $\langle e \rangle K \cap D = 1$. The maximality of K implies that that $e \in K$. Thus $a \in aD = eD \subseteq KD$. We proved that $a \in KD$ for all $a \in A$. Hence $A = KD$ and since $K \cap D = 1$ we get $A = K \oplus D$. □

5.3 Locally finite groups with finite involution centralizer

Remark 5.3.1. [**rm:equal commutator**] *Let G be a group, $g \in G$, $H \leq G$ and $a, b \in H$. Then*

$$[g,a] = [g,b] \iff g^a = g^b \iff ba^{-1} \in \mathrm{C}_H(a) \iff b \in \mathrm{C}_H(g)a \iff \mathrm{C}_H(g)a = \mathrm{C}_H(g)b$$

Proof.

$$\begin{aligned} & [g,a] = [g,b] \\ \iff\quad & g^{-1}g^a = g^{-1}g^b \\ \iff\quad & g^a = g^b \\ \iff\quad & g^{ba^{-1}} = g \\ \iff\quad & ba^{-1} \in \mathrm{C}_H(g) \\ \iff\quad & b \in \mathrm{C}_H(g) \\ \iff\quad & \mathrm{C}_H(g)b \in \mathrm{C}_H(g)a \end{aligned}$$

□

Proposition 5.3.2. [**brauer fowler**] *Let G be a finite group and $t \in G \setminus \mathrm{Z}(G)$ with $t^2 = 1$. Then there exists a non-trivial normal subgroup N of G with $N \leq [G,t]$ such that $G/C_G(N)$ is isomorphic to a subgroup of* $\mathrm{Sym}(2|C_G(t)|^2)$. *In particular,*

$$|G/\mathrm{C}_G(N)| \leq \left(2|\mathrm{C}_G(t)|^2\right)!$$

Proof. Put

$$\mathcal{C} := t^G \quad \text{and} \quad \mathcal{D} := \{(x,y) \in \mathcal{C} \times \mathcal{C} \mid x \neq y\}$$

We will count the element of $\mathcal{D}$ in two different ways. Put $n := |G|$ and $m := |\mathrm{C}_G(t)$. Then

$$|\mathcal{C}| = |t^G| = |G/\mathrm{C}_G(t)| = \frac{n}{m}$$

and so

$$(*) \qquad [\mathbf{1}]|\mathcal{D}| = |\mathcal{C}|(|\mathcal{C}|-1) = \frac{n}{m}\left(\frac{n}{m}-1\right) = \frac{n^2}{m^2} - \frac{n}{m}$$

Next we will partition the set $\mathcal{D}$ by the value of xy to obtain a second count. Observe that for $(x,y) \in \mathcal{D}$ we have $xy = xy^{-1} \neq 1$. Let $a \in G^\sharp$. Define

$$\mathcal{D}(a) =: \{x,y) \in \mathcal{D} \mid xy = a\} \quad \text{and} \quad k := \max_{a \in G^\sharp} |\mathcal{D}(a)|$$

Then $\mathcal{D}(a)$, $a \in G^\sharp$, is a partition of $\mathcal{D}$. Thus

$$(**) \qquad [\mathbf{2}]|\mathcal{D}| = \sum_{a \in G^\sharp} |\mathcal{D}(a)| \leq (n-1)k = nk - k$$

Suppose that $k = 0$. Then $\mathcal{D} = \varnothing$ and so $|\mathcal{C}| = 1$ and $\mathcal{C} = \{t\}$. Hence $t = t^g$ for all $g \in G$ and so $t \in \mathrm{Z}(G)$, a contradiction to the hypothesis. Thus

$$(***) \qquad [\mathbf{3}]k \geq 1.$$

From $(*)$and $(**)$we get

$$\frac{n^2}{m^2} - \frac{n}{m} \leq nk - k \leq k$$
$$\frac{n^2}{m^2} \leq nk + \frac{n}{m} = n(k + \frac{1}{m}) \overset{(***)}{\leq} n(k+k) = 2nk$$
$$\frac{n}{k} \leq 2m^2$$

Let $a \in G^\sharp$. We will now estimate $|\mathcal{D}(a)|$ in terms of $|\mathrm{C}_G(a)|$. Let $(x,y) \in \mathcal{D}(a)$. Then $a = xy$. As x and y have order two this gives $a^y = a^{-1}$. Let $(\tilde{x},\tilde{y}) \in \mathcal{D}(a)$. Then $a^y = a^{-1} = a^{\tilde{y}}$ and refrm:equal commutator gives $\tilde{y} \in \mathrm{C}_G(a)y$. So there are at most $|\mathrm{C}_G(a)|$ choices for $\tilde{y}$. Since $a = \tilde{x}\tilde{y}$ we have $\tilde{x} = a\tilde{y}^{-1}$. It follows that $\tilde{x}$ is determined by $\tilde{y}$, so

$$(+) \qquad [\mathbf{4}]|\mathcal{D}(a)| \leq |\mathrm{C}_G(a)|$$

Now choose $a \in G^\sharp$ with $|\mathcal{D}(a)| = k$. Then $k \leq |\mathrm{C}_G(a)|$ and so

$$|a^G| = \frac{|G|}{|\mathrm{C}_G(a)|} \le \frac{|G|}{k} = \frac{n}{k} \le 2m^2$$

Put $N := \langle a^G \rangle$. Note that $G/\mathrm{C}_G(N) = G/\mathrm{C}_G(a^G)$ and so $G/\mathrm{C}_G(N)$ is isomorphic to a subgroup $\mathrm{Sym}(a^G)$. Since $|a^G| \le 2m^2$ this implies that $G/\mathrm{C}_G(N)$ is isomorphic to a subgroup $\mathrm{Sym}(2m^2)$.

It remains to show that $N \le [t, G]$. Let $r, s \in G$ with $x = t^r$ and $y = t^s$. Then

$$a = xy = x^{-1}y = t^{-r}t^s = [t^{-1}, t^{sr^{-1}}]^r = [t, t^{sr^{-1}}]^r$$

Since $[t, G] \trianglelefteq G$ we conclude that $a \in [t, G]$ and also $N = \langle a^G \rangle \le [t, G]$. □

Corollary 5.3.3. [**simple bounded centralizer**] *Let $n \in \mathbb{Z}^+$ Then there exist only finitely isomorphism classes of finite simple groups G with a involution t such that $|\mathrm{C}_G(t)| = n$.*

Proof. Suppose first that $t \in \mathrm{Z}(G)$. Then $\langle t \rangle \trianglelefteq G$ and, since G is simple, $G = \langle t \rangle$. Thus $G \cong \mathrm{C}_2$. So suppose $t \notin \mathrm{Z}(G)$. Then by 5.3.2 there exists a $1 \ne N \trianglelefteq G$ such that $G/\mathrm{C}_G(N)$ is isomorphic to a subgroup of $\mathrm{Sym}(2n^2)$. Since G is simple we get $N = G$. Thus $\mathrm{C}_G(N) = \mathrm{Z}(G)$ and since $t \notin \mathrm{Z}(G)$ we know that $\mathrm{C}_G(N) \ne G$. Hence $\mathrm{C}_G(N) = 1$ and so G is isomorphic to a subgroup of $\mathrm{Sym}(2n^2)$. As $\mathrm{Sym}(2n^2)$ has only finitely many subgroups, the corollary holds. □

Lemma 5.3.4. [**brian**] *Let K be a finite group, $M \trianglelefteq K$, $\overline{K} = K/M$ and $h \in K$. Then $|\mathrm{C}_{\overline{K}}(\overline{h})| \le |\mathrm{C}_K(h)|$. Moreover if $|\mathrm{C}_{\overline{K}}(\overline{h})| = |\mathrm{C}_K(h)|$, then $Mh \subseteq h^K$.*

Proof. Let $A \le K$ with $M \le A$ and $A/M = \mathrm{C}_{\overline{K}}(\overline{h})$. Note that $\mathrm{C}_K(h) \le A$ and so $\mathrm{C}_A(a) = \mathrm{C}_K(a)$. Consider the function

$$\tau : \quad A \to H, \quad a \mapsto h^a$$

Since $\overline{h}^{\overline{a}} = \overline{h}$ for all $a \in A$ we have $h^a \in Ma$ and so $\mathrm{Im}\,\tau \subseteq Mh$.

Let $a, b \in A$. By 5.3.1 $a\tau = b\tau$ if and only if $b \in \mathrm{C}_A(h)a = \mathrm{C}_K(a)b$.

Thus for each $d \in \mathrm{Im}\,\tau$, the inverse image $d\tau^{-1}$ of d in A under τ has size $|\mathrm{C}_K(h)|$. Since $d\tau^{-1}, d \in \mathrm{Im}\,\tau$ is a partition of A we conclude that

$$|A| = |\mathrm{C}_K(h)||\mathrm{Im}\,\tau| \le |\mathrm{C}_K(h)||Mh| = |\mathrm{C}_K(h)|M|$$

and so

$$|\mathrm{C}_{\overline{K}}(\overline{h})| = |A/M| \le |\mathrm{C}_K(h)|$$

If $|\mathrm{C}_{\overline{K}}(\overline{h})| = |\mathrm{C}_K(h)|$ we conclude that $Mh = \mathrm{Im}\,\tau = h^A \subseteq h^K$. □

Definition 5.3.5. [**def:bounded**] *Let $\mathcal{D}$ be a class and $f, f_1, f_2, \ldots f_n$ be functions from $\mathcal{D}$ to $\mathbb{N}$. We say that f is bounded on $\mathcal{D}$ in terms of $f_1, f_2, \ldots f_n$ if there exists a function $g : \mathbb{N}^n \to \mathbb{N}$ such that*

$$f(D) \le g\big(f_1(D), \ldots f_n(D)\big)$$

for all $D \in \mathcal{D}$.

Lemma 5.3.6. [**h1 bounded**] *Let G be group acting on the group H. Then $|H/\mathrm{C}_H(G)|$ is bounded in terms of $|G/\mathrm{C}_G(H)|$ and $|[H, G]|$.*

Proof. Replacing G be $G/\mathrm{C}_G(H)$ we may assume that $\mathrm{C}_G(H) = 1$. Consider the function

$$\alpha : \quad H/\mathrm{C}_H(G) \to \bigtimes_{g\in G}[H,G], \quad h\mathrm{C}_H(G) \ \mapsto \ ([h,g])_{g\in G}$$

Let $a, b \in H$. Then

$$\begin{array}{rll} & aC_H(G) = bC_H(G) & \\ \iff & ba^{-1} \in \mathrm{C}_H(G) & \\ \iff & ba^{-1} \in \mathrm{C}_H(g) & \text{for all } g \in G \\ \iff & [a,g] = [b,g] & \text{for all } g \in \quad - 5.3.1 \end{array}$$

The forward direction shows that α is well-defined and the backward direction shows that α is injective. It follows that

$$|H/\mathrm{C}_H(G)| = |\operatorname{Im}\alpha| \le |[H,G]|^{|G|}$$

□

Recall from last semester: A prime is a positive prime integer. Let π be a set of primes. Then π' is the set of primes not contained in π. For $n \in \mathbb{Z}$, $\pi(n)$ is the set of prime divisors of n. Let G be a group and $g \in G$. Then g is called a π-element if g has finite order and $\pi(|g|) \subseteq \pi$. The group G is called a π-group if all elements of G are π-elements. Recall that if G is finite group and $p \in \pi(G)$, then G has a element of order p. In particular, a finite group G is a π-group if and only if $\pi(|G|) \subseteq \pi$.

Lemma 5.3.7. [**more coprime**] *Let P be a finite p-group acting on a finite p'-group Q.*

(a) [**a**] *Let $R \trianglelefteq S \le Q$ be P-invariant subgroups of Q. Then $\mathrm{C}_{S/R}(P) = \mathrm{C}_S(P)R/R$.*

(b) [**c**] *$Q = [Q,P]\mathrm{C}_Q(P)$. In particular, $[Q,P] = [Q,P;n]$ for all $n \in \mathrm{Z}^+$.*
Recall here that

$$[A,B;n] = [[\ldots[[A,\underbrace{B],B],\ldots],B]}_{n-\text{times}}$$

(c) [**b**] *Let $1 = Q_0 \trianglelefteq Q_1 \le Q_2 \trianglelefteq \ldots \trianglelefteq Q_n = Q$ be a P-invariant subnormal series of Q. Then*

$$|\mathrm{C}_Q(P)| = \prod_{i=1}^{n} |\mathrm{C}_{Q_i/Q_{i-1}}(P)|$$

Proof. (a): Clearly $\mathrm{C}_S(P)R/R \le \mathrm{C}_{S/R}(P)$. Let $\overline{s} \in \mathrm{C}_{S/R}(P)$ and pick $s \in S$ with $\overline{s} = sR$. Then sR is an P invariant subset of S. Note $|sR| = |R|$ and $|R|$ divides $|Q|$. Thus p does not divided $|sR|$. Since all non-trivial orbits of P on sR have size divisible by p we conclude that there an orbit $\{t\}$ of P on Rs of length 1. Then $t \in C_S(P)$ and $\overline{s} = tR \in \mathrm{C}_S(P)R$.

(b): Observe that P centralizes $Q/[Q,P]$. So (b)follows from (a).

(c): This clearly holds for $n = 1$. Suppose $n > 1$ and put $k = n-1$. Then

$$\begin{array}{rclcl} |\mathrm{C}_Q(P)| & = & |\mathrm{C}_Q(P)/\mathrm{C}_{Q_k}(P)||\mathrm{C}_{Q_k}(P)| & = & |\mathrm{C}_Q(P)/\mathrm{C}_Q(P)\cap Q_k||\mathrm{C}_{Q_k}(P)| \\ & = & ||\mathrm{C}_Q(P)Q_k/Q_k||\mathrm{C}_{Q_k}(P)| & \overset{(a)}{=} & |\mathrm{C}_{Q/Q_k}(P)||\mathrm{C}_{Q_k}(P)| \\ & \overset{\text{Ind}}{=} & |\mathrm{C}_{Q/Q_k}(P)|\prod_{i=0}^{k}|\mathrm{C}_{Q_i/Q_{i-1}}(P)| & = & \prod_{i=1}^{n}|\mathrm{C}_{Q_i/Q_{i-1}}(P)| \end{array}$$

□

Definition 5.3.8. [**def:oupg**] *Let G be a finite group and π a set of primes. Then*

$$O^\pi(G) = \langle x \in G \mid x \text{ is a } \pi'\text{-element}\rangle$$

Lemma 5.3.9. [**basic oupg**] *Let G be a finite group and π a set of primes.*

(a) [**a**] *Let $N \trianglelefteq G$. Then G/N is a π-group if an only if $\mathrm{O}^\pi(G) \le N$.*

(b) [**b**] $O^\pi(G) = \langle S \mid p \in \pi', S \in \mathrm{Syl}_p(G)\rangle$.

(c) [**c**] *Suppose G acts on the finite π-group Q. Then*

$$[Q, \mathrm{O}^\pi(G)] = [Q, \mathrm{O}^\pi(G); n]$$

for all $n \in \mathbb{Z}^+$.

Proof. (a): Suppose first that $\mathrm{O}^\pi(G) \le N$ and let $\overline{x} \in G/N$. Then $\overline{x}x = Nx$ for some $x \in G$. Note that $x = yz$ where y is a π'-element and z is p-element. Then $y \in \mathrm{O}^\pi(G) \le N$ and so So $\overline{x} = Nx = Nyz = Nz$. Since z is a π element in G, we know that Nz is a π-element in G/N. Thus G/N is a π-group.

Suppose next that G is a G/N is a π-group. Let $x \in G$ be a π'-element. Then xN/N is a π and a π' elements. So $xN = 1_{G/N}$ and thus $x \in N$. Thus $\mathrm{O}^\pi(G) \le N$.

(b)Put $H := \langle S \mid p \in \pi', S \in \mathrm{Syl}_p(G)\rangle$. Let $p \in \pi'$ and $S \in \mathrm{Syl}_p(G)$. Then S is a p-group and so a π'-group. Thus $S \le \mathrm{O}^\pi(G)$ and so $H \le \mathrm{O}^\pi(G)$.

Next let $x \in G$ be a π'-element. Then $x = \prod_{p\in\pi'} x_p$ where x_p is a p-element in G. The $x_p \in S_p$ for some $S_p \in \mathrm{Syl}_p(G)$, so $x \in H$ and thus $\mathrm{O}^\pi(G) \le H$.

(c)Let $p \in \pi'$ and $S \in \mathrm{Syl}_p(G)$. Then S is a p-group and Q a p'-group. So (5.3.7)(b)gives

$$[Q, S] = [Q, S; n] \le [Q, \mathrm{O}^\pi(G); n]$$

Using (b)we get

$$[Q, \mathrm{O}^\pi(G)] = [Q, \langle S \mid p \in \pi', S \in \mathrm{Syl}_p(G)\rangle] = \langle [Q, S] \mid p \in \pi', S \in \mathrm{Syl}_p(G)\rangle] \le [Q, \mathrm{O}^\pi(G); n] \le [Q, \mathrm{O}^\pi(G)]$$

Thus $[Q, \mathrm{O}^\pi(G)] = [Q, \mathrm{O}^\pi(G); n]$. □

Let G be a group and $n \in \mathbb{N}$. Recall that $\mathrm{Z}_n(G)$ is inductively defined by $\mathrm{Z}_0(G) = 1$ and

$$\mathrm{Z}_{n+1}(G)/\mathrm{Z}_n(G) = \mathrm{Z}(G/\mathrm{Z}_n(G))$$

G is nilpotent if $G = \mathrm{Z}_n(G)$ for some $n \in \mathbb{N}$. This holds if and only if $[G; G; n] = 1$. A finite group is nilpotent if and only its is the direct product of p-groups. For finite G, $\mathrm{Z}_*(G) := \mathrm{Z}_m(G)$ where $m \in \mathbb{N}$ with $\mathrm{Z}_m(G) = \mathrm{Z}_{m+1}(G)$. If N is a nontrivial normal subgroup of G contained in $\mathrm{Z}_*(G)$ then $N \cap \mathrm{Z}_*(G) \ne 1$.

Lemma 5.3.10. [**coprime action**] *Let p be a prime and G a finite group acting a finite p-group P. Then the following three statement are equivalent:*

(a) [**a**] *There exists $n \in \mathbb{Z}^+$ with $[P, G; n] = 1$.*

(b) [**b**] $[P, \mathrm{O}^p(G)] = 1$.

(c) [**c**] *$G/\mathrm{C}_G(P)$ is a p-group.*

Proof. (a)$\Longrightarrow$ (b): Suppose $[P, G; n] = 1$ for $n \in \mathbb{Z}^+$.

$$[P, \mathrm{O}^p(G)] = [P, \mathrm{O}^p(G); n] \leq [P, G; n] = 1$$

So (b)holds.

(b)$\Longrightarrow$ (c): Suppose $[P, \mathrm{O}^p(G)] = 1$. Then $\mathrm{O}^p(G) \leq \mathrm{C}_G(P)$ and (5.3.9)(b)shows that $P/\mathrm{C}_G(P)$ is a p-group.

(c)$\Longrightarrow$ (a): Suppose $G/\mathrm{C}_G(P)$ is a p-group. Replacing G by $G/C_G(P)$ we may assume that G is p-group. Let $H := P \ltimes G$ be the semidirect product of P by G. Then H is a finite p-group and so H is nilpotent. It follows that $[H, H; n] = 1$ for some $n \in \mathbb{Z}^+$, so also $[P, G; n] = 1$. □

Definition 5.3.11. [**def:invert**] *Let G be a group acting on a group H, $t \in G$ and $A \subseteq H$. Then t inverts A if $a^t = a^{-1}$ for all $a \in H$.*

Lemma 5.3.12. [**invert abelian**] *Let G be a group acting on a group H and $t \in G$.*

(a) [**a**] *If $t^2 \in \mathrm{C}_G(H)$ and $a \in H$, then t inverts $[a, t]$.*

(b) [**b**] *If t inverts H, then H is abelian and both t^2 and $[t, G]$ centralizes H.*

(c) [**c**] *If H is finite and $\mathrm{C}_H(t) = 1$, then t inverts H.*

Proof. Let $a, b \in H$.

(a)$[a,t]^t = (a^{-1}a^t)^t = (a^{-1})^t a^{t^2} = (a^t)^{-1}a = (a^{-1}a^t)^{-1} = [a,t]^{-1}$.

(b)$b^{-1}a^{-1} = (ab)^{-1} = (ab)^t = a^t b^t = a^{-1}b^{-1}$ and so H is abelian. Also

$$a^{t^2} = (a^t)^t = (a^{-1})^t = (a^{-1})^{-1} = a$$

and

$$a^{t^g} = ((a^{g^{-1}})^t)^g = ((a^{g^{-1}})^{-1})^g = ((a^{g^{-1}})^g)^{-1} = a^{-1} = a^t$$

Thus both t^2 and $[t, g]$ centralize H.

(c)Since $\mathrm{C}_H(t) = 1$, 5.3.1 shows that the function

$$\alpha : \quad H \to H, \quad a \mapsto [a, t]$$

is injective. Since H is finite, this implies that α is surjective. Hence every element of H is the form $[a, t]$ and (a)implies that t inverts every element of H.

□

Definition 5.3.13. [**def:gn**] *Let $(G, \cdot)$ be a group and $n \in \mathrm{Z}^+$.*

(a) [**a**] $G^{\cdot n} = \langle g^n \mid g \in G \rangle$ *and* $G_{\cdot n} := \langle g \in G \mid g^n = 1 \rangle$

(b) [**b**] *Suppose G is p-group, where p is a prime. Then $\Omega_n(G) := G_{\cdot p^n}$.*

(c) [**c**] $\mathrm{rank}(G) = \min\{ |I| \mid I \subseteq G, G = \langle I \rangle \}$.

Lemma 5.3.14. [**abelian p**] *Let p be a prime and G a finite abelian p-group.*

(a) [**a**] $G = \bigoplus_{i=1}^r G_i$ *, where $r \in \mathbb{N}$ and $G_i \leq G$ with $G_i \cong \mathrm{C}_{p^{n_i}}$, $n_i \in \mathrm{Z}^+$.*

(b) [**b**] *$\Omega_1(G) = \bigoplus_{i=1}^r \Omega_1(G_i)$ and $\Omega_1(G_i) \cong \mathrm{C}_p$. In particular, $|\Omega_1(G)| = p^r$.*

(c) [**e**] *$G^{\cdot p} = \bigoplus_{i=1}^r G_i^{\cdot p}$ and $G_i/G_i^{\cdot p} \cong \mathrm{C}_p$. In particular, $|G/G^{\cdot p}| = p^r$.*

(d) [**c**] *$\Phi(G) = G^{\cdot p}$ and $r = \mathrm{rank}(G)$.*

(e) [**d**] *If $m \in \mathbb{Z}^+$ with $G^{\cdot m} = 1$, then $p^{n_i} \big| m$ for all $1 \le i \le n$ and $|G| \le m^r$.*

Proof. (a)Any finite abelian groups is the direct sum of cyclic groups, so (a)holds.

(b)and (c)follow from (a).

(d)By (2.2.10)(b)$\Phi_p(G) = \Phi_p(G)$. By definition $\Phi_p(G) == \langle [x,y], z^p \mid x,y,z \in G\rangle$ and since G is abelian we get $\Phi_p(G) = G^{\cdot p}$. By 2.2.11 $G/\Phi(G) \cong \mathrm{C}_p^k$, where $k = \mathrm{rank}(G)$. On the otherhand by (c), $|G/\Phi(G)| = |G/G^{\cdot p}| = p^r$. So $r = k$.

(e)is obvious. □

Proposition 5.3.15. [**g mod zl**] *Let G be a finite group and $t \in G$ with $t^2 = 1$. Put $L := [t,G]$. Then $|G/\mathrm{Z}_*(L)|$ is bounded in terms of $|\mathrm{C}_G(t)|$*

Proof. The proof is by induction on $|\mathrm{C}_G(t)|$. Put $\widetilde{G} = G/\mathrm{Z}^*(L)$. Then $\widetilde{[L,t]} = [\widetilde{L},\widetilde{t}]$, $\mathrm{Z}(\widetilde{L}) = 1$ and by 5.3.4 $|\mathrm{C}_{\widetilde{G}}(\widetilde{t})| \le |\mathrm{C}_G(t)|$. So replacing G be $G/\mathrm{Z}_*(L)$ we may assume that $\mathrm{Z}(L) = 1$ and we need to bound G in terms of $\mathrm{C}_G(t)|$.

If $t \in \mathrm{Z}(G)$, then $|G| = |\mathrm{C}_G(t)|$. So we may assume that $t \notin \mathrm{Z}(G)$.

By 5.3.2 there exist a non-trvial normal subgroup N of G such that $N \le L$ and $|G/\mathrm{C}_G(N)|$ is bounded in terms of $|\mathrm{C}_G(t)|$-bounded. Without loss N is a minimal normal subgroup of G.

Suppose that t inverts N. By 5.3.12 we get $[t,G] \le \mathrm{C}_G(N)$. Since $N \le L = [t,G]$ this gives $N \le \mathrm{Z}(L) = 1$, a contradiction.

Hence there exists $n \in N$ with $n \ne n^{-1}$. Since $n = (nt)t$ we conclude that (nt) does not have order two. In particular,

$$nt \notin t^G$$

Put $\overline{G} := G/N$. By 5.3.4 we have $|\mathrm{C}_{\overline{G}}(t)| \le |\mathrm{C}_G(t)|$ and since $nt \notin t^G$ even $|\mathrm{C}_{\overline{G}}(t)| < |\mathrm{C}_G(t)|$. Induction on $|\mathrm{C}_G(t)|$ now shows that $\overline{G}/\mathrm{Z}_*(\overline{L})$ is bounded in terms of $|\mathrm{C}_{\overline{G}}(\bar{t})$, and so also in terms of $|C_G(t)$.

Let $Z \le L$ with $N \le L$ and $Z/N := \mathrm{Z}_*(\overline{L}))$. Then $|G/Z|$ is bounded in terms of $|\mathrm{C}_G(t)|$, and we need to bound $|Z|$.

Put $D := \mathrm{C}_Z(N)$. Then $|Z/D| = Z/Z \cap \mathrm{C}_G(N) = |Z\mathrm{C}_G(N)/\mathrm{C}_G(N)| \le |G/\mathrm{C}_G(N)|$. It follows that $|Z/D|$ and so also $|G/D|$ are bounded in terms of $|\mathrm{C}_G(t)|$.

It remains to bound the order of D. So suppose that $D \ne 1$ and let M be any non-trivial normal subgroup of G contained in D.

Assume for a contradiction that $M \cap N = 1$. Then

$$1 \ne M \cong M/M \cap N \cong MN/N = \overline{M} \le \overline{Z} = \mathrm{Z}_*(\overline{L})$$

and so $\mathrm{C}_{\overline{M}}(L) = \overline{M} \cap \mathrm{Z}(\overline{L}) \ne 1$. Since $M \cong \overline{M}$, this gives $\mathrm{C}_M(L) \ne 1$. As $M \le L$ we conclude that $\mathrm{Z}(L) \ne 1$, a contradiction.

Thus $M \cap N \ne 1$. Since N is a minimal normal subgroup of G this gives $N = N \cap M \le M$. For $M = D$ we conclude that $N \le D = \mathrm{C}_Z(N)$. Hence $N \le \mathrm{Z}(D)$, so N is abelian.

Thus $(nn^t)^t = (n^t n^{t^2}) = n^t n = nn^t$ and since $nn^{-1} \ne 1$ we get $\mathrm{C}_N(t) \ne 1$. So we can choose a prime p dividing $\mathrm{C}_N(t)$ and an element m of order p in $\mathrm{C}_N(t)$. Since N is a minimal normal subgroup of G this implies $N = \langle m^G \rangle$.

Recall that $Z/N = \mathrm{Z}_*(\overline{L})$ and so Z/N is nilpotent. Since $N \le \mathrm{Z}(D)$, also D is nilpotent. It follows that $D = D_1 \times D_2$, where D_1 is p-group and D_2 is a p'-group. Note that D_1 consist of all p-elements and D_2 of all the p'-elements of D. As $D \trianglelefteq G$ we conclude D_1 and D_2 are both normal in G. Since $m^p = 1$ we have $m \in D_1$ and so $N = \langle m^G \rangle \le D_1$. Hence $N \cap D_2 \le D_1 \cap D_2 = 1$. As $N \cap M \ne 1$ for any non-trivial normal subgroup M of G contained in M get $D_1 = 1$. Thus $D = D_1 D_2 = D_1$ and D is a p-group. In particular, N is an abelian p-group. Note that $n \in \Omega_1(N)$ and so also $N = \langle m^G \rangle \le \Omega_1(N)$. Thus $N = \Omega_1(N)$. $|m^G| = |G/\mathrm{C}_G(m)| \le |G/\mathrm{C}_G(N)$, we have $\mathrm{rank}(N) \le |G/\mathrm{C}_G(N)|$. Now (5.3.14)(b)shows that

$$|N| = |\Omega_1(N)| = p^{\mathrm{rank}(N)} \le p^{|G/\mathrm{C}_G(N)|} \le |\mathrm{C}_G(t)|^{|G/\mathrm{C}_G(N)|}.$$

Thus $|G/\mathrm{C}_G(N)|$ is bounded, we conclude that also $|N|$ is $|\mathrm{C}_G(t)|$-bounded.

Since $\overline{D} \le \overline{Z} \le \mathrm{Z}_*(\overline{L})$ know that $[\overline{D}, \overline{L}; k] = 1$ for some $k \in \mathbb{Z}^+$. Since D is p-group, (5.3.9)(c)shows that $[\overline{D}, \mathrm{O}^p(\overline{L})] = 1$. Thus $[D, \mathrm{O}^p(L)] \le N$. Put $E := \mathrm{C}_D(\mathrm{O}^p(L))$. By (5.3.9)(c)$[E, \mathrm{O}^p(L); l] = 1$ for some $l \in \mathbb{Z}^+$. Thus $E \le \mathrm{Z}_*(L) = 1$.

Note that

$$[[\mathrm{O}^p(L), D], D] \le [N, D] \le 1, \quad \textit{and} \quad [[D, \mathrm{O}^p(L)], D] = 1.$$

The Three Subgroup Lemma now shows that $[[D, D], \mathrm{O}^p(L)] = 1$. Thus $D' = [D, D] \le \mathrm{C}_D(\mathrm{O}^p(L)) = 1$, so D is abelian. In particular, $D \le \mathrm{C}_G(D)$ and

$$|\mathrm{O}^p(L)/\mathrm{C}_{\mathrm{O}^p(L)}(D)| \le |G/\mathrm{C}_G(D)| \le |G/D|$$

So, since $|G/D|$ is bounded in terms of $|\mathrm{C}_G(t)$, the same is true for $|\mathrm{O}^p(L)/\mathrm{C}_{\mathrm{O}^p(L)}(D)|$. Recall that $[D, \mathrm{O}^p(L)] \le N$ and since $|N|$ is bounded, we see that $[D, \mathrm{O}^p(L)$ is bounded in terms of $|\mathrm{C}_G(t)|$. By 5.3.6 $|D/\mathrm{C}_D(\mathrm{O}^p(L)|$ is bounded in terms of $|\mathrm{O}^p(L)/\mathrm{C}_{\mathrm{O}^p(L)}(D)|$ and $|[D, \mathrm{O}^p(L)]|$. As $\mathrm{C}_D(\mathrm{O}^p(L)) = 1$ we conclude that D is bounded in terms of $|\mathrm{C}_G(t)|$. □

Lemma 5.3.16. [**action on abelian i**] *Let G be a group acting on the abelian group A. Let $t \in G$.*

(a) [**a**] *The function*

$$\alpha: \quad A \to A, \quad a \mapsto [a, t]$$

is a homomorphism with $\mathrm{Ker}\alpha = \mathrm{C}_A(t)$ *and* $\mathrm{Im}\,\alpha = \{[a, t] \mid a \in A\} = [A, t]$.

(b) [**b**] $A/\mathrm{C}_A(t) \cong [A, t]$. *In particular,* $|A| = |\mathrm{C}_A(t)| \cdot |[A, t]|$.

(c) [**c**] *If $t^2 = 2$, then t inverts $[A, t]$.*

(d) [**d**] *If $t^2 = 1$ and $A^{\cdot 2} = 1$, then $[A, t] \le \mathrm{C}_A(t)$ and $|A| \le |\mathrm{C}_A(t)|^2$.*

Proof. (a)Let $a, b \in A$. Since A is abelian,

$$[a, t][b, t] = a^{-1} a^t b^{-1} b^t = (ab)^{-1} (ab)^t = [ab, t]$$

and so α is homomorphism. Clearly $\mathrm{Ker}\alpha = \mathrm{C}_A(t)$ and $\mathrm{Im}\,\alpha = \{[a, t] \mid a \in A\}$. Since α is a homomorphism, we know that $\mathrm{Im}\,\alpha$ is a subgroup of A, and so $\mathrm{Im}\,\alpha = \langle \{[a, t] \mid a \in A\} \rangle = [A, t]$.

(b)Since α is homomorphism, $A/\mathrm{Ker}\alpha \cong \mathrm{Im}\,\alpha$, so (b)holds.

(c)Since $t^2 = 1$ we know that t inverts $[a, t]$ for each $a \in A$. By (a)$[A, t] = \{[a, t] \mid a \in A\}$, so t inverts $[A, t]$.

(d)Let $a \in [A, t]$. Then $a^2 = 1$ and so $a^{-1} = a$. By (c)$a^t = a^{-1}$ so $a^t = a$ and $a \in \mathrm{C}_A(t)$. Thus $[A, t] \le \mathrm{C}_A(t)$. In particular, $|[A, t]| \le |\mathrm{C}_A(t)|$ and so (c)gives $|A| = |\mathrm{C}_A(t)||[A, t]| \le |\mathrm{C}_A(t)|^2$. □

Definition 5.3.17. [**def:FSG**] *Let G be a group. Then $\mathrm{SG}(G)$ is the set of subgroups of G and $\mathrm{FSG}(G)$ is the set of finitely generated subgroups.*

Definition 5.3.18. [**def:locally determined**] *Let G be a group and $\mathrm{D} : \mathrm{SG}(G) \to \mathrm{SG}(G)$ a function.*

(a) [**a**] D *has the union property if*

$$\mathrm{D}(H) = \bigcup_{F \in \mathrm{FSG}(H)} \mathrm{D}(F)$$

for all $H \le G$.

(b) [**b**] D *has the intersection property if*

$$\mathrm{D}(H) = \bigcap_{F \in \mathrm{FSG}(H)} \mathrm{D}(F)$$

for all $H \le G$.

(c) [**c**] D *is locally determined if for all $F \in \mathrm{FSG}(G)$ and $L \in \mathrm{SG}(G)$ exists $H \in \mathrm{FSG}(L)$ such that*

$$F \cap \mathrm{D}(L) = F \cap \mathrm{D}(K)$$

for all $K \le L$ with $H \le K$.

(d) [**d**] D *is increasing if $\mathrm{G}(H) \le \mathrm{D}(K)$ for all $H \le K \le G$.*

Remark 5.3.19. [**rm:union property**] *Let G be a group and $\mathrm{D} : \mathrm{SG}(G) \to \mathrm{SG}(G)$ be a function with the union property. Then D is increasing.*

Lemma 5.3.20. [**generation local**] *Let G be a group and $(I_d)_{d \in D}$ a family of subsets of G. For $E \subseteq D$ define $X_E = \langle X_e \mid e \in E \rangle$. Let $\mathcal{D}$ be the set of finite subsets of D. Then*

$$X_D = \bigcup_{E \in \mathcal{D}} X_E$$

Proof. Define $\mathcal{H} = \{X_E \mid E \in \mathcal{D}\}$. Let $E, F \in \mathcal{D}$. Then $E \cap F \in \mathcal{D}$ and both X_D and X_E in $X_{D \cup F}$. Now 5.1.4 shows that $\bigcup \mathcal{H}$ is a subgroup of G. If $d \in D$, the $d \in \{d\} \in \mathcal{D}$ and $X_d \subseteq X_{\{d\}}$. Also $X_E \subseteq X_D$ for $E \in \mathcal{D}$ and so

$$\bigcup_{d \in D} X_d \subseteq \bigcup \mathcal{H} \subseteq X_D = \langle X_d \mid d \in D \rangle = \left\langle \bigcup_{d \in D} X_d \right\rangle$$

Since $\bigcup \mathcal{H}$ is a subgroup of G this implies $X_D = \langle \mathcal{H} \rangle$. □

Example 5.3.21. [**ex:union**] Let G be a group.

(1) [**a**] The function $H \to H'$ has union property.

(2) [**b**] Let $n \in \mathbb{N}$. Then the function $H \to H^{\cdot n}$ has the unions property,

(3) [**c**] Let $T \subseteq G$. Then $H \to [H, T]$ has the union property.

(4) [**d**] $H \to \mathrm{C}_G(H)$ has the intersection property.

Proof. (3): Let $H \le G$. Let $\mathcal{H}$ be the set of finite subsets of H.

Note first that

(∗) [**1**]

$$\begin{aligned}[H,T] &= \langle [h,t] \mid h \in H, t \in T\rangle && \text{– Definition of } [H,T]\\ &= \Big\langle \langle [h,t] \mid t \in T\rangle \;\Big|\; h \in H\Big\rangle \\ &= \langle [h,T] \mid h \in H\rangle && \text{– Definition of } [h,T]\end{aligned}$$

and so

$$\begin{aligned}[H,T] &= \langle\, [h,T] \mid\; h \in H\,\rangle \\ &\subseteq \bigcup_{E\in\mathcal{H}} \langle [h,T] \mid h \in E\rangle && \text{– 5.3.20}\\ &\subseteq \bigcup_{E\in\mathcal{H}} \langle\, [h,T] \mid h \in \langle E\rangle\,\rangle \\ &\subseteq \bigcup_{F\in \mathrm{FSG}(H)} \langle\, [h,T] \mid h \in F\,\rangle && \text{– Definition of FSG}(H)\\ &\subseteq \bigcup_{F\in \mathrm{FSG}(H)} [F,T] && \text{– (}*\text{)applied to } F \text{ in place of} H\\ &\subseteq [H,T]\end{aligned}$$

So $[H,T] = \bigcup_{F\in\mathrm{FSG}(H)} [F,T]$ and (3)holds. □

Lemma 5.3.22. [**union determined**] *Let G be locally finite groups and $\mathrm{D} : \mathrm{SG}(G) \to \mathrm{SG}(G)$ function with the union property. Then* D *is locally determined.*

Proof. Let $F, L \leq G$ with F finite. Let $a \in F \cap \mathrm{D}(L)$. Since D has the union property we know that

$$\mathrm{D}(L) = \bigcup_{H\in\mathrm{FSG}(L)} \mathrm{D}(H)$$

and so there exists a finite subgroup H_a of L with $a \in \mathrm{D}(H_a)$. Since L is locally finite there exists finite subgroup H of L with

$$\bigcup_{a\in F\cap \mathrm{D}(L)} H_a \subseteq H$$

Let $K \leq L$ with $H \leq L$. Then $H_a \leq H \leq K \leq L$ and so

$$a \in \mathrm{D}(H_a) \leq \mathrm{D}(K) \leq \mathrm{D}(L)$$

Hence

$$F \cap \mathrm{D}(L) \leq F \cap \mathrm{D}(K) \leq F \cap \mathrm{D}(L)$$

and $F \cap \mathrm{D}(L) = F \cap \mathrm{D}(K)$. □

Lemma 5.3.23. [**determined and bounded**] *Let G be a group and $D : \mathrm{SG}(G) \to \mathrm{SG}(G)$ a locally determined functions. Suppose there exists $n \in \mathbb{Z}^+$ and $H \in \mathrm{FSG}(G)$ such that*

$$|F/F \cap \mathrm{D}(F)| \leq n$$

for all $F \in \mathrm{FSG}(G)$ with $H \leq F$. Then $|G/\mathrm{D}(G)| \leq n$.

Proof. Put $A := \mathrm{D}(G)$ and let $\mathcal{E}$ be a finite subset of G/A. We just need to show that $|\mathcal{E}| \le n$.

For $E \in \mathcal{E}$ choose $g_E \in G$ with $E = Ag_E$. Put

$$F := \langle g_E \mid E \in \mathcal{E} \rangle$$

and note that $F \in \mathrm{FSG}(G)$. Since D is locally determined there exists $L \in \mathrm{FSG}(G)$ such that

$$F \cap A = F \cap \mathrm{D}(G) = F \cap \mathrm{D}(K)$$

for all $K \le G$ with $L \le K$.

Define

$$K := \langle F, H, L \rangle$$

Since F, H and L are finitely generated, so is K. Since $H \le K$ we have $|K/K \cap \mathrm{D}(K)| \le n$ and since $L \le K$ we have $F \cap A = F \cap \mathrm{D}(K)$.

Note also that $g_E \in F$ for all $E \in \mathcal{E}$. Thus

$$\begin{aligned}|\mathcal{E}| &= |\{Ag_E \mid E \in \mathcal{E}\}| \le |\{Ae \mid e \in F\}| = |AF/A| = |F/F \cap A| = |F/F \cap \mathrm{D}(K)| \\ &= |F\mathrm{D}(K)/\mathrm{D}(K)| \le |K\mathrm{D}(K)/\mathrm{D}(K)| = |K/K \cap \mathrm{D}(K)| \le n\end{aligned}$$

□

Lemma 5.3.24. [**d circ e**] *Let G be group and $\mathrm{D}, \overline{E} : \mathcal{S}(G) \to \mathcal{S}(G)$ be locally determined function. Suppose that* E *is increasing. Then also* $\mathrm{D} \circ \mathrm{E}$ *is locally determined.*

Proof. Let $F, L \le G$ with F finitely generated. Since D is locally determined there exists $\widetilde{H} \in \mathrm{FSG}(\mathrm{E}(L))$ such that

$$(*) \qquad [\mathbf{1}] F \cap \mathrm{D}(\mathrm{E}(L)) = F \cap \mathrm{D}(\widetilde{K})$$

for all $\widetilde{K} \le \mathrm{E}(L)$ with $\widetilde{H} \le \widetilde{K}$. Since E is locally determined there exists $H \in \mathrm{FSG}(L)$ such that

$$(**) \qquad [\mathbf{2}] \widetilde{H} \cap \mathrm{E}(L) = \widetilde{H} \cap \mathrm{E}(K)$$

for all $K \le L$ with $H \le K$.

Let $K \in L$ with $H \le K$. By choice of $\widetilde{H}$ we have $\widetilde{H} \le \mathrm{E}(L)$. So $\widetilde{H} \cap \mathrm{E}(L) = \widetilde{H}$ and $(**)$gives $\widetilde{H} \le \mathrm{E}(K)$. Since $K \le L$ and E is increasing, $\mathrm{E}(K) \le \mathrm{E}(L)$. Thus

$$\widetilde{H} \le \mathrm{E}(K) \le \mathrm{E}(L)$$

and $(*)$applied with $\widetilde{K} = \mathrm{E}(K)$ shows that

$$F \cap \mathrm{D}(\mathrm{E}(L)) = F \cap \mathrm{D}(\mathrm{E}(K)$$

□

Recall from 2.1.15: Let G be a nilpotent group and A a maximal abelian normal subgroup of G. Then $\mathrm{C}_G(A) \le A$.

Lemma 5.3.25. [**2-group with small centralizer**] *Let P be a locally finite 2-group and $t \in P$ such that $t^2 = 1$ and $\mathrm{C}_P(t)$ is finite. Then there exists $n \in \mathbb{Z}^+$ such that t inverts $P^{\cdot n}$, $|P/P^{\cdot n}|$ is finite and n,* rank$(P^{\cdot n})$ *and $|P/P^{\cdot n}|$ all are bounded in terms of $|\mathrm{C}_P(t)|$*

Proof. Suppose the lemma holds for finite groups. Then there exists an $n \in \mathbb{Z}^+$ such that, for all finite subgroups F of P with $\mathrm{C}_P(t) \le F$, t inverts $F^{\cdot n}$ and both n and $|F^{\cdot n}|$ are bounded in terms of $|\mathrm{C}_F(t)|$. As $\mathrm{C}_P(t) \le F$ we have $\mathrm{C}_P(t) = \mathrm{C}_F(t)$. By 5.3.21 the function

$$\mathrm{D}: \quad \mathrm{SG}(P) \to \mathrm{SG}(P), \quad H \to H^{\cdot n}$$

as the union property. So by 5.3.22 $\overline{D}$ is locally determined. As $|F/F^{\cdot n}|$ is bounded, 5.3.23 now shows that $|P/P^{\cdot n}|$ is bounded. Let $g \in P^{\cdot n}$. Since D has the union property, there exists finite subgroup A of G with $g \in A^{\cdot n}$. Choose a finite subgroup F of G with $A^{\cdot n} \le F$ and $\mathrm{C}_G(t) \le F$. Then t inverts $F^{\cdot n}$ and so t inverts g and $P^{\cdot n}$. We proved, that if the theorem holds for finite groups it also holds for locally finite groups. So we may assume now that P is finite.

Let $m := |\mathrm{C}_P(t)|$. Since P is a 2-group, $m = 2^k$ for some $k \in \mathbb{N}$. Let A be a maximal normal abelian subgroup. Our main goal is the bounded the order of P/A. Since $A = \mathrm{C}_P(A)$ this amounts to bounding $|P/\mathrm{C}_P(A)$. Put

$$H := \mathrm{C}_P(A/A^{\cdot m}) \quad \text{and} \quad E := \mathrm{C}_H(A^{\cdot m})$$

We will bound $|P/H|$, $|H/E|$ and $|E/C_E(A)|$. Note that P/H is isomorphic to a subgroup of $\mathrm{Aut}(A/A^{\cdot m})$, so to bound $|P/H|$ it suffices to bound $|A/A^{\cdot m}|$.

By (5.3.16)(b)$|A| = |[A,t]| \cdot |\mathrm{C}_A(t)|$, so $|A/[A,t]| = |\mathrm{C}_A(t)$. Since $\mathrm{C}_A(t) \le \mathrm{C}_P(t)$ we conclude that $|A/[A,t]|$ divides m. It follows that $\overline{a}^m = 1$ for all $\overline{a} \in A/\mathrm{C}_A(t)$, thus

$$A^{\cdot m} \le [A,t]$$

As $t^2 = 1$ (5.3.16)(c)shows that t inverts $[A,t]$. Hence t also inverts $A^{\cdot m}$. Note that $b^2 = 1$ for all $b \in \Omega_1(A)$. Thus (5.3.16)(d)shows that

$$|\Omega_1(A)| \le |\mathrm{C}_{\Omega_1(A)}(t)|^2 \le |\mathrm{C}_P(t)|^2 = m^2 = 2^{2k}$$

Put $r = \mathrm{rank} A$. By (5.3.14)(b)$|\Omega_1(A)| = 2^r$, so $r \le 2k$ and r is bounded.

By (5.3.14)(e)

$$|A/A^{\cdot m}| \le m^r$$

Thus $|A/A^{\cdot m}|$ is bounded in terms of m. As mentioned above, this implies that $|P/H|$ is bounded in terms of m.

Note that t centralizes $H/[H,t]$. Hence by 5.3.4

$$|H/[H,t]| = |\mathrm{C}_{H/[H,t]}(t)| \le |\mathrm{C}_H(t)| \le |\mathrm{C}_P(t)| \le m$$

Since t inverts $A^{\cdot m}$ we know that $[H,t] \le \mathrm{C}_H(A^{\cdot m}) = E$, see 5.3.12. Thus

$$||H/E| \le |H/[H,t]| \le m$$

Let $a \in A$ and $e \in E$. Then

$$\begin{aligned} [a,e]^m &= [a^m, e] && - \text{since } a \mapsto [a,e] \text{ is a homomorphism} \\ &= 1 && - \text{since } e \in E = \mathrm{C}_H(A^{\cdot m}) \end{aligned}$$

It follows that $a^m = 1$ for all $a \in [A, E]$. Now (5.3.14)(e)gives $|[A, E]| \le m^{\text{rank}([A,E]} \le m^r$. Thus $|[A, E]|$ is bound in terms of m. Since $E = \mathrm{C}_H(A^{\cdot m})$ we have $A^{\cdot m} \le \mathrm{C}_A(E)$, so

$$|A/\mathrm{C}_A(E)| \le |A/A^{\cdot m}| \le m^r$$

We proved that $|A/\mathrm{C}_A(E)|$ and $|[A, E]|$ are bounded in terms of m. By 5.3.6 $|E/\mathrm{C}_E(A)|$ is bounded in terms $|A/C_A(E)|$ and $|[A, E]|$, so $|E/C_E(A)|$ is bounded in terms of m.

We proved that $|P/H|$, $|H/E|$ and $|E/\mathrm{C}_E(A)|$ all are bounded. Hence $|P/\mathrm{C}_P(A)|$ is bounded. Put $l = |P/A|$ and $n = lm$. Since $A = \mathrm{C}_P(A)$ we conclude that l and n are bounded terms of m. Note that $P^{\cdot l} \le A$. So $P^{\cdot n} = (P^{\cdot l})^{\cdot m} \le A^{\cdot m}$. Thus t inverts $P^{\cdot n}$. Observe that $A^{\cdot n} \le P^{\cdot n}$ and by (5.3.14)(e)$|A/A^{\cdot n}| \le n^{2k}$. Thus $|P/P^{\cdot n}| \le n^{2k}$. We bounded $|P/A|$ and $|A/P^{\cdot n}|$, so $|P/P^{\cdot n}|$ is bounded in terms of m.

□

Lemma 5.3.26. [**zn for locally finite**] *Let G be a locally finite group and $n \in \mathbb{N}$. Then the function*

$$\mathrm{Z}_n : \quad \mathrm{SG}(G) \to \mathrm{SG}(G), \quad K \mapsto \mathrm{Z}_n(K)$$

is locally determined.

Proof. Let F be a finite subgroup of G and $L \le G$. We need a finite subgroup H of L such that

$$F \cap \mathrm{Z}_n(L) = F \cap \mathrm{Z}_n(K)$$

for all $K \le L$ with $H \le K$.

For $a \in (F \cap L) \smallsetminus (F \cap \mathrm{Z}_{n+1}(L))$ choose $l_a \in L$ with $[a, l_a] \notin \mathrm{Z}_n(L)$. Since L is locally finite there exists a finite subgroup D of L with $F \cap L \le D$ and $l_a \in D$ for all $a \in (F \cap L) \smallsetminus (F \cap \mathrm{Z}_{n+1}(L))$. By induction there exists a finite subgroup $\widetilde{H}$ of L such that

$$D \cap \mathrm{Z}_n(L) = D \cap \mathrm{Z}_n(K)$$

for all $K \le L$ with $\widetilde{H} \le L$. As L is locally finite we can choose a finite $H \le L$ such that $D, \widetilde{H} \le H$.

It remains to show that $F \cap \mathrm{Z}_{n+1}(L) = F \cap \mathrm{Z}_{n+1}(K)$ for any $K \le G$ with $H \le K$. By (2.1.12)(a)we have

$$K \cap \mathrm{Z}_{n+1}(L) \le \mathrm{Z}_{n+1}(K)$$

In particular, since $F \cap L \le D \le H \le K$,

$$F \cap \mathrm{Z}_{n+1}(L) = (F \cap L) \cap \mathrm{Z}_{n+1}(L) \le F \cap (K \cap \mathrm{Z}_{n+1}(L_) \le F \cap \mathrm{Z}_{n+1}(K)$$

Let $a \in (F \cap L) \smallsetminus (F \cap \mathrm{Z}_{n+1}(L))$. Then $[a, l_a] \notin \mathrm{Z}_n(L)$. Recall that $a \in F \cap L \subseteq D \cap L$ and $l_a \in L \cap D \subseteq L \cap K$. Thus $[a, l_a] \in D \smallsetminus \mathrm{Z}_n(L)$ and $a, l_a \in K$. As $\widetilde{H} \le H \le K$ we have

$$D \cap \mathrm{Z}_n(L) = D \cap \mathrm{Z}_n(K)$$

and so $[a, l] \notin \mathrm{Z}_n(K)$. Since $a, l_a \in K$ this gives $a \notin \mathrm{Z}_{n+1}(K)$. We proved

$$(F \cap L) \smallsetminus (F \cap \mathrm{Z}_{n+1}(L)) \subseteq F \smallsetminus (F \cap L) \smallsetminus (F \cap \mathrm{Z}_{n+1})(K)$$

Thus

$$F \cap \mathrm{Z}_{n+1}(K) \le F \cap \mathrm{Z}_{n+1}(L)$$

and the lemma is proved. □

Lemma 5.3.27. [**bound nilpotency**] *Let L be finite group and $B \trianglelefteq L$ with $B \leq \mathrm{Z}_*(L)$. let $r \in \mathbb{N}$ be minimal with $B \leq \mathrm{Z}_r(B)$. For $i \in \mathbb{N}$ define $B_i := B \cap \mathrm{Z}_i(L)$. Let $i \in \mathbb{N}$.*

(a) [**a**] *Then $B_{i+1} = \mathrm{C}_{B/B_i}(L)$.*

(b) [**b**] *If $i < r$, then $B_{i+1} \neq B_i$.*

(c) [**c**] *$|B| \geq 2^r$. In particular, $r \leq \lfloor \log_2 |B| \rfloor$.*

Proof. (a)and (b)Put $\overline{L} := L/\mathrm{Z}_i(L)$ and note that

$$\overline{Z_{i+1}} = \mathrm{Z}(\overline{L})$$

We compute

$$\overline{B_{i+1}} = B_{i+1}Z_i/Z_i = B \cap \mathrm{Z}_{i+1})Z_i/Z_i = (BZ_i) \cap Z_{i+1}/Z_i = \overline{B} \cap \overline{Z_{i+1}} = \overline{B} \cap \mathrm{Z}(\overline{L}) = \mathrm{C}_{\overline{B}}(L)$$

Using the the isomorphism

$$B/B_i = B/B \cap \mathrm{Z}_i \cong BZ_i/Z_i = \overline{B}$$

we conclude that

$$B_{i+1}/B_i = C_{B/B_i}(L))$$

If $i < r$, then $B_i \neq B$ and so $\overline{B} \neq 1$. Also $\overline{B} \leq \overline{Z_*(L)} \leq Z_*(\overline{L})$ and 2.1.10 shows that $\overline{B} \cap \mathrm{Z}(\overline{L}) \neq 1$. Hence $\overline{B_{i+1}} \neq 1$ and so $B_{i+1} \neq B_i$.

(c)Note that $B = B_r$ and so $|B| = \prod_{i=0} r - 1 |B_{i+1}/B_i|$. By (b)$B_{i+1} \neq B_i$. Hence $|B_{i+1}/B_i| \geq 2$ and (c)holds □

Proposition 5.3.28. [**nilpotent by finite**] *Let G be a locally finite group and $t \in G$ with $t^2 = 1$. Then there exists a postive integer n such that n and $|G/\mathrm{Z}_n([G,t])|$ are bounded in terms of $|\mathrm{C}_G(t)|$. In particular, G is nilpotent by finite.*

Proof. Suppose the proposition holds for finite groups. Then there exists integers n and m such, for all finite subgroups F of G with $\mathrm{C}_G(t) \leq G$, both n and $|F/\mathrm{Z}_n([F,t])|$ are bounded in terms of $|C_G(t)|$. By (5.3.21)(3)the function $\overline{D} : H \to [H,t]$ has the union property. So by 5.3.19 $\overline{D}$ is increasing and by 5.3.22 $\overline{D}$ is locally determined. By 5.3.26 also $\mathrm{Z}_n : H \to \mathrm{Z}_n(H)$ is locally determined. Hence by 5.3.24 $H \to \mathrm{Z}_n([H,t]$ is locally determined. As $|F/\mathrm{Z}_n([F,t]|)$ is bounded for all F of G with $\mathrm{C}_G(t) \leq F$, 5.3.23 now shows that also $|G/\mathrm{Z}_n([F,t])|$ is bounded in terms of $|C_G(t)$.

So we may assume that G is finite. Put $L := [t,G]$ and $Z := \mathrm{Z}_*(L)$. Let n be minimal with $Z_n(L) = Z$. By 5.3.15 $|G/Z|$ is bounded in terms of $|\mathrm{C}_G(t)|$. So we just need to show that n is bounded. Since Z is nilpotent, there exists a unique 2-group A and a unique $2'$-group B with $Z = A \times B$.

Let r and s in $\mathbb{N}$ be minimal with $A \leq \mathrm{Z}_r(L)$ and $B \leq Z_s(L)$, respectively. Then $n = \max(r,s)$. Put $P := A\langle t \rangle$. Observe that observe P is a 2-group. By 5.3.25 there exists an integer m such that $P^{\cdot m}$ is inverted by t and both $|P/P^{\cdot m}|$ and $\mathrm{rank}(P^{\cdot m})$ are bounded in terms of $|\mathrm{C}_P(t)|$. As $P^{\cdot m}$ has bounded rank and by (5.3.14)(e)$|P^{\cdot m}/P^{\cdot 2m}| \leq 2^{\mathrm{rank}(P^{\cdot m}}|$, also $|P/P^{\cdot 2m}|$ is bounded in terms of $\mathrm{C}_P(t)$. Since $|P/A| \leq 2$ we have $P^{\cdot 2} \leq A$ and so $P^{\cdot 2m} \leq (P^{\cdot 2})^{\cdot m} \leq A^{\cdot m} \leq P^{\cdot m}$. It follows that t inverts A and that $|A/A^{\cdot m}|$ is bounded in terms of $|\mathrm{C}_G(t)|$. Note that L, Z, A and $A^{\cdot m}$ all are normal subgroups of G. As t inverts $A^{\cdot m}$ we conclude from 5.3.12 that L $(= [G,t])$ centralizes $A^{\cdot m}$. Thus $A^{\cdot m} \leq \mathrm{Z}(L)$. Put $l := \lfloor \log_2 |A/A^{\cdot m}| \rfloor$ and $\overline{L} := L/\mathrm{Z}(L)$. Then

$$|\overline{A}| = |AZ(L)/Z(L)| = |A/A \cap Z(L)| \leq |A/A^{\cdot m}| \quad \text{and} \quad \overline{A} \leq \overline{Z} = Z_*(L)$$

Thus 5.3.27 shows that $\overline{A} \leq Z_l(\overline{L})$ and so $A \leq Z_{l+1}(L)$. Thus $r \leq l+1$, and r is bounded in terms of $|C_G(t)|$.

For $i \in \mathbb{N}$ define $B_i := B \cap Z_i(L)$. Then by (5.3.27)(a)

$$B_{i+1}/B_i = C_{B/B_i}(L)$$

Let $s = 2u + \epsilon$ with $\epsilon = 0, 1$. Let $i \in \mathbb{B}$ with $i < u$. By (5.3.27)(b)$B_{2i+2} \neq B_{2i+2}$. Thus

$$C_{B_{2i+2}/B_{2i}}(L) = C_{B_{2i+1}/B_{2i}}(L) = B_{2i+1}/B_{2i} \neq B_{2i+2}/B_{2i}$$

Thus L does not centralizes B_{2i+2}/B_{2i}. It follows that t does not inverts B_{2i}/B_{2i-2} and (5.3.12)(c)shows that $C_{B_{2i+2}/B_{2i}}(t) \neq 1$. By (5.3.7)(a)

$$C_{B_{2i+2}/B_{2i}}(t) = C_{B_{2i+2}}(t)B_{2i}/B_{2i}$$

and so $C_{B_{2i+2}}(t) \nleq B_{2i+1}$. Thus

$$0 < C_{B_2}(t) < C_{B_4}(t) < \ldots < C_{B_{2u}}(t) \leq C_G(t)$$

Thus $2^u \leq |C_G(t)|$ and we conclude that s is bounded in terms of $|C_G(t)|$.

□

Corollary 5.3.29. [**infinite centralizer**] *Let H be an infinite locally finite simple group and t an involution in H. Then $C_H(t)$ is infinite.*

Proof. This follows immediately from 5.3.28 □

5.4 Locally finite groups with MIN

Definition 5.4.1. [**def:kegel cover**] *Let H be locally finite group. Then a Kegel cover $\mathcal{K}$ for H is a set of pairs of subgroup of H such that*

(i) [**1**] *If $(K, M) \in \mathcal{K}$ then $M \trianglelefteq K \leq H$, K is finite and K/M is simple.*

(ii) [**2**] *If F is a finite subgroup of H, then there exists $(K, M) \in \mathcal{K}$ with $F \leq K$ and $F \cap M = 1$.*

Theorem 5.4.2. [**kegel**] *Every locally finite simple group has a Kegel cover.*

Proof. Let H be a locally finite group. Define $\mathcal{K}$ to be the set of all pairs (K, M) such that $M \trianglelefteq K \leq H$, K is finite and K/M is simple. Let F be any non-trivial finite subgroup of H. Let $1 \neq f \in F$. Since H is simple $H = \langle f^H \rangle$ and so there exists a finite subset I_f of H with $F \leq \langle f^{I_F} \rangle$. Put $F^* = \langle F, I_f \mid f \in F^\sharp \rangle$ and note that $F \leq \langle f^{F^*} \rangle$ for all $f \in F^\sharp$. Put $K := \langle F^{F^{**}} \rangle$. Since $1 \neq F \subseteq F^*$ we have $F^* \leq \langle F^{F^{**}} \rangle \leq K$. Let N be the intersection of the maximal normal subgroups of K. Then N is characteristic subgroup of K and $N \neq K$. Since F^{**} normalizes K it also normalizes N. If $F \leq N$ we get $K = \langle F^{F^* *} \rangle \leq N$, a contradiction. Thus $F \nleq N$ and there exists a maximal normal subgroup M of K with $F \nleq M$. Note that $(K, M) \in \mathcal{K}$ and $F \leq K$. Suppose that $F \cap M \neq 1$ and pick $f \in F^\sharp \cap M$. The $F \leq \langle f^{F^*} \rangle \leq \langle M^K \rangle = M$, a contradiction. Thus $F \cap M = 1$ and $\mathcal{K}$ is a Kegel cover. □

Lemma 5.4.3. [**locally solvable**] *Let G be a locally solvable, locally finite, simple group. Then $G \cong C_p$, p a prime.*

Proof. Fix $1 \neq H \leq G$ and let $g \in G$. Since G is locally finite there exists a finite subgroup F of G with $F \leq H$ and $g \in F$. By 5.4.2 G has a Kegel cover $\mathcal{K}$ and so there exists $K \leq L$ and $M \trianglelefteq K$ such that K is finite, K/M is simple, $H \leq K$ and $H \cap M = 1$. Since K is finite and G is locally finite K is solvable. Hence also K/M is solvable and since K/M is simple we conclude that $K/M \cong C_p$, p a prime. Note that

$$1 \neq F \cong F/1 = F/F \cap M \cong F/M \leq F/M \cong \mathrm{C}_p$$

It follows that $F \cong \mathrm{C}_p$ and since $1 \neq H \leq F$ we get $F = H \cong \mathrm{C}_p$. In particular, $g \in F$ and so $F = G$. □

Lemma 5.4.4. [**periodic by periodic**]

(a) [**a**] *A periodic by periodic group is periodic.*

(b) [**b**] *A locally finite by locally finite group is locally finite.*

Proof. Let G be a group and $H \trianglelefteq G$ such that both H and G/H are periodic (in the proof of (a)) and locally finite (in the proof of (b)).

(a)Let $g \in G$. Since G/H is periodic there exists $m \in \mathbb{Z}^+$with $g^m \in H$. As H is periodic there exists $n \in \mathbb{Z}^+$ with $(g^m)^n = 1$. Thus $g^{mn} = *g^m)^n = 1$.

(b)Let I be a finite subset of G and put $F = \langle I \rangle$. We need to show that F is contained in a finite subgroup of G. Since G/H is locally finite, we know that FH/H is contained in a finite subgroup of G/H and so there exists a finite subset J of F with $F \subseteq JH$. Put $K := I \cap J \cap I^{-1} \cup J^{-1}$ and for $n \in \mathbb{N}$ define K^{*n} inductively by $K^{*0} = 1$ and $K^{*n+1} = KK^{*n}$. Note that

$$\langle K \rangle = \bigcup_{n \in \mathbb{N}} K^{*n}$$

Define

$$S := \langle abc \mid a, b, c \in K \mid abc \in H \rangle = K^{*3} \cap H$$

Then S is a finitely generated subgroup of the locally finite group H and so S is finite. We will now shows that $K^{*n} \subseteq KS$ for all $n \in \mathbb{N}$. For $n = 0, 1$ this is obvious. Let $b, c \in K$. Observe that $K \leq F \subseteq JH$ and so there exists $a \in K$ with $bc \in cH$. Note that $a^{-1} \in J^{-1} \subseteq K$ and $a^{-1}bc \in H$. Hence $a^{-1}bc \in S$ and so $bcSaS$. Thus $K^2 \subseteq KS$. Suppose inductively that $K^*n \leq KS$. Then

$$K^{*(n+1)} = KK^{*n} \subseteq K(KS) = (KK)S \leq KS$$

We proved that $K^{*n} \subseteq KS$ for all $n \in N$. So also $\langle K \rangle \leq KS$. Note that KS is a finite,and so $\langle K \rangle$ is a finite subgroup of G containing F. □

Definition 5.4.5. [**def:cernikov**] *A Cernikov group is an abelian by finite group which fulfills* MIN.

Lemma 5.4.6. [**abelian periodic**]

(a) [**a**] *Let G be an abelian by locally finite group. Then G is locally finite if and only G is periodic.*

(b) [**b**] *Cernikov groups are locally finite.*

Proof. (a)Any locally finite group is clearly periodic. So suppose that G is periodic. Let A be an abelian normal subgroup of G such that G/A is locally finite. Let I be a finite subset of A. Since A is periodic each $i \in I$ has finite order and so $\langle i \rangle$ is finite. Since A is abelian we have

$$\langle I \rangle = \prod_{i \in I} \langle i \rangle$$

and $\langle I \rangle$ is finite. Thus A is locally finite. Since also G/A is locally finite we conclude from (5.4.4)(b)that also G is locally finite.

(b)Let G be a Cernikov group. Then G fulfills MIN and so G is periodic (see (5.1.11)(b)). G is also abelian by finite and so (a)shows that G is locally finite. □

Lemma 5.4.7. [**abelian powers**] *Let A be an abelian group.*

(a) [**a**] *Let $m \in \mathbb{N}$. Then the function*

$$\alpha_m : \quad A \to A, \quad a \to a^m$$

is a homomorphism with $\mathrm{Ker}\alpha_m = \{a \in A \mid a^m = 1\} = A_{\cdot m}$ *and* $\mathrm{Im}\,\alpha_m = \{a^m \mid a \in A\} = A^{\cdot m}$.

(b) [**b**] *A is divisible if and only if $A = A^{\cdot m}$ for all $m \in \mathbb{Z}^+$.*

(c) [**c**] *Any subgroup of A generated by divisible subgroups is divisible. In particular, A a largest divisible subgroup* $\mathrm{Div}(A)$.

(d) [**d**] *There exists $K \le A$ with $A = \mathrm{Div}(A) \oplus K$ and* $\mathrm{Div}(K) = 0$.

Proof. (a)is obvious and (b)follows immediately from the definition of divisible.

(c)Let $\mathcal{D}$ be a set of divisible subgroups of A. Since α_m is a homomorphism we get

$$\langle D \mid D \in \mathcal{D} \rangle \alpha_m = \langle D\alpha_m \mid D \in \mathcal{D} \rangle = \langle D \mid D \in \mathcal{D} \rangle$$

and so $\langle \mathcal{D} \rangle$ is divisible.

(d)Since $\mathrm{Div}(A)$ is divisible and A is abelian, 5.2.4 shows that $A = \mathrm{Div}(A) \oplus K$ for some $K \le A$. Note that $\mathrm{Div}(K) \le K \cap \mathrm{Div}(A) = 1$ and so $\mathrm{Div}(K) = 1$. □

Lemma 5.4.8. [**min and direct sum**] *Let G be a group with* MIN *and $H \le G$ such that $H = \bigoplus_{i \in I} H_i$ where $1 \ne H_i \le H$. Then $|I|$ is finite.*

Proof. Suppose I is infinite. Then we may assume that $I = \mathbb{N}$. For $n \in \mathrm{N}$ define

$$K_n = \bigoplus_{\substack{i \in \mathbb{N} \\ i > n}} G_i$$

Then

$$K_0 \gneqq K_1 \gneqq K_2 \gneqq \ldots \gneqq K_n \gneqq K_{n+1} \gneqq \ldots$$

and so $\{K_n \mid n \in \mathrm{N}\}$ has no minimal element, a contradiction. □

Lemma 5.4.9. [**finite exponent**] *Cernikov groups of finite exponent are finite.*

Proof. Let D be a Cernikov group of finite exponent e. Since D fulfills MIN we may assume that all proper subgroups of D are finite. Since D abelian by finite there exists $A \trianglelefteq D$ such that A is abelian and D/A is finite.

Assume that $A \neq G$, then A is finite and so also G is finite.

Assume next that $D = A$. Then D is abelian.

Suppose that e is not a prime. Choose $a, b \in \mathbb{Z}^+$ with $e = ab$, $a > 1$ and $b > 1$. Then $D^{\cdot a} \neq 1$ and $D^{\cdot b} = 1$. As $(D^{\cdot a})^b = D^e = 1$ this gives $D^{\cdot a} \neq D$. Put $D_a = \{d \in D \mid d^a = 1\}$. The also $D_a \neq D$ and so both $D^{\cdot a}$ and D_a are finite. By (5.4.7)(a)we know that $D/D_a \cong D^{\cdot a}$ and we conclude that D is finite in the case.

Suppose next that e is a prime. Then D is an elementary abelian p-group and so (2.2.8)(a)shows that

$$D \cong \bigoplus_{i \in I} \mathrm{C}_e.$$

Since G fulfills MIN, 5.4.8 shows I is finite. So again G is finite. □

Lemma 5.4.10. [**basic cernikov**] *Let G be a Cernikov group.*

(a) [**a**] *$G^\circ = (G^\circ)^\circ$ and G/G° is finite.*

(b) [**b**] *G° is the largest divisible subgroup of G.*

(c) [**c**] *Let A be any divisible subgroup of G. Then there exist $n \in \mathbb{N}$ such that*

$$A = \oplus_{i=1}^n A_i \quad \textit{and} \quad A_i \cong \mathrm{C}_{p_i^\infty}$$

where $A_i \leq A$ and p_i is a prime.

(d) [**d**] *Let A be any divisible subgroup of G, π a set of primes and A_π the set of π-elements in A. Then $A = A_\pi \oplus A_{\pi'}$. In particular, A_π is divisible, and A/A_π is a π'-group.*

Proof. (a): Since G fulfill MIN, 5.1.10 shows that $G^\circ = (G^\circ)^\circ$ and G/G° is finite.

(b): Since G is abelian by finite, there exists a abelian normal subgroup A of G with G/A finite. Then $G^\circ \leq A$ and so G° is abelian. Let $m \in \mathrm{Z}^+$. Then $(G^\circ)/(G^\circ)^{\cdot m}$ is a Cernikov group of finite exponent, so 5.4.9 implies that $(G^\circ)/(G^\circ)^{\cdot m}$ is finite. As $G^\circ = (G^\circ)^\circ$ this gives $G^\circ) = (G^\circ)^{\cdot m}$ and so G° is divisible by (5.4.7)(b).

Let A be any divisible subgroup of G. Since G/G° is finite also AG°/G° and $A/A \cap G^\circ$ are finite. Thus $A/A \cap G^\circ$ is a finite divisible group, and so is trivial (see 5.2.3). Thus $A = A \cap G^\circ \leq G^\circ$ and G° is the largest divisible subgroup of G.

(c)Since A is divisible, Exercise 1 on Homework 1 shows that

$$A = \bigoplus_{i \in I} A_i$$

where $A_i \leq A$ and either $A_i \cong \mathrm{C}_{p_i^\infty}$, p_i a prime or $A_i \cong \mathbb{Q}$. Since G is periodic we have $A_i \not\cong \mathbb{Q}$. Also 5.4.8 shows that $|I|$ is finite. So (c)holds.

(d)follows from Exercise 1 on Homework 1. □

Recall that a Sylow p-subgroup of a groups G is maximal p-subgroup of G. Also $\mathrm{Syl}_p(G)$ is the set of Sylow p-subgroups of G.

Lemma 5.4.11. [**sylow for cernikov**] *Let G be periodic, abelian by finite group, Then G acts transitively on $\mathrm{Syl}_p(G)$,that is any two Sylow p-subgroups of G are conjugate in G.*

Proof. Let $A \trianglelefteq G$ such that G/A is finite. Let $i \in \{1,2\}$ and let $S_i \in \mathrm{Syl}_p(G)$. Since A is periodic and abelian, A has a largest p-subgroup E and A/E is a p'-group. Note that $E \trianglelefteq G$ and so ES_i is p-groups. Since S_i is a maximal p-subgroup of G this gives $S_i = ES_i$ and so $E \le S_i$. Thus, replacing G by G/E we may assume that $E = 1$, so A is a abelian p'-group. Put $\overline{G} = G/A$ and let $\overline{T}$ be a Sylow p-subgroup of $\overline{G}$. Note that $\overline{G}$ is finite and $\overline{S_i}$ is a p-subgroup of $\overline{G}$. Hence Sylow's Theorem shows that $\overline{S_i}^{g_i} \le \overline{T}$ for some $g_i \in G$. Replacing S_i by $S_i^{g_i}$ we may assume that $\overline{S_i} \le \overline{T}$. Let $H \le G$ with $A \le H$ and $\overline{T} = A/H$. Then $S_i \le H$. Since A is a p' group, the function $A \to A, a \mapsto a^p$ is a bijection. Also H/A is a p-group and so 1.3.11 shows that there exists a complement T to A in H. Thus

$$H = AT, \quad A \cap T = 1, \quad \text{and} \quad T \cong T/1 = T/(T \cap A) = TA/A = H/A = \overline{T}$$

So T is p-group. Define $T_i = AS_i \cap T$. Since $A \le AS_i \le H = AT$ we get

$$AS_i = A(AS_i \cap T) = AT_i$$

Since S_i is a p-group and A is p' we know that $A \cap S_i = 1$. Also $A \cap T_i \le A \cap T = 1$ and both S_i and T_i are complements to A in AS_i. 1.3.11 now shows that $S_i^{h_i} = T_i$ for some $h_i \in AS_i$. Replacing S_i by $S_i^{h_i}$ we may assume that $S_i = T_i$. Thus $S_i \le T$ and since S_i is maximal p-subgroup we conclude that $S_i = T$. Hence $S_1 = T = S_2$ and so S_1 and S_2 are indeed conjugate in G. □

Lemma 5.4.12. [**involution on divisible**] *Let G be a group acting on an divisible group D and let $t \in G$. Suppose that t^2 centralizes D and $\mathrm{C}_D(t)$ is finite. Then t inverts D.*

Proof. Observe that the function

$$\alpha : \quad D \to D, \quad d \to dd^t$$

is a homomorphism with $\mathrm{Im}\,\alpha \le \mathrm{C}_D(t)$. Thus $\mathrm{Im}\,\alpha$ is finite. Since $D/\mathrm{Ker}\alpha \cong \mathrm{Im}\,\alpha$ we conclude that $D/\mathrm{Ker}\alpha$ is a finite. As D is divisible also $D/\mathrm{Ker}\alpha$ is divisible. Thus 5.2.3 shows that $D/\mathrm{Ker}\alpha = 1$, so $D = \mathrm{Ker}a$. Thus $dd^t = 1$ for all $d \in D$. This gives $d^t = d^{-1}$ and so t inverts D. □

Recall that $H \lneqq \mathrm{N}_G(H)$ whenever G is a nilpotent group and $H \lneqq G$.

Lemma 5.4.13. [**inverts and transitive**] *Let G be a group and $a, g \in G$.*

(a) [**a**]
$$a^g = a^{-1} \iff ag = ga^{-1} \iff aga = g \iff (ga)^2 = g^2 \iff (a^{-1})^g = a \iff g^a = ga^2$$

(b) [**c**] *Suppose $g^2 = 1$. Then g inverts a if and only if $(ga)^2 = 1$.*

(c) [**b**] *Let $A \le G$ and suppose g inverts A. Then $g^A = gA^{\cdot 2}$ and if $A = A^{\cdot 2}$ then $g^A = gA$.*

Proof. (a)

$$a^g = a^{-1} \iff g^{-1}ag = a^{-1} \iff ag = ga^{-1} \iff aga = g \iff agag = gg \iff \iff (ag)^2 = g^2$$

Also

$$a^g = a^{-1} \iff (a^g)^{-1} = a \iff (a^{-1})^g = a \iff a^{-1}g = ga \iff a^{-1}ga = gaa \iff g^a = ga^2$$

(b)By (a)$a^g = a^{-1}$ if and only if $(ag)^2 = g^2$, that is $(ag)^2 = 1$.

(c)Since g inverts A we know that A is abelian, see (5.3.12)(b). It follows that

$$A^{\cdot 2} = \{a^2 \mid a \in A\}$$

and so (c)follows from (b). □

Lemma 5.4.14. [**basic locally nilpotent**] *Let G be a locally nilpotent group and H a proper, finitely generated subgroup of G. Then $H \lneq \mathrm{N}_G(H)$.*

Proof. Since $H \neq G$ we can choose $g \in G \setminus H$. Put $K = \langle H, g\rangle$. Then $H \lneq K$. Since H is finitely generated also K is finitely generated. It follows that K is nilpotent and so $H \lneq \mathrm{N}_K(H) \leq \mathrm{N}_G(H)$. □

Lemma 5.4.15. [**fg bounded**] *Let G be a group and $n \in \mathbb{Z}^+$. Suppose that $|H| \leq n$ for all finitely generated subgroups of G. Then $|G| \leq n$.*

Proof. Note that we can choose a finitely generated subgroups H of G with $|H|$ maximal. Then $H = \langle H, g\rangle$ for all $g \in G$ and so $H = G$. Thus $|G| \leq n$. □

Lemma 5.4.16. [**finite sylow**] *Let G be locally finite group, p a prime and $S \in \mathrm{Syl}_p(G)$. Suppose that S is finite. Then $\mathrm{Syl}_p(G) = \{T \leq G \mid |T| = |S|\}$ and G act transitively on $\mathrm{Syl}_p(G)$.*

Proof. Let T be a finite p-subgroup of G and put $H = \langle T, S\rangle$. Since G is locally finite, H is finite. Choose $\widetilde{T}, \widetilde{S} \in \mathrm{Syl}_p(G)$ with $T \leq \widetilde{T}$ and $S \leq \widetilde{S}$. Since S is maximal p-subgroup of G we get $S = \widetilde{S}$. By Sylow's Theorem there exists $h \in H$ with $\widetilde{T}^g = \widetilde{S} = S$ and so $T \leq S^g$. In particular, $|T| \leq |S|$. If $|T| = |S|$ we get $T = S^g$ and so $T \in \mathrm{Syl}_p(G)$. Let $R \in \mathrm{Syl}_p(G)$. Then $|T| \leq |S|$ for all finite subgroups T of R and so 5.4.15 shows that $|R| \leq |S|$. Thus $R \leq S^g$ for some $g \in G$ and since $R \in \mathrm{Syl}_p(G)$ we get $R = S^g$. Hence $|R| = |S|$ and G acts transitively on $\mathrm{Syl}_p(G)$. □

Definition 5.4.17. [**def:strongly p-embedded**] *Let H be a locally finite group, p a prime and M a subgroup of H. Then M is called strongly p-embedded if*

(i) [**i**] *M is not a p'-group and $M \neq G$.*

(ii) [**ii**] *$M \cap M^g$ is p'-group for all $g \in H \setminus M$.*

Proposition 5.4.18. [**char strongly p-embedded**] *Let G be a locally finite group, p a prime and $M \leq H$. Suppose that*

(i) [**i**] *M is not a p'- group and $M \neq G$.*

(ii) [**ii**] *$\mathrm{C}_G(x) \leq M$ for all $x \in M$ with $|x| = p$.*

(iii) [**iii**] *If $R \in \mathrm{Syl}_p(M)$ and $g \in G \setminus M$, then $R \neq R^g$ and $R \cap M^g$ is finite.*

Then M is a strongly p-embedded subgroup of H.

Proof. Suppose not. Then there exists $g \in G \setminus M$ such that $M \cap M^g$ is not a p' group. Let $T \in \mathrm{Syl}_p(M \cap M^g)$. Since $M \cap M^g$is not a p' group, we know that $T \neq 1$. Choose $R \in \mathrm{Syl}_p(M)$ and $S \in \mathrm{Syl}_p(M^g)$ with $T \leq R$ and $T \leq S$. Then $T \leq R \cap M^g$. By (iii)$R \cap M^g$ is finite, so T is finite.

Suppose that $T \neq S$. Since S is locally finite p-group, S is locally nilpotent. Thus 5.4.14 shows that $T \not\leq \mathrm{N}_S(T)$. Hence there exists a finite subgroup P of $\mathrm{N}_S(T)$ with $T \lneq P$. Then P is nilpotent and $1 \neq T \trianglelefteq P$. Thus $T \cap \mathrm{Z}(P) \neq 1$, see 2.1.10. In particular, there exists $x \in T \cap \mathrm{Z}(P)$ with $|x| = p$. Then $P \leq \mathrm{C}_G(x)$. Note that $x \in T \subseteq M$ and so (ii)shows that $\mathrm{C}_G(x) \leq M$. Thus $P \leq M$ and since $P \leq S \leq M^g$ we get

$$T \lneq P \leq M \cap M^g$$

a contradiction, since $T \in \mathrm{Syl}_p(M \cap M^g)$ (and so T is a maximal p-subgroup of G).

Thus $T = S \in \mathrm{Syl}_p(M^g)$. As T is finite, 5.4.16 shows that M^g acts transitively on $\mathrm{Syl}_p(M^g)$ and any subgroup of order $|T|$ in M^g is a Sylow p-subgroup of M^g. Since $T \leq M$ we have $T^g \leq M^g$. Also $|T^g| = |T|$.

So $T^g \in \mathrm{Syl}_p(M^g)$ and $T^{gh} = T$ for some $h \in M^g$. As $T^g \in \mathrm{Syl}_p(M^g)$ we have $T \in \mathrm{Syl}_p(M)$. Hence (iii)shows that $T \neq T^k$ for $k \in G \setminus M$. It follows that $gh \in M$. From $gh \in M$ and $h \in M^g$ we get

$$M = M^{gh} = (M^g)^h = M^g, \quad h \in M^g = M, \quad \text{and} \quad g = (gh)h^{-1} \in M$$

a contradiction to the choice of g. □

Remark 5.4.19. [**bar and normalizer**] *Let G be a group, $H \trianglelefteq G$ and $B \le G$ with $H \le G$. Put $\overline{G} := G/H$. Then G acts on $\overline{G}$ via $\overline{a}^g = \overline{a^g}$ for all $a, g \in G$. Moreover,*

$$\mathrm{N}_G(\overline{B}) = \mathrm{N}_G(B), \quad \overline{N_G(\overline{B})} = \mathrm{N}_{\overline{G}}(\overline{B}), \quad \textit{and} \quad \overline{\mathrm{C}_G(\overline{B})} = \mathrm{C}_{\overline{G}}(\overline{B})$$

Lemma 5.4.20. [**mod center**] *Let G be a finite group, $Z \le \mathrm{Z}(G)$ and $B \le G$ with $Z \le G$. Put $\overline{G} := G/Z$ and suppose that $\gcd(|Z|, |\overline{B}|) = 1$. The there exist a unique $A \le B$ with $B = A \times Z$. Moreover,*

$$\mathrm{C}_{\overline{G}}(\overline{B}) = \overline{C_G(A)} \quad \textit{and} \quad \mathrm{N}_{\overline{G}}(\overline{B}) = \overline{\mathrm{N}_G(A)}$$

Proof. Note that Z is abelian and $\gcd(|B/Z|, |Z|) = 1$. Hence Gaschütz' Theorem 1.3.11 shows that there exists complement A to Z in B. Since $Z \le \mathrm{Z}(G)$ we get $B = A \times Z$. As $\gcd(|B/A|, |A|)| = \gcd(|Z|, \overline{B}) = 1$ we see that A is the unique subgroup of order $|B/Z$ of B, so A is unique. It follows that

$$\mathrm{N}_G(B) \le \mathrm{N}_G(A) \le \mathrm{N}_G(AZ) = \mathrm{N}_G(B)$$

and so $\mathrm{N}_G(B) = \mathrm{N}_G(A)$. Since $Z \le B$ this gives $\mathrm{N}_{\overline{G}}(\overline{B}) = \overline{N_G(B)} = \overline{\mathrm{N}_G(A)}$.

Note that $\mathrm{C}_G((\overline{B}) \le \mathrm{N}_G(\overline{B}) = \mathrm{N}_G(B) = \mathrm{N}_G(A)$ and $[B, \mathrm{C}_G(\overline{B})] \le Z$. Hence

$$[A, \mathrm{C}_G(\overline{B})] \le A \cap Z = 1$$

and so

$$\mathrm{C}_G(\overline{B}) \le \mathrm{C}_G(A) \le \mathrm{C}_G(AZ) = \mathrm{C}_G(B) \le \mathrm{C}_G(\overline{B})$$

Hence $\mathrm{C}_G(\overline{B}) = \mathrm{C}_G(A)$ and so $\mathrm{C}_{\overline{G}}(\overline{B}) = \overline{\mathrm{C}_G(\overline{B})} = \overline{\mathrm{C}_G(A)}$ □

Definition 5.4.21. [**def:glnq**] *Let $\mathbb{K}$ be a field and V a $\mathbb{K}$-space (that is a vector space over $\mathbb{K}$.)*

(a) [**a**] $\mathrm{GL}_{\mathbb{K}}(V)$ *is the groups of $\mathbb{K}$-linear isomorphism of V.*

(b) [**b**] $\mathrm{ZGL}_{\mathbb{K}}(V) = \{\lambda \mathrm{id}_V \mid \lambda \in \mathbb{K}^\sharp\}$.

(c) [**c**] $\mathrm{PGL}_{\mathbb{K}}(V) = GL_{\mathbb{K}}(V)/\mathrm{ZGL}_{\mathbb{K}}(V).,$

(d) [**d**] *Suppose V is finite dimensional over $\mathbb{K}$. Then $\mathrm{SL}_{\mathbb{K}}(V) = \{g \in \mathrm{GL}_{\mathbb{K}}(V) \mid \det(g) = 1\}$, $\mathrm{ZSL}_{\mathbb{K}}(V) = \mathrm{SL}_{\mathbb{K}}(V) \cap \mathrm{ZGL}_{\mathbb{K}}(V)$ and $\mathrm{PSL}_{\mathbb{K}}(V) = \mathrm{SL}_{\mathbb{K}}(V)/\mathrm{ZSL}_{\mathbb{K}}(V)$.*

(e) [**f**] *Let $n \in \mathrm{N}$ and q a power of a prime. Suppose $\dim_{\mathbb{K}} V = n$ and $|\mathbb{K}| = q$. Then we write $\mathrm{GL}_n(q)$, $\mathrm{SL}_n(q)$, $\mathrm{PGL}_n(q)$ and $\mathrm{PSL}_n(q)$ for $\mathrm{GL}_{\mathbb{K}}(V)$, $\mathrm{SL}_{\mathbb{K}}(V)$, $\mathrm{PGL}_{\mathbb{K}}(V)$ and $\mathrm{PSL}_{\mathbb{K}}(V)$ respectively.*

Lemma 5.4.22. [**order glnq**] *Let $n \in \mathrm{N}$ and q a power of a prime.*

(a) [**a**] $|\mathrm{GL}_n(q)| = q^{\binom{n}{2}} \prod_{i=1}^n (q^i - 1)$.

(b) [**b**] $|\mathrm{SL}_n(q)| = |\mathrm{PGL}_n(q) = q^{\binom{n}{2}} \prod_{i=2}^n (q^i - 1)$.

(c) [**c**] $|\mathrm{PSL}_n(q| = q^{\binom{n}{2}} \prod_{i=2}^n (q^i - 1)\big/\gcd(q-1, n)$.

Proof. Let $\mathbb{K}$ be a field of order q and V an n-dimensional $\mathbb{K}$-space.

(a)Fix an (ordered) basis $\mathcal{A}$ of V. The for each basis $\mathcal{B} = (v_0, \ldots, v_{n-1})$ of V there exists a unique $g \in \mathrm{GL}_{\mathbb{K}}(V)$ with $\mathcal{A}g = \mathcal{B}$. Hence $|\mathrm{GL}_{\mathbb{K}}(V)|$ is the number of basis of V. For $0 \le i < n$ ler V_i be the $\mathbb{K}$-space generated by $v_0, \ldots, v_{i-1}$. Then $\dim V_i = i$ and so $|V_i| = q^i$. Note that v_i is an arbitray element of $V \smallsetminus V_i$. So there are $q^n - q^i$ choices for v_i. Thus

$$\begin{aligned} |\mathrm{GL}_n(q)| &= \prod_{i=0}^{n-1} q^n - q^i \\ &= \prod_{i=0}^{n-1} q^i(q^{n-i} - 1) \\ &= q^{\sum_{i=0}^{n-1}} \prod_{i=0}^{n-1} q^{n-i} - 1 \\ &= q^{\binom{n}{2}} \prod_{j=1}^{n} q^j - 1 \qquad \mid \quad j = n - i \end{aligned}$$

(b)Observe that $\mathrm{ZGL}_{\mathbb{K}}(V) = |\mathbb{K}^\sharp| = q-1$ and so $|\mathrm{PGL}_{\mathbb{K}}(V)| = |\mathrm{GL}_{\mathbb{K}}(V)|/(q-1)$. Also the function

$$\det : \quad \mathrm{GL}_{\mathbb{K}}(V) \to (\mathbb{K}^\sharp, \cdot), \quad g \mapsto \det g$$

is a surjective homomorphism and so $\mathrm{GL}_{\mathbb{K}}(V)/\mathrm{SL}_{\mathbb{K}}(V)| = |\mathbb{K}^\sharp| = q-1$. Together with (a)this gives (b).

(c)Note that $\det(\lambda \mathrm{id}_V) = \lambda^n$ and so $|\mathrm{ZSL}_{\mathbb{K}}(V)| = \{\lambda \in \mathbb{K}^\sharp \mid \lambda^n = 1\}$. Since $\mathbb{K}$ is a finite field we know that the multiplicative group $(\mathbb{K}^\sharp, \cdot)$ is cyclic of order $q-1$. Thus $\{\lambda \in \mathbb{K}^\sharp \mid \lambda^n = 1\}$ is a cyclic subgroup of $\mathbb{K}^\sharp$ of order $\gcd(q-1, n)$. Together with (b)this gives (c). □

Lemma 5.4.23. [**sylow structure of psl2**] *Let p be a prime and $k \in \mathrm{Z}^+$. Put $q = p^k$, $G = \mathrm{GL}_2(q)$, $L = \mathrm{PSL}_2(q)$, $H = \mathrm{SL}_2(q)$ and $\epsilon = \gcd(q-1, 2)$.*

(a) [**a**] $|L| = q(q^2-1)/\epsilon$ *and* $\epsilon = \begin{cases} 1 & \text{if } p = 2 \\ 2 & \text{if } p \neq 2 \end{cases}$

(b) [**b**] *Let $T \in \mathrm{Syl}_p(L)$ and $t \in T^\sharp$. Then $T \cong \mathrm{C}_p^k$, $\mathrm{C}_L(t) = \mathrm{C}_L(T) = T$ and $\mathrm{N}_G(T)/\mathrm{C}_L(T) \cong \mathrm{C}_{(q-1)/\epsilon}$.*

(c) [**c**] *Let R be a nontrivial cyclic p'-subgroup of L. Let $\hat{R} \le H$ with $Z \le \hat{R}$ and $R = \hat{R}/Z$. Then*

  (a) [**a**] *$\hat{R}$ is cyclic p'-group of order at least 3.*
  (b) [**b**] $\mathrm{C}_L(\hat{R}) \cong \mathrm{C}_{(q\pm1)/\epsilon}$, $\mathrm{N}_L(R) = \mathrm{N}_L(\hat{R}) \cong \mathrm{D}_{2(q\pm1)/\epsilon}$ *and* $|\mathrm{N}_L(\hat{R})/\mathrm{C}_L(\hat{R})| = 2$.
  (c) [**c**] $G = \mathrm{C}_G(\hat{R})H$
  (d) [**d**] *If $|R| > 2$, then $\mathrm{C}_L(R) = \mathrm{C}_L(\hat{R})$.*
  (e) [**e**] *If $|R| = 2$ and $\mathrm{C}_L(R) = \mathrm{N}_L(\hat{R})$.*

(d) [**d**] *Let r be a prime with $r \neq p$ and $r \neq 2$, and let R be a nontrivial r-subgroup of L. Then R is cyclic.*

Proof. (a)follows from 5.4.22.

Let $\mathbb{K}$ be a field of order q. Put $G = \mathrm{GL}_2(q)$, $H = \mathrm{SL}_2(q)$ and $Z = \mathrm{ZSL}_2(q)$. Then $L = H/Z$. Consider

$$S \in \left\{ \begin{bmatrix} 1 & 0 \\ s & 1 \end{bmatrix} \;\middle|\; s \in \mathbb{K} \right\} \subseteq G$$

Observe that

$$\begin{bmatrix} 1 & 0 \\ s & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ s+t & 1 \end{bmatrix}$$

and so S is a subgroup of G isomorphic to $(\mathbb{K}, +)$. Since $pa = 0$ for all $a \in \mathbb{K}$ we know that $\mathbb{K}$ is an elementary abelian p-group of order q and so $S \cong \mathbb{K} \cong C_{p^k}$.

Observe that $S \leq H$ and $S \cap Z = 1$. Put $T = SZ/Z$. Then $S \cong T$ and T is a subgroup of order q of L. By (a)q is the largest power of p dividing L. Hence SZ/Z is a Sylow p-subgroup of L. We now compute $\mathrm{C}_H([\begin{smallmatrix} 1 & 0 \\ s & 1 \end{smallmatrix}])$ for $s \in \mathbb{K}^\sharp$:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 0 \\ s & 1 \end{bmatrix} = \begin{bmatrix} a+sb & c \\ c+sd & d \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 \\ s & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ sa+c & sb+d \end{bmatrix}$$

So $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{C}_H([\begin{smallmatrix} 1 & 0 \\ s & 1 \end{smallmatrix}])$ if and only if $sb = 0$ and $sa = sd$ and so if and only if $b = 0$ and $a = d$. Thus

$$\mathrm{C}_H\left(\begin{bmatrix} 1 & 0 \\ s & 1 \end{bmatrix}\right) = \left\{ \begin{bmatrix} a & 0 \\ c & a \end{bmatrix} \;\middle|\; a, c \in \mathbb{K}, a^2 = 1 \right\} = \left\{ \begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix} \;\middle|\; c \in \mathbb{K} \right\}\left\{ \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \;\middle|\; a \in \mathbb{K}, a^2 = 1 \right\} = SZ$$

This holds for all $s \in \mathbb{K}^\sharp$, so $\mathrm{C}_H(S) = SZ$. Let $t \in T^\sharp$. Then $t = [\begin{smallmatrix} 1 & 0 \\ s & 1 \end{smallmatrix}]$ for some $s \in \mathbb{K}^\sharp$. Since $|Z|$ divides $q-1$ and $|S| = q$ we have $\gcd(|Z|, |S|) = 1$. Now 5.4.20 shows that

$$\mathrm{C}_L(t) = \mathrm{C}_L(T) = \mathrm{C}_H(S)/Z = SZ/Z = T$$

Put $W = \mathrm{C}_V(S)$ and note that $W = \mathbb{K}(1,0)$ is a 1 dimensional subspace of V. Also $S = \mathrm{C}_H(W) \cap \mathrm{C}_H(V/W)$. Thus

$$\mathrm{N}_H(S) \leq \mathrm{N}_H(\mathrm{C}_V(S)) = \mathrm{N}_H(W) \leq \mathrm{N}_H\big(\mathrm{C}_H(W) \cap \mathrm{C}_H(V/W)\big) = \mathrm{N}_H(S)$$

and

$$\mathrm{N}_H(S) = \mathrm{N}_H(W) = \left\{ \begin{bmatrix} a & 0 \\ c & a^{-1} \end{bmatrix} \;\middle|\; a \in \mathbb{K}^\sharp, c \in \mathbb{K} \right\}$$

It follows that $\mathrm{N}_H(S)/S \cong \mathbb{K}^\sharp \cong \mathrm{C}_{q-1}$. Since $Z \cap S = 1$ and Z is has order ϵ have $|SZ/Z| = \epsilon$. Any quotient of a cyclic group is cyclic, thus

$$N_H(S)/Z\big/SZ/Z \cong \mathrm{C}_{(q-1)/\epsilon}$$

By 5.4.20 $\mathrm{N}_L(T) = \mathrm{N}_H(S)/Z$, thus

$$\mathrm{N}_L(T)/T = \mathrm{N}_H(S)/Z\big/SZ/Z \cong \mathrm{C}_{(q-1)/\epsilon}$$

and (b)is proved.

(c:a)Observe that $|Z|\|2$ and so Z is cyclic. Since $Z \le \mathrm{Z}(R)$ and R is cyclic, $\hat{R}$ is abelian. If $2 \nmid |R|$ we conclude that also R is cyclic and $|\hat{R}| \ge |R| > 2$. So suppose $2\||R|$. Then $p \ne 2$ and so $|Z| = 2$ and $\hat{R}| > |Z| = 2$. Since R is cyclic we can choose $a \in \hat{R}$ with $R = \langle a\rangle Z$. Then a has even order. By Exercise 4a on Homework 2, all elements of order 2 in H are contained in Z. It follows that $Z \le \langle a\rangle$ and so $\hat{R} = \langle a\rangle$. Note that either p is odd or $Z = 1$. So since $p \nmid R$, also $p \nmid \hat{R}$ and so $\hat{R}$ is a p'-group.

(c:b)and (c:c)By (c:a)$\hat{R}$ is cyclic. $g \in \hat{R}$ with $\hat{R} = \langle g\rangle$. Then $|g| = |\hat{R}|$ and (c:a)mplies $|g| \ge 3$ and $p \nmid |g|$.

Case 1. [**q-1**] *There exists a non-zero eigenvector v for g in V.*

Extended v to a $\mathbb{K}$-basis (v, w) of V. The corresponding matrix of g is of the form

$$\begin{bmatrix} a & 0 \\ c & d \end{bmatrix}$$

with $a, c, d \in \mathbb{K}$. Since $\det(g) = 1$ we have $d = a^{-1}$. Suppose that $a = d$. Then $a^2 = 1$ and and so $g^2 = \left[\begin{smallmatrix} 1 & 0 \\ 2c & 1 \end{smallmatrix}\right]$. Thus $(g^2)^p = 1$ and since $p \nmid |g|$ we get $g^2 = 1$, a contradiction to $|g| > 2$.

Thus $a \ne d$ Hence we can choose w to be an eigenvector of g with eigenvalue d. It follows that $\mathbb{K}v$ and $\mathbb{K}w$ are the eigenspaces of g on V with eigenvalues a and d respectively. In particular $\mathbb{K}v$ and $\mathbb{K}w$ are the only g-invariant 1-dimensional $\mathbb{K}$-subspaces of V.

Let $h \in \mathrm{C}_H(g)$. Then vg and wg are eigenvectors of g with eigenvalues a and ad respectively.

It follows that $vg = \mathbb{K}v$ and $wg \in \mathbb{K}w$. Thus

$$\mathrm{C}_H(g) \le \left\{ \begin{bmatrix} e & 0 \\ 0 & e^{-1} \end{bmatrix} \,\middle|\, e \in \mathbb{K}^\sharp \right\}$$

Note that the group on the right side is cyclic for order $q-1$ and so contain in $\mathrm{C}_H(g)$. Thus

$$\mathrm{C}_H(\hat{R}) = \mathrm{C}_H(g) \cong \mathrm{C}_{q-1} \quad \text{and} \quad \mathrm{C}_H(g)/Z \cong \mathrm{C}_{(q-1)/\epsilon}$$

Note that $\mathbb{K}v$ and $\mathbb{K}w$ are the only are the only $\hat{R}$-invariant 1-dimensional $\mathbb{K}$-subspaces of V. Thus $\mathrm{N}_H(\hat{R})$ acts on $\{\mathbb{K}v, \mathbb{K}w\}$ and $\mathrm{C}_H(\{\mathbb{K}v, \mathbb{K}w\}) = \mathrm{C}_H(g) = \mathrm{C}_H(\hat{R})$. Thus $|\mathrm{N}_H(A)/\mathrm{C}_H(A)| \le 2$. Consider $h = \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$. Then $h \in H$, $h^2 = -\mathrm{id}_V \in \mathrm{Z}$ and $\mathbb{K}vh = \mathbb{K}w$ and $\mathbb{K}wh = \mathbb{K}v$. It follows that $h^{-1}\left[\begin{smallmatrix} e & 0 \\ 0 & e^{-1} \end{smallmatrix}\right] = \left[\begin{smallmatrix} e^{-1} & 0 \\ 0 & e \end{smallmatrix}\right]$. Thus h inverts $\mathrm{C}_H(\hat{R})$. In particular, $h \in \mathrm{N}_H(\hat{R})$ and so $\mathrm{N}_H(\hat{R}) = \mathrm{C}_H(\hat{R})\langle h\rangle$. We proved that hZ has order 2 in L, hZ inverts $\mathrm{C}_L(\hat{R})$ and $\mathrm{C}_L(\hat{R}) \cong \mathrm{C}_{(q-1)/\epsilon}$. Thus

$$\mathrm{N}_L(R) = \mathrm{N}_H(\hat{R})/Z = \mathrm{C}_H(\hat{E})\langle h\rangle/Z \cong \mathrm{D}_{2(q-1)/\epsilon}$$

Let $k \in \mathbb{K}^\sharp$. Then $\left[\begin{smallmatrix} k & 0 \\ 0 & k^{-1} \end{smallmatrix}\right]$ has determinant k and centralizes $\hat{R}$. It follows hat $G = \mathrm{C}_G(\hat{R})H$ and so (c:b)and (c:c)hold in this case.

Case 2. [**q+1**] *There does not exist a nonzero eigenvector for g on V.*

Then there also does not exists a proper, non-zero g-invariant $\mathbb{K}$-subspace of V. Put

$$\mathbb{D} := \{d \in \mathrm{End}_{\mathbb{K}}(V) \mid dg = gd\} = \{d \in \mathrm{End}_{\mathbb{K}}(V) \mid dh = hd \text{ for all } h \in \hat{R}\}$$

By Schur's Lemma $\mathbb{D}$ is a division ring. Note that

$$\mathrm{C}_H(\hat{R}) = \mathrm{C}_H(g) = \mathbb{D} \cap H = \{d \in \mathbb{D} \mid \det(d) = 1\}$$

Also,

$$2 = \dim_{\mathbb{K}} V = \dim_{\mathbb{K}} \mathbb{D} \cdot \dim_{\mathbb{D}} V$$

and so $\dim_{\mathbb{K}} \mathbb{D} = 2$ and $\dim_{\mathbb{D}} V = 1$. Identifying $e \in \mathbb{K}$ with $e\mathrm{id}_V \in \mathbb{D}$ we may assume that $\mathbb{K} \leq \mathbb{D}$. Observe that $g \in \mathbb{D} \smallsetminus \mathbb{K}$. It follows that $\mathbb{D} = \mathbb{K}[g]$ and so $\mathbb{D}$ is commutative. Note that $|\mathrm{Aut}_{\mathbb{K}}(\mathbb{D})| \leq \dim_{\mathbb{K}} \mathbb{D} = 2$. Define

$$\sigma : \quad \mathbb{D} \to \mathbb{D}, \quad d \mapsto d^g$$

Let $d, e \in \mathbb{D}$. Then $pd = 0$ and since $q = p^k$ we have $(d+e)^q = d^q + e^d$. Thus $q \in \mathrm{Aut}(\mathbb{D})$. Observe that

$$d^q = d \quad \Longleftrightarrow \quad d = 0 \text{ or } d^{q-1} = 1 \quad \Longleftrightarrow \quad d \in \mathbb{K}$$

Thus σ is the non-trivial element of $\mathrm{Aut}_{\mathbb{K}}(\mathbb{D})$ and $|\sigma| = 2$. We will now show that $\det(d) = dd^q$. If $d \in \mathbb{K}^\sharp$, then

$$\det(d) = \det(d\mathrm{id}_V) = d^2 = d^q d = d^{q+1}.$$

So suppose $d \in \mathbb{D} \smallsetminus \mathbb{K}$. Put $\alpha := d + d\sigma$ and $\beta := dd^q = d^{q+1}$. Since $|\sigma| = 2$ we have $\alpha^q = \alpha$ and $\beta^q = \beta$, so $\alpha, \beta \in \mathbb{K}$. Thus

$$m := (x-d)(x-d^q) = x^2 + \alpha x + \beta \in \mathbb{K}[x]$$

Note that $m(d) = 0$, so $d^2 = -\beta - \alpha d$. Fix $v \in V^\sharp$. Since d has no non-zero eigenvectors, we know that $dv \notin \mathbb{K}v$. Thus (v, dv) is $\mathbb{K}$-basis of V. As $dv = dv$ and $d(dv) = d^2 v = -\beta v - \alpha(dv)$, the matrix of d with respect to this basis is

$$\begin{bmatrix} 0 & 1 \\ -\beta & -\alpha \end{bmatrix}$$

Thus $\det d = \beta = d^{q+1}$.

It follows that $d \in H$ if and only of $d^{q+1} = 1$. Thus

$$\mathrm{C}_H(\hat{R}) = \{d \in \mathbb{D}^\sharp \mid d^{q+1} = 1\}$$

Since $\mathbb{D}^\sharp \cong \mathrm{C}_{q^2-1}$ this implies that

$$\mathrm{C}_H(\hat{R}) \cong \mathrm{C}_{q+1} \quad \text{and} \quad \mathrm{C}_L(\hat{R}) = \mathrm{C}_H(\hat{R})/Z \cong \mathrm{C}_{(q+1)/\epsilon}.$$

Also $\{\det(d) \mid d \in \mathbb{D}^\sharp\} = \{d^{q+1} \mid d \in \mathbb{D}^\sharp\} \cong \mathrm{C}_{q-1}$. As $\det(d) \in \mathbb{K}$ and $\mathbb{K}^\sharp = |q-1|$ we conclude that for each $k \in \mathbb{K}^\sharp$ there exists $d \in \mathbb{D}^\sharp$ with $\det(d) = k$. Then $d \in \mathrm{C}_G(\hat{R})$ and we conclude that $G = \mathrm{C}_G(\hat{R})H$.

Let $h \in \mathrm{N}_H(\hat{R})$. Then the function

$$\mathbb{D} \to \mathbb{D}, \quad d \to d^h$$

is an element of $\mathrm{Aut}_{\mathbb{K}}(\mathbb{D})$. It follows that

$$|\mathrm{N}_H(A)/\mathrm{C}_H(A)| \leq |\mathrm{N}_H(A)/\mathrm{C}_H(\mathbb{D})| \leq |\mathrm{Aut}_{\mathbb{K}}(\mathbb{D})| = 2$$

Define

$$\delta : \mathbb{D} \to V, \quad v \to vd$$

Since $\mathbb{D}$ is commutative, δ is a $\mathbb{D}$-linear isomorphism. In particular,

$$\rho := \delta^{-1} \circ \sigma \circ \delta \in \mathrm{GL}_{\mathbb{K}}(V)$$

Let $d, e \in \mathbb{D}$. Then $(vd)\rho = (d\delta)\rho = (d\sigma)\rho = v(d\sigma) = vd^q$ and so

$$(vd^q)e^\rho = ((((vd)\rho)\rho^{-1})e)\rho = ((vd)e)\rho = (v(de))\rho = v(de)^q = (vd^q)e^q$$

and so $e^\rho = e^q$. Thus $\rho \in \mathrm{N}_G(\hat{R})$. Note that $|\rho| = |\sigma| = 2$ and so $\rho \neq \mathrm{id}_V$. Thus $\det(\rho) = -1$. As seen above, we can choose $e \in \mathbb{D}$ with $\det(e) = -1$.

Put $h = e\rho$. Then $\det(h) = \det(e)\det(\rho) = -1 \cdot -1 = 1$ and so $h \in \mathrm{N}_H(\hat{R})$. We compute

$$h^2 = e\rho e\rho = ee^\rho = ee^q = e^{q+1} = \det(e) = -1$$

Let $d \in \mathrm{C}_H(\hat{R})$. Then $d \in \mathbb{D}$ and $d^{q+1} = 1$. So $d^q = d^{-1}$ and $d^h = d^{e\rho} = d^\rho = d^q = d^{-1}$. As in (Case 1)we now see that $\mathrm{N}_H(\hat{R}) = \mathrm{C}_H(\hat{R})\langle h\rangle$, hZ as order 2 in L, hZ inverts $\mathrm{C}_H(\hat{R})/Z$. Thus

$$\mathrm{N}_L(R) == \mathrm{N}_L(\hat{R}) = \mathrm{N}_H(\hat{)}/Z = \mathrm{C}_H(\hat{R})/Z \cdot \langle h\rangle Z/Z \cong \mathbb{D}_{2(q+1)}$$

This completes the treatment of (Case 2)and so (c:b)and (c:c)are proved.

(c:d)and (c:e)Observe that $\mathrm{C}_H(\hat{R}) \leq \mathrm{C}_H(R) \leq \mathrm{N}_H(R) = \mathrm{N}_H(\hat{R})$. By (c:a)$\mathrm{N}_H(\hat{R})/\mathrm{C}_H(\hat{R})| = 2$. Moreover, let $h \in \mathrm{N}_H(\hat{R})/\mathrm{C}_H(\hat{R})$, then h inverts $\hat{R}$. Hence h also inverts R. Recall that R is cyclic. If $|R| > 2$, then h does not centralizes R and if $|R| = 2$ then h centralizes R. In the first case we get $\mathrm{C}_L(R) = \mathrm{C}_L(\hat{R})$ and the second case $\mathrm{C}_L(R) = \mathrm{N}_L(\hat{R})$.

(d)Let r be a prime with $r \neq p$ and $r \neq 2$ and R a nontrivial r-subgroup of G. Then $\mathrm{Z}(R) \neq 1$ and we can choose non-trivial cyclic subgroup $U \leq \mathrm{Z}(R)$. Since $r \neq 2$ we have $|U| > 2$ and so (c:d)shows that shows that $\mathrm{C}_L(U) = \mathrm{C}_L(\hat{U})$. By (c:c)$\mathrm{C}_L(\hat{U}) \cong \mathrm{C}_{2(q\pm1)/\epsilon}$. Thus $\mathrm{C}_L(U)$ is cyclic. Note that $R \leq \mathrm{C}_L(U)$, so also R is cyclic. □

Lemma 5.4.24. **[sl3]** *Let L be a finite group and $t \in L$ with $|t| = 2$. Suppose that $L \cong \mathrm{PSL}_3(q)$ or $L \cong \mathrm{PSU}_3(q)$ where q is a power of prime. Then $|L| \leq q^{18}$ and $\mathrm{C}_L(t)$ has a normal subgroups isomorphic to $\mathrm{SL}_2(q)$.*

Proof. We treat the case $L \cong \mathrm{PSL}_3(q)$ and leave the $\mathrm{PSU}_3(q)$-case to the reader. Let $\mathbb{K}$ be a finite field of order q, V a 3-dimensional $\mathbb{K}$-space, $G = \mathrm{GL}_{\mathbb{K}}(V)$, $H = \mathrm{SL}_{\mathbb{K}}(V)$ and $Z = \mathrm{ZSL}_{\mathbb{K}}(V)$. Then $L \cong H/Z$ and we may assume $L = H/Z$. By 5.4.22 $|H| = q^3(q^3-1)(q^2-1)$ and so certainly $|L| \leq q^{18}$. Also $|Z| = \gcd(3, q-1)$ and so $\gcd(|Z|, 2) = 1$. Put $T = \langle t\rangle$ and let $T = \widehat{T}/Z$, where $\widehat{T} \leq H$ with $Z \leq \widehat{T}$. By 5.4.20 $\widehat{T} = Z \times S$ for some $S \leq \hat{T}$ and $\mathrm{C}_L(t) = \mathrm{C}_L(T) = \mathrm{C}_H(S)/Z$. Let $1 \neq s \in S$. Then $s^2 = 1$ and $\det s = 1$. Observe that the minimal polynomial of s divides $x^2 - 1$. As $x^2 - 1 = (x-1)(x+1)$ we conclude that t has basis consisting of eigenvectors with eigenvalues 1 and -1. Also $\det(s) = 1$. So s has matrix

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

It follows that

$$C_H(s) = \left\{ \begin{bmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & e \end{bmatrix} \middle| \; a,b,c,d,e \in \mathbb{K}, (ad-bc)e = 1 \right\}$$

Put

$$K := \left\{ \begin{bmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{bmatrix} \middle| \; a,b,c,d \in \mathbb{K}, ad-bc = 1 \right\}$$

Then $K \cong \mathrm{SL}_2(q)$, $K \trianglelefteq C_H(s)$ and $K \cap Z = 1$. Thus

$$\mathrm{SL}_2(q) \cong K \cong K/(K \cap Z) \cong KZ/Z \trianglelefteq C_H(s)/Z = C_H(S)/Z = C_L(t)$$

□

Reminder: Let $G = H \ltimes K$ where $n \in \mathbb{Z}^+$ with $n \geq 2$, $H = \langle h \rangle \cong C_{4n}$, $K = \langle k \rangle \cong C_2$ and K acts on H via $h^k = h^{2n-1}$. Then G is denoted by QD_{8n}. Any group isomorphic to QD_{8n} is called an quasidihedral group of order $8n$.

Lemma 5.4.25. [**basic quasidihedral**] *Let G be a quasidihedral group of order $8n$, $n \geq 2$, and let D be the subgroup G generated by the involutions in G. Then D is a dihedral goup of order $4n$ and D is the largest dihedral subgroup of G.*

Proof. Since any dihedral group is generated by involutions, the first statement implies the second. According to the definition of quasidihedral group we can choose $H = \langle h \rangle \trianglelefteq G$ and $K = \langle k \rangle \leq G$ such that $G = HK, H \cap K = 1$, $h^k = h^{2n-1}$, $|h| = 4n$ and $|k| = 2$.

Let $g \in G$. Then either $g \in H$ or $g = h^m k$ with $m \in \mathbb{N}$. Since H is cyclic of order $4n$, H has a unique involution, namely h^{2n}. Since $k^2 = 1$, (5.4.13)(b)shows that $h^m k$ is an involutions if and only if k inverts h^m. We have

$$(h^m)^k = (h^m)^{-1} \iff (h^k h)^m = 1 \iff (h^{2n-1}h)^m = 1 \iff h^{2nm} = 1.$$

Since $|h| = 4n$, this holds if and only if $4n | 2nm$ and so if and only if $2|m$. Thus the involutions in G are h^{2n} and $h^{2l}k$, $l \in \mathbb{Z}$. Thus $D = \langle h^2 \rangle \langle k \rangle = \langle h^2 k, k \rangle$. Note that both $h^2 k$ and k have order 2, and $(h^2 k)k$ has order $2n$ so D is a dihedral group of order $4n$. □

Lemma 5.4.26. [**char quasidihedral**] *Let P be a finite 2-group and $A \leq P$. Suppose that $A \cong C_2 \times C_2$ and $C_P(A) = A$. Then P is a dihedral or quasidihedral group.*

Proof. Observe that $1 \neq Z(P) \leq C_P(A) = A$. Hence either $Z(P) = A$ or $|Z(P)| = 2$

Suppose first that $Z(P) = A$. Then

$$P = C_P(Z(P)) = C_P(A) = A \cong C_2 \times C_2 \cong D_4$$

So suppose $|Z(P)| = 2$. Choose $1 \neq z \in Z(P)$ and $a \in A \setminus Z(P)$. Then $A = \langle a, z \rangle$ and $C_P(a) = C_P(\langle a, z \rangle) = C_P(A) = A$.

Choose $D \leq P$ maximal subject to $A \leq D$ and D is a dihedral group of order, say, $4n$.

If $P = D$ we are done. So suppose $D \lneq P$ and put $Q = \mathrm{N}_P(D)$. Since P is nilpotent we get $D \lneq Q$. Since $z \in \mathrm{Z}(D)$ and D is a dihedral group where exists a cyclic subgroup $X \leq D$ such that $z \in X$ and $|D/Z| = 2$. Then $X \cong \mathrm{C}_{2n}$, a inverts X, $D \setminus X| = 2n$ and all elements in $D \setminus X$ have order 2. We claim that $X \trianglelefteq Q$. If $n = 1$, $X = \langle z \rangle \trianglelefteq Q$. If $n \neq 1$, then X is the unique cyclic subgroup of order $2n$ in D and since $D \trianglelefteq Q$ we again get $X \trianglelefteq Q$.

So indeed $X \trianglelefteq Q$. In particular, Q acts on $D \setminus X$ by conjugation. As $|D/X| = 2$ we have $D \setminus X = Xa$. Thus $a^Q \subseteq XaD \setminus X$ and so

$$2n = |X| = |Xa| \geq | \geq |a^Q| = |Q/\mathrm{C}_Q(a)| = |Q/A| = |Q/D||D/A| = |Q/D|\frac{4n}{4} = |Q/D| \cdot n$$

As $Q \neq D$ we have $|Q/D| \geq 2$. It follows that $|Q/D| = 2$ and $Xa = a^Q$. Let $x \in X$ with $\langle x \rangle = X$. Put $b := xa$ and choose $t \in Q$ with $a^t = b$. Note that $\langle a, b \rangle = D$. Consider $\overline{Q} = Q/X^{\cdot 2}$. Then $\overline{D} = \langle \overline{a}, \overline{b} \rangle \cong \mathrm{C}_2 \times \mathrm{C}_2$. Since $\overline{a}^t = \overline{b}$ and $\overline{t}$ centralizes $\overline{t}^2$ we conclude that $\overline{t}^2 = \overline{1}$ or $\overline{t}^2 = \overline{x}$. Suppose $\overline{t}^2 = 1$. Note that $\overline{t}$ does not inverts $\overline{a}$. Thus $\overline{at}$ does not have order 2. Hence, replacing t by at if neceassy we may assume that $\overline{t}^2 \neq 1$. Hence $\overline{t}^2 = \overline{x}$. It follows that $t^2 = x^l$ for some odd integer l. As $|x|$is a power of 2 we conclude that $|t^2| = |x|$. Hence $|t| = 2|x| = 4n$ and $x = t^{2m}$ for some odd integer m with $1 \leq m < 2n$. We compute

$$t^a = a{-}1ta = ata = tt^{-1}ata = ta^ta = txaa = tx = tt^{2m} = t^{2m+1}$$

and so

$$t = (t^a)^a = (t^{2m+1})^{2m+1} = t^{(2m+1)^1} = t^{4m(m+1)+1}$$

Thus $t^{4m(m+1)} = 1$. As t has order $4n$ this shows that $4n|4m(m+1)$. Hence $n|m(m+1)$. Since m is odd and n is a power of 2, this gives $n|m+1$. Since $0 \leq m < 2n$ we get $m+1 = n$ or $m+1 = 2n$. Note that $t^2 = t^{2m+1} = t^{2(m+1)-1}$ and so either

$$\left(m+1 = m \text{ and } t^a = t^{2n-1}\right) \quad \text{or} \quad \left(m+1 = 2n \text{ and } t^a = t^{4n-1} = t^{-1}\right)$$

Note that $Q = \langle t \rangle \langle a \rangle$. If $t^a = t^{-1}$ we conclude that Q is a dihedral group, a contradiction the maximality of D. Thus $m+1 = n$ and $t^a = t^{2n-1}$. As $m \geq 1$ we have $n \geq 2$ and so Q is a quasidihedral group.

Put $E = \langle D^{\mathrm{N}_P(Q)} \rangle$. Then $D \leq E \leq Q$ and E is generated by involutions. 5.4.25 Q is not generated by involutions. Since $|Q/D| \leq 2$ this gives $E = D$ and so $D \trianglelefteq N_P(Q)$. Thus $\mathrm{N}_P(Q) \leq \mathrm{N}_P(D) = Q$ and so $Q = P$. ☐

Definition 5.4.27. [**def:char**] *Let G be a group and $H \leq G$. Then H is called a characteristic subgroup of G and we write $H \operatorname{char} G$ if $H = H^\alpha$ for all $\alpha \in \mathrm{Aut}(H)$.*

Remark 5.4.28. [**rm:char**] *Let G be a group, $N \trianglelefteq G$ and $H \operatorname{char} N$. Then $H \trianglelefteq G$.*

Proof. Let $g \in G$. Then the function $N \to N, a \to a^g$ is an automorphism of G. ☐

Lemma 5.4.29. [**divisible and char**] *Let D be a divisible Chernikov group.*

(a) [**a**] *Let $m \in \mathbb{Z}^+$. Then $D_{\cdot m}$ is finite.*

(b) [**b**] *Any finite subset of D is contained in a finite characteristic subgroup of D.*

Proof. (a): Observe that $D_{\cdot m}$ is a Chernikov group of finite exponent and so by 5.4.9 $D_{\cdot m}$ is finite.

(b)Let A be a finite subset of D and put $m = \mathrm{lcm}_{a \in A} |a|$. Then $D_{\cdot m}$ is a characteristic subgroup of G containing A. By (a)$D_{\cdot m}$ is finite. ☐

Lemma 5.4.30. [**finite on finite**] *Let H be a non-trivial Cernikov p-group*

(a) [**a**] *Let G be a group on group H. If $G/C_G(H)$ is a finite p-group then* $\mathrm{C}_H(G) \neq 1$.

(b) [**b**] $\mathrm{Z}(H) \neq 1$.

Proof. (a)Suppose first that H is finite. Then $K := H \ltimes G/\mathrm{C}_G(H)$ is a finite p-group Since $H \trianglelefteq K$ we get $H \cap \mathrm{Z}(K) \neq 1$ ad so $\mathrm{C}_H(G) \neq 1$..

(b)If H is finite then certainly $\mathrm{Z}(H) \neq 1$. So suppose H is infinite. Then $H^\circ \neq 1$ and since H° is abelian, $H/\mathrm{C}_H)H^\circ)$ is finite. Thus (a)shows that $C_{H^\circ}(H) \neq 1$. So also $\mathrm{Z}(H) \neq 1$. □

Lemma 5.4.31. [**action on abelian ii**] *Let G be group acting on the abelian group A.*

(a) [**a**] *Let $g \in G$ and $n \in \mathbb{Z}^+$ with $a^n \in \mathrm{C}_G(A)$. Then $[A, g] \cap \mathrm{C}_A(g) \leq A_{\cdot n}$.*

(b) [**b**] *Suppose A is a divisible p-group and G is periodic. Then $\mathrm{C}_G(\Omega_2(A)) = \mathrm{C}_G(A)$.*

Proof. (a)Let $b \in [A, g] \cap \mathrm{C}_A(g)$. Since A is abelian there exists $a \in A$ with $b = [a, g]$. Then $[b, g] \in \mathrm{C}_A(g)$ and hence $b = [a, g]^n = [a^n, g] = 1$.

(b)Otherwise we can choose $g \in \mathrm{C}_G(\Omega_2(A))$ and a prime r such that $g^r \in \mathrm{C}_G(A)$ but $g \notin \mathrm{C}_G(A)$. Then $\mathrm{C}_A(g) \neq A$ and since $[A, g] \cong A/\mathrm{C}_A(g)$ we conclude that $[A, g]$ is a non-trivial divisible p-group. It follows that $[A, g]$ has elements of order p. But

$$\Omega_2([A, g]) \leq \mathrm{C}_A(g) \cap [A, g] \overset{(a)}{\leq} A_{\cdot r}$$

a contradiction □

Definition 5.4.32. [**def:mp**] *Let G be a group. Then*

$$\mathrm{m}_p(G) = \sup\{k \mid |A \leq G, A \cong \mathrm{C}_p^k\}$$

Lemma 5.4.33. [**order of sz**] *Let p be a prime, $k \in \mathrm{Z}^+$ and $G \cong \mathrm{PSL}_2(q), PSU_3(q)$ of $\mathrm{Sz}(q)$ (where $p = 2$ in the last case). Then $\mathrm{m}_p(G) = k$ and $|G| \leq q^{18} = p^{18k}$.*

Assumed Theorem 5.4.34 (Brauer)**.** [**brauer triple**] *Let G be a finite simple group, $S \in \mathrm{Syl}_2(G)$ and $x_0, x_1, x_2 \in S$ with $|x_1| = |x_2| = 2$. Suppose there do not exist $y_i \in S \cap x_i^G$, $0 \leq i \leq 2$ such that $y_0 = y_1 y_x$ and $\mathrm{C}_S(y_0) \in \mathrm{Syl}_2(\mathrm{C}_G(y_0))$. Then $|G|$ is bounded in terms of the triple*

$$\Big(\ \big|\mathrm{C}_G(x_i)/\mathrm{O}_{2'}(\mathrm{C}_G(x_i))\big| \ \mid 0 \leq i \leq 2\Big)$$

Assumed Theorem 5.4.35 (Feit-Thompson)**.** [**odd order**] *Every finite group of odd order is solvable.*

Recall that for a finite group G, $\mathrm{O}_{2'}(G)$ is largest normal subgroup of odd order of G

Assumed Theorem 5.4.36 (Bender)**.** [**strongly 2**] *Let G be a finite groups with a strongly 2-embedded subgroup. Suppose that $\mathrm{m}_2(G) \geq 2$. Then*

$$G/\mathrm{O}_{2'}(G) \cong \mathrm{PSL}_2(q), \mathrm{Sz}(q), \mathrm{PSU}_3(q)$$

where q is a power of 2 and $q > 2$.

Assumed Theorem 5.4.37 (Glauberman)**.** [**z***] *Let G be a finite group and $t \in G$ with $|t| = 2$. Suppose that $t^G \cap \mathrm{C}_G(t) = \{t\}$ and put $\overline{G} = G/\mathrm{O}_{2'}(G)$. Then $\bar{t} \in \mathrm{Z}(\overline{G})$.*

Assumed Theorem 5.4.38. [**d and qd**] *Let G be a finite simple group and $S \in \mathrm{Syl}_2(G)$.*

(a) [**a**] *Suppose that S is a dihedral group. Then $G \cong \mathrm{PSL}_2(q)$ or* Alt(7)*, where q is a power of an odd prime and $q > 3$.*

(b) [**b**] *Suppose that S is a quasidihedral group of order at least 16. Then $G \cong \mathrm{PSL}_3(q), \mathrm{PSU}_3(q)$ or* Mat_{11}*, where q is a power of an odd prime.*

Theorem 5.4.39. [**lf with min**] *Every locally finite group with* MIN *is a Cernikov group.*

Proof. Suppose the theorem is false. Then there exists locally finite group G_0 with MIN which is not Cernikoóv. Since G_0 fulfills MIN the set of all subgroups of G_0 which are not Cernikov has a minimal element G. It follows that

(**Step 1.**) [**1**] *G is not Cernikov, but all proper subgroup of G are Cernikov.*

Next we show

(**Step 2.**) [**2**] *$G = G^\circ$.*

Suppose for a contradiction that $G^\circ \neq G$. Then G° is Cernikov, and so G° is abelian by finite. Thus G° is abelian. Since G fulfills MIN, 5.1.11 shows that $G^{\circ\circ} = G^\circ$ and G/G° is finite. Thus G is abelian by finite. But then G is a Cernikov group, a contradiction.

(**Step 3.**) [**3**] *$G/\mathrm{Z}(G)$ is infinite.*

Otherwise G is abelian by finite and so G is Cernikov.

(**Step 4.**) [**4**] *Let $A, B \trianglelefteq G$. Put $\overline{G} = G/B$ and suppose that $\overline{A}$ is finite or that $[\overline{a}, G]$ is finite for all $a \in A$. Then $\overline{A} \le \mathrm{Z}(\overline{G})$ and $[A, G] \le B$.*

Let $a \in A$. Note that $[\overline{a}, G] \le \overline{A}$. So if $\overline{A}$ is finite, also $[\overline{a}, G]$ is finite. Also $|G/\mathrm{C}_G(\overline{a})| \le [\overline{a}, G]$ and so $G/\mathrm{C}_G(\overline{a})$ is finite. Thus $G^\circ\mathrm{C}_G(\overline{a})$. By (Step 2)we know that $G = G^\circ$, so $G \le \mathrm{C}_G(\overline{a})$. Hence $[a, G] \le B$, $[A, G] \le B$ and $\overline{A} \le \mathrm{Z}(\overline{G})$.

(**Step 5.**) [**5**] $\mathrm{Z}(G)$ *is the largest proper normal subgroup of G. In particular, $G/\mathrm{Z}(G)$ is simple.*

By (Step 3)$\mathrm{Z}(G)$ is a proper subgroup of G. Let M be any proper normal subgroup of G. Then G is Cernikov. Thus (5.4.10)(a)shows that M° is divisible. Let $a \in M^\circ$. Then by (5.4.29)(b)a is contained in a finite characteristic subgroup A of M°. Then $A \trianglelefteq G$ and since A is finite,we can apply (Step 3)with $B = 1$. Thus $A \le \mathrm{Z}(G)$ and so $M^\circ \le \mathrm{Z}(G)$. By (5.4.10)(a)we know that M/M° is finite. So (Step 3)applied with $A = M$ and $B = M^\circ$ shows that $[M, G] \le M^\circ \le \mathrm{Z}(G)$. Let $a, b \in M$ and $g \in G$. Then

$$[ab, g] = [a, g]^b[a, b] = [a, b][b, g]$$

and so with $m = |a|$

$$[a, b]^m = [a^m, g] = 1$$

Thus $[a, G] \le (M^\circ)_{\cdot m}$. Since M° is a divisible Cernikov-group (5.4.29)(a)shows that $(M^\circ)_{\cdot m}$ is finite. Thus $[a, G]$ is finite and (Step 4)applied that $A = M$ and $B = 1$ shows that $M \le \mathrm{Z}(G)$.

(**Step 6.**) [**6**] *$G/\mathrm{Z}(G)$ is a infinite simple locally finite group with* MIN*, $G/\mathrm{Z}(G)$ is not Cernikov and proper subgroup of $G/Z(G)$ are Cernikov.*

By (Step 3)$G/Z(G)$ is infinite. By (Step 4)$G/\mathrm{Z}(G)$ is simple. Since G is a locally finite group with MIN, also $G/\mathrm{Z}(G)$ is a locally finite group with MIN. As every proper subgroup of G is Cernikov, also every proper subgroup of $G/Z(G)$ is Cernikov. An infinite simple group cannot have a abelian normal subgroup of finite index, so $G/\mathrm{Z}(G)$ is not Cernikov.

In view of (Step 6)$G/\mathrm{Z}(G)$ has the same properties as G, so replacing G by $G/Z(G)$ we may assume that

(Step 7.) [**7**] *G is simple.*

Next we show that:

(Step 8.) [**step 2**] *G is not a $2'$-group.*

Otherwise the Odd Order Theorem implies that all finite subgroups of G are solvable. So G is locally solvable and 5.4.3 shows that $G \cong \mathrm{C}_p$, p a prime. But G is infinite, a contradiction.

Notation: Let $\mathcal{P}$ be the set of all positive primes and $\pi \subseteq \mathcal{P}$. Let $\mathcal{D}$ be the set of $\mathcal{D}_\pi$ be the set of divisible a π-subgroups of G, and $\mathcal{D} := \mathcal{D}_{\mathcal{P}}$. So $\mathcal{D}$ is the set of maximal divisible subgroups of G. For $H \lneqq G$ let

$$H_\pi = \{x \in H^\circ \mid x \text{ is a } \pi-\text{element}\}.$$

(Step 9.) [**step 3**] *Let $H \lneqq G$. Then H_π is the largest divisible π-subgroup of H divisible abelian π-subgroup of H and H_π contained in every maximal π-subgroup of H.*

Since H° is divisible, H_π is a divisible π subgroup of H. Let D be any divisible π-subgroup of H. Then $D = D^\circ \leq H^\circ$ and so $D \leq H_\pi$.

Let M be maximal π-subgroup of H. Note that H_π is normal in H and so $H_\pi M$ is π-subgroup of G. Thus $M = H_\pi M$ by maximality of M and so $H_\pi \leq M$.

(Step 10.) [**step 4**] *Let $D \in \mathcal{D}_\pi$ and $H \lneqq G$ with $D \leq H$. Suppose that $D \neq 1$. Then $D = H_\pi$ and $H \leq \mathrm{N}_G(D)$. In particular, $\mathrm{N}_G(D)$ is the largest proper subgroup of containing D, and $\mathrm{N}_G(D)$ is a maximal subgroup of G.*

By (Step 9)we have $D \leq H_\pi$ and so the maximality of D gives $D = H_\pi$. Since $H_\pi \trianglelefteq H$ this implies $H \leq \mathrm{N}_G(D)$.

(Step 11.) [**step 5**] *Let $D \in D_\pi$ and E a divisible π subgroup of G. Then $E \leq D$ or $E \cap D = 1$.*

Assume that $E \cap D \neq 1$. Then $D \neq 1$. Put $H = \mathrm{C}_G(E \cap D)$. Since G is simple, $E \cap D \trianglelefteq G$ and so $H \neq G$. Since both E and D are abelian, both E and D are contained in H. Hence (Step 10)gives $D = H_\pi$, thus $E \leq D$ be (Step 9).

(Step 12.) [**step 6**] *Let A be a non-trivial divisible subgroup of G. Then there exists largest divisible subgroup $\overline{A}$ of G containing A. If, in addition A is a π-group, then $\overline{A}_\pi$ is the largest divisible π-subgroup of G containing A and $\overline{A}_\pi \in \mathcal{D}_\pi$.*

Note that the union of a totally ordered set of divisible subgroups of G is a divisible subgroup of G. Thus Zorn's Lemma shows that A is contained in a maximal divisible subgroup $\overline{A}$ of G. Let E be any divisible subgroup of G with $A \leq E$. Then $1 \neq A \leq \overline{A} \cap E$ and (Step 11)shows that $E \leq \overline{A}$.

Suppose now in addition that A is a π-group and let F be divisible π-subgroup of G with $A \leq F$. Then $F \leq \overline{A}$, and since F is a π-group we get $F \leq \overline{A}_\pi$.

(Step 13.) [**step 7**] *Let D be non-trivial divisible abelian subgroup of G. Then $\mathrm{N}_G(D) \leq \mathrm{N}_G(\overline{D})$ and if $D \in \mathcal{D}_\pi$, then $N_G(D) = N_G(\overline{D})$.*

Note that $\overline{D}$ is uniquely determined by D, so $\mathrm{N}_G(D) \leq \mathrm{N}_G(\overline{D})$. Suppose that $D \in \mathcal{D}_\pi$. Then (Step 12)gives $D \leq \overline{D}_\pi$, thus $D = \overline{D}_\pi$ by maximality of D. Note that $\overline{D}_\pi$ is characteristic subgroup of G, so $D = \overline{D}_\pi \trianglelefteq \mathrm{N}_G(\overline{D})$. Thus $\mathrm{N}_G(\overline{D}) \leq \mathrm{N}_G(D)$ and (Step 13)is proved.

(Step 14.) [**step 19**]

(a) [**19:a**] *Every maximal subgroup of G is infinite.*

(b) [**19:b**] *Let R be proper infinite subgroup G. Then there exists a largest proper subgroup $\widetilde{R}$ of G containing R, namely $\tilde{R} = \mathrm{N}_G(\overline{R^\circ})$.*

(c) [**19:c**] *If M_1 and M_2 are maximal subgroups of G with $M_1 \cap M_2$ infinite, then $M_1 = M_2$.*

(d) [**19:d**] *Let M be a maximal subgroup of G and $H \leq G$ with $M \cap H$ infinite. Then $H \leq M$.*

(a)Suppose F be a finite subgroup of G. Since G is infinite, there exists $g \in G \setminus F$. As G is locally finite, $\langle F, g\rangle$ is finite. Thus $F \lneqq \langle F, g\rangle \lneqq G$ and so F is not a maximal subgroup of G.

(b)Let $M \lneqq G$ with $R \leq M$. Since R is infinite, $R^\circ \neq 1$. Thus $\overline{R^\circ}$ is a the largest divisible subgroup of G containing R°. Note that $R^\circ \leq M^\circ$ and since M° is divisible we get $M^\circ \leq \overline{R^\circ}$. This implies $\overline{R^\circ} \leq \overline{M^\circ}$ and then $\overline{R^\circ} = \overline{M^\circ}$. Thus $M \leq \mathrm{N}_G(\overline{R^\circ})$.

(c)By (b)$M_1 \cap M_2$ is contained in a unique maximal subgroup and so $M_1 = M_2$.

(d)By (b)H lies in a maximal subgroup $\tilde{H}$ of G. Then $H \cap M \leq M \cap \tilde{H}$ and so by (c), $M = \tilde{H}$. Thus $H \leq M$.

(Step 15.) [**char max**] *Let $M < G$. Then following are equivalent.*

(a) [**char max:a**] *M is a maximal subgroup of G.*

(b) [**char max:b**] *$1 \neq M^\circ \in \mathcal{D}$ and $M = \mathrm{N}_G(M^\circ)$.*

(c) [**char max:c**] *$M = N_G(D)$ for some set of prime π and some $1 \neq D \in D_\pi$.*

(a)$\Longrightarrow$ (b): Suppose M is maximal in G. By (a)M is infinite and so $M^\circ \neq 1$. Since $M \leq \mathrm{N}_G(\overline{M^\circ})$ the maximality of M gives $M = \mathrm{N}_G(\overline{M^\circ})$. In particular, $\overline{M^\circ} \leq M$ and thus $M^\circ = \overline{M^\circ} \in \mathcal{D}$.

(b)$\Longrightarrow$ (c): Just set $\pi = \mathcal{P}$ and $D = M^\circ$.

(c)$\Longrightarrow$ (a): Just recall from (Step 10)that $\mathrm{N}_G(D)$ is a maximal subgroup of G.

(Step 16.) [**step 10**] *Let p be a prime and $1 \neq D \in \mathcal{D}_p$. Let T be p-subgroup of G with $\Omega_2(D) \leq T$. Then $T \leq \mathrm{N}_G(D)$ and $|T/T \cap D| \leq |\mathrm{N}_G(D)/\overline{D}|_p$.*

Since $D \leq \mathrm{N}_G(\Omega_2(D))$, (Step 10)implies $\mathrm{N}_G(\Omega_2(D)) \leq \mathrm{N}_G(D)$. Since T is a Cernikov-p-group we know that $1 \neq \mathrm{Z}(T)$ (see (5.4.30)(b)). Since $\Omega_2(D) \leq T$ we have $[\Omega_2(D), \mathrm{Z}(T)] = 1$ and so $\mathrm{Z}(T) \leq \mathrm{N}_G(\Omega_2(D)) \leq \mathrm{N}_G(D)$. Thus $\mathrm{Z}(T)$ is a periodic group acting on the divisible p group D. As $\mathrm{Z}(T)$ centralizes $\Omega_2(D)$, we conclude from (5.4.31)(b)that $[D, \mathrm{Z}(T)] = 1$. Hence $D \leq \mathrm{C}_G(\mathrm{Z}(T)) \lneqq G$ and (Step 10)gives $\mathrm{C}_G(\mathrm{Z}(T)) \leq \mathrm{N}_G(D)$. Thus also $T \leq \mathrm{N}_G(D)$. Since $D \in \mathcal{D}_p$ we know that $D = (\overline{D})_p$ and so (5.4.10)(d)implies that $\overline{D}/D_p$ is p'-group. It follows that $T \cap \overline{D} \leq D$ and so

$$T/T \cap D = T/T \cap \overline{D} \cong T\overline{D}/\overline{D} \leq \mathrm{N}_G(D)/\overline{D}$$

Hence $|T/T \cap D| \leq |\mathrm{N}_G(D)/\overline{D}|_p$ and (Step 16)is proved.

(Step 17.) [**scirc**] *Let* $S \in \mathrm{Syl}_p(G)$*. Then* $S^\circ = (\overline{S^\circ})_p$ *and* $S^\circ \in \mathcal{D}_p$*.*

Suppose first that S is finite and let be any divisible p-subgroup of G. Then D is contained a Sylow p-subgroup T of G. As S is finite 5.4.16 shows that $|T| = |S|$. So D is finite and this $D = 1$. It follows that S°, $(\overline{S^\circ})_p = 1$ and $A = 1$ for all $A \in \mathcal{D}_p$. Thus (Step 17)holds in this case.

Since S is infinite, S° is a non-trivial divisible p-goup. It follows that $S^\circ \le \overline{S^\circ}_p$. Note that S normalizes $\overline{S^\circ}_p$ and so $S\overline{S^\circ}_p$ is a p-group. Since S is a maximal p-subgroup of G we conclude that $\overline{S^\circ}_p \le S$ and so also $\overline{S^\circ}_p \le S^\circ$ Thus $S^\circ = \overline{S^\circ}_p$. By (Step 12)$\overline{S^\circ}_p \in \mathcal{D}_p$ and so (Step 17)is proved.

(Step 18.) [**transitive on syl**] *Let* $H \le G$*. Then* H *acts transitively on* $\mathrm{Syl}_p(H)$*. In particular, if* P *is a* p*-subgroup of* H *and* $T \in \mathrm{Syl}_p(H)$*, then* $R^g \le T$ *for some* $h \in H$*.*

We first show that H is transitively on $\mathrm{Syl}_p(H)$. If $H \ne G$, then H is a Cernikoóvgroup an this holds by 5.4.11.

So suppose $G = H$ and let S_1 and S_2 be Sylow p-subgroups of G. If S_1 or S_2 is finite we are done by 5.4.16. So we may assume that $S_i^\circ \ne 1$ for $i = 1$ and 2. Put $E_i = \Omega_2(S_i^\circ)$ and $L = \langle E_1, E_2\rangle$. Then L is a finite group. Choose $T_i \in \mathrm{Syl}_p(L)$ with $E_i \le S_i$. Then there exists $g \in L$ with $T_1^g = T_2$. By (Step 17)we have $S_2^\circ \in \mathcal{D}_p$ and since $E_2 = \Omega_2(S_2^\circ) \le T_2$ we conclude from (Step 16)shows that $T_2 \le \mathrm{N}_G(S_2^\circ)$. In particular, T_2 is contained in a Sylow 2-subgroup R_2 of $\mathrm{N}_G(S_2^\circ)$. Note that also $S_2 \in \mathrm{Syl}_2(\mathrm{N}_G(S_2^\circ)$ and since $\mathrm{N}_G(S_2^\circ)$ is Cernikov, there exists $h \in \mathrm{N}_G(S_2^\circ)$ with $R_2^h = S_2$. Then

$$E_1^{gh} \le T_1^{gh} = T_2^h \le R_2^h = S_2$$

As above, as $S_1^\circ \in \mathcal{D}_p$ and $\Omega_2((S_1^{gh})^\circ) = E_1^{gh} \le S_2$ conclude from (Step 16)that $S_2 \le \mathrm{N}_G(S_1^{\circ gh})$. Hence both S_1^{gh} and S_2 are Sylow p-subgroups of $\mathrm{N}_G(S_1^{\circ gh})$. The latter group is Cernikov, so there exist $k \in \mathrm{N}_G(S_1^{\circ gh})$ with $S_1^{ghk} = S_2$.

We proved that H acts transitively on $\mathrm{Syl}_p(H)$. Note that P is contained in Sylow p-subgroup R of H. Then $R^h = T$ for some $h \in H$ and so $P^h \le R^g = T$.

(Step 19.) [**step 9**] *Let* p *be a prime. Let* $D \in \mathcal{D}_p$ *and* $S \in \mathrm{Syl}_p(G)$ *with* $D \le S$*. Then* $D = S^\circ$*. In particular,* $\mathcal{D}_p = \{S^\circ \mid S \in \mathrm{Syl}_p(G)\}$ *and* G *acts transitively on* $\mathcal{D}_p$*.*

By (Step 10)we have $D = S_p$. Since S is a p-group, we know that $S_p = S^\circ$, so $D = S^\circ$. By (Step 17)$T^\circ \in \mathcal{D}_p$ for all $T \in \mathrm{Syl}_p(G)$. Thus $\mathcal{D}_p = \{S^\circ \mid S \in \mathrm{Syl}_p(G)$. As G acts transitively in $\mathrm{Syl}_p(G)$ (see (Step 18)) we conclude that G acts transitively on $\mathcal{D}_p$.

(Step 20.) [**step 12**] *Let* p *be prime. Then* $\mathrm{m}_p(G)$ *is finite.*

Fix $S \in \mathrm{Syl}_p(G)$. Let $A \le G$ with $A \cong C_p^k$. Then $A \le T$ for some $T \in \mathrm{Syl}_p(G)$. By (Step 18)there exists $g \in G$ with $T^g \le S$. Thus $A^g \le S$

Since S is Cernikov, both S/S° and $\Omega_1(S^\circ)$ are finite (see (5.4.10)(a)and 5.4.9 and Note that $A^g \cap S^\circ \le \Omega_1(S^\circ)$ and so

$$p^k = |A| = |A^g| = |A^g \cap S^\circ||A^g S^\circ/S^\circ| \le |\Omega_1(S^\circ)||S/S^\circ|$$

Thus

$$\mathrm{m}_p(G) \le \log_p(\Omega_1(S^\circ)||S/S^\circ|)$$

(Step 21.) [**step 14**] *There exists a finite subgroup* Q *of* G *such that* $M = 1$ *for all finite subgroups* M *of* G *with* $Q \le N_G(M)$ *and* $Q \cap M = 1$*.*

Suppose not. Then for each finite subgroup Q of G we can choose a non-trivial finite subgroup Q^* of G such that $Q \leq \mathrm{N}_G(Q^*)$ and $Q \cap Q^* = 1$. Put $L_0 := 1$ and for $i \in \mathbb{Z}^+$ define M_i and L_i inductively by $M_i = L_{i-1}^*$ and $L_i = L_{i-1}M_i$. For $n \in \mathbb{Z}^+$ define

$$H_n = \langle M_i \mid i \in \mathbb{Z}^+, i \geq n \rangle$$

Then clearly

$$H_1 \geq H_2 \geq H_3 \geq \ldots$$

Fix $n \geq 2$. We will now show that $L_{n-1} \cap H_n = 1$. Let $x \in L_{n-1} \cap H_n$. For $m \geq n$ define

$$R_m = \langle M_i \mid n \leq i \leq m \rangle$$

Note that $R_n \leq R_{n+1} \leq \ldots \leq R_m \leq R_{m+1}$ and so

$$H_n = \bigcup_{m=n}^{\infty} R_m$$

Thus $x \in R_m$ for some $m \in \mathrm{Z}^+$ with $m \geq n$. Choose a minimal such m. Suppose that $m > n$. Then $R_m = \langle R_{m-1}, M_m \rangle$. Since $M_i \leq L_i \leq L_{i+1}$ we have that $R_{m-1} \leq L_{m-1} \leq \mathrm{N}_G(M_m)$. Thus $R_m = R_{m-1}M_n$. Since $x \in L_{n-1} \leq L_{m-1}$ and $R_{m-1} \leq L_{m-1}$ we get

$$x \in L_{n-1} \cap R_m = L_{m-1} \cap R_{m-1}M_n = R_{m-1}(L_{m-1} \cap M_n) = R_{m-1},$$

a contradiction to the minimal choice of m. Thus $m = n$. Hence $x \in R_n = M_n$ and so $x \in L_{n-1} \cap M_n = 1$.

Thus shows that $L_{n-1} \cap H_n = 1$. As $M_{n-1} \leq L_{n-1}$ this gives $M_{n-1} \cap H_n - 1$. Note that $1 \neq M_{n-1} \leq H_{n-1}$ and we conclude $H_n \neq H_{n-1}$. We proved that

$$H_1 \gneqq H_2 \gneqq H_3 \gneqq \ldots$$

a contradiction since G fulfills MIN.

(Step 22.) [**simple cover**] *Let F be a finite subgroup of G and $m \in \mathbb{Z}^+$. Then there exists a finite simple subgroup K of G with $F < K$ and $|K| \geq m$.*

Let Q be as in (Step 21). Since G is infinite there exists $I \subseteq G$ with $|I| = m$. Put $H := \langle I, F, Q \rangle$. Then H is finite subgroup of G, $F \leq H$, $Q \leq H$ and $|H| \geq m$. Since G is a locally finite simple group, 5.4.2 shows that there exists Kegel cover $\mathcal{K}$. Hence we can choose $(K, M) \in \mathcal{K}$ with $H \leq K$ and $H \cap M = 1$. By definition of a Kegel cover, K is a finite subgroup of G, $M \trianglelefteq K$ and K/M is simple. Hence $Q \leq H \leq\leq N_G(M)$ and $Q \cap M \leq H \cap M = 1$. Thus (Step 21)gives $M = 1$. Hence $K \cong K/M$ and so K is simple. Note $F \leq H \leq K$ and $|K| \geq |H| \geq m$. So (Step 22)holds.

From now on S is a Sylow 2-subgroup of G.

(Step 23.) [**s is not dihedral**] *$S \ncong D_{2^k}$ for $k \in \mathbb{Z}^+ \cup \infty$ with $k \geq 2$.*

Suppose that $S \cong \mathrm{D}_{2^k}$ for some $k \in \mathrm{Z}^+ \cup \infty$ with $k \geq 2$. Then there exists $A \leq S$ with $A \cong \mathrm{C}_2 \times \mathrm{C}_2$. By (Step 22)we an choose finite simple subgroups $H_i \leq G$ such that

$$R \leq H_1 \lneqq H_2 \lneqq H_3 \lneqq \ldots \lneqq H_n \lneqq H_{n+1} \lneqq \ldots$$

and $|H_i| \geq 7!$.

Let $S_i \in \mathrm{Syl}_2(H_i)$ with $A \le S_i$. Then $S_i \in T_i$ for some $T_i \in \mathrm{Syl}_2(G_i)$. By (Step 18)there exists $g \in G$ with $S_i = T_i^{g_i}$. Hence $T_i \cong \mathrm{D}_{2^k}$ and since $A \le S_i \le T_i$ we conclude that S_i is a Dihedral group. Hence (5.4.38)(a)implies that $H_i \cong \mathrm{Alt}(7)$ or $L_2(p_i^{k_i})$, p_i an odd prime. Since $|H_i| \ge |7!$ we have $H_i \ncong \mathrm{Alt}(7)$ and $p_i^{k_i} > 5$.

Let $p = p_1$ and choose $R \in \mathrm{Syl}_p(H_1)$. Then by (5.4.23)(b)implies $|\mathrm{N}_{H_1}/\mathrm{C}_{H_1}(R)| = \frac{p^{k_1}-1}{2} > \frac{5-1}{2} = 2$. Hence also $|\mathrm{N}_{H_i}(R)/\mathrm{C}_{H_i}(R)| > 2$. Suppose that $p \ne p_i$ for some i. Then (5.4.23)(c)implies that $|\mathrm{N}_{H_i}/\mathrm{C}_{H_i}(A)| = 2$, a contradiction.

Thus $p_i = p$ and so $H_i \cong \mathrm{PSL}_2(p^{k_i})$. Since $H_i \lneqq H_{i+1}$ we get $k_i < k_{i+1}$. Note that $\mathrm{m}_p(G) \ge \mathrm{m}_p(H_i) = k_i$ and we conclude $\mathrm{m}_p(G) = \infty$ a contradiction to (Step 20)

(Step 24.) [**step 13**] *G has no proper strongly 2-embedded subgroup.*

Suppose for a contradiction that M is a strongly 2 embedded subgroup of G. Then $M \ne G$, M is not a $2'$-group and $M \cap M^g$ is a $2'$-group for all $g \in G \smallsetminus M$. In particular we can choose $x \in G \smallsetminus M$ and a 2-element y in $M^\sharp$. By (Step 20)$\mathrm{m}_2(G)$ is finite and so by (Step 22)we can choose a finite simple subgroup K of G with $\langle x, y\rangle \le K$ and $|K| \ge 2^{18 \cdot \mathrm{m}_2(G)} + 1$. Since $x \in K$ we have $K \nleq M$ and since $y \in K \cap M$, $K \cap M$ is not a $2'$-group. Let $k \in K \smallsetminus (K \cap M)$. Then $k \notin M$ and so $M \cap M^k$ is a $2'$ group. Hence $M \cap K$ is a strongly 2-embedded subgroup of K.

Note that K has even order and since K is simple we conclude that $\mathrm{O}_{2'}(K) = 1$.

Let $a \in K$ with $|a| = 2$. Suppose that $\mathrm{C}_K(a) \cap a^K = \{a\}$. Then Glauberman's Z^*- Theorem shows that $\hat{a} \in \mathrm{Z}(\hat{K})$ where $\hat{K} = K/\mathrm{O}_{2'}(K)$. As $\mathrm{O}_{2'}(K) = 1$, this gives $a \in \mathrm{Z}(K)$. Hence $\langle a\rangle \trianglelefteq K$, contradiction, since K is simple and $|K| > 2 = |\langle a\rangle|$.

Thus there exists $b \in a^K$ with $a \ne b$. Then $\langle a, b\rangle \cong \mathrm{C}_2 \times \mathrm{C}_2$ and so $\mathrm{m}_2(K) \ge 2$. Then Bender's strongly 2-embeded theorem shows that

$$K \cong K/\mathrm{O}_{2'}(K) \;\cong \mathrm{PSL}_2(q), \mathrm{Sz}(q), \text{ or } PSU_3(q),$$

where $q = 2^k$ for some $k \in \mathbb{Z}^+$. Now 5.4.33 shows that $k = \mathrm{m}_2(K)$ and $|K| \le 2^{18k}$. Observe that $\mathrm{m}_2(K) \le \mathrm{m}_2(G)$ and thus $|K| \le 2^{18 \cdot \mathrm{m}_2(G)}$, a contradiction to the choice of K.

(Step 25.) [**hz**] *Let $z \in G$ with $|z| = 2$. Then $\mathrm{C}_G(z)$ is infinite. In particular, there exists a largest proper subgroup H_z of G containing $\mathrm{C}_G(z)$.*

Since G is an infinite, locally finite, simple group, 5.3.29 shows that $\mathrm{C}_G(z)$ is infinite. The existence of H_z now follows from (Step 14)(b).

(Step 26.) [**step 15**] *Let $z \in G$ with $|z| = 2$ and $M \lneqq G$ with $z \in M$ and $M \nleq H_z$. Then $\mathrm{C}_{M^\circ}(z)$ is finite and z inverts M°.*

Suppose that $\mathrm{C}_{M^\circ}(z)$ is infinite. As $\mathrm{C}_{M^\circ}(z) \le H_z \cap M$ we see that $M \cap H_z$ is infinite. Note that H_z is a maximal subgroup of G and so (Step 14)(d)gives $M \le H_z$.

Thus $\mathrm{C}_{M^\circ}(z)$ is finite. Since M° is divisible we concluded from 5.4.12 z inverts M°.

(Step 27.) [**step 16**] *Let $A \le G$ with $A \cong \mathrm{C}_2 \times \mathrm{C}_2$.*

(a) [**16:a**] *Let M a proper infinite subgroup of G containing A. Then $M \le H_x$ for some $x \in A^\sharp$.*

(b) [**16:b**] *Suppose that $\mathrm{C}_G(A)$ is infinite and let $x, y \in A^\sharp$. Then $H_x = H_y$ and H_x is the unique maximal subgroup of G containing A.*

(a): Assume for a contradiction that $M \nleq H_z$ for all $z \in A^\sharp$. Let $z \in A^\sharp$. Then (Step 25)shows that $C_{M^\circ}(z)$ is finite and z inverts M°. Let $a, b \in A^\sharp$ with $a \neq b$. Then a and b inverts M° and so ab centralizes M°. Thus $M^\circ = C_{M^\circ}(ab)$ and so M° is finite. But then also M is finite, a contradiction.

(b)Note that $C_G(A) \leq C_G(x) \cap C_G(y) \leq H_x \cap H_y$ and so $H_x \cap H_y$ is infinite. As H_x and H_y are maximal subgroups of G, this gives $H_x = H_y$, see (Step 14)(c). Let M be any maximal subgroup of G containing A. Then A is infinite and (a)shows that $M \leq H_z$ for some $z \in A^\sharp$. Since M is maximal subgroup of G we get $M = H_z$. Also $H_z = H_x$ and so (b)holds.

(**Step 28.**) [**bhz**] *Let M be a maximal subgroup of G.*

(a) [**bhz:a**] $N_G(M) = M$

(b) [**bhz:b**] *Let $g \in G \setminus M$. Then $M \neq M^g$ and $M \cap M^g$ is finite.*

(c) [**bhz:c**] *Let $T \in \mathrm{Syl}_2(M)$ and suppose that $N_G(T) \leq M$. Then there exists $a \in T$ with $|a| = 2$ and $C_G(a) \nleq M$.*

(a)Since G is simple and $1 \neq M \lneq G$ we have $M \ntrianglelefteq G$. Thus $M \leq N_G(M) \lneq M$ and since M is a maximal subgroup we get $M = N_G(H_z)$.

(b)By (a)we have $g \notin M = M_G(M)$ and so $M \neq M^g$. As both M and M^g are maximal subgroups of G (Step 14)(c)shows that $M \cap M^g$ is finite.

(c)Suppose for a contradiction that $C_G(a) \leq M$ for all $a \in T$ with $|a| = 2$. Let $b \in M$ with $|b| = 2$. Since M acts transitively on $\mathrm{Syl}_p(M)$ we get $b \in T^m$ for some $m \in M$ and so $C_G(b) = C_G(a)^m \leq M^m = M$. Let $g \in G \setminus M$. Let $R \in \mathrm{Syl}_2(M)$. Since $N_G(T) \leq M$ and M acts transitively on $\mathrm{Syl}_2(M)$ we get $N_G(R) \leq M$ and so $R \neq R^g$. By (b)$M \cap M^g$ is finite, so also $R \cap M^g$ is finite. Since $N_G(T) \leq M$ we know that $T \neq 1$, so M is not a $2'$-group. Now 5.4.18 shows that M is a strongly 2-embedded subgroups of G, a contradiction to (Step 24).

(**Step 29.**) [**cga not in hz**] *Let $S \in \mathrm{Syl}_2(G)$ and $z \in Z(S)$ with $|z| = 2$. There exists $a \in S$ such that $|a| = 2$, $a \neq z$, $C_G(a) \nleq H_z$ and $H_a \neq H_z$.*

Suppose first that $N_G(S) \nleq H_z$ and pick $g \in N_G(S) \setminus H_z$. By (Step 28)$H_z \neq H_z^g$. Put $a = z^g$. Then $a \in S^g = S$ and $H_a = H_z^g \neq H_z$. In particular, $a \neq z$. By (Step 25)$C_G(a)$ is the unique maximal subgroup of G containing a, so $C_G(a) \nleq H_z$.

Suppose next that $N_G(S) \leq H_z$. Note that $S \leq C_G(z) \leq H_z$ and so $S \in \mathrm{Syl}_2(H_z)$. Then (Step 28)(c)shows that there exists $a \in S$ with $|a| = 2$ and $C_G(a) \nleq H_z$. As $C_G(a) \leq H_a$ we have $H_a \neq H_z$ and since $C_G(z) \leq H_z$ we get $a \neq z$.

(**Step 30.**) [**rank less than 2**] $m_2(S^\circ) \leq 1$.

Suppose for a contradiction that $m_2(S^\circ) > 1$. Then there exists $A \leq S^\circ$ with $A \cong C_2 \times C_2$. Put $M = N_G(S^\circ)$. By (Step 17)$S^\circ \mathrm{inn} \mathcal{D}_p$ and so by (Step 15)(c)M is maximal subgroup of G. Note that $S \in \mathrm{Syl}_2(M)$ and $N_G(S) \leq N_G(S^\circ) \leq M$. Thus (Step 28)(c)show that there exists $y \in S$ with $|y| = 2$ and $C_G(y) \nleq M$. If $M \leq H_a$ we get $M = H_y$, and $C_G(y) \leq M$. Thus $M \nleq H_y$ and (Step 26)shows that y inverts M°. Let $a \in A$. Then $a^2 = 1$ and $a \in A \leq S^\circ \leq M^\circ$. Thus $a^y = a^{-1} = a$ and so $A \leq C_G(y) \leq H_y$. Note that $S^\circ \neq 1$ and S° is divisible. Thus S° is infinite. Also $S^\circ \leq C_G(A)$, so $C_G(A)$ is infinite. Thus (Step 27)(b)shows that A is contained in a unique maximal subgroup of G,a contradiction since $A \leq M \cap H_y$ and $M \neq H_y$.

(**Step 31.**) [**step 18**] *Define $\mathcal{I} : \{x \in G \mid x^2 = 1, (H_x)_2 \neq 1\}$. Suppose* $m_2(S^\circ) \geq 1$. *Then* $m_2(S^\circ) = 1$, *$\mathcal{I} \neq \varnothing$ and G acts transitively on $\mathcal{I}$.*

By (Step 30)we know that $\mathrm{m}_2(S^\circ) \leq 1$. Thus $\mathrm{m}_2(S^\circ) = 1$ and so S° has a unique involution x. Since S° is abelian, we have $S^\circ \leq \mathrm{C}_G(x) \leq H_x$. By (Step 17)$S^\circ \in \mathcal{D}_2$ and so (Step 10)gives $S^\circ = (H_x)_2$. In particular, $(H_x)_2 \neq 1$. Thus $x \in \mathcal{I}$ and so $\mathcal{I} \neq 1$.

Suppose for a contradiction that G does not act transitively on $\mathcal{I}$. Then we can chose $y \in \mathcal{I}$ with $y \notin x^G$. Since G is simple and $H_y \neq 1$ we know that $\bigcap_{g \in G} H_y^g = 1$. Hence there exists $g \in G$ with $x \notin H_y^g = H_{y^g}$. x. Replacing y by y^g we may assume that $x \notin H_y$. Put $D = \langle x, y \rangle$. Then $D \cong \mathrm{D}_{2m}$ for some $m \in \mathbb{Z}^+$. Since $y \notin x^G$ we have $y \notin x^D$. So Homework 1 shows that m is even, $u := (xy)^{\frac{m}{2}} \in \mathrm{Z}(D)$ and $|u| = 2$. Then $u \leq \mathrm{C}_G(y) \leq H_y$. By (Step 18)$G$ acts transitively on $\mathrm{Syl}_2(G)$ and so $(H_y)_2 \leq S^h$ for some $h \in G$. By (Step 15)$H_y^\circ \in \mathcal{D}$ and so (Step 12)shows that $(H_y)_2 \in \mathcal{D}_2$. Now (Step 11)implies that $(H_y)_2 = (S^h)^\circ = S^{\circ h}$. In particular, x^h is the unique involution in $(H_y)_2$ and so $x^h \in \mathrm{Z}(H_y)$. Note that $\langle u, y \rangle \leq \mathrm{C}_G(y) \leq H_y$ and so both u and y centralizes x^h. Put $A := \langle y, x^h \rangle$. Since $y \notin x^G$ we have $y \neq x^h$. As $[y, x^h] = 1$ this implies that $A \cong \mathrm{C}_2 \times \mathrm{C}_2$. By (Step 25)$\mathrm{C}_G(y)$ is infinite and since H_y/H_y° is finite we conclude that $\mathrm{C}_{H^\circ}(y)$ is infinite. As $x^h \in (H_y)_2 \leq H_y^\circ$ and H_y° is abelian, we know that x^h centralizes $\mathrm{C}_{H^\circ}(y)$. It follows that $\mathrm{C}_{H^\circ}(y) \leq \mathrm{C}_G(A)$, and so $\mathrm{C}_G(A)$ is infinite. Thus by (Step 27), A lies in a unique maximal subgroup of G. Note that $A \leq H_y$ and $A \leq \mathrm{C}_G(u) \leq H_u$. Thus $H_y = H_u$ and $x \leq \mathrm{C}_H(u) \leq H_u = H_y$, a contradiction to the choice of y.

(Step 32.) [**s is finite**] *S is finite.*

Suppose S is infinite. Then $S^\circ \neq 1$ and so $\mathrm{m}_2(S^\circ) \geq 1$. Thus (Step 31)shows that $\mathrm{m}_2(S^\circ) = 1$, so $S^\circ \cong \mathrm{C}_2^\infty$. We will show that $S \cong \mathrm{D}_2^\infty$, which will be a contradiction to (Step 23). Let $x \in S^\circ$ with $|x| = 2$.

Assume that $\mathrm{C}_S(S^\circ) \neq S^\circ$. Then there exists $T \leq \mathrm{C}_S(S^\circ)$ with $S^\circ \leq T$ and $|T/S^\circ| = 2$. Then $T/\mathrm{Z}(T)| \leq 2$ and so T is abelian. Hence 5.2.4 shows that $T = S^\circ \times K$ for some $K \leq T$. Let $y \in K$ with $|y| = 2$. Then $S^\circ \leq H_x^\circ \cap H_y^\circ$. By (Step 10)$S^\circ \in \mathcal{D}_2$ and so (Step 10)shows that $1 \neq S^\circ = (H_x)_2 = (H_y)_2$. In particular, $x, y \in \mathcal{I}$. By (Step 31)acts transitively on $\mathcal{I}$, so $y = x^g$ for some $g \in G$. But then

$$y = x^g \in (S^\circ)^g = ((H_x)_2)^g = (H_y)_2 = S^\circ$$

and contradiction to $1 \neq y \in K$ and $K \cap S^\circ = 1$.

Hence $\mathrm{C}_S(S^\circ) = S^\circ$. Put $S_0 = \Omega_2(S^\circ) \cong \mathrm{C}_4$. (5.4.31)(b), $\mathrm{C}_S(S_0) = \mathrm{C}_S(S^\circ) = S^\circ$ and since $|\mathrm{Aut}(S_0)| = 2$ we conclude that $|S/S^\circ| \leq 2$. Note that $x \in \mathrm{Z}(S)$. By (Step 29)there exists $a \in S$ with $a \neq x$ and $H_a \neq H_z$. By (Step 26)a inverts H_x°. In particular, a inverts S°. It follows that $a \notin S^\circ$ and so $S = S^\circ\langle a \rangle \cong \mathrm{D}_{2^\infty}$, a contradiction to (Step 23).

(Step 33.) [**mcirc odd**] *All 2-subgroups of G are finite. In particular, K° is a $2'$-group for all $K \lneqq G$.*

By (Step 32)all Sylow 2-subgroups of G are finite. Any 2-subgroup of G is contained in Sylow 2-subgroup of G and so is finite. In particular, K_2 is finite. As K_2 is divisible this implies $K_2 = 1$. Hence K° is a $2'$-group.

Notation Fix $x \in \mathrm{Z}(S)$ with $|z| = 2$.

(Step 34.) [**brauer step**] *For all $1 \neq x_0 \in S$ there exists $y_1, y_2 \in S \cap z^G$ and $y_0 \in S \cap y_0^G$ with $y_1 y_2 = y_0$ and $\mathrm{C}_S(y_0) \in \mathrm{Syl}_2(\mathrm{C}_G(y_0))$.*

Put $x_i = z$ for $i = 1, 2$ and for $0 \leq i \leq 2$ define $t_i = \mathrm{C}_G(x_i)/\mathrm{C}_G(x_i)^\circ|$. By 5.4.34 there exists $m \in \mathrm{Z}$ such that $|H| \leq m$ whenever H is a finite simple group such that

(i) [**br:i**] $S \in \mathrm{Syl}_2(H)$;

(ii) [**br:ii**] $|\mathrm{C}_{H_i}/\mathrm{O}_{2'}(\mathrm{C}_H(x_i)))| \leq t_i$;and

(iii) [**br:iii**] There do not exist$y_i \in x_i \cap H$, $0 \leq i \leq 2$ such that $y_0 = y_1 y_2$ and $\mathrm{C}_S(y_0) \in \mathrm{Syl}_2(\mathrm{C}_H(y_0))$.

Pick $T \in \mathrm{Syl}_2(\mathrm{C}_G(x_0))$. By (Step 33)$T$ and S are finite.. So also $\langle S, T\rangle$ is finite. Thus by (Step 22)we can choose finite simple subgroup H of G with $\langle S, T\rangle \le H$ and $|H| \ge m+1$. Since $S \in \mathrm{Syl}_2(G)$ and $S \le G$ we have $S \in \mathrm{Syl}(2)$.

Put $s_i := |\mathrm{C}_H(x_i)/O_{2'}(\mathrm{C}_H(x_i))$. By (Step 33)$\mathrm{C}_G(x_i)^\circ$ is a $2'$-group and so $\mathrm{C}_H(x_i) \cap \mathrm{C}_G(x_i)^\circ \le \mathrm{O}_{2'}(\mathrm{C}_H(x_i))$. Hence

$$s_i = |\mathrm{C}_H(x_i)/\mathrm{O}_{2'}(\mathrm{C}_H(x_i))| \le |\mathrm{C}_H(x_i)/\mathrm{C}_H(x_i) \cap \mathrm{C}_G(x_i)^\circ| = |\mathrm{C}_H(x_i)\mathrm{C}_G(x_i)^\circ/\mathrm{C}_G(x_i)^\circ| \le |\mathrm{C}_G(x_i)/\mathrm{C}_G(x_i)^\circ| \le t_i$$

We proved that (i)and (ii)hold for H. If also (iii)would hold, then $|H| \le m$, a contradiction to $H| \ge m+1$. Thus (iii)fails and so there exists $y_i \in S \cap x_i^H$ such that $y_1 y_2 = y_0$ and $\mathrm{C}_S(y_0) \in \mathrm{Syl}_2(\mathrm{C}_H(y_0))$. Since $T \in \mathrm{Syl}_2(\mathrm{C}_G(x_0)$ and $T \le C_H(x_0) \le \mathrm{C}_G(x_0)$ we know that $T \in \mathrm{Syl}_2(\mathrm{C}_H(x_0))$. Let $h \in H$ with $x_0^h = y_0$. Then $T^h \in \mathrm{Syl}_2(\mathrm{C}_G(y_0))$ and so $T^{hl} = \mathrm{C}_S(y_0)$ for some $l \in \mathrm{C}_H(y_0)$. Note that $x_0^{hl} = y_0^l = y_0$ and since $T \in \mathrm{Syl}_2(\mathrm{C}_G(x_0))$ we conclude that $\mathrm{C}_S(y_0) = T^{hl} \in \mathrm{C}_G(y_0$. Thus (Step 34)is proved.

(Step 35.) [**2 central fours group**] *There exists $E \le S$ such that $E \cong \mathrm{C}_2 \times \mathrm{C}_2$, $z \in E$ and $E^\sharp \in z^G$.*

By (Step 34)applied with $x_0 = z$, there exists $y_i \in z^G \cap S$ with $y_1 y_2 = y_0$. Put $F := \langle y_1, y_2\rangle$. Since $|y_i| = |z| = 2$ we have $F \cong \mathrm{D}_4 \cong \mathrm{C}_2 \times C_2$ and so $F^\sharp = \{y_0, y_1, y_2\} \subseteq z^G$. Let $g \in G$ with $y_1^g = z$. Then $z = y_1^g \in z \in F^g \le \mathrm{C}_G(z)$. Note that $S \le \mathrm{C}_G(z)$ and since $S \in \mathrm{Syl}_2(G)$ we conclude that $S \in \mathrm{Syl}_2(\mathrm{C}_G(z))$. By (Step 18)there exists $h \in \mathrm{C}_G(z)$ with $E := F^{gh} \le S$. Then $E \cong F \cong \mathrm{C}_2 \times \mathrm{C}_2$ and $z = z^h \in F^{gh} = E$.

(Step 36.) [**centralizer of hyper planes**] *Let B be non-trivial finite elementary abelian p group acting on a periodic p'-group D. Then $D = \langle \mathrm{C}_D(X) \mid X \le B, |B/X| = p\rangle$.*

Let $e := \frac{|B|}{p}$. By Exercise 2 on Homework 2 we have

$$D^{\cdot e} = \langle \mathrm{C}_D(Y) \mid Y \le B, B/Y \text{ is cyclic}\}$$

Note that e is a power of p. As D is a p'-groups we conclude that $D = D^{\cdot e}$. Let $Y \le B$ such that B/Y is cyclic. As B is elementary abelian p-group we have $a^p = 1$ for all $a \in B/X$. It follows that $B/Y = 1$ or $|B/Y| = p$. If $|B/Y| = p$ let $X = Y$. If $B/Y = 1$ and we can choose $X \le Y$ with $|B/X| = p$. In either case $|B/X| = p$ and $\mathrm{C}_B(X) \le \mathrm{C}_B(Y)$. Thus

$$D = D^{\cdot e} = \langle \mathrm{C}_D(Y) \mid Y \le B, B/Y \text{ is cyclic}\rangle \le \langle \mathrm{C}_D(X) \mid X \le B, |B/X| = p\rangle.$$

(Step 37.) [**CGA**] *Let $A \le S$ with $A \cong \mathrm{C}_2 \times \mathrm{C}_2$. Suppose that A is contained in at more than one maximal subgroup of G. Then $\mathrm{C}_G(A)_{\cdot 2} = A$ and there exists $d \in z^G \cap S$ with $d \notin \mathrm{C}_S(A)$. In particular, $A \nleq \mathrm{Z}(S)$.*

Suppose for contradiction that there exists an involution $b \in \mathrm{C}_G(A) \setminus A$. Put $B = \langle A, b\rangle$. Then $B \cong \mathrm{C}_2^3$. Let M_1 and M_2 be two distinct maximal subgroups of G containing A. By (Step 27), $M_i = H_{a_i}$ for some $a_i \in A$. Thus $B \le \mathrm{C}_G(a_i) \le M_i$. By (Step 36)$M_1^\circ = \langle \mathrm{C}_{M_1^\circ}(X) \mid X \le B, |B/X| = 2\rangle$. Since M_1° is infinite, we conclude that there exists $X \le B$ with $|B/X| = 2$ and $\mathrm{C}_{M_1^\circ}(X)$ infinite. In particular, $\mathrm{C}_G(X)$ is infinite and (Step 27)shows B_i is contained in a unique maximal subgroup of G, a contradiction to $X \le M_1 \cap M_2$.

Thus $\mathrm{C}_G(A)_{\cdot 2} = A$. Suppose that $x^2 = 1$ for all $x \in S$. Then $S \le \mathrm{C}_G(A)_{\cdot 2} = A$ and so $S = A \cong \mathrm{C}_2 \times \mathrm{C}_2 \cong \mathrm{D}_4$, a contradiction to (Step 23). Hence there exists $x_0 \in S$ with $|x_0| > 2$. By (Step 34)there exists $y_1, y_2 \in S \cap z^G$ and $y_0 \in S \cap x_0^G$ with $y_1 y_2 = y_0$. Suppose both y_1 and y_2 are in $\mathrm{C}_S(A)$. Then $y_0 \in \langle y_1, y_2\rangle \le \mathrm{C}_S(A)_{\cdot 2} = A$ and so $y_0^2 = 1$, a contradiction. Thus one of y_1 and y_2 is not in $\mathrm{C}_S(A)$.

(Step 38.) [**s in a unique maximal**] *H_z is the unique maximal subgroup of G containing S.*

Suppose for a contradiction that M is a maximal subgroup of G with $S \leq M$ and $M \neq H_z$. If $|\Omega_1 Z(S)| \geq 4$, we can choose $A \leq \Omega_1 Z(S)$ with $A \cong C_2 \times C_2$, a contradiction to (Step 37). Thus $|\Omega_1 Z(S)| \leq 2$ and so $\Omega_1 Z(S) = \langle z \rangle$. Since $z \in M \nleq H_z$, (Step 26)shows that z inverts M°. Thus $\Omega_1 Z(S) \cap C_S(M^\circ) = 1$. Since $C_S(M^\circ)$ is normal in S this implies $C_S(M^\circ) = 1$. Let E be as in (Step 35)and let $E \setminus \langle z \rangle = \{a, b\}$. If a inverts M° we get $b = az \in C_S(M^\circ) = 1$, a contradiction. Thus a does not invert M° and by (Step 27), $M = H_a$. In particular, $H_a \neq H_z$ and since $a \in C_G(z) \leq H_z$ we conclude from (Step 26)that a inverts H_z°. By symmetry, $M = H_b$ and b inverts H_z°. Note that $z = ab$ and we conclude that z centralizes H_z°. Since $a \in E^\sharp \subseteq\in z^G$, this implies that a centralizes H_a°. But this is a contradiction to $H_a^\circ = M^\circ$ and $C_S(M^\circ) = 1$.

Notation According to (Step 29)we can choose $e \in S$ with $|e| = 2$, $e \neq z$, and $H_e \neq H_z$. If $H_e \in H_z^G$, put $x := e$. If $H_e \notin H_z^G$, then according to (Step 34)we can choose $g, h \in G$ with $e := z^g z^h$ and put $x := e^{g^{-1}}$. In either case, put $A := \langle x, z \rangle$, $y := zx$ and $\mathcal{A} = \{a \in A \mid H_a \in H_z^G\}$.

(Step 39.) **[basic a]** *$A \cong C_2 \times C_2$, $H_x \neq H_z$ and $|\mathcal{A}| \geq 2$.*

Suppose first that $H_e \in H_z^G$. Then $x = e \in \mathcal{A}$, $H_x = H_e \neq H_z$ and $x \in S \leq C_G(z)$. Also $x = e \neq z$ and so $A = \langle x, z \rangle \cong C_2 \times C_2$.

Suppose next that $H_e \notin H_z^G$. Then $x = e^{g^{-1}} = (z^g z^h)^{g^{-1}} = z z^{hg^{-1}}$ and so $y = zx = z^{hg^{-1}} \in z^G$. Thus zx has order two and $A = \langle x, z \rangle \cong D_4 \cong C_2 \times C_2$. Also $H_y = H_z^{hg^{-1}} \in H_z^G$ and so $y \in \mathcal{A}$. Since $H_x = H_e^{g^{-1}} \notin H_z^G$ we have $H_x \neq H_z$.

Notation Note that $A \leq C_G(x) \cap C_G(z) \leq H_x \cap H_z$. So we can choose $T \in \mathrm{Syl}_2(H_x \cap H_z)$ with $A \leq T$. For $a \in A^\sharp$ pick $S_a \in \mathrm{Syl}_2(H_a)$ with $T \cap H_a \leq S_a$ and define $T_a := N_{S_a}(C_{S_a}(A))$.

(Step 40.) **[omega t]** *Let $a \in A^\sharp$.*

(a) **[tz]** $S_a \in \mathrm{Syl}_2(G)$.

(b) **[ta]** $\mathcal{A} = A^\sharp \subseteq z^G$.

(c) **[tb]** $A = \Omega_1 Z(T) = \Omega_1(T)$ *and* $C_{S_a}(A) = T$

(d) **[tc]** $\Omega_1 Z(S_a) = \Omega_1 Z(T_a) = \langle a \rangle$

(e) **[td]** $T_a = N_{S_a}(T) = N_{S_a}(A)$ *and* $|T_a/T| = 2$.

(f) **[te]** $N_G(T)/N_G(T) \cap C_G(A) \cong \mathrm{Sym}(A^\sharp)$

Let $d \in \mathcal{A}$. Since $S \in \mathrm{Syl}_2(G)$ and $S \leq C_G(z) \leq H_z$ we have $S \in \mathrm{Syl}_2(H_z)$. By definition of $\mathcal{A}$, H_d is conjugate to H_z in G. As $S_d \in \mathrm{Syl}_2(H_d)$ and $S \in \mathrm{Syl}_2(H_z)$ we conclude that S_d and S are conjugate in G. Since $S \in \mathrm{Syl}_2(G)$ this implies that also

$$S_d \in \mathrm{Syl}_2(G)$$

Since $A \leq C_G(d) \leq H_d$ and $A \leq T$ we have $A \leq T \cap H_d \leq S_d$. Also A lies in two different maximal subgroups of G, namely H_x and H_z and so (Step 37)shows that $A \nleq Z(S_d)$ and $A = C_G(A)_{\cdot 2}$. Thus $C_{S_d}(A) \lneq S_d$ and $A = C_{S_d}(A)_{\cdot 2} \operatorname{char} C_{S_d}(A)$. It follows that

$$C_{S_d}(A) \lneq N_{S_d}(C_{S_d}(A)) = T_d \quad \text{and} \quad A \trianglelefteq N_{S_d}(C_{S_d}(A)) = T_d$$

In particular, $1 \neq A \cap Z(T_d) \neq A$. As $|A| = 4$ we get $|A \cap Z(T_d)| = 2$ and so there exists a unique $d^* \in A^\sharp \cap Z(T_d)$. Thus

$$|T_d / C_{S_d}(A)| = 2$$

and T_d acts as the 2-cycle with fixed point d^* on the three elements of $A^\sharp$.

Note that both $\Omega_1 Z(S_d)$ and $\Omega_1 Z(T_d)$ are contained in $C_G(A)_{\cdot 2}$, that is in A. Since $T_d \le S_d$, both $\Omega_1 Z(S_d)$ and $\Omega_1 Z(T_d)$ are centralized by T_d. Hence

$$\langle \Omega_1 Z(S_d), \Omega_1 Z(T_d)\rangle \le C_A(T_d) = A \cap Z(T_d) = \langle d^* \rangle \quad \text{and so} \quad \Omega_1 Z(S_d) = \Omega_1 Z(T_d) = \langle d^* \rangle$$

Then $S_d \le C_G(d^*)$. Since $S_d \in \mathrm{Syl}_2(G)$, (Step 38)shows that S_d is contained in a unique maximal subgroup of G. As $S \le H_d \cap H_{d^*}$ this gives $H_{d^*} = H_d$.

Assume for a contradiction that $d \ne d^*$ and let $t \in T \smallsetminus C_{T_d}(A)$. Then $A^\sharp = \{d^*, d, d^t\}$. As $t \in T_d \le H_d$ we have $H_{d^t} = H_d^t = H_d = H_{d^*}$, a contradiction to $H_x \ne H_z$. Thus $d = d^*$.

Since $|\mathcal{A}| \ge 2$, we can choose $b \in \mathcal{A}$ with $b \ne d$. Note T_d acts as the two cycle with fix-point d on $A^\sharp$ and T_b as the 2 cycle with fix point b. Thus

$$\langle T_d, T_b\rangle \text{ acts as } \mathrm{Sym}(A^\sharp) \text{ on } A^\sharp.$$

Hence all elements in $A^\sharp$ are conjugate in G and so $A^\sharp \subseteq z^G$. If $a \in A \cap z^G$, then $H_a \in H_z^G$ and so $a \in \mathcal{A}$. Thus $A^\sharp = A \cap z^G = \mathcal{A}$.

In particular, the results we proved for $d \in \mathcal{A}$ hold for all elements $a \in A^\sharp$.

Note that $T \le H_x \cap H_z$. Let $c \in A^\sharp$ with $T \le H_c$. Then $T = T \cap H_c \le S_c \le C_G(c)$ and so $c \in C_A(T) \le Z(T)$. This holds for $c = x$ and z and so $A = \langle x, z\rangle \le Z(T)$. It follows that

$$T = C_T(A) \quad \text{and} \quad \Omega_1 Z(T)) \le \Omega_1(T) \le C_G(A)_{\cdot 2} = A$$

and so

$$A = \Omega_1 Z(T) = \Omega_1(T)$$

Observe that $T \le S_c \cap H_x \cap H_z$. Since $T \in \mathrm{Syl}_s(H_x \cap H_z)$ this implies $T = S \cap H_x \cap H_z$ and so

$$C_{S_c}(A) \le S_c \cap C_G(x) \cap C_G(y) \le S_c \cap H_x \cap H_y = T$$

As $T \le S_c$ this gives

$$C_{S_c}(A) = C_T(A) = T.$$

In particular, $A \,\mathrm{char}\, T$ and so $N_{S_c}(T) \le N_{S_c}(A)$. Moreover,

$$N_{S_c}(A) \le N_{S_c}(C_{S_c}(A))) = N_{S_c}(T))$$

and so

$$N_{S_c}(T) = N_{S_c}(A) = N_{S_c}(C_{S_c}(A)) = T_c$$

In particular, $\langle T_c, T_b\rangle \le N_G(T) \le N_G(Z(T)) = N_G(A)$. As $\langle T_c, T_b\rangle$ induces $\mathrm{Sym}(A^\sharp$ on $A^\sharp$ we conclude that also $N_G(T)$ induces $\mathrm{Sym}(A^\sharp)$ on $A^\sharp$. Thus $N_G(T)/N_G(T) \cap C_G(A) \cong \mathrm{Sym}(A^\sharp)$ and $N_G(A)$ acts transitively on $A^\sharp$. Let $h \in N_H(T)$. As $T \le H_z$ we get $T = T^h \le H_z^h = H_{z^h}$. It follows $T \le H_a$ for all $a \in A^\sharp$, so all the results we proved for c hold for all $a \in A^\sharp$.

(Step 41.) [**step semidihedral**] *S is not a quasidihedral group.*

Suppose S is a quasidihedral group. By (Step 22)we can choose finite simple subgroup L of G with $S \le L$ and

$$|L| \ge |H_z/H_z^\circ|^{18}.$$

Then $S \in \mathrm{Syl}_2(L)$. By (Step 23)S is not a dihedral group. So $|S| \geq 16$. Now (5.4.38)(b)gives

$$L \cong \mathrm{Mat}_{11}, \mathrm{PSL}_3(q), \text{ or } \mathrm{PSU}_3(q)$$

By (Step 33)H_z° is a $2'$-group. As $z \in H_z$ this gives $H_z \neq H_z^\circ$ and so $|L| \geq |H_2/H_z^\circ|^{18} \geq 2^{18}$. As $|M_{11}| = 11 \cdot 10 \cdot 9 \cdot 8 < 2^{18}$, this gives $L \ncong \mathrm{Mat}_{11}$. Now 5.4.24 shows that $|L| \leq q^{18}$ and there exists $K \leq \mathrm{C}_L(z)$ with $K \cong \mathrm{SL}_2(q)$. In particular, $|H_z/H_z^\circ|^{18} \leq |L| \leq q^{18}$ and so that

$$|H_z/H_z^\circ| \leq q$$

By Homework 3 K has no non-trivial normal subgroup of odd order. As $K \leq \mathrm{C}_G(z) \leq H_z$ and H_z° is a $2'$-group, this gives $K \cap H_z^\circ = 1$.

Thus

$$|H_z/H_z^\circ| \geq |KH_z^\circ/H_z^\circ| = |K/K \cap K_z^\circ| = |K| = |\mathrm{SL}_2(q)| = q(q^2-1) > q$$

a contradiction.

(Step 42.) [**t not a**] *$T \neq A$.*

Suppose that $T = A$ and let $a \in A^\sharp$. By (Step 40)(c)$\mathrm{C}_{S_a}(A) = T = A$ and so by 5.4.26 shows that S_a is a dihedral or quasidihedral group. By (Step 40)(a)we know that $S_a \in \mathrm{Syl}_2(G)$ and we obtain a contradiction to (Step 23)and (Step 41)

(Step 43.) [**z centralizes hz**] *Let $a, b \in A^\sharp$ with $a \neq b$.*

(a) [**za**] *$H_a \neq H_b$.*

(b) [**zb**] *a centralizes H_a°.*

(c) [**zc**] *Let $\mathrm{C}_G^*(H_a^\circ)$ be the set of elements in G which centralize or invert H_a°. Let $t \in a^G \cap H_a$. Then $t \in \mathrm{C}_G^*(H_a^\circ)$ and $[H_a, t] \leq \mathrm{C}_G(H_a^\circ)$.*

(d) [**zd**] *$\mathrm{C}_G(H_a^\circ) \cap \mathrm{C}_G(H_b^\circ) = 1$.*

(a)By (Step 40)(f)$\mathrm{N}_G(T)$ acts as $\mathrm{Sym}(A^\sharp)$ on $A^\sharp$. Thus there exists $g \in \mathrm{N}_G(T)$ with $x^g = a$ and $z^g = b$. By (Step 39)$H_x \neq H_z$ and so also $H_a \neq H_b$.

(b)Put $c = ab$. By (a)$H_a \neq H_b$ and $H_a \neq H_c$. Hence (Step 26)implies that b and c invert H_a°. As $a = cb$ this shows that a centralizes H_a°.

(c)By (b)a centralizes H_a°. As $t \in a^G$, this proves that t centralizes H_t. So if $H_t = H_a$, then t centralizes H_a. If $H_t \neq H_b$, then (Step 26)shows that t inverts H_a°. In either case $t \in \mathrm{C}_G^*(H_a^\circ)$.

If t inverts H_a°, then (5.3.12)(b)shows that $[t, H_a]$ centralizes H_a°. Thus (c)holds.

(d)Suppose that $X := \mathrm{C}_G(H_a^\circ) \cap \mathrm{C}_G(H_b^\circ) \neq 1$. Then $\langle H_a^\circ, H_b^\circ \rangle \leq \mathrm{C}_G(X) \lneqq G$. By (Step 15)$H_a \in \mathcal{D}$ and so (Step 10)shows that $H_a^\circ = X^\circ$. By symmetry $H_b^\circ = X^\circ$, so $H_a^\circ = H_b^\circ$. Thus $H_a = \mathrm{N}_G(H_a^\circ) = H_b$, contradiction to (a).

Notation: For $a \in A^\sharp$ define

$$\mathcal{T}_a := \begin{cases} (z^G \cap T_a) \smallsetminus T & \text{if } S_a = T_a \\ \{t \in (z^G \cap T_a) \smallsetminus T \mid [T,t] \leq \langle a \rangle\} & \text{if } S_a \neq T_a \end{cases}$$

(Step 44.) [**ngt i**] *Let $a \in A^\sharp$. Then $\mathcal{T}_a \neq \varnothing$.*

Suppose first that $S_a = T_a$. Note that A is contained in two maximal subgroups of G, namely H_x and H_z. Hence (Step 37)shows that there exists $t \in z^G \smallsetminus C_{S_a}(A)$ By (Step 40)(d)we have $C_{S_a}(T)$ and so $t \in S_a \smallsetminus T = T_a \smallsetminus T$ and $t \in \mathcal{T}_a$.

Suppose next that $T_a \lneq S_a$. Then $T_a \lneq N_{S_a}(T_a)$ and we can choose $s \in N_{S_a}(T_a) \smallsetminus T_a$. By (Step 40)(e)$T_a = N_{S_a}(A)$. Hence $A \neq A^s$ and so also $A^s \nleq A$. By (Step 40)(c)$A = \Omega_1(T)$ and so $A^s \neq \Omega_1(T)$. As $A = \Omega_1(A)$ this implies $A \nleq T$ and so also $A^s \neq T$. Thus we can choose $t \in A^s \smallsetminus T$. Since $A^\sharp \in z^G$ we have $t \in z^G$ and since $A \leq T_a$, also $A^s \leq T_a$. Hence $t \in (z^G \cap T_a) \smallsetminus T$. Since $a \in Z(S_a)$ we now that $a^s = a$ and T_a normalizes $\langle a \rangle$. Thus T_a acts on $A/\langle \alpha >$. As $|A/\langle a > | = 2$ it follows hat T centralizes $A/\langle a \rangle$ and so $[T_a, A] \leq \langle a \rangle$. Conjugation with s gives $[T_a, A^s] \leq \langle a \rangle$. Thus also $[t, T] \leq \langle a \rangle$ and so $tin\mathcal{T}_a$ and (Step 44)hold.

Notation $A^\sharp = \{a, b, c\}$ and for $x \in A^\sharp$, $t_x \in \mathcal{T}_x$.

(Step 45.) [**ngt ii**] $T = [T, t_a][T, t_b]$

Recall from (Step 40)that $T_a = N_{S_a}(A)$ and $T = C_{S_a}(A)$. Since $t \in T_a \smallsetminus T$ this implies that t_a acts non-trivially on $A^\sharp$. As $|t_a| = 2$ and t_a fixes a, we conclude that t_a acts as the cycle (b, c) on $A^\sharp$. Also t_b acts as the cycle (a, c). Put

$$k = t_a t_b \quad \text{and} \quad K = \langle k \rangle.$$

Since $(b, c)(a, c) = (a, c, b)$ in $\mathrm{Sym}(A^\sharp)$ we conclude that k acts as (a, c, b) on $A^\sharp$. In particular, $k^3 \in C_G(A) \leq H_a$. By (Step 43)(c)$[H_a, t_a] \leq C_G(H_a^\circ)$ and so $[k^3, t_a]$.

Since both t_a and t_b have order 2, t_a inverts k. Thus $[k^3, t_a] = (k^3)^{-1}(k^3)^{t_a} = k^{-6}$ and so $k^6 = [k^3, t_a]^{-1} \in C_G(H_a^\circ)$. By symmetry, $k^6 \in C_G(H_b^\circ)$. By (Step 43)(d)we have $C_G(H_a^\circ) \cap C_G(H_b^\circ) = 1$. Thus $k^6 = 1$ and so $k^3 \in C_G(A)_{\cdot 2} = A$. Observe that $C_A(k) = 1$ and k centralizes k^3. Hence $k^3 = 1$.

Put $K = \langle k \rangle$. Recall that $\Omega_1(T) = A$ and $C_A(K) = C_A(k) = 1$. Thus $C_T(K)$ contains no element of order 2 and so $C_T(K) = 1$

As $t_a \in T_a = N_{S_a}(T) \leq N_G(T)$ we have $K = \langle t_a t_b \rangle \leq \langle t_a, t_b \rangle \leq N_G(T)$. Since $|K| = 3$ and T is 2-group we have $\gcd(|K|, |T|) = 1$. Thus (5.3.7)(b)shows that $T = C_T(K)[T, K] = [T, K]$. Hence

$$T = [T, K] \leq [T, \langle t_a, t_b \rangle] = [T, t_a][T, t_b] \leq [T, N_G(T)] \leq T$$

It follows that $T = [T, t_a][T, t_b]$ and (Step 45)holds.

(Step 46.) [**t normal in s**] $S_a = T_a$.

Since $T_a \leq S_a$, we have $S_a = T_a$ if and only if $|T_a| = |S_a|$. As $|T_a/T| = 2$ this holds of and only if $|S_a| = 2|T|$ and so if and only if $|S| = 2|T|$.

Suppose now that $S_a \neq T_a$ for some $a \in A^\sharp$. Then $|S| \neq 2|T|$ and so $T_x \neq S_x$ for all $x \in A^\sharp$. As $T_x \neq S_x$, the definition of $\mathcal{T}_x$ implies that $[T, t_x] \leq \langle x \rangle$. Now (Step 45)gives

$$T = [T, t_a][T, t_b] \leq \langle a \rangle \langle b \rangle = A$$

Thus $T = A$, a contradiction to (Step 42).

Notation: For $x \in A^\sharp$ define $C_x := C_T(H_x^\circ)$.

(Step 47.) [**step c**] $T = C_a \times C_b$ *and* T *is abelian.*

By (Step 43)(c)$[H_a,t_a] \le \mathrm{C}_G(H_a^\circ)$. Since $T \le H_a$ this gives $[T,t_a] \le \mathrm{C}_G(H_a^\circ)$ and since $t_a \in T_a = \mathrm{N}_{S_a}(T)$ we have $[T,t_a] \le T$, Thus $[T,t_a] \le C_a$. By (Step 45)$T = [T,t_a][T,t_b]$ and so $T = C_aC_b$. By (Step 43)(d)$\mathrm{C}_G(H_a^\circ) \cap \mathrm{C}_G(H_b^\circ) = 1$, so $C_a \cap C_b = 1$. Since $T \le H_a$, T normalizes H_a° and so both C_a and C_b are normal in T. Thus $[C_a,C_b] \le C_a \cap C_c = 1$. Hence $T = \mathrm{C}_a \times \mathrm{C}_b$. By symmetry, $[C_a,C_c] = 1 = [C_b,C_c]$. As $T = C_aC_b$ this gives $C_c \le \mathrm{Z}(T)$. By symmetry $C_a \le Z(T)$ and $\mathrm{C}_b \le \mathrm{Z}(T)$. Thus $T = \mathrm{C}_a\mathrm{C}_b \le \mathrm{Z}(T)$ and so T is abelian.

(Step 48.) [**sz**] $\mathrm{Z}(S) = \langle z \rangle$.

Let $x_0 \in \mathrm{Z}(S)$. Then $S \le \mathrm{C}_G(x_0)$. Thus $|C_G(x)|_p = |S| = |G|_p$. By (Step 34), there exists $y_1, y_2 \in a^G \cap S$ and $y_0 \in x_0^G$ with $y_0 = y_1y_2$ and $\mathrm{C}_S(y_0) \in \mathrm{Syl}_2(\mathrm{C}_G(y_0)$. In particular,$|\mathrm{C}_S(y_0)| = |\mathrm{C}_G(y_0)| = |\mathrm{C}_G(x_0)| = |S|$ and so $\mathrm{C}_S(y_0) = S$. It follows that $y_0^{y_1} = y_0$. On the otherhand, as $y_0 = y_1y_2$ and $|y_1| = |y_2| = |z| = 2$ we know that y_1 inverts y_0. So $y_0^{-1} = y_0^{y_1} = y_0$ and $y_0^2 = 1$. Hence also $x_0^2 = 1$ and so $x_0 \in \Omega_1\mathrm{Z}(S)$. Thus $\mathrm{Z}(S) = \Omega_1\mathrm{Z}(S)$. By (Step 40)(d)$\Omega_1\mathrm{Z}(S_a) = \langle a \rangle$, so also $\Omega_1\mathrm{Z}(S) = \langle z \rangle$ and (Step 48)is proved.

(Step 49.) [**contradiction**] *The final contradiction.*

Let $d \in C_a$. Since T is abelian and t_c has order 2, dd^{t_c} is centralized by $T\langle t_c \rangle$. As $|T_c/T| = 2$ we have $T\langle t_c \rangle = T_c = S_c$. Thus $dd^{t_c} \in \mathrm{Z}(S_c) = \langle c \rangle$ and so $d^{t_c} = d^{-1}(dd^{t_c}) \in C_a\langle c \rangle$. Note that $a^{t_c} = b$. So $C_b = C_a^{t_c} \le C_a\langle c \rangle$ and then $T = C_aC_b \le \mathrm{C}_a\langle c \rangle$. Thus $C_c \le C_a\langle c \rangle$ and since $\langle c \rangle \le C_c$, the modular law gives

$$C_c = (C_c \cap C_a)\langle c \rangle = \langle c \rangle$$

By symmetry, $C_a = \langle a \rangle$ and so $T = C_aC_c = \langle a \rangle\langle c \rangle = A$, a contradiction to (Step 42). □

Chapter 6

Local Group Theory

6.1 Quadratic Action

Definition 6.1.1. [**def:quadratic**] *Let G be a group acting on a group H and let $A \subseteq G$.*

(a) [**a**] *A acts nilpotently on H if there exists $n \in \mathrm{N}$ with $[H, G; n] = 0$. The smallest such n is called the nilpotency height of the of action G in A.*

(b) [**b**] *A acts quadratically on H if $[H, A], A] = 1$.*

Note that A acts quadratically on V if and only if A acts nilpotently of height at most 2 on H.

Lemma 6.1.2. [**quadratic and abelian**] *Let G be a group acting on a group V and let $a, b \in G$. Suppose that*

(i) [**i**] *V is abelian.*

(ii) [**ii**] *$[b, b^a] = 1$ and $[b, b^{a^2}] = 1$.*

(iii) [**iii**] *a acts quadratically on V.*

Then $[b, a]$ acts quadratically on $V^{\cdot 2}$.

Proof. We will use additive notation for V. In particular, the hypothesis $V = V^{\cdot 2}$ then translates to $V = 2V$.

Without loss we assume that $\mathrm{C}_G(V) = 1$ and that $G \leq \mathrm{Aut}(V)$. In particular, G is a subset of the ring $\mathrm{End}(V)$. If $\alpha \in \mathrm{End}(V)$ and $v \in V$ we denote that image of v under α by $v\alpha$. If $n \in \mathbb{Z}$ then the function $f_n : V \to V, v \to nv$ is in $\mathrm{End}(V)$. Abusing notation we just write n for f_n, even though the function $n \to f_n$ does not need to be injective. Then $nv = vn$ and $\alpha n = \alpha n$ for all $n \in \mathrm{Z}, v \in V$ and $\alpha \in \mathrm{End}(V)$.

Since we view G is subsets of G we have

$$[v, g] = -v + vg = vg - v = v(g-1) \quad \text{and so} \quad [[v, g], g] = v(g-1)^2$$

for $v \in V$ and $g \in G$. As a acts quadratically on V we get $v(a-1)^2 = 0$ and so $(a-1)^2 = 0$ in $\mathrm{End}(V)$. Define $d := a - 1$. Then $d^2 = 0$ and $a = 1 + d$. In particular, $(1+d)(1-d) = 1 - d^2 = 1$ and so $a^{-1} = 1 - d$. Let $n \in \mathbb{N}$. Note that $d^i = 0$ for all $2 \leq i \leq n$ and so

$$a^{\pm n} = (1 \pm d)^n = 1 \pm nd + \frac{n}{2}d^2 \pm \frac{n}{3}d^3 \ldots = 1 \pm na$$

It follows that

(∗) $$[\mathbf{1}]b^{a^n} = a^{-n}ba^n = (1-nd)b(1+nd) = (1-nd)(b+nbd) = b + nbd - ndb - n^2dbd$$

Let $n \in \{1,2\}$. By Hypothesis we have $[b, b^{a^n}] = 1$ and so $bb^{a^n} = b^{a^n}b$. We have and so

$$bb^{a^n} = b^2 + nb^2d - nbdb - n^2bdbd \quad \text{and} \quad b^{a^n}b = b^2 + nbdb - ndb^2 - n^2dbdb$$

Thus

$$\begin{aligned} nb^2d - nbdb - n^2bdbd &= nbdb - ndb^2 - n^2dbdb \\ nb^2d - nbdb - nbdb + ndb^2 &= n^2bdbd - n^2dbdb \\ n(b^2d + db^2 - 2bdb) &= n^2(bdbd - dbdb) \end{aligned}$$

Let $\alpha, \beta \in \mathrm{End}(V)$. We will write $\alpha \equiv \beta$ if the images of α and β in $2V$ are the same. Suppose that $n\alpha = n\beta$. Then also $2\alpha = 2\beta$. Thus for all $v \in V$, $(2v)\alpha = v(2\alpha) = v(2\beta) = (2v)\beta$ and so $\alpha \equiv \beta$. It follows that

$$b^2d + db^2 - 2bdb \equiv n(bdbd - dbdb)$$

for $n = 1, 2$. Thus

$$\begin{aligned} b^2d + db^2 - 2bdb &\equiv bdbd - dbdb \\ b^2d + db^2 - 2bdb &\equiv 2(bdbd - dbdb) \end{aligned}$$

Subtracting the first from the second equation gives

$$0 \equiv bdbd - dbdb$$

In particular, $bdbd \equiv dbdb$. Also substitution in the first equation shows that

$$b^2d + db^2 - 2bdb \equiv 0$$

Multiplying with d from the right and using that $d^2 = 0$ gives $db^2d - 2bdbd \equiv 0$. Since $bdbd \equiv dbdb$ we get

(∗∗) $$[\mathbf{2}]db^2d - bdbd - dbdb \equiv 0$$

Define $f := [b,a] - 1 = b^{-1}b^a - 1$. To show that $[b,a]$ acts quadratically on $2V$ we need to prove that $f^2 \equiv 0$. Put $e := bf = b^a - b$. Since $[b, b^a] = 1$ we have $bf = fb$ and so $e^2 = (fb)^2 = b^2f^2$ and $f^2 = b^{-2}e^2$. So it remains to shows that $e^2 \equiv 0$. By (∗)for $n = 1$ we have $b^a = b + bd - db - dbd$ and so $e = b^a - b = bd - db - dbd$. Thus (using that $d^2 = 0$),

$$\begin{aligned} e^2 &= (bd - db - dbd)(bd - db - dbd) \\ &= bdbd - 0 + 0 \\ &\quad - db^2d + dbdb + \cancel{dbdbd} \\ &\quad - \cancel{dbdbd} + 0 + 0 \\ &= bdbd + dbdb - db^2d \\ &\overset{(**)}{\equiv} 0 \end{aligned}$$

□

Corollary 6.1.3. [**quadratic on abelian**] *Let G be a group acting on a group V and let p be a odd prime. Let $B \le G$ and $A \subseteq \mathrm{N}_G(B)$. Suppose that*

(i) [**i**] *V is an abelian p-group.*

(ii) [**ii**] *B is abelian p'-group.*

(iii) [**iii**] *A acts quadratically in V.*

Then $[B, A]$ centralizes V.

Proof. Let $b \in B$ and $a \in A$. It suffices to show $[b, a]$ centralizes V. We will first show that the Hypothesis of 6.1.2 is fulfilled. Since p is an odd prime we have $\gcd(p, 2) = 1$. As V is a p-group we conclude that the function $V \to V, v \to v^2$ is a bijection. Thus $V = V^{\cdot 2}$. Since $A \subseteq \mathrm{N}_G(B)$ we have $b^a \in B$ and $b^{a^2} \in B$. As B is abelian this gives $[b, b^a] = 1$ and $[b, b^{a^2}] = 1$. By Hypothesis A acts quadratically in V, so also a acts quadratically on V. Thus the Hypothesis of 6.1.2 is fulfilled and we conclude that $[b, a]$ acts quadratically on V. Put $c = [b, a]$ and $C = \langle c \rangle$. Since $a \in \mathrm{N}_G(B)$ we have $C \le B$ and since B is a p'-group also C is p'-group. As V is a p-group we conclude fro (5.3.7)(b)that $V = [V, C]\mathrm{C}_V(C)$. Since c acts quadratically on V we know that $[[V, c], c] = 1$ and so $[V, c] \le \mathrm{C}_V(c)$. Observe that $\mathrm{C}_V(c) = \mathrm{C}_V(C)$ and by (2.1.2)(e)$[V, c] = [V, C]$. Thus $[V, C] \le \mathrm{C}_V(C)$ and so $V = [V, C]\mathrm{C}_V(C) \le \mathrm{C}_V(C) = \mathrm{C}_V(c)$. So c indeed centralizes V. □

Definition 6.1.4. [**def:stably**] *Let G be a group acting on a group H.*

(a) [**a**] *We say that G acts stably on H if $\langle A^G \rangle$ acts nilpotently on H, whenever $A \subseteq G$ acts quadratically on H.*

(b) [**b**] *We say that G acts super stably on H if A acts trivially on H, whenever $A \subseteq G$ acts quadratically on H.*

Example 6.1.5. [**ex:quadratic on abelian**]

(a) [**a**] Let $\mathbb{F}$ be a field and V a 2-dimensional $\mathbb{F}$-space. Then $\mathrm{SL}_{\mathbb{F}}(V)$ does not act stably on V.

(b) [**b**] 6.1.3 is false if $p = 2$. In particular, if $V \ne V^{\cdot 2}$, the element $[b, a]$ in 6.1.3 does not have to act quadratically on V.

(c) [**c**] 6.1.3 can be false of B is a non-abelian p' group.

(a): Put $H := \mathrm{SL}\mathbb{F}(V)$ and $X := \{\left[\begin{smallmatrix} 1 & 0 \\ a & 1 \end{smallmatrix}\right] \mid a \in \mathbb{F}\}$. Let $a, b \in \mathbb{F}$. In $\mathrm{End}_{\mathbb{F}}(V)$

$$\begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix} - 1 = \begin{bmatrix} 0 & 0 \\ a & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 0 \\ a & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 \\ b & 0 \end{bmatrix} = 0$$

Thus X acts quadratically on V. Note that $V = [V, H]$, so H does not act nilpotently in V. By Exercise 1 in Homework 3 we have $H = \langle X^H \rangle$. We proved that X acts quadratically on V and $\langle X^H \rangle$ does not act nilpotently on V. So H does not act stably on V.

(b)and (c): Put $p := |\mathbb{F}|$ and suppose that $p = 2$ or 3. Put $a = \left[\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right]$ and $Z := \mathrm{ZSL}_{\mathbb{F}}(V)$. Then a acts quadratically in V. By Exercise 1 on Homework 3 H has a normal p-complement B, B/Z is elementary abelian of order $p + 1$ and $[B/Z, a] \ne 1$

Suppose $p = 2$. Then $Z = 1$ and B is cyclic of order 3. So B is an a-invariant $2'$-group. Since $[B, a] \neq 1$ we see that 6.1.3 is false for $p = 2$.

Suppose $p = 3$. Then $|Z| = 2$ and $B/Z \cong \mathrm{C}_2 \times \mathrm{C}_2$. Since all involution in H are contained in Z (see Homework 2) we conclude that $B \cong \mathrm{Q}_8$. In particular, B is a non-abelian a-invariant p' with $[B, a] \neq 1$. So 6.1.3 can be false of B is a non-abelian.

Definition 6.1.6. [**def:p-stable**] *Let G be a finite group and p a prime.*

(a) [**a**] *We say that G has characteristic p if $\mathrm{C}_G(\mathrm{O}_p(G)) \leq \mathrm{O}_p(G)$.*

(b) [**b**] *We say that G is p-stable if G has characteristic p and G acts stably on each of the normal p-subgroup of G.*

Definition 6.1.7. [**def:dg**] *Let G be a group. Then $\mathcal{D}(G)$ is the set of all abelian normal subgroup A of G such that G acts super-stably on A; and $\mathrm{D}(G) := \langle \mathcal{D}(G) \rangle$.*

Lemma 6.1.8. [**basic cg**] *Let G be a group and $H \leq G$.*

(a) [**a**] *Let $A \in \mathcal{D}(G)$ with $A \leq H$. Then $A \in \mathcal{D}(H)$.*

(b) [**b**] *Suppose that $\mathrm{D}(G) \subseteq H$. Then $\mathcal{D}(G) \leq \mathcal{D}(H)$ and $\mathrm{D}(G) \leq \mathrm{D}(H)$.*

Proof. (a)follows immediately from the defintion of $\mathcal{D}(H)$, and (b)follows from (a). □

Lemma 6.1.9. [**closure of cg**] *Let G be a group.*

(a) [**a**] *Let $D \leq \mathrm{Z}(G)$. Then $D \in \mathcal{D}(G)$. In particular, $\mathrm{Z}(G) \leq \mathrm{D}(G)$.*

(b) [**b**] *Let A and H be normal subgroups of G. Suppose that A is abelian. Then A acts quadratically on H.*

(c) [**c**] *Let A and H be normal subgroups of G. If A is abelian and G acts super stably on V, then A centralizes H.*

(d) [**d**] *Let $\mathcal{E} \subseteq \mathcal{D}(G)$ and put $H = \langle \mathcal{E} \rangle$. Then H is abelian and $H \in \mathcal{D}(G)$.*

(e) [**e**] $\mathrm{D}(G)$ *abelian and* $\mathrm{D}(G)$ *is the largest element of* $\mathcal{D}(G)$.

Proof. (a)Since $D \leq \mathrm{Z}(G)$ we know that $A \trianglelefteq G$ and A is abelian. Also $[D, A] = 1$ for all $A \leq G$ and so G acts super-stably in G. Thus $A \in \mathcal{D}(G)$.

(b)Since H normalizes A we have $[H, A] = 1$. As A is abelian, this gives $[[H, A], A] = 1$

(b)Let $E, F \in \mathcal{E}$. Then E, F are normal abelian subgroup of G and (b)shows that F acts quadratically on E. As G acts super stably on F this implies that $[E, F] = 1$. As $H = \langle \mathcal{E} \rangle$ this implies that H is abelian. Let $A \leq G$ be acting quadratically on H. Then A also acts quadratically on each $E \in \mathcal{E}$. It follows that $[E, A] = 1$ and so $[H, A] = 1$. Thus G acts super stably on H, so $H \in \mathcal{D}$.

(e)Note that (d)applied with $\mathcal{E} = \mathcal{D}(G)$ shows that $\mathrm{D}(G)$ is abelian and $\mathrm{D}(G) \in \mathcal{D}(G)$. As $D \leq \mathrm{D}(G)$ for all $D \in \mathcal{D}(G)$, this means that $\mathrm{D}(G)$ is the largest elements of $\mathcal{D}(G)$. Thus (e)holds. □

Lemma 6.1.10. [**basic p-stable**] *Let p be a prime, G a finite p-stable group and $A \leq G$.*

(a) [**a**] *Suppose that A acts quadratically on $\mathrm{O}_p(G)$. Then $A \leq \mathrm{O}_p(G)$.*

(b) [**b**] *Suppose that A is abelian and $\mathrm{O}_p(G)$ normalizes A. Then $A \leq \mathrm{O}_p(G)$.*

Proof. (a)Put $H = \langle A^G \rangle$. Since G is p-stable, G acts stably on $O_p(G)$. As A acts quadratically on $O_p(G)$ this shows that H acts nilpotently on $O_p(G)$. Hence 5.3.10 implies that $H/C_H(O_p(G))$ is a p-group. Since G has characteristic p, $C_G(O_p(G)) \le O_p(G)$. Thus $C_H(O_p(G))$ is a p-group. So H is a p-group. As $H \trianglelefteq G$ this implies $H \le O_p(G)$, so also $A \le O_p(G)$.

(b)Since $O_p(G)$ normalizes A we have $[O_p(G), A] \le A$. Since A is abelian this implies $[[O_p(G), A], A] \le [A,, A] = 1$. Thus A acts quadratically on $O_p(G)$ and (a)implies that $A \le O_p(G)$. □

Lemma 6.1.11. [**sylow and normal**] *Let p be a prime, G a finite group, $S \in \mathrm{Syl}_p(G)$ and $H \trianglelefteq G$.*

(a) [**a**] *$S \cap H \in \mathrm{Syl}_p(G)$ and $SH/H \in \mathrm{Syl}_p(G)$.*

(b) [**b**] *The following statements are equivalent:*

  (1) [**a**] $O^p(G) \le H$

  (2) [**b**] *G/H is a p-group.*

  (3) [**c**] $G = SH$.

Proof. (a)We compute

$$|SH/H||S \cap H| = |S/S \cap H||S \cap H| = |S| = |G|_p = (|G/H||H|)_p = |G/H|_p|H|_p$$

As $|SH/H|_p \le |G/H|_p$ and $|S \cap H| \le |H|_p$ this implies $|SH/H|_p = |G/H|_p$ and $|S \cap H| = |H|_p$. Hence $SH/H \in \mathrm{Syl}_p(G/H)$ and $S \cap H \in \mathrm{Syl}_p(H)$.

(b)By (5.3.9)(a)the first two statement in (b)are equivalent. By (a)$SH/H \in \mathrm{Syl}_p(G)$. Hence G/H is a p-group if and only if $G/H = SH/H$ and so if and only if $G = SH$. □

Lemma 6.1.12. [**closed under normal closure**] *Let G be a finite group, p a prime, $S \in \mathrm{Syl}_p(G)$ and $D \in \mathcal{D}(S)$. Put $B := \langle D^G \rangle$. Suppose that $D \le O_p(G)$ and G acts stably on B. Then $B \in \mathcal{D}(G)$.*

Proof. Let $g \in G$. Since $D \le O_p(G)$, (6.1.8)(a)implies that $D \in \mathcal{D}(O_p(G))$. Hence also $D^g \in \mathcal{D}(O_p(G))$. Now (6.1.9)(e)(applied with $(D^G, O_p(G), B)$ in place of $(\mathcal{E}, G, H)$) shows that B is abelian. Let $A \le G$ such that A acts quadratically on B. Put $H = \langle A^G \rangle$. By Hypothesis G acts stably on B, and so H acts nilpotently on B. Thus $[B, H; n] = 1$ for some $n \in \mathbb{Z}^+$ and (5.3.10)(c)shows that $H/C_H(B)$ is a p-group. By 6.1.11 $S^g \cap H$ and $S \cap H$ are Sylow p-subgroups of H, and $H = (H \cap S)C_H(B)$. Note that H acts transitively on $\mathrm{Syl}_p(H)$ and $H = (S \cap H)C_H(B) = N_H(S \cap H)C_H(B)$. Thus $C_H(B)$ acts transitively on $\mathrm{Syl}_p(H)$ and we can choose $h \in C_H(B)$ with $(S^g \cap H)^h = S \cap H$. Since $D \in \mathcal{D}(S)$ we have $D^{gh} \in \mathcal{D}(S^{gh})$. As $D^g \le B$ and $h \in C_G(B)$ we know that $D^{gh} = D^g$, so $D^g \le \mathcal{D}(S^{gh})$. Thus $D^g \trianglelefteq S^{gh}$ and S^{gh} acts super stably on D^g. Note that $H = (S^{gh} \cap H)C_G(B)$ and so we can choose $X \le S^{gh}$ with $AC_G(B) = XC_G(B)$. Since $D^g \le B$ we get $[D^g, X, X] \le [B, X, X] = [B, A, A] = 1$. As $X \le S^{gh}$ and S^{gh} acts super-stably on G this implies $[D^g, X] = 1$. Hence also $[D^g, A]$. This holds holds for all $g \in G$ and so $[B, A] = 1$. Hence $B \in \mathcal{D}(S)$. □

Theorem 6.1.13. [**zj**] *Let p be prime, G a finite p-stable group and $S \in \mathrm{Syl}_p(G)$. Then $D(S) = D(G) \trianglelefteq G$.*

Proof. By (6.1.9)(e)both $D(S)$ and $D(G)$ are abelian. Observe that $O_p(G) \le S$ and $D(S) \trianglelefteq S$. Thus $O_p(G)$ normalizes $D(S)$ and $D(G)$. Thus (6.1.10)(b)implies that $D(S) \le O_p(G)$ and $D(G) \le O_p(G)$. Put $B = \langle D(S)^G \rangle$. Then B is a normal p-subgroup of G and since G is p-stable, G acts stably on B. Now 6.1.12 shows that $B \in \mathcal{D}(G)$. In particular, $B \le D(G)$ and so also $D(S) \le D(G)$. As $D(G) \le O_p(G) \le S$, (6.1.8)(b)implies that $D(G) \le D(S)$. Thus $D(S) = D(G)$.

□

Example 6.1.14. [**ex:not p-stable**] Let p be a prime, $k \in \mathbb{Z}^+$, $q = p^k$, $|\mathbb{F}|$ a finite field of order q, V a 2-dimensional $\mathbb{F}$-space, $H = \mathrm{SL}_{\mathbb{F}}(V)$, $T \in \mathrm{Syl}_p(H)$ and let G be an internal semidirect product of V by H. Then

(a) [**a**] G is of characteristic p and G is not p-stable.

(b) [**b**] $S \in \mathrm{Syl}_p(G)$ and $\mathrm{D}(S) = \mathrm{Z}(S) \ntrianglelefteq G$.

(a)Observe that $\mathrm{C}_G(V) = V$ and V is a normal p-subgroup of G. So G has characteristic p. By (6.1.5)(a)H does not acts stably on G. Since $G = HV$ and V centralizes V this implies that G does not act quadratically on V. So G is not p-stable.

(b)Note that V is a p-group. Also $T \in \mathrm{Syl}_p(G)$, $G = VH$ and $S = VT$. So $S \in \mathrm{Syl}_p(G)$. By (6.1.5)(a)T acts quadratically on V. As V and T are abelian this implies that $S' \le \mathrm{Z}(S)$ and so S has class 2. If $D \in \mathcal{D}(S)$ we conclude that $[D, S, S] = 1$ and then $[D, S] = 1$. Thus $D \le \mathrm{Z}(S)$. As $\mathrm{Z}(S) \le \mathrm{D}(S)$. Thus $\mathrm{D}(S) = \mathrm{Z}(S)$. Since $\mathrm{C}_G(V) = V$ we have $\mathrm{Z}(S) = V \cap \mathrm{Z}(S) = \mathrm{C}_V(T)$. Hence $\mathrm{Z}(S)$ is a 1-dimensional $\mathbb{F}$-subspace of V. In particular, $\mathrm{Z}(S)$ is not invariant under H, so $\mathrm{Z}(S) \ntrianglelefteq G$.

6.2 Weakly Closed Subgroups

Definition 6.2.1. [**def sqcap**] *Let A be a set and $\mathcal{B}$ a set of sets. Then $A \sqcap \mathcal{B} := \{B \in \mathcal{B} \mid B \subseteq A\}$*

Note that $A \sqcap \mathcal{B} = \mathcal{P}(A) \cap \mathcal{B}$ where $\mathcal{P}(A)$ is the set of subsets of A.

Definition 6.2.2. [**def:weakly closed**] *Let G be a group, $H \le G$ and $Z \le H$. Then Z is called weakly closed in H with respect to G if $H \sqcap Z^G = \{Z\}$ (i.e. $Z^g = Z$ for all $g \in G$ with $Z^g \le H$).*

Lemma 6.2.3. [**weakly closed and fusion**] *Let G be a finite group, $S \in \mathrm{Syl}_p(G)$, $Z \le \mathrm{Z}(S)$ and $A \subseteq S$. Suppose that Z is weakly closed in S with respect to G. Then $S \sqcap A^G = S \sqcap A^{\mathrm{N}_G(Z)}$*

Proof. Clearly $S \sqcap A^{\mathrm{N}_G(Z)} \subseteq S \sqcap A^G$. Let $g \in G$ with $A^g \in S$. It remains to find $h \in \mathrm{N}_G(Z)$ with $A^g = A^h$. Since $A \in S$ we have $A^g \in S \cap S^g$. As $Z \le \mathrm{Z}(S)$ this implies that both Z and Z^g are contained in $\mathrm{C}_G(A^g)$. Sylow's Theorem applied to $\mathrm{C}_G(A^g)$ shows that there exists $c \in \mathrm{C}_G(A^g)$ such that $\langle Z^c, Z^g\rangle$ is a p-group. Now Sylow's Theorem applied to H shows that there exist $l \in G$ with $\langle Z^c, Z^g\rangle^l \le S$. Thus both Z^{cl} and Z^{gl} are contained in S. As Z is weakly closed in S with respect to G this implies $Z^{cl} = Z$ and $Z^{gl} = Z$. Put $h := gc^{-1} = (gl)(cl)^{-1}$. Then $Z^h = Z^{(gl)(cl)^{-1}} = Z$ and so $h \in \mathrm{N}_G(Z)$. As $c^{-1} \in \mathrm{C}_G(A^g)$ we have $A^h = (A^g)^{c^{-1}} = A^g$ and the lemma is proved. □

Lemma 6.2.4. [**burnside**] *Let G be a finite group, $S \in \mathrm{Syl}_p(G)$ and A and B normal subsets of S. If $B \in A^G$, then $B \in A^{\mathrm{N}_G(S)}$.*

Proof. Let $g \in G$ with $B = A^g$. Since $S \le \mathrm{N}_G(A)$ we get $S^g \le \mathrm{N}_G(B)$. Also $S \le \mathrm{N}_G(B)$. It follows that both S and S^g are Sylow p-subgroups of G, so $S = S^{gh}$ for some $h \in \mathrm{N}_G(B)$. Then $gh \in \mathrm{N}_G(S)$ and $A^{gh} = B^h = B$. □

Lemma 6.2.5. [**char weakly closed**] *Let G be a finite group, p a prime, $S \in \mathrm{Syl}_p(G)$ and $Z \le S$ with $Z \trianglelefteq \mathrm{N}_G(S)$. The the following two statements are equivalent:*

(a) [**a**] *Z is weakly closed in S with respect to G.*

(b) [**b**] *If $R \in \mathrm{Syl}_p(G)$ with $Z \le R$,then $Z \trianglelefteq R$.*

Proof. (a)$\Longrightarrow$ (b): Suppose that $S \sqcap Z^G = \{Z\}$ and let $R \in \mathrm{Syl}_p(G)$ with $Z \le R$. By Sylow's Theorem $S = R^g$ for some $g \in G$. Then $Z^g \le R^g = S$ and so $Z^g = Z$. Hence also $Z^{g^{-1}} = Z$. From $Z \trianglelefteq S$ implies $Z^{g^{-1}} \trianglelefteq S^{g^{-1}}$, so $Z \trianglelefteq R$.

(b)$\Longrightarrow$ (a): Suppose (b)holds and let $g \in G$ with $Z^g \le S$. Observe that (b)holds also holds for Z^g in place of Z, thus $Z^g \trianglelefteq S$. Hence both Z and Z^g are normal in S. 6.2.4 now implies that $Z^g = Z^h$ for some $h \in$
$N_G(S)$. As $Z \trianglelefteq \mathrm{N}_G(S)$, this gibes $Z^g = Z$ and so Z is weakly closed in S with respect to G. □

Lemma 6.2.6. [**weakly complement**] *Let G be a group, $N \trianglelefteq G$, K a complement to N in K and $Z \subseteq K$. Then*

$$K \sqcap Z^G = Z^K$$

In particular, Z is a weakly closed subgroup of K with respect to G if and only if $Z \trianglelefteq K$.

Proof. Clearly $Z^K \subseteq K \sqcap Z^G$. For the reverse inclusion let $g \in G$ with $Z^g \subseteq K$. Note that $G = NK$ and so there exists $n \in N$ and $g \in K$ with $g = nk$. Since $Z^g \subseteq K$ we have

$$Z^n = Z^{gk^{-1}} \le K^{k^{-1}} = K$$

Let $z \in Z$. $z^1 z^n \in K$. As $N \trianglelefteq G$ we have $[z,n] \in N$. Also $[z,n] = z^{-1}z^n$ we conclude that $[z,n] \in N \cap K = 1$. Thus $[Z,n] = 1$ and $Z = Z^g = Z^{nk} = Z^k \in Z^K$. □

6.3 The Transfer Homomorphism

Lemma 6.3.1. [**transfer hom**] *Let G be finite group and $H \le G$. Let $\mathcal{B}$ be the set of transversals to H in G. For $R, S \in \mathcal{B}$ define*

$$(R|S) = \prod_{\substack{(r,s)\in R\times S\\ Hr=Hs}} H'rs^{-1} \in H/H'$$

and

$$\tau_{G\to H}: \quad G \to H/H', \quad g \to (Rg|R)$$

Then $\tau_{G\to H}$ does not depended on the choice of $R \in \mathcal{B}$ and is a homomorphism of groups.

Proof. Let $R, S, T \in \mathcal{B}$. Recall from 1.3.9 that

(∗) $$[\mathbf{2}](R|S)(S|T) = (R|T)$$

and

(∗∗) $$[\mathbf{3}](R|S)^{-1} = (S|R)$$

Let $g \in G$. Observe that the function $R \to Rg, r \to rg$ is a bijection. Thus

$$\begin{aligned}(Rg|Sg) &= \prod_{\substack{(\tilde{r},\tilde{s})\in Rg\times Sg\\ H\tilde{r}=H\tilde{s}}} H'\widetilde{r}\widetilde{s}^{-1}\\ &= \prod_{\substack{(r,)\in R\times S\\ Hrg=Hsg}} H'(rg)(sg)^{-1}\\ &= \prod_{\substack{(r,)\in R\times S\\ Hr=Hs}} H'rs^{-1}\\ &=(R|S)\end{aligned}$$

We proved

$$(***) \qquad [\mathbf{4}](Rg|Sg) = (R|S)$$

We compute

$$\begin{aligned}(Sg|S) &= (Sg|S)(S|R)(R|S) && -(**)\\ &= (Sg|R)(R|S) && -(*)\\ &= (R|S)(Sg|R) && -H/H' \text{ is abelian}\\ &= (Rg|Sg)(Sg|R) && -(***)\\ &= (Rg|R) && -(*)\end{aligned}$$

We proved that $(Sg|S) = (Rg|R)$ and so $\tau_{G\to H}$ is independent of the choice of $R \in \mathcal{B}$. To show that $\tau_{G\to H}$ is a homomorphism, let $g, h \in G$. Then

$$(Rgh|R) \overset{(*)}{=} (Rgh|Rh)(Rh|R) = ((Rg)h|Rh)(Rh|R) \overset{(***)}{=} (Rg|R)(Rh|R)$$

and so $\tau_{G\to H}$ is a homomorphism. □

Lemma 6.3.2. [**cyclic transitive**] *Let G be group acting on a finite set A, let $x \in G$ and let B be an orbit of $\langle x\rangle$ on Ω. Let $n = |B|$ and $b \in B$. Then $B = \{bx^m \mid 0 \le m < n\}$ and $x^n \in G_b$.*

Proof. Put $X := \langle x\rangle$. Recall that $X_b = \{y \in X \mid by = b\}$ and by 1.1.9 the function

$$\phi: \quad X/X_b \to B, \quad X_b y \mapsto by$$

is a well-defined X-isomorphism. In particular, $|X/X_b| = |B| = n$ and since X is cyclic we get $X_b = \langle x^n\rangle$. Thus $x^n \in X_b \le G_b$ and $X/X_b = \{X_n x^j \mid 0 \le j < n\}$. Applying ϕ shows that $B = \{bx^j \mid 0 \le j < n\}$. □

Lemma 6.3.3. [**compute tau**] *Let G be a finite group, $H \le G$ and $x \in G$. Let $\Omega_1, \dots, \Omega_k$ be the distinct orbits of $\langle x\rangle$ on G/H. For $1 \le i \le k$ put $n_i := |\Omega_i|$ and let $g_i \in G$ with $Hg_i \in \Omega_i$. Then*

(a) [**a**] $|G/H| = \sum_{i=1}^k n_i$.

(b) [**b**] $x^{n_i g_i^{-1}} \in H$ *for all* $1 \le i \le k$.

(c) [**c**] $x\tau_{G\to H} = \prod_{i=1}^k H' x^{n_i g_i^{-1}}$.

Proof. By 6.3.2 we conclude that

$$\Omega_i = \{Hg_ix^j \mid 0 \mid j < n_i\}$$

and

$$x^{n_i} \in G_{Hg_i} = G_H^{g_i} = H^{g_i}$$

In particular,

$$Hg_i = Hg_ix^{n_i}$$

and

$$x^{n_ig_i^{-1}} \in H.$$

Thus (b)holds.

Also

$$G/H = \dot{\bigcup_{i=1}^{k}} \Omega_i = \{Hg_ix^j \mid 1 \le i \le k, 0 \le j < n_i\}$$

Thus $|G/H| = \sum_{i=1}^{k} n_i$ and

$$R := \{g_ix^j \mid 1 \le i \le k, 0 \le j < n_i\}$$

is a transversal to H in G. We will use this transversal to compute $x\tau_{G\to H}$.

Observe that

$$Rx = \{g_ix^j \mid 1 \le i \le k, 1 \le j \le n_i\}$$

so if $(r,s) \in (Rx,R)$ with $Hr = Hs$, then either $r = s = g_ix^j$ for some $1 \le i \le k$ and $1 \le j < n$ or $r = g_ix_i^{n_i}$ and $s = g_ix_i^0 = g_i$ for some $1 \le k \le n$. In the first case $rs^{-1} = 1$ and in the second case $rs^{-1} = g_ix^{n_i}g_i^{-1} = x^{n_ig_i^{-1}}$. Hence

$$\begin{aligned} x\tau_{G\to H} &= (Rg|R) \\ &= \prod_{\substack{(r,s)\in Rg\times R \\ Hr=Hs}} H'rs^{-1} \\ &= \prod_{i=1}^{k} H'x^{n_ig_i^{-1}} \end{aligned}$$

□

Definition 6.3.4. [**def:abc**] *Let G be a group and A, B and C subgroups of G define*

$$[A,_C B] = \langle [a,b] \mid a \in A, b \in B, [a,b] \in C\rangle$$

Lemma 6.3.5. [**basic abc**] *Let G be a group and A, B, C subgroups of G.*

(a) [**a**] $[A,_C, B] \le C \cap [A,B]$.

(b) [**b**] *If $A \le B \cap C$, then $A' \le [A,_C, B]$.*

(c) [**c**] *If $B \le N_G(A)$, then $[A,_A B] = [A,B]$.*

(d) [**d**] $[A,_A B] = \langle a^{-1}a^b \mid a \in A, b \in B, a^b \in A\rangle$.

Proof. (a)and (b)are obvious.

(c)If $B \le N_G(A)$, then $[A,B] \le A$ and so (c)holds.
(d)Let $a \in A$ and $b \in B$. Then $[a,b] = a^{-1}a^b$ and so $[a,b] \in A$ if and only if $a^b \in A$. Thus (d)holds. □

Lemma 6.3.6. [**transfer modulo h***] *Let G be a finite group and $H \le G$. Put $H^* := [H,_H G]$ and let $x \in H$. Then $H^*(x\tau_{G\to H}) = H^* x^{|G/H|}$.*

Proof. Retain the notation from 6.3.3. Since $x \in H$ also $x^{n_i} \in H$. Also by (6.3.3)(b)$x^{n_i g_i^{-1}} \in H$. Thus

$$(x^{n_i})^{-1} x^{n_i g_i^{-1}} \in H^*$$

and so

$$H^* x^{n_i g_i^{-1}} = H^* x^{n_i}$$

It follows that

$$H^*(x\tau_{G\to H}) = H^* \prod_{i=1}^{k} H' x^{n_i g_i^{-1}} = \prod_{i=1}^{k} H^* x^{n_i g_i^{-1}} = \prod_{i=1}^{k} H^* x^{n_i} = H^* x^{\sum_{i=1}^{k} n_i} = H^* x^{|G/H|}$$

□

Theorem 6.3.7 (Grün)**.** [**gruen**] *Let G be a finite group, π a set of primes and H a Hall π-subgroup of G. Put $H^* = [H,_H, G]$. Then*

$$H^* = H \cap G' = H \cap O^\pi(G)G' \quad \textit{and} \quad H/H^* \cong G/O^\pi(G)G'.$$

Proof. Put $K = O^\pi(G)G'$. Note that G' is the smallest normal subgroup of G such that G/G' is abelian, and $O^\pi(G)$ is the smallest normal subgroup of G such that $G/O^\pi(G)$ is π-group. Thus K is the smallest normal subgroup of G such that G/K is an abelian π-group. As G/G' is abelian, it is the direct product of a π and π' group, namely $G/G' = (G/G')_\pi \times (G/G')_{\pi'}$. It follows that $KG'/G' = (G/G')_\pi$. In particular, $K/K \cap G'$ is a π'-group. As H is a π-group, this implies

(∗) $$[\mathbf{1}] H \cap K = H \cap G'.$$

As H is a Hall π-subgroup of G, and G/K is a π-group we have $G = HK$ and so

(∗∗) $$[\mathbf{2}] H/H \cap K \cong HK/K = G/K$$

So it remains to show that $H^* = H \cap G'$.

Put $\tau := \tau_{G\to H}$. Since $\mathrm{Im}\,\tau \le H/H'$, $\mathrm{Im}\,\tau$ is an abelian π'-group. Hence $G/\mathrm{Ker}\tau$ is an abelian π'-group and so

(∗ ∗ ∗) $$[\mathbf{3}] K \le \mathrm{Ker}\tau$$

Let $x \in H$. By 6.3.6

$$H^*(x\tau) = H^* x^{|G/H|}$$

Since H is a Hall π-subgroup of G, we know that $\gcd(|H|, |G/H|) = 1$. Hence also $\gcd(|x|, |G/H|)$ and so $\langle x^{|G/H|} \rangle = \langle x \rangle$. If $x \in \mathrm{Ker}\tau$ we conclude that $x \in H^*$ and so

(+) $$[\mathbf{4}] H \cap \mathrm{Ker}\tau \le H^*$$

Thus

$$H^* \overset{(6.3.5)(a)}{\le} H \cap G' \overset{(*)}{=} H \cap K \overset{(***)}{\le} H \cap \mathrm{Ker}\tau \overset{(+)}{\le} H^*$$

and so indeed $H^* = H \cap G'$. □

Corollary 6.3.8. [**char g=opg**] *Let p be a prime, G a finite group and $S \in \mathrm{Syl}_p(G)$. Let $S^* = [S,_S, G]$. Then $G = \mathrm{O}^p(G)$ if and only if $S = S^*$.*

Proof. Note that $G = \mathrm{O}^p(G)$ if and only if $G/\mathrm{O}^p(G) = 1$. As $G/\mathrm{O}^p(G)$ is a p-group, this holds if and only if $(G/\mathrm{O}^p(G))' = 1$ and so if only if $G/\mathrm{O}^p(G)G' = 1$. By 6.3.7 this is equivalent to $S/S^* = 1$, and so to $S = S^*$. □

Lemma 6.3.9. [**abelian gruen**] *Let G be a finite group, p a prime and $S \in \mathrm{Syl}_p(G)$. Put $S^* := [S,_S G]$ and $H := \mathrm{N}_G(S)$. Suppose that S is abelian. Then*

$$S^* = [S, H] = [S,_H H] \quad \textit{and} \quad S/S^* \cong H/\mathrm{O}^p(H)H' \cong G/\mathrm{O}^p(G)G'$$

Proof. By (6.3.5)(d)we have

$$S^* = [S,_S G] = \langle a^{-1}a^g \mid a \in S, g \in G, a^g \in S\rangle$$

As S is abelian, any subset of S is normal. Thus 6.2.4 shows that

$$S^* = \langle a^{-1}a^g \mid a \in S, h \in H, a^h \in S\rangle$$

and so $S^* = [S,_S H]$. So we can apply 6.3.7 to (S,H) and (S,G) in place of (H,G). It follows that

$$S/S^* \cong H/\mathrm{O}^p(H)H' \quad \text{and} \quad S/S^* \cong G/\mathrm{O}^p(G)G'$$

As $H = \mathrm{N}_G(S)$, (6.3.5)(c)shows that $[S,_S, H] = [S, H]$ and so 6.3.9 is proved. □

Lemma 6.3.10. [**weakly gruen**] *Let G be a finite group, p a prime, $S \in \mathrm{Syl}_p(G)$ and $Z \le \mathrm{Z}(S)$. Put $S^* := [S,_S G]$ and $H := \mathrm{N}_G(Z)$. Suppose that Z is weakly closed in S with respect to G. Then $S^* = [S, H] = [S,_H H]$. In particular,*

$$S/S^* \cong H/\mathrm{O}^p(H)H' \cong G/\mathrm{O}^p(G)G'$$

and

$$S = S^* \iff H = \mathrm{O}^p(H) \iff G = \mathrm{O}^p(G)$$

Proof. Let $a \in S$. By 6.2.3 $S \cap a^G = S \cap a^H$. Hence

$$\langle a^{-1}a^g \mid a \in S, g \in G, a^g \in S\rangle = \langle a^{-1}a^g \mid a \in S, h \in G, a^h \in S$$

By (6.3.5)(d)applied to G and to H, the group on the left side of this equation is $[S,_S G]$ and the group on the right side is $[S,_S H]$. So indeed $S^* = [S,_S H]$.

So we can apply 6.3.7 to (S,H) and to (S,G) in place of (H,G). It follows that

$$S/S^* \cong H/\mathrm{O}^p(H)H' \quad \text{and} \quad S/S^* \cong G/\mathrm{O}^p(G)G'$$

Also 6.3.8 applied to H and to G gives

$$S = S^* \iff H = \mathrm{O}^p(H) \iff G = \mathrm{O}^p(G)$$

□

6.4 Normal p-complements

Definition 6.4.1. [**def:p-complement**] *Let G be a finite group, p a prime and $K \leq G$. Then K is called a p-complement of G if there exists $S \in \mathrm{Syl}_p(G)$ such that HK is a complement to S in G.*

Lemma 6.4.2. [**basic p-complement**] *Let G be a finite group, p a prime and $K \leq G$.*

(a) [**a**] *The following statements are equivalent:*

   (1) [**a**] *K is a p-complement of G.*

   (2) [**b**] *K is a Hall p' subgroup of G.*

   (3) [**c**] *For each $S \in \mathrm{Syl}_p(G)$, K is a complement to S in G.*

(b) [**b**] *K is a normal p-complement if and only if $K = \mathrm{O}^p(G)$ and $\mathrm{O}^p(G)$ is a p'-group.*

(c) [**c**] *Suppose G has a normal p-complement. Then every section of G has normal p-complement*

Proof. (a): (a:1)$\Longrightarrow$ (a:2): Suppose K is a p-complement of G. Then there exists $S \in \mathrm{Syl}_p(G)$ with $G = KS$ and $K \cap S = 1$. Then $|G| = |K||S| = |K||G|_p$ and so $|K| = |G|_{p'}$. Thus K is a Hall p'-subgroup of G.

(a:2)$\Longrightarrow$ (a:3): Suppose K is a Hall p'-subgroup of G and let $S \in \mathrm{Syl}_p(G)$. Then K is p'-group and S is a p-group. Hence $K \cap S = 1$. It follows that

$$|KS| = \frac{|K||S|}{|K \cap S|} = \frac{|G|_{p'}|G|_p}{1} = |G|$$

Hence $G = KS$. Thus K is a complement to S in G and (a:3)holds.

(a:3)$\Longrightarrow$ (a:1): Just observe that where does exists a Sylow p-subgroup of G.

(b)Suppose first that K is a normal p-complement of G. Let $S \in \mathrm{Syl}_p(G)$. Then $G = KS$ and so G/K is a p–group. Hence $\mathrm{O}^p(G) \leq K$. By (a)K is a p'-group, and so $K = O^p(K) \leq \mathrm{O}^p(G)$. Thus $K = \mathrm{O}^p(G)$ and $\mathrm{O}^p(G)$ is a p'-group.

Suppose next that $K = \mathrm{O}^p(K)$ and $\mathrm{O}^p(G)$ is a p'-group. Then $K = \mathrm{O}^p(G) \trianglelefteq G$ and K is p'-group. Let $S \in \mathrm{Syl}_p(G)$. By (6.1.11)(b)$G = \mathrm{O}^p(G)S = KS$. As K is a p'-group and S is a p-group we have $K \cap S = 1$. So K is a complements to S in G. Thus K is indeed a normal p-complement of G.

(c)By (b)$\mathrm{O}^p(G)$ is a p'-group. Observe that $\mathrm{O}^p(K) \leq \mathrm{O}^p(G)$, so also $\mathrm{O}^p(K)$ is a p'-group. Another application of (a)now shows that K has a normal p-complement.

Suppose now that $K \trianglelefteq G$. Then $\mathrm{O}^p(G/K) = \mathrm{O}^p(G)K/K \cong \mathrm{O}^p(G)/O^p(G) \cap K$ and so $\mathrm{O}^p(G/K)$ is a p'-group. Hence also G/K has normal p-complement. □

Lemma 6.4.3. [**burnside p-complement**] *Let G be finite group, p a prime and $S \in \mathrm{Syl}_p(G)$. Suppose that $\mathrm{C}_G(S) = \mathrm{N}_G(S)$. Then G has a normal p-complement.*

Proof. Since $S \leq \mathrm{N}_G(S) = \mathrm{C}_G(S)$ we know that S is abelian. Hence 6.3.9 shows that

$$[S,_S G] = [S, \mathrm{N}_G(S)] = [S, \mathrm{C}_G(S)] = 1$$

and by 6.3.7 $[S,_S G] = S \cap \mathrm{O}^p(G)G'$. Thus $S \cap \mathrm{O}^p(G) = 1$. As $G = S\mathrm{O}^p(G)$ this implies that $\mathrm{O}^p(G)$ is normal p-complement of G. □

Lemma 6.4.4. [**cyclic p-complement**] *Let G be finite group, p the smallest prime dividing $|G|$ and $S \in \mathrm{Syl}_p(G)$. Suppose that S is cyclic. Then G has a normal p-complement.*

Proof. In view of 6.4.3 we just need to show that $\mathrm{N}_G(S) = \mathrm{C}_G(S)$. Since S is cyclic we have $S \le \mathrm{C}_(S)$ and so $\mathrm{N}_G(S)/C_G(S)$ is a p'-group. $\mathrm{N}_G(S)/\mathrm{C}_G(S)$. Any prime dividing $|\mathrm{N}_G(S)/\mathrm{C}_G(S)|$ also divides $|G|$ and so is smaller that p. We conclude that any prime dividing $|\mathrm{N}_G(S)/\mathrm{C}_G(S)|$ is larger than p.

Let $|S| = p^k$. Then $\mathrm{Aut}(S)| = p^{k-1}(p-1)$. As $\mathrm{N}_G(S)/\mathrm{C}_G(S)$ is a isomorphic to a subgroup of G, this implies that $|\mathrm{N}_G(S)/\mathrm{C}_G(S)|$ divides $p^{k-1}(p-1)$. We conclude ny prime dividing $|\mathrm{N}_G(S)/\mathrm{C}_G(S)|$ is less or equal to p. Thus $|\mathrm{N}_G(S)/\mathrm{C}_G(S)| = 1$, so $\mathrm{N}_G(S) = \mathrm{C}_G(S)$. □

Definition 6.4.5. [**def:p-local**] *Let G a group, $H \le G$ and p a prime. Then H is called a p-local subgroup of G, if there exists a non-trivial p-subgroup Q of G with $H = \mathrm{N}_G(Q)$.*

Lemma 6.4.6. [**frobenius p-complement**] *Let G be a finite group and p a prime. Suppose that all p-local subgroups of G have a normal p-complement. Then G has a normal p-complement*

Proof. The proof is by induction on G. Let $S \in \mathrm{Syl}_p(G)$. If $S = 1$, then $G = GS$ and $G \cap S = 1$ and so G is a normal p-complement of G. So we may assume that $S \ne 1$. Put $Z = \mathrm{Z}(S)$ and observe that $Z \ne 1$. We will first show

(∗) [**1**] *Z is a weakly closed subgroup of S with respect to G.*

Suppose not. Then 6.2.5 shows that there exists $R \in \mathrm{Syl}_p(G)$ with $Z \le R$ and $Z \ntrianglelefteq R$. We choose such an R with $\mathrm{N}_R(Z)$ maximal. Define $T := \mathrm{N}_R(Z)$. Then $T \le \mathrm{N}_G(Z)$ and we can choose $P \in \mathrm{Syl}_p(\mathrm{N}_G(Z))$ with $T \le P$. Observe that $S \le \mathrm{N}_G(Z)$ and it follows that $|P| = |S| = |R|$. Since $Z \ntrianglelefteq R$ we know that $T \nleq R$. Thus $|T| < |R| = |P|$ and so also $T \lneq P$. Since both P and S are nilpotent we conclude that

$$T \lneqq \mathrm{N}_R(T) \quad \text{and} \quad T \lneqq \mathrm{N}_P(T).$$

Put $M := \mathrm{N}_G(T)$. Then both $\mathrm{N}_R(T)$ and $\mathrm{N}_P(T)$ are p-subgroups of M and we can choose $R_1, P_1 \in \mathrm{Syl}_p(M)$ with

$$\mathrm{N}_R(T) \le R_1 \quad \text{and} \quad \mathrm{N}_P(T) \le P_1$$

Let $P_2 \in \mathrm{Syl}_p(G)$ with $P_1 \le P_2$. Since $P \le \mathrm{N}_G(Z)$ and $\mathrm{N}_P(T) \le P_1 \le P_2$ we have $\mathrm{N}_P(T) \le \mathrm{N}_{P_2}(Z)$. As $\mathrm{N}_R(Z) = T \lneqq \mathrm{N}_P(T)$ this shows that $\mathrm{N}_R(Z) \lneqq \mathrm{N}_{P_2}(Z)$. Recall that $\mathrm{N}_R(Z)$ was maximal with respect to $Z \trianglelefteq R$. It follows that $Z \trianglelefteq P_2$ and so also $Z \trianglelefteq P_1$. As $1 \ne Z \le T$ we have $T \ne 1$. Thus M is a p-local subgroup of G. By Hypothesis this means that M has normal p-complement K. Then $M = KP_1$ and $K \cap P_1 = 1$. As $Z \trianglelefteq P_1$ we conclude from 6.2.6 that Z is weakly closed in P_1 with respect to M. Since $Z \le T \le R_1 \in \mathrm{Syl}_p(M)$ we can apply 6.2.5 and conclude that $Z \trianglelefteq R_1$. Since $\mathrm{N}_R(T) \le T_1$, this implies that $\mathrm{N}_R(T) \le \mathrm{N}_R(Z)$. Hence

$$\mathrm{N}_R(Z) = T \lneqq \mathrm{N}_R(T) \le \mathrm{N}_R(Z)$$

a contradiction. This contradiction completes the proof of (∗).

Put $H = \mathrm{N}_G(Z)$. As $Z \ne 1$, H is a p-local subgroup of G and so has a normal p-complement. It follows that $\mathrm{O}^p(H)$ is p'-group. Since $1 \ne S \le H$, H is not a p'-group so $\mathrm{O}^p(H) \ne H$. By $(*)Z$ is weakly closed in S with respect to G, so 6.3.10 shows that $G \ne \mathrm{O}^p(G)$. Put $L = \mathrm{O}^p(G)$. We claim that every p-local subgroup of L has a normal p-complement. Indeed let Q be a non-trivial normal p-subgroup of L. Then $\mathrm{N}_L(Q) \le \mathrm{N}_G(Q)$ and $\mathrm{N}_G(Q)$ has a normal p-complement. Hence (6.4.2)(c)shows that $\mathrm{N}_L(Q)$ has a normal p-complement. Since $|L| < |G|$ induction on $|G|$ now implies that L has a normal p-complement. Thus $\mathrm{O}^p(L)$ is a p'-group. Note that $\mathrm{O}^p(L) = \mathrm{O}^p(\mathrm{O}^p(G)) = \mathrm{O}^p(G))$, so $\mathrm{O}^p(G)$ is a p'-group and G has a normal p-complement.

□

Lemma 6.4.7. [**coprime and invariant**] *Let p, q be a prime, and S a finite p-group acting on a finite p' H. Then there exists an S-invariant Sylow q-subgroup of H.*

Proof. Let Q be any Sylow q-subgroup of H and let $G = H \ltimes S$ be the semidirect product of H and S. We view H and S as subgroups of G. Since H is a p' group and S a p-group, S is a Sylow subgroup of G. Since $H \trianglelefteq G$ and $Q \in \mathrm{Syl}_p(H)$, the Frattini argument shows that $G = H\mathrm{N}_G(Q)$. Let $T \in \mathrm{Syl}_p(\mathrm{N}_G(Q))$. Since $|H|$ is a p'-group and $G = H\mathrm{N}_G(Q)$ we see that $|G|_p = |\mathrm{N}_G(Q)|_p$. Thus $T \in \mathrm{Syl}_p(G)$ and so $T^g = S$ for some $g \in G$. As Q is a T-invariant Sylow q-subgroup of H, conjugation with g shows that Q^g is a S-invariant Sylow q-subgroup of H. □

Lemma 6.4.8. [**nilpotent and section**] *Let G be a group action on a group V. Let*

$$1 = V_0 \trianglelefteq V_1 \trianglelefteq \dots V_{n-1} \trianglelefteq V_n = V$$

be a G-invariant subnormal series of V.

(a) [**a**] *Let $m \in \mathbb{N}$ and suppose that $[V, G; m] = 1$. Then $[V_i/V_{i-1}, G; m]$ for all $1 \le i \le n$.*

(b) [**b**] *G acts nilpotently on V if and only if G acts nilpotently for each V_i/V_{i-1}, $1 \le i \le n$.*

(c) [**c**] *Suppose G acts stably on each V_i/V_{i-1}. Then G acts stably on V.*

Proof. (a)

$$[V_i, G; m] \le [V, G; n] = 1 \quad \text{and} \quad [V_i/V_{i-1}, G; n] = [V_i, G; n]V_{i-1}/V_{i-1} = V_{i-1}/V_{i-1} = 1.$$

(b)The forward direction follows from (a). So suppose G acts nilpotently on each V_i/V_{i-1}, $1 \le i \le n$. By induction on n we may assume that G acts nilpotently on V/V_1. So there exists $k, l \in \mathbb{N}$ with $[V/V_1, G, k] = 1$ and $[V_1, G; l] = 1$. Then $[V, G; k] \le V_1$ and so

$$[V, G; l + k] = [[V, G; l], G; k] \le [V_1, G, k] = 1$$

Thus G acts nilpotently on V.

(c)Let $A \le G$ such that A acts acting quadratically on V. Let $1 \le i \le n$. By (a)(applied with $G = A$ and $m = 2$) we know that A acts quadratically on V_i/V_{i-1}. As G acts stably on V_i/V_{i-1} we conclude that $\langle A^G\rangle$ acts nilpotently on V_i/V_{i-1}. Now (c)shows that $\langle A^G\rangle$ acts nilpotently on V. Thus G acts stably on V. □

Lemma 6.4.9. [**modulo pprime**] *Let G be a finite group, p a prime, P a p-subgroup of G and K a normal p'-subgroup of G. Put $\overline{G} = G/K$.*

(a) [**a**] *The function $P \to \overline{P}, a \to aK$ is an isomorphism of groups.*

(b) [**b**] $\mathrm{N}_{\overline{G}}(\overline{P}) = \overline{\mathrm{N}_G(P)}$.

(c) [**c**] *G has a normal p-complement if and only if $\overline{G}$ has a normal p-complement.*

Proof. (a)The function is surjective homomorphism with kernel $P \cap K$. Since P is a p-group and p'-group we have $P \cap K = 1$ and so (a)holds.

(b)Clearly $\overline{\mathrm{N}_G(P)} \le \mathrm{N}_{\overline{G}}(\overline{P})$. Let $H \le G$ with $K \le H$ and $H/K = \mathrm{N}_{\overline{G}}(\overline{P})$. Note that $\overline{P} \trianglelefteq \overline{H}$ and so $KP \trianglelefteq H$. Since K is a p'-group we see that $P \in \mathrm{Syl}_p(KP)$. The Frattini argument gives $H = (KP)\mathrm{N}_H(P)$ and since $P \le \mathrm{N}_H(P)$ we get $H = K\mathrm{N}_H(P) \le K\mathrm{N}_G(P)$. Thus $\overline{H} \le \overline{\mathrm{N}_G(P)}$ and (b)is proved.

(c)Recall that G has a normal p-complement if and only if $\mathrm{O}^p(G)$ is a p'-group. Since K is a p'-group, $K \le \mathrm{O}^p(G)$ and $\mathrm{O}^p(G)$ is a p'-group of only if $\mathrm{O}^p(G)/K$ is p'-group. As $\mathrm{O}^p(\overline{G}) = \mathrm{O}^p(G)/K$, this hold if and only if $\mathrm{O}^p(\overline{G})$ is a p'-group and so if and only if $\overline{G}$ is a normal p-complement. □

Lemma 6.4.10. [**char p-stable**] *Let G be a finite group, p an odd prime and $S \in \mathrm{Syl}_p(G)$. Suppose that*

(i) [**i**] *G is p-separable.*

(ii) [**ii**] $\mathrm{O}_{p'}(G) = 1$.

(iii) [**iii**] *Either $G = S$ or S is a maximal subgroup of G.*

Then G is p-stable.

Proof. Since G is p-separable and $\mathrm{O}_{p'(G)} = 1$, 3.2.10 shows that $\mathrm{C}_G(\mathrm{O}_p(G)) \le \mathrm{O}_p(G)$. Thus G has characteristic p. Let V be a normal p-subgroup of G. To show that G is p-stable it remains to show that G acts stably on V. Let W be normal p-subgroup of G maximal with $W \lneq V$. By induction on $|V|$ we may assume that G acts stably on V. Put $U = V/W$. Let $A \le G$ such that A acts quadratically in U. Put $\overline{G} := G/\mathrm{C}_G(U)$ and observe that $\overline{G}$ acts faithfully on U.

Suppose for a moment that we can prove that $\overline{A} = 1$. Then $\langle A^G \rangle$ centralizes U. In particular, $\langle A^G \rangle$ acts nilpotently on U. So G acts stably on U and (6.4.8)(c)shows that G acts stably on V and we will be done.

Note that $\mathrm{O}_(\overline{G})$ is a finite p-group acting on the non-trivial finite p-group $\overline{U}$. Thus $\mathrm{C}_U(\mathrm{O}_p(\overline{G})) \ne 1$ (see for example (5.4.30)(b)Observe that $\mathrm{C}_{\mathrm{C}_U(\mathrm{O}_p(\overline{G}))}$ is G-invariant and so the maximal choice of W implies that $U = \mathrm{C}_U(\mathrm{O}_p(\overline{G}))$. Thus $\mathrm{O}_p(\overline{G})$ centralizes U and so $\mathrm{O}_p(\overline{G}) = 1$. Since G is p-separable, also $\overline{G}$ is p-separable and p'-separable. Put $\overline{K} = \mathrm{O}_{p'}(\overline{G})$. Then 3.2.10 shows that $\mathrm{C}_{\overline{G}}(\overline{K}) = 1$. If $\overline{K} = 1$ we get $\overline{G} = 1$ and so also $\overline{A} = 1$ and we are done. So suppose that $\overline{K} \ne 1$ and choose a prime q with $q||\overline{K}|$. By 6.4.7 there exists an $\overline{S}$ invariant Sylow q-subgroup, $\overline{Q}$ of $\overline{K}$. Then $\overline{Q} \ne 1$ and so also $\mathrm{Z}(Q) \ne 1$. Note that $\mathrm{Z}(\overline{Q})$ is $\overline{S}$ invariant and so

$$\overline{S} \lneq \mathrm{Z}(\overline{Q})S \le \overline{G}$$

By Hypothesis, $G = S$ or S is a maximal subgroup of G. Hence also $\overline{G} = \overline{S}$ or $\overline{S}$ is maximal subgroup of $\overline{G}$. It follows that $\overline{G} = \mathrm{Z}(\overline{Q})\overline{S}$. As $\mathrm{Z}(\overline{Q}) \le \overline{K}$ and $\overline{K}$ is a p'-group this implies that $\overline{K} = \mathrm{Z}(\overline{Q})$. Thus $\overline{K}$ is abelian.

Note that $\overline{A}$ acts quadratically on the p-group U and $\overline{K}$ is an abelian p'-group normalizes by $\overline{A}$. Now 6.1.3 shows that $[\overline{K}, \overline{A}]$ centralizes U. As $\overline{G}$ is faithful on U, this gives $[\overline{K}, \overline{A}] = 1$ and so $\overline{A} \le \mathrm{C}_{\overline{G}}(\overline{K}) = \overline{K}$. Thus $\overline{A}$ is a p' group. On the otherhand $\overline{A} \cong A/\mathrm{C}_A(U)$ and A acts nilpotently on the p-group U and so by (5.3.10)(c)$\overline{A}$ is a p-group. Hence $\overline{A} = 1$ and the lemma is proved. ☐

Example 6.4.11. [**ex:pstable**] Let $p = 2$ or 3, a finite field of order p, V a 2-dimensional $\mathbb{F}$-space, $H = \mathrm{SL}_{\mathbb{F}}(V)$ and let G be internal semidirect product of V by H. Then

(a) [**a**] G has characteristic p, G is not p-stable, G is p-separable and $\mathrm{O}_{p'}(G) = 1$.

(b) [**b**] Let $S \in \mathrm{Syl}_p(G)$. If $p = 2$, the S is maximal subgroup of G.

(c) [**c**] 6.4.10 can be false if $p = 2$ and if S is not a maximal subgroup of G.

(a)By 6.1.14 G has characteristic p and G is not-p-stable. Note that $\mathrm{O}_p(G) \cap \mathrm{O}_{p'}(G) = 1$. Thus $\mathrm{O}_{p'}(G) \le \mathrm{C}_G(\mathrm{O}_p(G)) \le \mathrm{O}_p(G)$ and then $\mathrm{O}_{p'(G)} = 1$. Since $|\mathbb{F}| \le 4$, Exercise 1 on Homework 3 shows that H has normal p-complement K. Then $1 \trianglelefteq V \trianglelefteq VK \trianglelefteq VH = G$, V and H/VK are p-groups and VK/V is p'-group.

(b): If $p = 2$, the $|G| = 4 \cdot 6 = 24$ and $|S| = 8$. Thus $|G/S| = 3$ and so S is a maximal subgroup if G.
(c)follows from (a)and (b).

Theorem 6.4.12 (Thompson's p-complement theorem)**.** [**thompson p-complement**] *Let G a finite group, p an odd prime and $S \in \mathrm{Syl}_p(G)$. Suppose that $\mathrm{N}_G(\mathrm{D}(S))$ has a normal p-complement. Then G has a normal p-complement.*

Proof. The proof is by induction on $|G|$.

Case 1. [**1**] *Suppose* $\mathrm{O}_{p'}(G) \neq 1$.

Put $\overline{G} := G/\mathrm{O}_{p'}(G)$. Since $\mathrm{O}_{p'}(G) \neq 1$ we have $|\overline{G}| < |G|$. Note that $\overline{S} \in \mathrm{Syl}_p(\overline{G})$. By (6.4.9)(c)the function $S \to \overline{S}, s \to \overline{s}$ is isomorphism and so $\overline{\mathrm{D}(S)} = \mathrm{D}(\overline{S})$. Hence, using (6.4.9)(b),

$$\mathrm{N}_{\overline{G}}(\mathrm{D}(\overline{S})) = \mathrm{N}_{\overline{G}}(\overline{\mathrm{D}(S)}) = \overline{\mathrm{N}_G(\mathrm{D}(S))}$$

Thus $\mathrm{N}_{\overline{G}}(\mathrm{D}(\overline{S}))$ is a isomorphic a quotient of $\mathrm{N}_G(\mathrm{D}(S))$. As $\mathrm{N}_G((\mathrm{D}(S))$ has a normal p-complement we conclude from (6.4.2)(c)that also $\mathrm{N}_{\overline{G}}(\mathrm{D}(\overline{S}))$ has a normal p-complement. By induction, $\overline{G}$ has a normal p-complement and so by (6.4.9)(b)also G has a normal p-complement.

Case 2. [**2**] *Suppose* $\mathrm{O}_{p'}(G) = 1$

In this case we put $\overline{G} := G/\mathrm{O}_p(G)$. We will first show

$(*)$ [**1**] *Let $H \lneqq G$ with $S \leq H$. Then H has a normal p-complement.*

Observe that $S \in \mathrm{Syl}_p(H)$ and $\mathrm{N}_H(\mathrm{D}(S)) \leq \mathrm{N}_G(\mathrm{D}(S))$. Since $\mathrm{N}_G(\mathrm{D}(S))$ has a normal p-complement, also $\mathrm{N}_H(\mathrm{D}(S))$ has a normal p-complement. Induction on $|G|$ now shows that H has a normal p-complement.

$(**)$ [**2**] $\mathrm{O}_p(\overline{G}) = 1$. *In particular, every p-local subgroup of $\overline{G}$ is a proper subgroup of $\overline{G}$.*

Let $H \leq G$ with $\mathrm{O}_p(G) \leq H$ and $H/\mathrm{O}_p(G) = \mathrm{O}_p(\overline{G})$. Then H is normal p-subgroup of G and so $H \leq \mathrm{O}_p(G)$. Thus $\mathrm{O}_p(\overline{G}) = 1$.

$(***)$ [**3**] *Every proper subgroup of $\overline{G}$ has a normal p-complement.*

Let $\overline{H} \lneqq \overline{G}$. We will prove $(***)$by induction on $\frac{|G|_p}{|H|_p}$.

Suppose first that $|H|_p = |G_p$ and let $H \leq G$ with $\mathrm{O}_p(G) \leq H$ and $H/\mathrm{O}_p(G) = \overline{H}$. Then H contains a Sylow p-subgroup of G and $(*)$shows that H has a normal p-complement. Thus also $\overline{H}$ has a normal p-complement.

Suppose next that $|\overline{H}|_p = 1$. Then $\overline{H}$ is a p'-group and so $\overline{H}$ is a normal p-complement of $\overline{H}$.

Suppose finally that $1 \neq |H|_p < |G|_p$. Let $\overline{T} \in \mathrm{Syl}_p(\overline{H})$ and choose $\overline{R} \in \mathrm{Syl}_p(\overline{G})$ with $\overline{T} \lneqq \overline{R}$. Then $1 \neq \overline{T} \lneqq \mathrm{N}_{\overline{R}}(\overline{T})$. Since $\mathrm{D}(\overline{T})$ is a characteristic subgroup of $\overline{T}$ we have $\mathrm{D}(\overline{T}) \trianglelefteq \mathrm{N}_{\overline{R}}(\overline{T})$ and so $|\mathrm{N}_{\overline{H}}(\mathrm{D}(\overline{T}))|_p > |\overline{H}|_p$. By (6.1.9)(a)$\mathrm{Z}(\overline{T}) \leq \mathrm{D}(\overline{T})$, so $\mathrm{D}(\overline{T}) \neq 1$ and $(***)$implies that $\mathrm{N}_{\overline{H}}(\mathrm{D}(\overline{T}))$ is a proper subgroup of $\overline{G}$. Induction on $\frac{|G|_p}{|H|_p}$ now shows that $\mathrm{N}_{\overline{H}}(\mathrm{D}(\overline{T}))$ has a normal p-complement. Note that $|\overline{H}| < |\overline{G}| \leq |G|$, so induction on G implies that $\overline{H}$ has a normal p-complement.

$(+)$ [**4**] *$\overline{G}$ has a normal p-complement.*

By $(**)$the p-local subgroups of $\overline{G}$ are proper subgroups, and so by $(**)$have a normal p-complement. Frobenius' p-complement Theorem 6.4.6 now shows $\overline{G}$ as normal p-complement.

$(++)$ [**5**] *G is p-stable.*

By $(+)\overline{G}$ has a normal p-complement. Thus $O^p(\overline{G})$ is a p'-group. Note that $O_p(G)$ and $\overline{G}/O^p(\overline{G})$ are p-groups. Thus G is p-separable. In view of 6.4.10 it remains to show that $G = S$ or S is maximal subgroup. So let $H \lneq G$ with $S \le H$. We need to show that $H = S$. By $(*)H$ has a normal p-complement K.

By Hypothesis of the current case, $O_{p'}(G) = 1$, so3.2.10 shows that $C_G(O_p(G)) \le O_p(G)$. Note that $O_p(G) \le S \le H$. It follows that that $O_p(G)$ and K are normal subgroup of H. Thus

$$[K, O_p(G)] \le K \cap O_p(G)$$

Since K is p'-group and $O_p(G) = 1$ we get $K \cap O_p(G) = 1$. Thus $[K, O_p(G)] = 1$ and so $K \le C_G(O_p(G)) = O_p(G)$. Hence $K \le K \cap O_p(G) = 1$ and $H = KS = S$.

We proved that G is p-stable. Thus Theorem 6.1.13 shows that $D(S) \trianglelefteq G$. Thus $G = N_G(D(S))$ and so G has a normal p-complement. □

Example 6.4.13. [**ex:thompson**] Sym(4) is counterexample to Thompson's p-complement Theorem for $p = 2$.

Indeed, let $G = \mathrm{Sym}(4)$ and $S \in \mathrm{Syl}_2(G)$. By 6.4.11 $D(S) = Z(S) \ntrianglelefteq G$. As S is a maximal subgroup of G this gives $N_G(Z(S)) = S$. Since S is a 2-group, S has a normal 2-complement (namely 1). So $N_G(D(S))$ has a normal 2-complement. But $O_{2'}(G) = 1$ and G is not a 2-group. Thus G does not have a normal 2-complement.

Example 6.4.14. [**ex:thompson ii**] Let G be a group of order $3^3 \cdot 5^2 \cdot 11$. Show that G is not simple.

Let $S \in \mathrm{Syl}_3(G)$. We will show that either $D(S) \trianglelefteq G$ or G has a normal 3-complement. Note that in either case G is not simple and so we are done. We may assume that $D(S) \ntrianglelefteq G$. Put $B := N_G(S)$ and $D := N_G(D(S))$. Since $D(S) \ntrianglelefteq G$ and $D(S) \operatorname{char} S$ we have

$$(*) \qquad [\mathbf{2}] S \le B \le D \lneq G$$

In particular, $S \ntrianglelefteq G$.

Let $H \le G$ and m_H the number of Sylow 2-subgroups of H. Then $m \big| |H|$ and $m \equiv 1 \pmod 3$. Then m divides $5^2 \cdot 11$ and so $m_H = 5^a \cdot 11^b$ where $0 \le a \le 2$ and $0 \le b \le 1$. Note that $5 \equiv -1 \pmod 3$ and $11 \equiv -1 \pmod 3$. So $1 \equiv m_H \equiv (-1)^{a+b} \pmod 3$. Thus $a + b$ is even. It follows that

$$(**) \qquad [\mathbf{1}] m_H \in \{1, 5^2, 5 \cdot 11\}$$

Since $S \ntrianglelefteq G$, we have $m_G \ne 1$. Hence $m_G = 5^2$ or $5 \cdot 11$. Note that $|G/B| = m_G$ and that $|D/B|$ divides $|G/B|$. As $G \ne D$ we have $|D/B| \ne |G/B$, thus $|D/B| = 1, 5$ or 11. As $|D/B| = m_D$ we conclude from $(**)$that $|D/B| = 1$. Thus $D = B$.

Note that $|D| = |B| = \frac{|G|}{|G/B|}$ and so $D = 3^3 \cdot q$ where $q = 5$ or $q = 11$. Let $Q \in \mathrm{Syl}_q(D||)$. We have $3 - 1 = 2, 3^2 - 1 = 8, 3^3 - 1 = 26$ and so q does not divide $3^e - 1$ for $1 \le e \le 3$. So $3^e \not\equiv 1 \pmod q$ and $|\mathrm{Syl}_q(D| = 1$. Thus $Q \trianglelefteq D$. As $D = QS$ we conclude that Q is a normal 3-complement of D. Thompson's p-complement Theorem now shows that G has a normal 3-complement.

6.5 Frobenius Groups

Lemma 6.5.1. [**partition and centralizer**] *Let G be a finite group acting on an abelian group V. Let $\mathcal{A}$ be a set of subgroups of G such that $(A^\sharp)_{A\in\mathcal{A}}$ is a partition of G. Put $k := |\mathcal{A}| - 1$. Then*

$$V^{\cdot k} \leq \langle C_V(A) \mid A \in \mathcal{A}\rangle$$

Proof. See Exercise 1 on Homework 2. REMARK: Exercise 1 on Homework 2 was a special case : $G = C_p \times C_p$ and $\mathcal{A} = \{A \leq G \mid |A| = p\}$. But the proof is exactly the same. □

Recall that G is Frobenius group with complement H and kernel K if G is a group, $1 \neq H \lneqq G$, $H \cap H^g = 1$ for all $g \in G$ and

$$K = G \setminus \bigcup H^{\sharp G} = \{g \in G \mid g \neq h^l \text{ for all } h \in H^\sharp, l \in G\}$$

Corollary 6.5.2. [**frobenius on abelian**] *Let G be finite group acting an finite group V. Suppose that*

(i) [**i**] *G is Frobenius group with complement H and kernel K.*

(ii) [**ii**] *K is a subgroup of G and $C_V(K) = 1$.*

(iii) [**iii**] *V is abelian, $V \neq 1$ and $\gcd(|V|, |K|) = 1$.*

Then $C_V(H) \neq 1$.

Proof. Put $\mathcal{A} := \{K\} \cup H^G$ and $k := |K|$. Since G is a Frobenius group with complement H and kernel G we see that $(A^\sharp)_{A\in\mathcal{A}}$ is a partition of $G^\sharp$. By (1.4.4)(d)we have $|H^G| = |K| = k$. Thus $|\mathcal{A}| = k+1$. Hence 6.5.1 shows that

$$V^k \leq \langle C_V(A) \mid A \in \mathcal{A}\rangle$$

As $\gcd(|V|, k) = 1$ we have $V^k = V \neq 1$ and so $C_V(A) \neq 1$ for some $A \in \mathcal{A}$. Since $C_V(K) = 1$ we get $A \neq K$ and so $A = H^g$ for some $g \in G$. Then $C_V(H^g) \neq 1$, so also $C_V(H) \neq 1$. □

Theorem 6.5.3. [**fixed point free p**] *Let G be a finite group, p a prime and $\alpha \in \mathrm{Aut}(G)$. Suppose that $|\alpha| = p$ and $C_G(\alpha) = 1$. Then G is nilpotent.*

Proof. The proof is by induction on $|G|$. If $G = 1$, the theorem holds. So suppose $G \neq 1$. Put $A = \langle\alpha\rangle$. Then A acts on G and $|A| = p$.

We first show

(∗) [**1**] *$p \big| |G^\sharp|$. In particular, G is a p'-group.*

Since $|A| = p$ we have $C_G(A) = C_G(\alpha) = C_G(a)$ for all $a \in A^\sharp$. Thus all orbits of A on $G^\sharp$ of length p and (∗)holds.

(∗∗) [**2**] *Let H be α-invariant subgroup of G. If $H \neq G$, then $C_H(\alpha) = 1$ and H is nilpotent. If $H \neq 1$ and $H \trianglelefteq G$, then $C_{G/H}(\alpha) = 1$ and G/H is nilpotent.*

Note that $C_H(\alpha) \leq C_G(\alpha) = 1$. So the first statement holds by induction on G.

Since A is a p-group and G is a p'-group (5.3.7)(a)shows that

$$C_{G/H}(\alpha) = C_{G/H}(A) = C_G(A)H/H = H/H = 1$$

So again induction on $|G|$ shows that G/H is nilpotent.

$(***)$ **[3]** *There exists a non-trivial, α-invariant, nilpotent, normal subgroup N of G.*

If G is a 2-group, then G is nilpotent and we can choose $N = G$.

So suppose G is not a 2-group and let q be an odd prime with $q \| |G|$. Since G is p'-group, 6.4.7 shows that there exists an α-invariant Sylow q-subgroup S of G. Then also $\mathrm{D}(S)$ is αinvariant.

Suppose that $\mathrm{D}(S) \trianglelefteq G$. Note that $S \neq 1$, so also $\mathrm{D}(S) \neq 1$ and we can choose $N = \mathrm{D}(S)$.

Suppose that $\mathrm{D}(S) \ntrianglelefteq G$. Then $\mathrm{N}_G(\mathrm{D}(S))$ is proper α-invariant subgroup of G and so $\mathrm{N}_G(\mathrm{D}(S))$ is a nilpotent. It follows that $\mathrm{N}_G(\mathrm{D}(S))$ has normal p-complement and so Thompson's p-complement Theorem shows that G has a normal p-complement K. As K is a q'-group, $K \neq G$. As $K \operatorname{char} G$, K is α-invariant, so K is nilpotent by $(**)$and we can choose $N = K$.

According to $(***)$we can choose a minimal non-trivial α-invariant, nilpotent normal subgroup B of G. Note that $\mathrm{Z}(B) \neq 1$, $\mathrm{Z}(B) \trianglelefteq G$ and $\mathrm{Z}(B)$ is α-invariant. The minimality of B implies that $B = \mathrm{Z}(B)$ so B is abelian. By $(**)G/B$ is nilpotent. If $B \leq \mathrm{Z}(G)$, we conclude that G is nilpotent and we are done. So suppose that $B \nleq \mathrm{Z}(G)$.

Then $D := \mathrm{C}_G(B) \lneqq G$. Let $K = G \ltimes A$ and as usual view G and A as subgroup of K. Since G is a p'-group, $A \in \mathrm{Syl}_p(K)$. Also $[D, a] \neq 1$ for all $a \in A^\sharp$, so $\mathrm{C}_K(B)$ is a p'-group. Since K/G is a p-group this implies $\mathrm{C}_K(B) = \mathrm{C}_G(D) = B$. Put $\overline{K} := K/D$. Then $\overline{K}$ is the semidirect product of $\overline{G}$ by $\overline{A}$. Also by $(**)\mathrm{C}_{\overline{G}}(a) = 1$ for all $\overline{a} \in \overline{A}^\sharp$. Hence (1.4.9)(f)shows that $\overline{K}$ is a Frobenius group with complement $\overline{A}$ and kernel $\overline{G}$. Let r be a prime with $r \| |\overline{G}|$. Since $\overline{G}$ and B are nilpotent, $\overline{G}$ has a unique Sylow p-subgroup $\overline{T}$, and B has a unique Sylow r-subgroup R. It follows that $1 \neq \overline{T} \trianglelefteq \overline{K}$ and $R \trianglelefteq K$. Thus $B \neq \mathrm{C}_B(\overline{T}) \trianglelefteq K$. The minimal choice of B implies that $\mathrm{C}_B(\overline{T}) = 1$. In particular, $\mathrm{C}_R(\overline{T}) = 1$. Since $\overline{T}$ and R are r-groups this implies $R = 1$. Thus B is an r'-group. It follows that $\gcd(|B|, |\overline{G}|) = 1$ and $\mathrm{C}_B(\overline{G}) = 1$. Now 6.5.2 shows that $\mathrm{C}_B(\overline{A}) \neq 1$. But then also $\mathrm{C}_G(\alpha) \neq 1$, a contradiction. □

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