Group Theory
Lecture Notes for MTH 912/913
08/09

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Contents

Chapter 1

Basic Concepts for Infinite Groups

1.1 Classes of Groups and Operators

Definition 1.1.1. [**class of groups**] *A class of groups is class $\mathcal{X}$ such that*

(i) [**i**] *Each member of $\mathcal{X}$ is a group.*

(ii) [**ii**] *If $G \in \mathcal{X}$ and $H \cong G$ then $H \in \mathcal{X}$.*

(iii) [**iii**] *All trivial groups are in $\mathcal{X}$.*

For example each of the following are classes of groups:

- [**a**] $\mathcal{F}$, the class of finite groups.
- [**b**] $\mathcal{F}_\pi$, the class of finite π-groups (here π is a set of primes, and a finite group G is a π-group if all prime divisors of $|G|$ are in π.
- [**c**] $\mathcal{C}$, the class of cyclic groups.
- [**d**] $\mathcal{A}$, the class of abelian groups.
- [**e**] $\mathcal{G}$, the class of finitely generated groups.
- [**f**] $\mathcal{T}$, the class of trivial groups.

Definition 1.1.2. [**extensions**] *Let $\mathcal{X}$ and $\mathcal{Y}$ be classes of groups.*

(a) [**a**] *The members of $\mathcal{X}$ are called $\mathcal{X}$-groups.*

(b) [**c**] *We say that $\mathcal{X}$ is a subclass of $\mathcal{Y}$ and write $\mathcal{X} \leq \mathcal{Y}$ if $A \in \mathcal{Y}$ for all $A \in \mathcal{X}$.*

(c) [**b**] *$\mathcal{X}\mathcal{Y}$ denotes the class of all groups G such that there exists $A \trianglelefteq G$ with $A \in \mathcal{X}$ and $G/A \in \mathcal{Y}$. A $\mathcal{X}\mathcal{Y}$-group is also called a $\mathcal{X}$-by-$\mathcal{Y}$ group.*

Consider the subnormal series

$$1 \trianglelefteq \langle (12)(34) \rangle \trianglelefteq \langle (12)(34), (13)(24) \rangle \trianglelefteq \mathrm{Alt}(4) \trianglelefteq \mathrm{Sym}(4)$$

The factors of this series are isomorphic to

$$C_2, C_2, C_3, C_2$$

Thus Sym(4) is a member of $((\mathcal{C}\mathcal{C}), \mathcal{C})\mathcal{C}$.

Note that Sym(4) has no non-trivial cyclic subgroup. It follows that Sym(4) is not a member of $\mathcal{C}((\mathcal{C}(\mathcal{C}\mathcal{C})))$. hence the associate law does not hold for products of classes og groups. To save parentheses we use the following convention for products. Let $a_1, a_2, \ldots a_n$ in a set with a binary operation. Then

$$a_1 \cdot a_2 \cdot a_3 = a_1(a_2 a_3)$$

and inductively

$$a_1 \cdot a_2 \cdot a_3 \cdot \ldots \cdot a_n = a_1(a_2 \cdot a_3 \cdot \ldots \cdot a_n)$$

Lemma 1.1.3. [**char ext**] *Let $\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_n$ be classes of groups and G a group.*

(a) [**a**] *$G \in \mathcal{X}_1\mathcal{X}_2 \ldots \mathcal{X}_n$ if and only if there exists a subnormal series*

$$1 \trianglelefteq G_1 \trianglelefteq G_2 \trianglelefteq \ldots G_{n-1} \trianglelefteq G_n$$

of G such that $G_i/G_{i+1} \in \mathcal{X}_i$ for all $1 \leq i \leq n$.

(b) [**b**] *$G \in \mathcal{X}_1 \cdot \mathcal{X}_2 \cdot \ldots \cdot \mathcal{X}_n$ if and only if there exists a normal series*

$$1 \trianglelefteq G_1 \trianglelefteq G_2 \trianglelefteq \ldots G_{n-1} \trianglelefteq G_n$$

of G such that $G_i/G_{i+1} \in \mathcal{X}_i$ for all $1 \leq i \leq n$. (Recall here that "normal series" means that each G_i is normal in G.

(c) [**c**] $\mathcal{X}_1 \cdot \mathcal{X}_2 \cdot \ldots \cdot \mathcal{X}_n \leq \mathcal{X}_1\mathcal{X}_2 \ldots \mathcal{X}_n$

Proof. (a) and (b) follows easily from the definitions. Since every normal series is a subnormal series, (c) follows from (a) and (b). □

Definition 1.1.4. [**operation**] *An operation **A** on the classes of groups is a rule which assigns to each class of group $\mathcal{X}$ a class of group $\mathbf{A}\mathcal{X}$ such that*

(i) [**a**] $\mathbf{A}\mathcal{T} = \mathcal{T}$.

(ii) [**b**] *$\mathcal{X} \leq \mathbf{A}\mathcal{X}$ for each class of groups $\mathcal{X}$.*

(iii) [**c**] *$\mathbf{A}\mathcal{X} \leq \mathbf{A}\mathcal{Y}$ for each classes of groups $\mathcal{X}, \mathcal{Y}$ with $\mathcal{X} \leq \mathcal{Y}$.*

For a class of group $\mathcal{X}$ let $\mathbf{S}\mathcal{X}$ the class of all groups which are isomorphic to a subgroup of $\mathcal{X}$-group.

For a class of group $\mathcal{X}$ let $\mathbf{H}\mathcal{X}$ the class of all groups which are isomorphic to a homomorphic image of a $\mathcal{X}$-group.

Then both $\mathbf{S}$ and $\mathbf{H}$ are operations.

Define $\mathcal{X}^0 := \mathcal{T}$ and inductively, $\mathcal{X}^{n+1} := \mathcal{X}^b\mathcal{X}$. Also put $\mathbf{P}\mathcal{X} := \bigcup_{n=0}^{\infty} \mathcal{X}^n$. Then $\mathbf{P}$ is an operation. Then members of $\mathbf{P}\mathcal{X}$ are called poly-$\mathcal{X}$-groups.

Lemma 1.1.5. [**char solvable**] *Let G be a group and $n \in \mathrm{N}$. Then the following are equivalent.*

(a) [**a**] $G \in \mathcal{A}^n$.

(b) [**b**] $G^{(n)} = 1$.

(c) [**c**] $G \in \underbrace{\mathcal{A} \cdot \mathcal{A} \cdot \ldots \cdot \mathcal{A}}_{n-times}$.

Here $G^{(n)}$ is inductively defined as $G^{(0)} := G$ and $G^{n+1} = [G^n, G^n]$. Also we often use G' for $G^{(1)}$, G'' for $G^{(2)}$ and so on.

Proof. (a) $\Longrightarrow$ (b): Suppose $G \in \mathcal{A}^n$. Since $\mathcal{A}^n = \mathcal{A}^{n-1}\mathcal{A}$ there exists $H \trianglelefteq G$ with $H \in \mathcal{A}^{n-1}$ and $G/H \in \mathcal{A}$. Hence G/H is abelian and so $G' \leq H$. By induction on n, $H^{(n-1)} = 1$ and so

$$G^{(n)} = (G')^{(n-1)} \leq H^{(n-1)} = 1$$

(b) $\Longrightarrow$ (c): Suppose $G^{(n)} = 1$ and consider the normal series

$$1 = G^{(n)} \trianglelefteq G^{(n-1)} \trianglelefteq \ldots G^{(1)} \leq G^0 = G$$

Since $G^{(i-1)}/G^{(i)}$ is abelian, 1.1.3(b) shows that $G \in \underbrace{\mathcal{A} \cdot \mathcal{A} \cdot \ldots \cdot \mathcal{A}}_{n-\text{times}}$.

(c) $\Longrightarrow$ (a): Suppose that $G \in \underbrace{\mathcal{A} \cdot \mathcal{A} \cdot \ldots \cdot \mathcal{A}}_{n-\text{times}}$. Then by 1.1.3(c), $G \in \mathcal{A}^n$. □

Definition 1.1.6. [**def:solvable**] *A group G is called in solvable if and only if its is polyabelian, that is if $G \in \mathbf{P}\mathcal{A}$.*

Combining 1.1.5 and 1.1.3 we see G is solvable iff G has a subnormal series with abelian quotients, iff $G^{(n)} = 1$ for some $n \in \mathrm{N}$ and iff G has a normal series with abelian factors.

Definition 1.1.7. [**A-closed**] *Let $\mathbf{A}$ and $\mathbf{B}$ be operations.*

(a) [**a**] *A class of groups $\mathcal{X}$ is called $\mathbf{A}$-closed if $\mathbf{A}\mathcal{X} = \mathcal{X}$.*

(b) [**b**] *The operation $\mathbf{AB}$ is defined by $(\mathbf{AB})\mathcal{X} = \mathbf{A}(\mathbf{B}\mathcal{X}$ for all classes of groups $\mathcal{X}$.*

(c) [**c**] **A** *is called an closure operation if for all classes of groups* $\mathcal{X}$*,* $\mathbf{A}\mathcal{X}$ *is* **A***-closed.*

$\mathcal{X}$ is **S** closed if and only if every subgroup of an $\mathcal{X}$-group is a $\mathcal{X}$-group.
The classes of groups $\mathcal{F}, \mathcal{G}, \mathcal{A}, \mathcal{F}_\phi$, all are **S** and **H** closed.
A is a closure operator iff $\mathbf{A}(\mathbf{A}\mathcal{X}) = \mathbf{A}\mathcal{X}$ for all classes of groups $\mathcal{X}$ and so iff $\mathbf{A} = \mathbf{A}^2$.

Definition 1.1.8. [**def: subdirect product**]

(a) [**a**] *Let* $(G_i, i \in I)$ *be a family of groups and* H *a subgroup of* $\bigtimes_{i\in I} G_i$ *such that for all* $i \in I$ *the projection of* H *onto* G_i *is onto. Then* H *is called a subdirect product of* $(G_i, i \in I)$*. More generally we will also call any group isomorphic to a subdirect product a subdirect product.*

(b) [**b**] *Let* $\mathcal{X}$ *be a class of groups. Then* $\mathbf{R}\mathcal{X}$ *is the class of all groups which are isomorphic to subdirect product of a family of* $\mathcal{X}$*-groups. The members of* $\mathbf{R}\mathcal{X}$ *are called residually* $\mathcal{X}$*-groups.*

Lemma 1.1.9. [**subdirect product**] *let* G *be a group.*

(a) [**a**] *Let* $(G_i, i \in I)$ *be a family of normal subgroups of* G*. Then* $G/\bigcap_{i\in I} G_i$ *is a subdirect product of* $(G/G_i, i \in I)$*.*

(b) [**b**] *Let* $(H_i, i \in I)$ *be a family of groups Then* G *is isomorphic to a subdirect product of* $(G_i, i \in I)$ *iff there exists a family of* $(G_i, i \in I)$ *of normal subgroups of* G *such that* $\bigcap_{i\in I} G_i = 1$ *and* $G/G_i \cong G_i$ *for all* $i \in I$*.*

(c) [**c**] G *is a residually* $\mathcal{X}$ *group iff for all* $1 \neq a \in G$ *there exists a normal subgroup* G_a *of* G *such that* $a \notin G_a$ *and* $G/G_a \in \mathcal{X}$*.*

Proof. (a) Define $\alpha : G \to \bigtimes_{i\in I} G/G_i, h \to (aG_i, i \in I)$. Then $\ker \alpha = \bigcap_{i\in I} G_i = 1$. Also the image of α is clearly of subdirect product of $(G/G_i, i \in I)$. So $G/\bigcap_{i\in I} G_i \cong G/\ker \alpha \cong \operatorname{Im} \alpha$ is a subdirect product of $(H_i, i \in I)$.

(b) Suppose there exists a family of $(G_i, i \in I)$ of normal subgroups of G such that $\bigcap_{i\in I} G_i = 1$ and $G/G_i \cong G_i$ for all $i \in I$. Then by (a) $G \cong G/\bigcap_{i\in I} G_i$ is a subdirect product of $(G/G_i, i \in I)$. Since $\bigtimes_{i\in I} G/G_i \cong \bigtimes_{i\in I} H_i$, G is also a subdirect product of $(H_i, i \in I)$

Suppose next that G is a subdirect product of $(H_i, i \in I)$. Let G_i be the kernel of the project of H onto G_i. Then clearly $\bigcap_{i\in I} G_i = 1$ and $G/G_i \cong H_i$.

(c) Suppose G is a residually $\mathcal{X}$ groups. G is a subdirect product of a family $(H_i, i \in I)$ of $\mathcal{X}$ groups. By (b) there exists a family $(G_i, i \in I)$ of normal subgroups of G with $\bigcap_{i\in I} G_i = 1$ and $G/G_i \cong H_i$. Thus G/G_i is an $\mathcal{X}$ groups. Let $1 \neq a \in G$. Since $\bigcap_{i\in I} G_i = 1$ there exists $i \in I$ with $a \notin G_i$. So the second statement in (c) holds with $G_a = G_i$.

Suppose next that for each $1 \neq a \in G$ there exists a normal subgroup G_a of G such that $a \notin G_a$ and $G/G_a \in \mathcal{X}$. Then $\bigcap_{a\in G^\sharp} G_a = 1$ and so by (b), G is a subdirect product of the family of $\mathcal{X}$-groups, $(G_a, a \in G^\sharp)$. Thus G is residually $\mathcal{X}$. □

1.2 Varieties

We will consider classes of groups which are **R** and **H** closed. It will turn out that these are exactly the so called varieties of groups:

Let I be a set. Recall that a free group on I is a groups generated by a family $x = (x_i, i \in I)$ of elements such that for each group G and each family of elements $y = (y_i, i \in I) \in G^I$, there exists a unique homomorphism $\alpha_y : F \to G$ with $\alpha_y(x_i) = y_i$ for all $i \in I$. The call the elements of F words in $(x_i, i \in I)$. Note that each word $\theta \in F$ can be uniquely written as

$$\theta = x_{m_1}^{i_1} \dots x_{i_k}^{m_k}$$

where k is a non-negative integer, $i_l \in I, i_l \neq i_{l+1}$ and m_l is a non-zero integer. Also

$$\alpha_y(\theta) = y_{m_1}^{i_1} \dots y_{i_k}^{m_k}$$

We will also write $\theta(y)$ for $\alpha_y(\theta)$.

If θ is a word and G is group define

$$\theta(G) := \langle \alpha_y(\theta) \mid y \in G^I \rangle = \langle \theta(y) \mid y \in G^I$$

For example $1_F(G) = 1$, $x_1(G) = G$, $[x_1, x_2](G) = G'$, and $[[x_1, x_2], [x_3, x_3](G) = G''$

More generally if $W \subseteq F$ is a set of words we define

$$W(G) = \langle G^\theta \mid \theta \in W = \langle \alpha_y(\theta) \mid y \in G^I, \theta \in W \rangle$$

The variety $\mathcal{V}(\theta)$ defined by θ is the class of all groups G such that $\theta(G) = 1$, so $G \in \mathcal{V}(\theta)$ if and only if

$$y_{i_1}^{m_1} \dots y_{i_k}^{m_k} = 1 \text{ for all } y \in G^I$$

For example $\mathcal{V}(1)$ is the class $\mathcal{D}$ of all groups, $\mathcal{V}(x_1)$ is the class $\mathcal{T}$ of trivial groups and $\mathcal{V}([x_1, x_2])$ is the class $\mathcal{A}$ of abelian groups.

More generally if W is a set of words the variety $\mathcal{V}(W)$ defined by W is the class of all groups G such that $W(G) = 1$. And a variety of groups is the variety defined by some sets of words.

Lemma 1.2.1. [**onto hom**] *Let I be a set, $J \subseteq I$, F a free group on I, H a groups and $y \in H^J$. Suppose that $|I \setminus J| \geq |H|$. Then there exists an onto homomorphism $\beta : F \to H$ with $\beta(x_j) = y_j$ for all $j \in J$.*

Proof. Since $|I \setminus J| \geq |H|$ there exists an onto function $\tau : I \setminus J \to J$. Define $z \in H^I$ by $z_i = \tau(i)$ of $i \notin J$ and $z_i = y_i$ if $i \in J$. Then the lemma holds with $\beta = \alpha_z$. □

Definition 1.2.2. [**def:wx**] *Let $\mathcal{X}$ be a class of groups and F a free group of infinite rank on $(x_i, i \in \mathbb{Z}^+)$.*

$$W(\mathcal{X}) = \{w \in F \mid w(G) = 1 \textit{ for all } G \in \mathcal{X}\}$$

Proposition 1.2.3. [**char variety**] *Let $\mathcal{X}$ be class of groups. The the following are equivalent:*

(a) [**a**] *$\mathcal{X}$ is* **H** *and* **R** *closed.*

(b) [**b**] $\mathcal{X} = \mathcal{V}(W(\mathcal{X}))$

(c) [**c**] *$\mathcal{X}$ is a variety of groups.*

Proof. It is easy to verify that a variety of groups is **H** and **R** closed (see Homework 1). Also (b) implies (c). So we just need to show that (a) implies (b). Assume $\mathcal{X}$ is **H** and **R** closed and put $W = W(\mathcal{X})$. Clearly $\mathcal{X} \leq \mathcal{V}(W)$. So we just need to show that any $G \in \mathcal{V}(W)$ is an $\mathcal{X}$-group. Note that for any $\theta \in F \setminus W$ there exists a $\mathcal{X}$-group H_θ with $\theta(H_\theta) \neq 1$. Let I be an infinite set with cardinality larger that $|G|$ and any $|H_\theta|$, $\theta \in F \setminus W$ (For example $J = \biguplus_{\theta \in T} H_\theta \uplus \mathbb{N} \uplus G$.) Let F_I be a free group on $(z_i, i \in I)$. By 1.2.1 there exists an onto homomorphism $\alpha : F_I \to G$. Put $M = \ker \alpha$. We will now show

1°. [**1**] *Let $a \in F_I \setminus M$, then there exists $K_a \trianglelefteq F_I$ with $F_I/K_a \in \mathcal{X}$ and $a \notin K_a$.*

Indeed let $a = z_{i_1}^{m_1} \dots z_{i_k}^{m_k}$ with $i_l \in I$ and $m_k \in \mathbb{Z}^\sharp$. Since $\mathbb{Z}^+$ is infinite, there exists $j_1, \dots, j_k \in I$ with $i_s = i_t$ if and only if $j_s = j_t$. Put

$$\theta := x_{j_1}^{m_1} \dots x_{j_k}^{m_k} \in F$$

$u_i = z_i M \in F_I/M$ and $u = (u_i)_{i \in I} \in (F_I/M)^I$.

Then

$$\theta(u) = u_{j_1}^{m_1} \dots u_{j_k}^{m_k} = z_{i_1}^{m_1} \dots z_{i_k}^{m_k} M = aM \neq 1_{F/M}$$

Hence $\theta(F_I/M) \neq 1$ and since $F_I/M \cong G$ also $\theta(G) \neq 1$. As $\rho(G) = 1$ for all $\rho in W$ this implies that $\theta \in F \setminus W$. Since $\theta(H_\theta) \neq 1$ there exists $y \in H_\theta^I$ with $\theta(y) \neq 1$. Since I is infinite

$$|I \setminus \{i_l \mid 1 \leq l \leq k\}| = |I| \geq |H_\theta|$$

Thus 1.2.1 there exists an onto homomorphism $\beta : F_I \to H_\theta$ with $\beta(z_l) = y_l$ for all $l \in \{i_1, \dots i_k\}$. Then

$$\beta(a) = y_{j_1}^{m_1} \dots y_{j_k}^{m_k} = \theta(y) \neq 1$$

and so $a \notin \ker \beta$. Also $F_I / \ker \beta \cong \operatorname{Im} \beta = H_\theta \in \mathcal{X}$ and so (1°) holds with $K_a := \ker \beta$.

Put $K := \bigcap_{a \in F_i \setminus M} K_a$. If $a \in F_I \setminus M$, then $a \notin K_a$ and so also $a \notin K$. Thus $K \leq M$. By 1.1.9(a), F_I/K is a subdirect product of the family of $\mathcal{X}$groups $(F_I/K_a, a \in F_I \setminus M)$.

Since $\mathcal{X}$ is **R**-closed this means that F_I/K is a $\mathcal{X}$-group. Since $\mathcal{X}$ is **H**-closed, any quotient of F_I/K is also a $\mathcal{X}$ -group. As

$$G \cong F_I/M \cong F_I/K/M/K$$

we conclude that $G \in \mathcal{X}$ and so $\mathcal{X} = \mathcal{V}(W)$. □

Definition 1.2.4. [**def:hom**] *Let H and G be groups.*

(a) [**a**] $\mathrm{Hom}(H, G)$ *is the set of homomorphism from H to G.*

(b) [**b**] $\mathrm{End}(G)$ *is the set of endomorphism of G, that is* $\mathrm{End}(G) = \mathrm{Hom}(G, G)$*.*

(c) [**c**] *A subgroup A of G is called fully invariant in G, if $\alpha(A) \leq A$ for all $\alpha \in$* $\mathrm{End}(G)$*.*

(d) [**d**] *A subgroup A of G is called characteristic in G if $\alpha(A) \leq A$ for all $\alpha \in$* $\mathrm{Aut}(G)$*.*

See Homework 1 for example if subgroups which are characteristic but not fully invariant.

Lemma 1.2.5. [**hom fg**] *Let F be a free group on the set I, $W \subseteq F$ and G a group.*

(a) [**a**] $\mathrm{Hom}(F, G) = \{\alpha_y \mid y \in G^I\}$.

(b) [**b**] $\mathrm{End}(F) = \{\alpha_y \mid y \in G^I\}$.

(c) [**c**] $W(G) = \langle \beta(W) \mid \beta \in \mathrm{Hom}(F, G)\rangle$.

(d) [**d**] $W(F) = \langle \beta(W) \mid \beta = \mathrm{End}(F)\}\rangle$.

Proof. (a) follows immediately from a definition a free group. (b) is the special case $F = G$ in (a). (c) follows from (a) and the definition of $W(G)$. (d) is the special case $F = G$ in (c). □

Lemma 1.2.6. [**full invariant**] *Let F be a free group and $W \leq F$. Then the following are equivalent.*

(a) [**a**] $W = W(F)$.

(b) [**b**] *W is fully invariant in F.*

Proof. By definition, W is full invariant in F iff $\beta(W) \leq W$ for all $\beta \in \mathrm{End}(\mathbb{F})$ and so if and only if $\langle \beta(W) \mid \beta \in \mathrm{End}(\mathbb{F})\rangle \leq W$. Since $W = \mathrm{id}_F(W) \leq \langle \beta(W) \mid \beta \in \mathrm{End}(\mathbb{F})\rangle$, this holds iff $W = \langle \beta(W) \mid \beta \in \mathrm{End}(\mathbb{F})\rangle$ and so by 1.2.5(d), iff $W = W(F)$. □

1.3 Series

Definition 1.3.1. [**def:action**]

(a) [**a**] *An actions (of groups) is a triple (A, G, α), where A and G are groups and $\alpha : A \to \mathrm{Aut}(G)$ is a homomorphism. We usually will write g^a for $g.a\alpha$ and call (A, G) an action. We also will say that say that A acts on G and that G is an A-group.*

(b) [**b**] *Suppose A acts on G. A subgroup H of G is called A-invariant if $H^a = H$ for all $a \in A$. We also will say that H is an A-subgroup*

(c) [**c**] *We say that an action of A on G is simple, if there exists no proper normal A-subgroup of G. In this case we call G a simple A-group.*

(d) [**d**] *An action is called faithful if α is 1-1.*

(e) [**e**] *If G is an A-group, $S \subseteq G$ and $T \subseteq A$, then $C_S(T) = \{s \in S \mid s^t = s \text{ for all } t \in T\}$ and $C_T(S) = \{t \in T \mid s^t = s \text{ for all } s\}$. $C_A(G)$ is called the kernel of the action. Note here that $C_A(G) = \ker \alpha$.*

Definition 1.3.2. [**def:series**] *Let G be a group, A a group acting on G, H an A-invariant subgroup of G and H am A-invariant subgroup of G. An A-series from H to G is set $\mathcal{N}$ such that*

(i) [**i**] *If $D \in \mathcal{N}$ then D is an A- subgroup of G containing H.*

(ii) [**ii**] *$H \in \mathcal{N}$ and $G \in \mathcal{N}$.*

(iii) [**iii**] *$\mathcal{N}$ is totally ordered with respect to inclusion, that is if $D, E \in \mathcal{N}$ then $D \leq E$ or $E \leq D$.*

(iv) [**iv**] *$\mathcal{N}$ is closed under intersections and unions, that is if $\emptyset \neq \mathcal{M} \subseteq \mathcal{N}$, then $\bigcap \mathcal{M} \in \mathcal{N}$ and $\bigcup \mathcal{M} \in \mathcal{N}$.*

(v) [**v**] *For $D \in \mathcal{N} \setminus H$ define $D^- : \bigcup\{E \in \mathcal{N} \mid E < D\}$. Then $D^- \trianglelefteq D$.*

A A-series of G is a A-series from 1 to G.
A series from H to G is a 1-series from H to G.

Observe that a finite series of G is such a set of subgroups $\{N_0, N_1, N_2, \dots N_k\}$ of G such

$$1 = N_0 \trianglelefteq N_1 \trianglelefteq N_2 \trianglelefteq \dots N_{k-1} \trianglelefteq N_k = G$$

Let $\mathbb{K}$ be a field, Ω a set and V a $\mathbb{K}$-space with basis $(v_i, i \in \Omega)$, Observe that $\mathrm{Sym}(\Omega)$ acts on V via $v_i^g = v_{ig}$ for all $i \in \Omega$, $g \in \mathrm{Sym}(\Omega)$. Let $V_0 = \{\sum_{i\in\Omega} \lambda_i v_i \mid \sum_{i\in\Omega} \lambda_i = 0\}$. Then

$$0 \leq V_0 \leq V$$

is a normal Sym(Ω)-series on V. Let p be a prime, then

$$0 \ldots p^{k+1}\mathrm{Z} \leq p^k\mathrm{Z} \leq \ldots p^2\mathrm{Z} \leq p\mathrm{Z} \leq \mathrm{Z}$$

is a normal series of Z.

Definition 1.3.3. [**def:basic series**] *Let G be a group, A a group acting on G, H an A-subgroup of G, and $\mathcal{N}$ an A-series from H to G*

(i) [**a**] *If $D \in \mathcal{N} \setminus \{H\}$ with $D \neq D^-$ then D/D^- is called a factor of $\mathcal{N}$ and (D^-, D) is called a jump of $\mathcal{N}$*

(ii) [**b**] *$\mathcal{N}$ is called a normal if $D \trianglelefteq$ in G for all $D \in \mathcal{N}$.*

(iii) [**c**] *$\mathcal{N}$ is called an A-composition series from H to G if each factor of $\mathcal{N}$ is a simple A-group,*

(iv) [**d**] *$\mathcal{N}$ is called an A-chief series from H to G if $\mathcal{N}$ is a normal and no proper subgroup of a factor of $\mathcal{N}$ is invariant under A and G.*

(v) [**e**] *$\mathcal{N}$ is called ascending if $\mathcal{N}$ is well-ordered with respect to inclusion, that is every non empty subset of $\mathcal{N}$ has a minimal element.*

(vi) [**f**] *$\mathcal{N}$ is called descending if $\mathcal{N}$ is well-ordered with respect to reverse inclusion, that is every non empty subset of $\mathcal{N}$ has maximal element.*

The series

$$0 \ldots p^{k+1}\mathrm{Z} \leq p^k\mathrm{Z} \leq \ldots p^2\mathrm{Z} \leq p\mathrm{Z} \leq \mathrm{Z}$$

is a descending compositions series for Z. We claim that Z does not have an ascending compositions series. Indeed, let $\mathcal{N}$ be any ascending series of Z and let D be the minimal element of $\mathcal{N} \setminus \{1\}$. Then $D^- = 1$ and so $D \cong D/D^-$ is isomorphic to a factor of $\mathcal{N}$. Since D is a non-trivial subgroup of Z, $D \cong \mathrm{Z}$ and so D is not simple. Thus $\mathcal{N}$ is not a composition series.

Lemma 1.3.4. [**easy jumps**] *Let $\mathcal{N}$ be a series from H to G.*

(a) [**a**] *Let $B, T \in \mathcal{N}$ with $B < T$, then (B, T) is a jump of $\mathcal{N}$ if and only if $C = B$ or $C + T$ for any $C \in \mathcal{N}$ with with $B \leq C \leq T$.*

(b) [**b**] *Let $X \subseteq G$ with $X \nsubseteq H$. Put $B_X := \bigcup\{D \in \mathcal{N} \mid X \nsubseteq D\}$ and $T_x = \bigcap\{E \in \mathcal{N} \mid X \subseteq E\}$. Then $B_X \cup X \subseteq T_X$ and one of the following holds:*

1. [**1**] *$X \subseteq B_X = T_X$ and X is infinite.*

2. [**2**] $X \nsubseteq B_X < T_X$ *and* (B_X, T_X) *is the unique jump of* $\mathcal{N}$ *with* $X \subseteq T_X$ *and* $X \nsubseteq B_X$.

Proof. (a) Let (B,T) is a jump and suppose $C \in \mathcal{N}$ with $B \leq C \leq T$. Since (B,T) is a jump, $B = T^-$. If $C \neq T$ then $C \leq T^- = B$ by definition of T^-. Thus $C = B$.

Suppose now that $C = B$ or $C = T$ for all $C \in \mathcal{N}$ with $B \leq C \leq T$. Let $D \in \mathcal{N}$ with $D < T$. The $B \leq D$ or $D \leq B$. In the former case we have $B \leq D < T$ and so the assumption of $(B.T)$ implies $B = D$. So in any case $D \leq B$ and thus $T^- \leq B$. Since $B < T$, we also have $B \leq T^-$ and so $B = T^-$ and $(B,T) = (T^-, T)$ is a jump of $\mathcal{N}$.

(b) Let $D \in \mathcal{N}$ with $X \nsubseteq D$ and $E \in \mathcal{N}$ with $X \subseteq E$. Then $E \nsubseteq D$ and so $D \subseteq E$. Thus $B_X \subseteq T_X$. Clearly $X \subseteq T_X$.

Suppose that $X \subseteq B_X$. Then $T_X \subseteq B_X$ and so $T_X = B_X$. Moreover for each $x \in X$ there exists $D_x \in \mathcal{N}$ with $x \in D_x$ but $X \nsubseteq D_x$. Let $D = \bigcup_{x \in X} D_x$. Then $X \subseteq D$ and so $D \neq D_x$ for all $x \in X$. Since $\mathcal{N}$ is totally ordered this implies that X is infinite.

Suppose next that $X \nsubseteq B_X$. Then $B_X \subset T_X$. Let $D \in \mathcal{N}$ with $B_X \leq D \leq T_X$. If $X \subseteq D$, then $T_X \leq D$ and so $D = T_X$. If $X \nsubseteq D$, then $D \leq B_X$ and so $D = B_X$. Hence by (a), (B_X, T_X) is a jump.

Now let (B,T) be any jump with $X \subseteq T$ and $X \nsubseteq B$. Then by definition of B_X and T_X,

$$B \leq B_X < T_X \leq T$$

Since (B,T) is a jump, (a) implies $B = B_X$ and $T = T_X$. □

Lemma 1.3.5. [**completion**] *Let* S *be a set and* $\mathcal{N}$ *a chain of subsets of* § *(That is every member of* $\mathcal{N}$ *is a subset of* S *and if* $D, E \in \mathcal{N}$ *then* $D \subseteq E$ *or* $E \subseteq D$*) . Let* $\mathcal{N}^* = \{\bigcap \mathcal{M}, \bigcup \mathcal{M} \mid \emptyset \neq \mathcal{M} \subseteq \mathcal{N}\}$*. Then* $\mathcal{N}^*$ *complete chain of subsets of* S*, that is* $\mathcal{N}^*$ *is a chain of subsets of* $\mathcal{N}$ *and is closed under unions and intersections.*

Proof. Let $D \in \mathcal{N}^*$. Then there exists $\mathcal{D} \subseteq \mathcal{N}$ with $D = \bigcap \mathcal{D}$ or $D = \bigcup \mathcal{D}$. In the first case put $\tilde{D} = \{A \in \mathcal{N} \mid D \subseteq A\}$ and note that $D = \bigcap \tilde{\mathcal{D}}$. In second case put $\tilde{D} = \{A \in \mathcal{N} \mid A \subseteq D\}$ and notet that $D = \bigcap \tilde{\mathcal{D}}$. D is either the intersection of a subset of $\mathcal{N}$ which is closed under supersets or the unions of subset of $\mathcal{N}$ which is closed under subsets.

We will first show that

1°. [**1**] $\mathcal{N}^*$ *is a chain.*

For this let $D, E \in \mathcal{N}^*$. Suppose first that $D = \bigcap \mathcal{D}$, $E = \bigcap \mathcal{E}$ with $\mathcal{D}, \mathcal{E}$ subsets of $\mathcal{N}$. Suppose $D \nsubseteq \mathcal{E}$. Then there exists $B \in \mathcal{E}$ with $D \nsubseteq B$. Since $D \subseteq A$ for all $A \in \mathcal{D}$, we get $A \nsubseteq B$ and so $B \subseteq A$ for all $A \in \mathcal{D}$. Thus $B \subseteq \bigcap \mathcal{D}$ and so also $E \subseteq D$.

Suppose next that $D = \bigcap \mathcal{D}$ and $E = \bigcup \mathcal{E}$ with $\mathcal{D}, \mathcal{E}$ subsets of $\mathcal{N}$. Suppose $D \nsubseteq E$. Then $D \nsubseteq B$ for all $B \in \mathcal{E}$. Thus $A \nsubseteq B$ for all $A \in \mathcal{D}$ and so $B \subseteq A$. Since this holds for all $A \in \mathcal{D}$ and all $B \in \mathcal{E}$, $E = \bigcup \mathcal{E} \subseteq \mathcal{D} = D$.

Suppose next that $D = \bigcup \mathcal{D}$ and $E = \bigcup \mathcal{E}$ with $\mathcal{D}, \mathcal{E}$ subsets of $\mathcal{N}$. Suppose $D \nsubseteq E$. Then $A \nsubseteq E$ for some $A \in \mathcal{E}$. It follows that $A \nsubseteq B$ for all $B \in \mathcal{B}$ and so $B \subseteq A$. Thus $E = \bigcup \mathcal{R} \subseteq A$ and so also $E \subseteq D$. Thus (1°) holds.

Next let $\mathcal{M}$ be a nonempty chain in $\mathcal{N}^*$. Let $\mathcal{M} = \{D_i \mid i \in I\} \cup \{E_j \mid j \in J\}$ such that $D_i = \bigcap \mathcal{D}_i$, where $\mathcal{D}_i \subseteq \mathcal{N}$ is closed under supersets, and $E_j = \bigcup \mathcal{E}_j$, where $\mathcal{E}_j \subseteq \mathcal{N}$ is closed under subsets.

2°. [**2**] $\quad \bigcap \mathcal{M} \in \mathcal{N}^*$.

Put $D = \bigcap_{i\in I} D_i$ and $E = \bigcap_{j\in J} E_j$. Then $\bigcap \mathcal{M} = D \cap E$. Observe that $D = \bigcap \left(\bigcup_{i\in I} \mathcal{D}_i\right)$ and so $D \in \mathcal{N}^*$. If $E \in \mathcal{N}^*$, the since $\mathcal{N}^*$ is a chain $D \cap E = D$ or $D \cap E = E$. In either case $D \cap E \in \mathcal{N}^*$. So to complete the proof of (2°) to show that $E \in \mathcal{N}^*$.

Put $\mathcal{E} = \bigcap_{j\in J} \mathcal{E}_j$. We claim that

$$(*) \qquad \bigcup \mathcal{E} \leq E \leq \bigcap (\mathcal{N} \setminus \mathcal{E})$$

Indeed let $A \in \mathcal{E}$. Then $A \in \mathcal{E}_j$ for all $j \in J$ and so $A \leq \bigcap \mathcal{E}_j = E_j$ and $A \leq \bigcap_{j\in J} E_j = E$. Thus $\bigcup \mathcal{E} \leq E$.

Also if $B \in \mathcal{N} \setminus \mathcal{E}$, the $B \notin \mathcal{E}_k$ for some $k \in J$. Since $\mathcal{E}_k$ is closed under subsets, this means $B \nsubseteq X$ and $X \subseteq B$ for all $X \in \mathcal{E}_k$. Thus $E_k = \bigcup \mathcal{E}_k \leq B$ and $E = \bigcap_{j\in J} E_j \leq E_k \leq B$. Since thus holds for all $B \in \mathcal{N} \setminus \mathcal{E}$, $E \leq \bigcap (\mathcal{N} \setminus \mathcal{E})$. So (*) is proved.

If $\bigcap \mathcal{N} \setminus \mathcal{E} \subseteq E$ we conclude that $E = \bigcap \mathcal{N} \setminus \mathcal{E} \in \mathcal{N}^*$.

So suppose that $\bigcap \mathcal{N} \setminus \mathcal{E} \nsubseteq E$. Since $E = \bigcap_{j\in J} E_j$ this means that $\bigcap \mathcal{N} \setminus \mathcal{E} \subseteq E_k$ for some $k \in J$. Let $A \in \mathcal{N} \subseteq \mathcal{E}$. It follows that $A \nsubseteq E_k$ and hence $A \nsubseteq B$ for $B \in \mathcal{E}_k$. In particular, $A \notin \mathcal{E}_k$. We proved that $\mathcal{N} \setminus \mathcal{E} \subset \mathcal{N} \setminus \mathcal{E}_k$ and so $\mathcal{E}_k \subseteq \mathcal{E}$. As $\mathcal{E} \subseteq \mathcal{E}_k$, we have $\mathcal{E}_k = \mathcal{E}$. Thus

$$E = \bigcap_{j\in J} E_j \leq E_k = \bigcup \mathcal{E}_k = \bigcup \mathcal{E}$$

and (*) gives $E = \bigcup \mathcal{E} \in \mathcal{N}^*$.

3°. [**3**] $\quad \bigcup \mathcal{M} \in \mathcal{N}^*$.

Put $D = \bigcup_{i\in I} D_i$ and $E = \bigcup_{j\in J} E_j$. Then $\bigcup \mathcal{M} = D \cup E$. Observe that $E = \bigcup \bigcup_{i\in I} \mathcal{E}_i$ and so $E \in \mathcal{N}^*$. If $D \in \mathcal{N}^*$, then since $\mathcal{N}^*$ is a chain $D \cup E = D$ or $D \cup E = E$. In either case $D \cup E \in \mathcal{N}^*$. So to complete the proof of (3°) to remains show that $D \in \mathcal{N}^*$.

Put $\mathcal{D} = \bigcap_{i\in I} \mathcal{D}_i$. We claim that

$$(**) \qquad \bigcup (\mathcal{N} \setminus \mathcal{D}) \leq D \leq \bigcap \mathcal{D}$$

Indeed let $A \in \mathcal{D}$. Then $A \in \mathcal{D}_i$ for all $i \in I$ and so $D_i \bigcup \mathcal{D}_i \leq A$. Thus $D = \bigcup \mathcal{D} \leq A$ and so $D \leq \bigcap \mathcal{D}$.

Also if $B \in \mathcal{N} \setminus \mathcal{D}$, then $B \notin \mathcal{D}_k$ for some $k \in I$. Since $\mathcal{D}_k$ is closed under supersets, this means $X \nsubseteq B$ and $B \subseteq X$ for all $X \in \mathcal{D}_k$. Thus $B \leq \bigcap \mathcal{D}_k = D_k$ and $B \leq D_k \leq \bigcup_{i \in I} D_i = D$. Since thus holds for all $B \in \mathcal{N} \setminus \mathcal{E}$, $\bigcup(\mathcal{N} \setminus \mathcal{D}) \leq D$. So (**) holds.

If $D \leq \bigcup(\mathcal{N} \setminus \mathcal{D})$ we conclude that $D = \bigcup \mathcal{N} \setminus \mathcal{D} \in \mathcal{N}^*$.

So suppose that $D \nleq \bigcup \mathcal{N} \setminus \mathcal{D}$. Since $D = \bigcup_{i \in I} D_i$ this means that $D_k \nleq \bigcup \mathcal{N} \setminus \mathcal{D}$ for some $k \in I$. Let $A \in \mathcal{N} \subseteq \mathcal{D}$. It follows that $D_k \nsubseteq A$. Since $D_k = \bigcap \mathcal{D}_k$, $B \nsubseteq A$ for $B \in \mathcal{D}_k$. In particular, $A \notin \mathcal{D}_k$. We proved that $\mathcal{N} \setminus \mathcal{D} \subset \mathcal{N} \setminus \mathcal{D}_k$ and so $\mathcal{D}_k \subseteq \mathcal{D}$. As $\mathcal{D} subseteq \mathcal{D}_k$, we have $\mathcal{D}_k = \mathcal{D}$. Thus

$$D = \bigcup_{i \in I} D_k \geq D_k = \bigcap \mathcal{D}_k = \bigcap \mathcal{D}$$

and (**) gives $D = \bigcap \mathcal{D} \in \mathcal{N}^*$. □

Lemma 1.3.6. [**char comp**] *Let G be an A-group and $\mathcal{N}$ an A-series from H to G. Order the set of A-series from H to G by inclusion.*

(a) [**a**] *If $\mathcal{N}$ is a maximal A-series from H to G, then $\mathcal{N}$ is an A-composition series from H to G.*

(b) [**b**] *Suppose $\mathcal{N}$ is normal. Then $\mathcal{N}$ is a maximal normal series from H to G if and only if $\mathcal{N}$ is a chief-series from H to G.*

(c) [**c**] *There exists a maximal A-series from H to G containing $\mathcal{N}$. In particular, there exists a A-composition series from H to G containing $\mathcal{N}$.*

(d) [**d**] *Suppose $\mathcal{N}$ is normal. There exists a maximal normal A-series from H to G containing $\mathcal{N}$. In particular, there exists a A-series from H to G containing $\mathcal{N}$.*

Proof. (a) Suppose cN is a maximal A-series from H to G. Let (B, T) be a jump of $\mathcal{N}$ and let $\overline{D}$ be a A-invariant normal subgroup of T/B. Then $\overline{D} = D/B$ for normal A-subgroup of G with $B \leq D \leq T$. It is readily verified that $\mathcal{N} \cup \{D\}$ is an A-series from H to G. So the maximality of $\mathcal{N}$ shows that $D \in \mathcal{N}$ and so $D = B$ or $D = T$. Thus T/B is a simple A-group and $\mathcal{N}$ is an A-composition series.

(b) If $\mathcal{N}$ is a maximal normal series from H to G, then the argument in (a) shows that $\mathcal{N}$ a chief-series. (Alternatively let$A*G$ be the free product of A and G. Then $A*G$ acts on G and a normal A-series from H to G is the same as $A*G$ series. Also an $A*G$-composition series is the same an A-chiefseries.)

Now let $\mathcal{N}$ be a A-chief series from H to Gb and $\mathcal{M}$ a normal A-series from H to G with $\mathcal{N} \subseteq \mathcal{M}$. Let $M \in \mathcal{M} \setminus \{H\}$. Put $T = \bigcap\{E \in \mathcal{N} \mid M \leq E\}$ and $B = \bigcup\{D \in \mathcal{N} \mid M \nleq D\}$. Since $\mathcal{N}$ is totally order $M \nleq D$ for $E \in \mathcal{N}$ implies $D \leq M$. Thus $B \leq M \leq T$. If $M = T$, then $M \in \mathcal{N}$. So suppose $M \neq T$. Then also $B \neq T$ and by **??**(**??**), (B, T) is a jump of $\mathcal{N}$. Since $\mathcal{M}$ is normal, M/B is G and A-invariant subgroup of T/B. Since $\mathcal{N}$is a A-chiefseries, this implies $M/B = 1$ and so $M = B \in \mathcal{N}$.

Thus $\mathcal{M} = \mathcal{N}$.

(c) By (a) it suffices to proof that $\mathcal{N}$ is contained in a maximal A-series from H to G. Let $(\mathcal{M}_i, i \in I)$ be a chain of A-series from H to G. Let $\mathcal{M} = \bigcup_{i \in I} \mathcal{M}_i$ and observe that $\mathcal{M}$ is a chain of A subgroups of G containing H and G. Let $\mathcal{M}^*$ be the set of intersection and unions of non-subsets of $\mathcal{M}$. Using 1.3.5 we conclude that $\mathcal{M}^*$ is a set of A-invariant subgroups of G which is closed under intersection and unions. We claim that $\mathcal{M}^*$ is an A-series. 1.3.2(i)-iv are obvious. So let (B, T) be a jump of $\mathcal{M}^*$. We need to show that $B \trianglelefteq T$. For $i \in I$ define $B_i := \bigcup\{D \in \mathcal{N}_i \mid T \nleq D\}$ and $T_i = \bigcup\{E \in \mathcal{N}_i \mid T \nleq E\}$. Since $\mathcal{M}^*$ is a chain, $B_i = \bigcup\{D \in \mathcal{N}_i \mid D < T\}$. Thus $B_i \leq B < T \leq T_i$. Thus by 1.3.4(b), (B_i, T_i) is a jump of $\mathcal{N}_i$ and so $B_i \trianglelefteq T_i$. In particular, $B_i \trianglelefteq T$. By definition of $\mathcal{M}^*$, $B = \bigcup \mathcal{B}$ or $B = \bigcap \mathcal{B}$ for non-empty subset $\mathcal{B}$ of $\mathcal{M}$. Suppose first that $B = \bigcup \mathcal{B}$. Let $D \in \mathcal{B}$, then $D \in \mathcal{N}_i$ for some $i \in I$. Since $D \leq B < T$ we get $B \leq B_i$. It follows that

$$B = \bigcup \mathcal{B} \leq \bigcup_{i \in I} B_i \leq D$$

and so $B = \bigcup_{i \in I} B_i$. Since each B_i is normal in T we conclude that $B \trianglelefteq T$.

Suppose next that $B = \bigcap \mathcal{B}$. Since $T \nleq B$, there exists $D \in \mathcal{B}$ with $T \nleq D$. Since $\mathcal{M}^*$ is chain this gives $D < T$ and so $D \leq B$. Thus $D \leq B = \bigcap \mathcal{B} \leq D$ and $B = D$. So B is a union of members of $\mathcal{M}$ and so we are done by the previous case.

(d) Either use the same argument as in (c) or apply (c) to $A * G$. □

Definition 1.3.7. [**def:class of action**]

(a) [**b**] *Two actions (A, G) and (A^*, G^*) are called isomorphic and we write $(A, G) \cong (A^*, G^*)$ if there exist isomorphisms $\beta : A \to A^*$ and $\gamma :\to G^*$ with $g^a\gamma = (g\gamma)^{a\beta}$ for all $g \in G$ and $a \in A$.*

(b) [**c**] *A class of actions is class $\mathcal{X}$ such that*

- *(a)* [**a**] *The members of $\mathcal{X}$ are faithful actions*
- *(b)* [**b**] *If $D \in \mathcal{X}$ and $D^* \cong D$ then $D^* \in \mathcal{X}$.*
- *(c)* [**c**] $(1, 1) \in \mathcal{X}$.

(c) [**d**] *If $\mathcal{X}$ and $\mathcal{Y}$ are classes of groups, then $[\mathcal{X}, \mathcal{Y}]$ denotes of class of all faithful actions (A, G) with $A \in \mathcal{X}$ and $H \in \mathcal{Y}$*

Definition 1.3.8. [**def:xa series**] *Let $\mathcal{X}$ be a class of actions.*

(a) [**z**] *We say that A acts $\mathcal{X}$ on a group G, or that G is a $\mathcal{X} - A$ group, if $(A/C_A(G), G) \in \mathcal{X}$.*

(b) [**a**] *An A-series $\mathcal{N}$ from H to G is called called $\mathcal{X}$-A-series if each factor of $\mathcal{N}$ is an $\mathcal{X} - A$-group.*

(c) [**b**] *We say that A acts poly-$\mathcal{X}$ on G, or that G is poly $\mathcal{X} - A$group, if there exists G is exists a finite normal $\mathcal{X} - A$-series on G.*

(d) [**c**] *We say that A acts hyper-$\mathcal{X}$ on G, or that G is hyper $\mathcal{X} - A$-group, if there exists an ascending normal $\mathcal{X} - A$-series on G.*

(e) [**d**] *We say that A acts hypo-$\mathcal{X}$ on G, or that G is hypo $\mathcal{X}$-group, if there exists G is exists descending normal $\mathcal{X} - A$-series on G.*

(f) [**e**] *If $A = G$ acting by conjugation on G we drop the prefix A in (b) to (c).*

We usually write $[\mathcal{X}, *]$ in place of $[\mathcal{X}, \mathcal{D}]$ and $[\mathcal{X}, 1]$ in place of $[\mathcal{X}, \mathcal{T}]$. Recall here that $\mathcal{T}$ denotes the calls of trivial groups and $\mathcal{D}$ the class of all groups.

If $\mathcal{X}$ is the calls of simple actions, then an $\mathcal{X} - A$-series is just an A-composition series.

If $\mathcal{X}$ is a class of groups, then a poly $[*, \mathcal{X}] - 1$-group is just a poly-$\mathcal{X}$-group. So a poly $[*, \mathcal{A}] - 1$-group, is a poly abelian group, that is a solvable group. A hyper $[*, \mathcal{X}]$-group, is called an hyper $\mathcal{X}$-group and a hypo $[*, \mathcal{X}] - 1$-group, is called an hypo $\mathcal{X}$-group. Note that a hyper $\mathcal{X}$-group is a group with normal ascending series all of whose factors are $\mathcal{X}$-groups.

A poly $[1, *]$-groups is called nilpotent. So a group is nilpotent if and only if there exists a finite normal ascending series

$$N_0 = 1 \leq N_1 \leq N_2 \leq \ldots \leq N_{k-1} \leq N_k = G$$

such that $(G/C_G(E) \in [1, *]$ for all factors E of the series. Note that thus just means that $G/C_G(E) = 1$, that is G centralizes E. In other words, $[N_i, G] \leq N_{i-1}$ for all $1 \leq i \leq k$.

A hyper $[1, *]$-groups is called a hypercentral group and a hypo $[1, *]$-group is called a hypocentral group. So a hypercentral group is a group G with a normal series all of whose factors are centralized by G.

Consider the chief-series

$$1 \trianglelefteq \mathrm{Alt}(3) \trianglelefteq \mathrm{Sym}(3)$$

of Sym(3). The factors of this series are $E_1 = \mathrm{Alt}(3)/1 \cong \mathrm{C}_3$ and $E_2 = \mathrm{Sym}(3)/\mathrm{Alt}(3) \cong \mathrm{C}_2$. Moreover, $\mathrm{C}_{\mathrm{Sym}(3)}(E_1) = \mathrm{Alt}(3)$, $\mathrm{Sym}(3)/\mathrm{C}_{\mathrm{Sym}(3)}(E_1) \cong \mathrm{C}_2$, $\mathrm{C}_{\mathrm{Sym}(3)}(E_2) = \mathrm{Sym}(3)$ and $\mathrm{Sym}(3)/\mathrm{C}_{\mathrm{Sym}(3)}(E_2) = 1$. So the group induced on each of the factors is abelian and so Sym(3) is an poly-$[\mathcal{A}, *]$-group.

Consider the chief-series

$$1 \trianglelefteq K := \langle (12)(34), (13)(23) \rangle \trianglelefteq \mathrm{Alt}(4) \trianglelefteq \mathrm{Sym}(4)$$

of Sym(4). The factors of this series are $E_1 := K/1 \cong \mathrm{C}_2 \times \mathrm{C}_2$, $E_1 = \mathrm{Alt}(4)/K \cong \mathrm{C}_3$ and $E_2 = \mathrm{Sym}(4)/\mathrm{Alt}(4) \cong \mathrm{C}_2$. Moreover, $\mathrm{C}_{\mathrm{Sym}(4)}(E_1) = K$,$\mathrm{Sym}(4)/\mathrm{C}_{\mathrm{Sym}(4)}(E_1) \cong \mathrm{Sym}(3)$, $\mathrm{C}_{\mathrm{Sym}(4)}(E_2) = \mathrm{Alt}(4)$, $\mathrm{Sym}(4)/\mathrm{C}_{\mathrm{Sym}(4)}(E_2) \cong \mathrm{C}_2$, $\mathrm{C}_{\mathrm{Sym}(4)}(E_3) = \mathrm{Sym}(4)$ and $\mathrm{Sym}(4)/\mathrm{C}_{\mathrm{Sym}(4)}(E_3) = 1$. Since the group induced on E_1 is not abelian, we conclude that Sym(4) is not poly-$[\mathcal{A}, *]$-group.

We will later see that every poly-$[\mathcal{A}, *]$ group is solvable. So the class of poly-$[\mathcal{A}, *]$ groups is a proper subclass of $\mathcal{S}$.

Lemma 1.3.9. [**factors of an ascending series**]. *Let $\mathcal{N}$ be an A-series from H to G, and M an A-subgroup of G.*

(a) [**a**] *Define $\mathcal{N} \wedge M := \{D \cap M \mid D \in \mathcal{N}\}$. Then $\mathcal{N}$ is an A-series from $H \cap M$ to M. If $(\tilde{B}, \tilde{T})$ is a jump of $\mathcal{N} \wedge M$ then there a jump (B, T) of M such that $\tilde{B} = B \cap M$, $\tilde{T} = T \cap M$ and $\tilde{T}/\tilde{B} \cong (T \cap M)B/B$ as an A-group. In particular, any factor of $\mathcal{N} \wedge M$ is isomorphic to an A-subgroup of a factor of $\mathcal{N}$.*

(b) [**b**] *Suppose $M \trianglelefteq G$ and $\mathcal{N}$ is ascending. Then $\overline{\mathcal{N}} := \mathcal{N}M/M := \{DM/M \mid D \in \mathcal{N}\}$ is an ascending A-series from HM/M to G/M. Moreover, if $(\overline{B}, \overline{T})$ is a jump of $\overline{\mathcal{N}}$, then there exists a jump (B, T) of $\mathcal{N}$ with $\overline{B} = BM/M, \overline{T} \cong TM/M$ and $\overline{T}/\overline{B} \cong T/(T \cap M)B$. In particular, each factor of $\overline{\mathcal{N}}$ is isomorphic to an A-quotient of a factor of $\mathcal{N}$.*

Proof. (a) Readily verified.

(b) The first three axioms of an A series are obvious. Let $\overline{\mathcal{M}}$ be an non-empty subset of $\overline{\mathrm{N}}$ and define $\mathcal{M} = \{D \in \mathcal{N} \mid DN/N \in \overline{\mathcal{M}}$.

1°. [**1**] *Put $B = \bigcup \mathcal{M}$. Then $\bigcup \mathcal{M} = BM/M$.*

Let $x \in BM/M$, then $x = eM$ for some $e \in B$. Pick $D \in \mathcal{M}$ with $e \in D$. Then $x = eM \in DM/M \in \overline{\mathcal{M}}$. and so $BM/M \subseteq \bigcup \overline{\mathcal{M}}$.

Conversely if $\overline{e} \in \bigcup \overline{\mathcal{M}}$, the $\overline{e} \in \overline{D}$ for some $\overline{D} \in \overline{\mathcal{M}}$. Note that $\overline{D} = DM/M$ for some $D \in \mathcal{M}$ and then $\overline{e} = eM$ for some $e \in D$. Thus $e \in B$ and $\overline{e} \in BM/M$. Hence $\bigcup \mathcal{M} \subseteq BM/M$ and (1°) holds.

2°. [**2**] *Let T be the minimal element $\mathcal{M}$ (which exists since $\mathcal{N}$ is well ordered). Then $\bigcap \overline{\mathcal{M}} = TM/M$.*

Let $\overline{D} \in \overline{\mathcal{M}}$. Then $\overline{D} = DM/M$ for some $D \leq \mathcal{M}$. Since T is the minimal element of $\mathcal{M}$ we get $T \leq D$ and so $TM/M \leq DM/M = \overline{D}$ and $TM/M \leq \bigcap \overline{\mathcal{M}}$.

Conversely, $T \in \mathcal{M}$ and so $TM/M \leq \overline{\mathcal{M}}$. Hence $\bigcap \overline{\mathcal{M}} \leq TM/M$ and (2°) is proved.

By (1°) and (2°), $\overline{\mathcal{M}}$ is closed under unions and intersection.

Noe let $(\overline{B}, \overline{T})$ be a jump of $\overline{c}M$. Let $B = \bigcup \{D \in \mathcal{N} \mid DM/M = \tilde{B}$. Then (for example by (1°) applied with $\overline{\mathcal{M}} = \{\overline{B}\}$, $BM/M = \overline{B}$. Let T be minimal in $\mathcal{N}$ with $TM/M = \overline{T}$. Since $BM/M = \phi B < \overline{T} = TM/M$ we have $BM < TM$ and so $T \not\leq B$. Since $\mathcal{N}$ is totally ordered, $B < T$. We claim that (B, T) is a jump of $\mathcal{N}$ so let $D \in \mathcal{N}$ with $B \leq D \leq T$. Then $\overline{B} = BM/M \leq DM/M \leq TM/M = \overline{T}$ and since $(\overline{B}, \overline{T})$ is a jump of $\overline{\mathcal{N}}$ we conclude that $DM/M = \overline{B}$ or $DM/M = \overline{T}$. In the first case the definition of B shows that $D \leq B$ and so $D = B$. In the second case the minimality of T gives, $T \leq D$ and so $D = T$. Hence (B, T) is a jump. Since $\mathcal{N}$ is a series this implies that $B \trianglelefteq T$. Hence also $\overline{B} = BM/M \trianglelefteq TM/M = \overline{T}$ and so $\overline{\mathcal{N}}$ is a series.

We compute

$$\overline{B}/\overline{T} = TM/M\big/BM/M \cong TM/BM = T(BM)/BM$$

$$\cong T/T \cap BM = T/(T \cap B)M \cong T/B\big/(T \cap M)B/B$$

and so also the remaining statements in (b) are proved. □

Definition 1.3.10. [**def:s for action**] *Let $\mathcal{X}$ be a class of actions.*

(a) [**a**] $[\mathrm{id}, \mathbf{S}]\mathcal{X}$ *denotes the class of all actions isomorphic to an action $(A/C_A(H), H)$, where $(A, G) \leq \mathcal{X}$ and H is an A-subgroup of G.*

(b) [**c**] $[\mathbf{S}, \mathrm{id}]\mathcal{X}$ *denotes the class of all actions isomorphic to an action (B, G), where $(A, G) \leq \mathcal{X}$ and B is a A-subgroup of G.*

(c) [**d**] $\mathbf{S}\mathcal{X}$ *denotes the class of all actions isomorphic to an action $(B/C_B(H), H)$, where $(A, G) \leq \mathcal{X}$, $B \leq A$ and H is an B-subgroup of G.*

(d) [**b**] $\mathbf{H}\mathcal{X}$ *denotes the class of all actions isomorphic to an action $(A/C_A(H), G/H)$, where $(A, G) \leq \mathcal{H}$ and H is a normal A-subgroup of G.*

Note that $\mathbf{S}\mathcal{X} = [\mathrm{id}, \mathbf{S}][\mathbf{S}, \mathrm{id}]\mathcal{X}$, but in general $\mathbf{S}\mathcal{X} \neq [\mathbf{S}, \mathrm{id}][\mathrm{id}, \mathbf{S}]\mathcal{X}$.

Corollary 1.3.11. [**s h a hyp**] *Let $\mathcal{X}$ be a class of actions, A a group, G a hyper $\mathcal{X} - A$-group and M an A-subgroup of G.*

(a) [**a**] *If $\mathcal{X}$ is* $[\mathrm{id}, \mathbf{S}]$ *closed, then M is a hyper $\mathcal{X} - A$-group.*

(b) [**b**] *If $\mathcal{X}$ is* $\mathbf{H}$*-closed and $M \trianglelefteq G$, then G/M is a hyper $\mathcal{X} - A$-group.*

Proof. This follows immediately from 1.3.9. □

Corollary 1.3.12. [**s hyp**] *Let $\mathcal{X}$ be class of groups, G a hyper $\mathcal{X}$-group and $M \leq G$.*

(a) [**a**] *If $\mathcal{X}$ is* $\mathbf{S}$*-closed, then* $\mathrm{Hyp}(\mathcal{X})$ *is* $\mathbf{S}$*-closed.*

(b) [**b**] *If $\mathcal{X}$ is* $\mathbf{H}$*-closed, then* $\mathrm{Hyp}(\mathcal{X})$ *is* $\mathbf{H}$*-closed.*

Proof. (a) Since $\mathcal{S}$ is $[\mathbf{S}, \mathrm{id}]$ closed, M acts hyper $\mathcal{X}$ on G. So (a) follows from 1.3.11(a).

(b) By **??**(**??**), G acts hyper $\mathcal{X}$ on G/M. Since M acts trivially on G/M, also G/M acts hyper $\mathcal{X}$ in G/M. □

Corollary 1.3.13. [**zg cap n**]

(a) [**a**] *Subgroups and quotients of hypercentral groups are hypercentral.*

(b) [**b**] *Let M be a normal subgroup of the hypercentral group G, then G acts hyper centrally on G. In particular, $M \cap \mathrm{Z}(G) \neq 1$.*

Proof. Since $[1, *]$ is $\mathbf{S}$ and $\mathbf{H}$ closed, we can apply the previous two corollaries. □

1.4 Hyper Sequences

Definition 1.4.1. [**def:ascending sequence**] *Let G be an A-group, H an A-subgroup of G. Then an A-sequence from H to G is a a sequence $(G_\alpha)_{\alpha\in\mathrm{Ord}}$ of A-subgroups of G such that*

(a) [**a**] $G_0 = H$ *and there exists* $\delta \in \mathrm{Ord}$ *with* $G_\beta = G$ *for all* $\beta \geq \delta$.

(b) [**b**] $G_\alpha \trianglelefteq G_{\alpha+1}$

(c) [**c**] *If α is limit ordinal, then* $G_\alpha = \bigcup_{\alpha<\beta} G_\beta$.

Lemma 1.4.2. [**ascending ord**] *Let $\mathcal{N}$ be an ascending A-series from H to G. Then there exists an A-sequence $(G_\alpha)_{\alpha\in\mathrm{Ord}}$ from H to G with $\mathcal{N} = \{G_\alpha \mid \alpha \in \mathrm{Ord}\}$.*

Proof. Since $\mathcal{N}$ is well ordered with respect to inclusion we conclude from Homework 3, that there exists an ordinal δ and an isomorphism of ordered sets, $F : \delta \to \mathcal{N}, \alpha \to G_\alpha$. Define $\Phi : \mathrm{Ord} \to \mathcal{N}$ by $\Phi(\alpha) = H_\alpha$ if $\alpha < \delta$ and $\Phi(\beta) = G$ if $\delta < \beta$. Since 0 is the element of δ and H the minimal element of $\mathcal{N}$ we have $G_0 = F(0) = H$. Since F preserved the order we have $\alpha \leq \beta$ if and only if $G_\alpha \leq G_\beta$. Since either $\beta \leq \alpha$ or $\alpha+ \leq \beta$ we conclude that either $G_\alpha = G_{\alpha+1}$ or $(G_\alpha, G_{\alpha+1})$ is a jump of $\mathcal{N}$. In both cases $G_\alpha \trianglelefteq G_{\alpha+1}$.

Now let α be a limit ordinal and put $M := \bigcup_{\beta<\alpha} G_\beta$. Then $M \in \mathcal{N}$ and $M \leq G_\alpha$ and so $M = G_\gamma$ for some γ in $\gamma \in \delta$. Since $G_\gamma \leq G_\alpha$ we have $\gamma \leq \alpha$. If $\gamma = \alpha$ we are done. So suppose $\gamma < \alpha$. Then also $\gamma + 1 < \alpha$ and so $G_{\gamma+1} \leq M \leq G_\gamma \leq G_{\gamma+1}$. Thus $G_\gamma = G_{\gamma+1}$. Since F is a bijection, this gives $\gamma + 1 \notin \delta$. Thus $G = G_{\gamma+1} = M \leq G_\alpha \leq G$. So again $M = G = G_\alpha$ and all parts of the definition of a A-sequence from H to G are verified. □

Lemma 1.4.3. [**ord ascending**] *Let G be an A-group, H an A-subgroup of G and and $(G_\alpha)_{\alpha\in\mathrm{Ord}}$ a sequence of A-sequence from A to G. Then $\mathcal{N} := \{G_\alpha \mid \alpha \in \mathrm{Ord}\}$ is an ascending A-series from H to G. Moreover, the jumps of $\mathcal{N}$ are exactly the pairs $(G_\alpha, G_{\alpha+1})$, where α is an ordinal with $G_\alpha \neq G_{\alpha+1}$.*

Proof. Note that $\mathcal{N} = \{G_\alpha \mid \alpha \leq \delta\}$, so $\mathcal{N}$ is the image of a set under function and thus a set. From (**??**) and (**??**) we have $G_\alpha \leq G_\beta$ for all $\alpha \leq \beta$ and so $\mathcal{N}$ is totally ordered with respect to inclusion. So (**??**) gives $H \in \mathcal{N}$, $G \in \mathcal{N}$ and $H \leq G_\alpha$ for all $\alpha \in \mathrm{Ord}$.

Let $\mathcal{M}$ be a non empty subset $\mathcal{N}$ and let $M = \{\alpha \in \mathrm{Ord} \mid \alpha \in \mathcal{M}$. Then M has minimal element m and so $\bigcup \mathcal{M} = G_m \in \mathcal{N}$

Suppose that $\delta \leq \beta$ for some $\beta \in \mathrm{Ord}$. Then $\bigcup \mathcal{M} = G \in \mathcal{N}$.

Suppose that $\beta < \delta$ for all $\beta \in \mathrm{Ord}$. Then M has a least upper bound γ. If $\gamma \in M$, then $\bigcup \mathcal{M} = G_\gamma \in \mathcal{N}$. If $\gamma \notin M$ the for all $\beta < \delta$ there exists $\beta^* \in \delta$ with $\beta < \beta^* < \delta$. In particular δ is limit ordinal and

$$G_\gamma = \bigcup_{\beta<\delta} G_\beta \leq \bigcup_{\beta<\delta} G_{\beta^*} \leq \bigcup \mathcal{M} \leq \bigcup_{\beta<\delta} G_\beta = G_\gamma$$

Hence again $\bigcup_{\beta<\delta} = G_\gamma \in \mathcal{N}$. We show that $\mathcal{N}$ is closed under intersections.

Noe let $D \in \mathcal{N}$ with $D \neq H$ and let $\alpha \in \mathrm{Ord}$ be minimal with G_α. The $G_\beta < D$ if and only if $\beta < \alpha$. Thus

$$D^- = \bigcup\{E \in \mathcal{N} \mid E < D\} = \bigcup_{\beta<\alpha} G_\beta$$

If α is a limit ordinal, the latter unions is G_α and if α is a successor it is $(G_{\alpha-1}$. So if (D^-, D) is a jump then α is a successor, $(D, D^-) = (G_{\alpha-1}, G_\alpha)$, $G_{\alpha-1} \neq G_\alpha$ and $D^- = G_{\alpha-1} \trianglelefteq G_\alpha = D$. In particular, $\mathcal{N}$ is an ascending series.

If α is an ordinal with $G_\alpha \neq G_{\alpha+1}$ the clearly $(G_\alpha, G_{\alpha+1})$ is a jump of $\mathcal{N}$. So also the second statement of the lemma holds. □

Note that we allow $G_\alpha = G_\beta$ for distinct $\alpha, \beta \in \mathrm{Ord}$. So a given ascending A-series corresponds to more than then one A-sequence. We will use all the notation introduces from ascending A -series. For example an hyper A-sequence is a normal A-sequence, that is a A-sequence with $G_\alpha \trianglelefteq G$ for all $\alpha \in \mathrm{Ord}$.

Definition 1.4.4. [**def:strongly hyper**] *Let $\mathcal{X}$ be class of groups and G an A-group. We say that A acts strongly hyper-$\mathcal{X}$ on G or that G is a strongly-hyper $\mathcal{X} - A$ group, if for all normal A-subgroups, M of G with $M \neq G$ there exists an normal A-subgroup M^* of G with $(A/C_A(M^*/M), M^*/M) \in \mathcal{X}$.*

Lemma 1.4.5. [**strong hyper**] *Let $\mathcal{X}$ be a class of actions and G an A-group.*

(a) [**a**] *If A acts strongly hyper-$\mathcal{X}$ on G, then A acts hyper-$\mathcal{X}$ on G.*

(b) [**b**] *If $\mathcal{X}$ is* **H***-closed that A acts strongly hyper-$\mathcal{X}$ on G iff A act hyper $\mathcal{X}$ on G.*

Proof. (a) By the definition of strongly-hyper and the axiom of choice we can choose a function $M \to M^*$ on the normal subgroups of G such that $M^* = G$ if $M = G$ and $M < M^*$ with $(A/C_A(M^*/M), M^*/M) \in \mathcal{X}$ if $M \neq G$. If f is any function which is a set, define $\tau(f) = \bigcup\{f(M)*\rangle \mid M \in \mathrm{Dom}(f)\rangle$ provided that all members of $\mathrm{Dom}(f)$ are normal A-subgroups A and $\tau(f) = 0$ otherwise.

By the 'Recursion' Theorem **??** for each ordinal α there exists function F such that $\tau(F \mid (\mathrm{Ord}_\alpha)) = F(\alpha)$ for all ordinals α. Put $N_\alpha = F(\alpha)$. Then a moments thought reveals that

$$\begin{cases} N_\alpha = 1 & \text{if } \alpha = 0 \\ N_\beta^* & \text{if } \alpha = \beta + 1 \\ \bigcup_{\beta<\alpha} N_\beta & \text{if } \alpha \text{ is a limit ordinal} \end{cases}$$

Let α be an ordinal with $|\alpha| > |G|$. If $G \neq M_\beta$ for all $\beta \leq \alpha$ we get $|G| \leq |\alpha$, a contradiction. Thus $G_\alpha = G$ and it follows that $\mathcal{N} = \{G_\alpha \mid \alpha\}$ is an hyper A-series on G with factors $N_{\alpha+1}/N_\alpha = N_\alpha^*/N_\alpha$. Thus A acts $\mathcal{X}$ in each factor of $\mathcal{N}$ and so $\mathcal{N}$ is hyper $\mathcal{X} - A$-series.

(b) Suppose A acts hyper $\mathcal{X}$ on G and let M be a normal A-subgroup of G. By **??** G/M is a hyper $\mathcal{X} - A$-group. In particular, G/M has a non-trivial normal $\mathcal{X} - A$-subgroup, M^*/M. Thus A acts strongly $\mathcal{X}$ on G. Together with (a) this gives (b). □

Notation 1.4.6. [**not:f**] *F denotes the free group on $(x_i)_{i\in 1}^{\infty}$. The elements of F are called words.*

Definition 1.4.7. [**almost decreasing**] *Let $W = (W_i)_{i=1}^{\infty} \in \mathcal{W}^{\infty}$ be a sequence of sets of words.*

(a) [**a**] *W is decreasing if $W_{i+1}(F) \leq W_i(F)$ for all i.*

(b) [**b**] *W is almost decreasing if for all $i, j \in \mathbb{Z}^+$ there exists $k \geq j$ with $W_k(F) \leq W_i(F)$.*

(c) [**c**] *$\mathcal{V}(W) = \bigcup_{i=1}^{\infty} \mathcal{V}(W_i)$.*

Lemma 1.4.8. [**trivial dec**] *Let G be group.*

(a) [**a**] *Let V, W be sets of words with $V(F) \leq W(F))$. Then $V(G) \leq V(W)$.*

(b) [**b**] *Let $W = (W_i)I = 1^{\infty}$ be almost decreasing sequence of sets words. Then $(W_i(G))_{i=1}^{\infty}$ is almost decreasing, that is for $i, j \in \mathbb{Z}^+$ there exists $k \geq j$ with $W_k(G) \leq W_i(G)$.*

Proof. (a) Let $g \in V(G)$. Then $g \in V(H)$ for some finitely generated subgroup H of G. Since H is countable, there exists an onto homomorphism $\alpha : F \to H$. Then

$$g \in V(H) = \alpha(V(F))) \leq \alpha(W(V)) = W(H) \in W(G)$$

(b) follows from (a) □

Lemma 1.4.9. [**sdp**] *Let G be an A-group then there exists a group H such that $A \leq H$, $G \trianglelefteq H$, $H = GA$, $A \cap G = 1$ and the actions of G on A is the same as the action of G on A by conjugation in H. Moreover, H is unique up to an isomorphism centralizing A and G.*

Proof. Suppose first that H is such a group. Let $x, y \in H$. Then there exists $a, b \in A$ and $g, h \in H$ with $x = ga$ and $y = bh$. Then $xy = (ga)(hb) = gaha^{-1}ab = gh^{a^{-1}}ab$ and so the multiplication on H is unique determined.

Conversely, let $H = G \times A$ as a set and define the multiplication on $H \times A$ by

$$(g, a)(h, b) = (gh^{a^{-1}}, ab)$$

Identify g with $(g, 1)$ and a with $(1, a)$. Then is readily verified that H has all the required properties. □

Lemma 1.4.10. [**largest normal**] *Let $\mathcal{V}$ be an variety and G an A-group. Then there exists unique largest normal A-subgroups M of G such that $A/C_A(M) \in \mathcal{V}$.*

Proof. Let $H = GA$ be the semidirect product of A and G. Let $W = \mathrm{W}(\mathcal{V})$ and put $M = \langle \mathrm{C}_G(\langle W(A)^H \rangle)$. □

Definition 1.4.11. [**def:h class**] *Let G be an A-group and $W = (W_i)_{i \in \mathrm{Z}^+}$ a sequence of sets of words.*

(a) [**a**] *Define* $H_\alpha = \mathrm{Hyp}^W_\alpha(A,G)$ *inductively as follows:*

$$\begin{array}{rcl} H_\alpha = & 1 & \text{if } \alpha = 0 \\ H_\alpha = & \bigcup_{\beta<\alpha} H_\beta & \text{if } 0 \neq \alpha \text{ is a limit ordinal} \\ H_\alpha/H_{\alpha-1} = & \mathrm{C}_{H_\alpha/H_{\alpha-1}}(\langle W_k(A)^G\rangle) & \text{if } \alpha = \beta + k \text{ with} \\ & & \alpha = 0 \text{or limit ordinal and } k \in \mathrm{Z}^+. \end{array}$$

(b) [**b**] $\delta = \delta^W(A,G)$ *is the least ordinal such that* $H_\delta = H_\beta$ *for all* $\beta \geq \delta$. *Moreover,* $\mathrm{Hyp}^W(A,G) := H_\delta$

Note that if $\alpha = \beta + k$, ($\beta = 0$ or a limit ordinal and $k \in \mathbb{Z}^+$), then $H_\alpha/H_{\alpha-1}$ is the largest normal $(\mathcal{V}(W_k),*)$-A-subgroup of $G/H_{\alpha-1}$

Define $\mathrm{Hyp}^W_\alpha(G) = \mathrm{Hyp}^W_\alpha(G,G)$, where G is acting on G by conjugation and $\mathrm{Hyp}^W(G) = \mathrm{Hyp}^W(G,G)$. As above if there is no doubt about the group action (A,G) and the sequence W in question we write H_α for $\mathrm{Hyp}^W_\alpha(A,G)$.

Proposition 1.4.12. [**g=s**] *Let* (AG) *be a group action and* $W = (W_i)_{i\in \mathrm{Z}^+}$ *a sequence of sets of words.*

(a) [**a**] $(H_\alpha)_\alpha$ *is a hyper-*$(\mathcal{X}(W),*) - A$ *sequence for* G *on* $\mathrm{Hyp}^W(G)$.

(b) [**b**] *Let* M *be a normal-A-subgroup and* $(M_\alpha)_\alpha$ *be a hyper-*$(\mathcal{X}(W),*) - A$*-sequence on* M *such that each* M_α *is normal in* G.

  (a) [**a**] *For every ordinal* α *there exists an ordinal* α^* *with* $M_\alpha \leq H_{\alpha^*}$. *In particular,* $M \leq \mathrm{Hyp}^W(A,G)$.

  (b) [**b**] *If* W *is almost decreasing we can choose* α^* *such that* $\alpha^* = \alpha + n_\alpha$ *for some* $n_\alpha \in \mathbb{N}$ *and* $n_\alpha = 0$ *if* α *is a non-successor.*

(c) [**c**] G *is a hyper-*$(\mathcal{X}(W),*)$*-A-group if and only if* $G = \mathrm{Hyp}^W(A,G)$.

Proof. (a) Let $\alpha = \beta + k$ for some non-successor β and some $k \in \mathbb{Z}^+$. Then $W_k(A)$ centralizes $H_\alpha/H_{\alpha-1}$. Hence $A/\mathrm{C}_A(H_\alpha/H_{\alpha-1}) \in \mathcal{V}(W_k) \subseteq \mathcal{X}(W)$ and (a) holds.

(b) By induction we may assume that for all $\beta < \alpha$ there exists β^* with $M_\beta \leq H_{\beta^*}$. Moreover if W is almost decreasing we assume that $\beta^* = \beta + n_\beta$ for some $n \in \mathrm{N}$ with $n_\beta = 0$ if β is a non-successor.

Suppose first that α is a limit ordinal. Put $\alpha^* = \bigcup_{\beta<\alpha} \beta^*$. Then α^* is an ordinal and

$$M_\alpha = \bigcup_{\beta<\alpha} M_\beta \subseteq \bigcup_{\beta<\alpha} H_{\beta^*} \leq H_{\alpha^*}.$$

Moreover, if for all $\beta < \alpha$, $\beta^* = \beta + n_\beta$ for some $n_\beta \in \mathbb{N}$ then $b^* < \alpha^*$ and so $\alpha^* = \alpha$. So (b:a) and (b:b) hold for α.

Suppose next that $\alpha = \beta + k$ for some non-successor β and some $k \in \mathbb{Z}^+$. Since $(M_\alpha)_\alpha$ is hyper-$(\mathcal{X}(W),*)$, $A/\mathrm{C}_A(M_\alpha/M_{\alpha-1}) \in \mathcal{X}(W)$ and so $A/\mathrm{C}_A(M_\alpha/M_{\alpha-1}) \in \mathcal{V}(W_i)$ for some $i \in \mathrm{Z}^+$. Thus $[M_\alpha, W_i(A)] \leq M\alpha - 1$.

Assume that W is almost decreasing. By induction we may assume $M_{\alpha-1} \leq H_{\alpha-1+n_{\alpha-1}}$ for some $n_{\alpha-1} \in \mathbb{Z}^+$. Since W is almost decreasing there exists $n \in \mathbb{Z}^+$ with $n \geq k + n_{\alpha-1}$ and $W_n(A) \leq W_i(G)$. Then

$$[M_\alpha, W_n(A)] \leq [M_\alpha, W_i(A)] \leq M_{\alpha-1} \leq H_{\alpha-1+n_{\alpha-1}} = H_{\beta+k-1+n_{\alpha-1}} \leq H_{\beta+n-1}.$$

Since M_α and $H_{\beta+n-1}$ are normal in G, this gives $[M_\alpha, \langle W_n(A)^G\rangle] \leq H_{\beta+n-1}$ and so $M_\alpha \leq H_{\beta+n} = H_{\alpha+n-k}$. Hence (b:b) holds with $n_\alpha = n-k$.

Assume next that W is not almost decreasing. Let γ be the smallest limit ordinal with $(\alpha-1)^* \leq \gamma$. Then

$$[M_\alpha, W_i(G)] \leq M_{\alpha-1} \leq H_{(\alpha-1)^*} \leq H_\gamma \leq H_{\gamma+i-1}$$

and so $M_\alpha \leq H_{\gamma+i}$. Thus (b:a) holds.

(c) Follows from (a) and (b). □

If $W_i = \{x_1\}$ for all i, then $\mathcal{X}(W) = \mathcal{T}$ and so $(H_\alpha)_\alpha$ is a hypercentral series for A on $\mathrm{Hyp}^W(G,A)$. If $A = G$ acting by conjugation we write $\mathrm{Z}(G_\alpha)$ for H_α. ($\mathrm{Z}(G_\alpha)_\alpha$ is called the hypercentral series for G and $\mathrm{Z}_{\mathrm{Ord}}(G) := \mathrm{Hyp}^W(G,A)$) is called the hypercenter of G. If $G = \mathrm{Z}_{\mathrm{Ord}}(G)$, then G is called hypercentral. Note that $\mathrm{Z}_1(G) = \mathrm{Z}(G)$, $\mathrm{Z}_2/\mathrm{Z}(G) = \mathrm{Z}(G/\mathrm{Z}_2(G))$ and $\mathrm{Z}_\omega(G) = \bigcup_{i<\omega} \mathrm{Z}_i(G))$.

For a prime p let $\mathrm{C}_{p^\infty} = \{x \in \mathrm{C} \mid x^{p^k} = 1 \text{ for some } k \in \mathrm{N}\}$. The set C_{p^k} of elements of order dividing p^k is a cyclic group of order p^k. So C_{p^∞} can is union of the countable sequence

$$1 \leq \mathrm{C}_p \leq \mathrm{C}_{p^2} \leq \mathrm{C}_{p^3} \leq \dots$$

From $C_{p^{k+1}}/\mathrm{C}_p \cong \mathrm{C}_{p^k}$ we conclude that $\mathrm{C}_{p^\infty}/\mathrm{C}_p \cong \mathrm{C}_{p^\infty}$. So C_{p^∞} is isomorphic to a proper quotient of itself.

Let $\tau \in \mathrm{Aut}(C_{p^\infty})$ with $x^\tau = x^{-1} = \overline{x}$ for all $x \in \mathrm{C}_{p^\infty}$ and let D_{2p^∞} be the semidirect product of C_{p^∞} with $\langle tau\rangle$. Note that $\mathrm{D}_{2p^k} := \mathrm{C}_{p^k}\langle\tau\rangle$ is a dihedral group of order $2p^k$. So

So D_{p^∞} can be viewed as union of the countable sequence

$$1 \leq \mathrm{D}_p \leq \mathrm{D}_{p^2} \leq \mathrm{D}_{p^3} \leq \dots$$

If p is odd, then $\mathrm{Z}(\mathrm{D}_{2p^\infty}) = 1$ and so also $\mathrm{Z}_{\mathrm{Ord}}(\mathrm{D}_{2p^\infty}) = 1$.

If $p = 2$, then $\mathrm{Z}(\mathrm{D}_{2p^\infty}) = \mathrm{C}_2$. Also $\mathrm{D}_{2p^\infty}/\mathrm{C}_2 \cong \mathrm{D}_{2p^\infty}$ and inductively we conclude that

$$\mathrm{Z}_k(\mathrm{D}_{2p^\infty}) = \mathrm{C}_{p^k}$$

for all $i > \omega$. Thus

$$\mathrm{Z}_\omega(\mathrm{D}_{2p^\infty}) \bigcup_{i\in\omega} \mathrm{C}_{p^k} = \mathrm{C}_{p^\infty}$$

Since $\mathrm{D}_{2p^\infty}/\mathrm{C}_{p^\infty} \cong \langle\tau\rangle = \mathrm{C}_2$ we have

$$Z_{\omega+1}(D_{2p^\infty}) = D_{2p^\infty}$$

So D_{2p^∞} is hypercentral with hypercentral length $\omega + 1$.

Define $\phi_1 = x_1$, $\phi_2 = [x_1, x_2]$, $\phi_3 = [[x_1, x_2], [x_3, x_4]]$ and so on. Also let $W_i = \{\phi_i\}$. Then $W_i(G) = G^{(i-1)}$, the $i-1$'th commutator group of W_i. So $\mathcal{X}(W)$ is the class of solvable groups. The series $(H_\alpha)_\alpha$ is called the hyper (solvable,*)-series for G.

Suppose p is odd. Then $W_1(D_{2p^\infty}) = D_{2p^\infty}$, $W_2(D_{2p^\infty} = D'_{2p^\infty} = C_{p^\infty}$ and $W_3(D_{2p^\infty}) = D''_{2p^\infty} = 1$. So

$H_1 = Z(\overline{D}2p^\infty) = 1$, $H_2 = \langle C_{D_{2p^\infty}}(C_{p^\infty}) = C_{p^\infty}$ and $H_3 = D_{2p^\infty}$.

So D_{2p^∞} is a hyper-(solvable,*) group.

Lemma 1.4.13. [**direct sums**] *Let $\mathcal{X}$ be a class of groups and G an A- group. Suppose that there exists a hyper A-series $\mathcal{N}$ on G such that for each factor E of $\mathcal{N}$ there exists a G-invariant hyper-$\mathcal{X} - A$ series on E. Then A acts hyper-$\mathcal{X}$ on G.*

Proof. Let $\mathcal{N}$ be a hyper A-series on F. By assumption and the axiom of choice, the exists a function $E \to \mathcal{N}_E$ which associates to each factor E of $\mathcal{N}$ a G-invariant hyper $\mathcal{X} - A$-series on of E. If E is factor of $\mathcal{N}$ then $E = T/B$ for a unique jump (B, T) of $\mathcal{N}$. Put

$$\mathcal{M}_E = \{D \mid B \le D \le T, D/B \in \mathcal{N}_E\}$$

and $\mathcal{M} = \mathcal{N} \cup \bigcup\{\mathcal{M}_E \mid E$ a factor of $\mathcal{N}$.

Note that $\mathcal{M}$ is a set.

1°. [**0**] *Let (B, T) be a jump of cN and $E = T/B$. Then $\mathcal{M}_E$ is a G-invariant hyper $\mathcal{X} - A$ series from B to T.*

Since $\mathcal{N}_E$ is G-invariant hyper $\mathcal{X} - A$ series from 1 to E, this follows from the homomorphism theorems.

Recall that for $N \in \mathcal{N}$, $N^- = \bigcup\{E \in \mathcal{N} \mid E < N\}$. For each $D \in \mathcal{M}$ pick $\tilde{D} \in \mathcal{M}$ minimal with $D \le \tilde{D}$.

2°. [**.1**] *Let (B, T) be a jump of $\mathcal{N}$ and $D \in \mathcal{M}$ with $B \le D \le T$. then either $D = B = \tilde{D}$ or $B \ne D$ and $(B, T) = (\tilde{D}^-, \tilde{D})$.*

If $D = B$, then $B = \tilde{D}$. So suppose $B < D \le T$. Since $D \le T$, the minimality of $\tilde{D}$ gives $\tilde{D} \le T$. So $B < \tilde{D} \le T$ and since (B, T) is a jump, $\tilde{D} = T$. Hence $B = T^- = \tilde{D}^-$.

3°. [**.2**] *$D^- \le D \le \tilde{D}$ and either $D = \tilde{D} = \tilde{D}^-$ or $D^- < D \le \tilde{D}$ and $D \in \mathcal{M}_{\tilde{D}/\tilde{D}^-}$.*

If $D \in \mathcal{N}$, then clearly $\tilde{D} = D$ and (2°) holds. So suppose $D \notin \mathcal{N}$. Then $D \in \mathcal{M}_{T/B}$ for some jump $(B, T) \in \mathcal{T}$. Then $B \le D \le T$ and since $D \notin \mathcal{N}$, $B \ne D$. So by (3°), $(B, T) = (\tilde{D}, \tilde{D})$ and (3°) holds.

4°. [**1**] *$\mathcal{M}$ is totally ordered.*

Let $D, E \in \mathcal{M}$. Suppose first that $\tilde{D} = \tilde{E}$. Then $\tilde{D}^- \leq E \leq \tilde{D}$. If $\tilde{D}^- = \tilde{D}$ this gives $D = E$ and if $\tilde{D}^- \neq \tilde{D}$, then by **??** both D and E are in $\mathcal{M}_{\tilde{D}/\tilde{D}^-}$. So by (1°), $D \leq E$ or $E \leq D$.

Now suppose that $\tilde{D} \neq \tilde{E}$ and without loss $\tilde{D} < \tilde{E}$. Then $D \leq \tilde{D} \leq \tilde{E}^- \leq E$ and so $D \leq E$.

Let $\mathcal{D}$ be a non-empty subsets of $\mathcal{M}$.

5°. [**2**] *$\mathcal{D}$ has a minimal element D^*. In particular, $\bigcup \mathcal{D} = D^* \in \mathcal{M}$.*

Let M be the minimal element of $\{\tilde{D} \mid D \in \mathcal{D}\}$ and pick $E \in \mathcal{D}$ with $M = \tilde{E}$. If $D \in \mathcal{D}$, then $M \leq \tilde{D}$ and since $\tilde{D}^- \leq D$, $M^- \leq D$. If $M^- = M$, then $E = M^-$ and E is the minimal element of $\mathcal{D}$. If $M^- \neq M$, then by (1°) the non empty set $\{E \in \mathcal{D} \mid M^- \leq E \leq M\}$ has a minimal element D^*. But then D^* is also a minimal element of $\mathcal{D}$.

6°. [**3**] $\bigcup \mathcal{D} \in \mathcal{M}$

Put $M = \bigcup_{D \in \mathcal{D}} \tilde{D}$. Then $M \in \mathcal{N}$. Let $E \in \mathcal{N}$ with $E < M$. The there exists $D \in \mathcal{D}$ with $\tilde{D} \nleq E$. So $E < \tilde{D} \leq D$. It follows that $M^- \leq \bigcup \mathcal{D}$. If $M^- = \bigcup \mathcal{D}$ we are done. If $M^- = \bigcup \mathcal{D}$. Then $\mathcal{E} := \{E \in \mathcal{D} \mid E \nleq M^-\}$ is not empty. Observe that $M^- < E \leq M$ for all $E \in \mathcal{E}$. Thus $\bigcup \mathbb{E} = \bigcup \mathcal{D}$ and $\mathcal{E} \in \mathcal{M}_{M/M^-}$. By (1°), $\mathcal{M}_{M/M^-}$ is closed under unions and so $\bigcup \mathcal{D} = \bigcup \mathcal{E} \in \mathcal{M}_{M/M^-} \subseteq \mathcal{M}$. Thus (6°) holds.

7°. [**4**] *Let (B, T) be a jump of $\mathcal{M}$. Then (B, T) is jump of some $\mathcal{M}_E$, E a factor of $\mathcal{N}$. In particular, $B \trianglelefteq T$ and T/B is an $\mathcal{X} - A$-group.*

Suppose first that $\tilde{T}^- \neq T$. Then $\tilde{T}^- < T$ and since (B, T) is a jump of $\tilde{T}^- \leq B \leq T \leq \tilde{T}$. Thus by (3°) both B and T are in $\mathcal{M}_{\tilde{T}/\tilde{T}^-}$ and so (B, T) is a jump of $\mathcal{M}_{\tilde{T}/\tilde{T}^-}$

Suppose next that $\tilde{B} \neq B$. Then $B < \tilde{B}$ and since (B, T) is a jump $T \leq \tilde{B}$. Thus $B^- \leq T \leq B$ and so by (3°) both B and T are in $\mathcal{M}_{\tilde{B}/\tilde{B}^-}$ and so (B, T) is a jump of $\mathcal{M}_{\tilde{B}/\tilde{B}^-}$.

Suppose finally that $\tilde{T}- = T$ and $\tilde{B} = B$. Then both B and T are in $\mathcal{N}$ and so (B, T) is a jump of $\mathcal{N}$, but then $T^- = B \neq T$, a contradiction.

The lemma is now a direct consequence of (4°)-(7°). □

Lemma 1.4.14. [**direct hyp**]*Let $\mathcal{X}$ be a class of actions, A a group and G an A-group. Let $(G_i, i \in I)$ a non empty family normal hyper-$\mathcal{X} - A$ groups of G with $G = \langle G_i \mid i \in I\rangle$. Suppose that either $\mathcal{X}$ is $\mathbf{H}$ closed or $G = \bigoplus_{i \in I} G_i$. Then G is a hyper-$\mathcal{X} - A$-group.*

Proof. Without loss $G_i \neq 1$ for all $i \in I$. Pick $m \in I$ and choose some well ordering on $I \setminus m$. Well order I such that I has a maximal element. For $i \in I$ define $G_i^+ = \langle G_j \mid j \leq i\rangle$ and $G_i^- = \langle G_j \mid j < i\rangle$. We claim that $\mathcal{N} = \{G_i^-, G_i^+ \mid i \in I\}$ is hyper A-series on $\bigoplus_{i \in I} G_i$ with factors all the $G_i^+/G_i^- \cong G_i/G_i \cap G_i^-$, where $i \in I$ with $G_i \nleq G_i^-$.

Let $i < j \in I$. Then $G_i^- \leq G_i^+ \leq G_j^- \leq G_j^+$ and so $\mathcal{N}$ is totally ordered. Let $\mathcal{M}$ be non-empty subset of $\mathcal{N}$. Let i be minimal in I with $G_i^\epsilon \in \mathcal{D}$ for some $\epsilon \in \{\pm\}$. If $G_i^- \in \mathcal{N}$ choose $\epsilon = -$. Then G_i^ϵ is the minimal element of $\mathcal{M}$ and $G_{i^\epsilon} = \bigcup \mathcal{D}$.

Next let k be minimal with $\bigcup \mathcal{D} \leq G_k^+$. Let $i < k$. Then $\bigcup \mathcal{D} \not\leq G_i^+$ and so the exists $j \in I$ and $\delta \in \{\pm\}$ with $G_j^\delta \in \mathcal{D}$ and $G_j^\delta \not\leq G_i^+$. Thus $i \leq j$ and so $G_i^- \leq G_j^\delta \leq \bigcup \mathcal{D}$.

Suppose first that $\{l \in I \mid l < k\}$ has no maximal element. Let $g = \prod_{i \in I} g_i \in G_k^-$ (where $g_i \in G_i$ and only finitely many g_i are non trivia. Let t be maximal with $g_t \neq 1$. Then $t < l$ and so there exists $l \in I$ with $t < l < k$. Then $g \in G_t^- \leq \bigcup \mathcal{D}$. Hence $G_k^- \leq \bigcup \mathcal{D} \leq G_i^+$. If $G_k^+ \in \mathcal{D}$ we get $\bigcup \mathcal{D} = G_k^+$ and if $G_k^+ \notin \mathcal{D}$ we get $\bigcup \mathcal{D} = G_k^-$.

Suppose $\{l \in I \mid l < k\}$ has maximal element j. Since $\bigcup \mathcal{D} \not\leq G_j^+$ we must have $G_k^- \in \mathcal{D}$ or $G_k^+ \in \mathcal{D}$. In either case we again have $\bigcup \mathcal{D} = G_k^+$ and $\bigcup \mathcal{D} = G_k^-$.

Thus $\mathcal{N}$ is closed under unions. Let $D \in \mathcal{N}$ with $D \neq D^- := \{\bigcup E \in \mathcal{N} \mid E < D\}$. Pick $k \in I$ minimal with $D = G_k^\epsilon$ for some $\epsilon \in \{pm\}$, where we choose $\epsilon = -$ if $D = G_k^-$ for some $\epsilon \in \{\pm\}$. By minimality of k, $G_j^+ < D$ for all $j < k$. Thus

$$G_k^i = \langle G_j \mid j < k \rangle \leq \langle G_j^+ \mid j < k \rangle \leq D^-$$

In particular, $G_k^- < D$ and so $G_k^- = D^-$, $D = G_k^+$, $G_k \not\leq G_k^-$ and

$$D/D^- \cong G_k^+/K_k^- = G_k G_k^-/G_k^- \cong G_k/G_k \cap G_k^-$$

Conversely if $k \in I$ with $G_k \not\leq G_k^-$, then (G_k, G_k^-) is clearly a jump of $\mathcal{N}$.

This proves the claim. If $\mathcal{X}$ is **H** closed then by **??**(**??**), $G_k/G_k \cap G_k^-$ is an hyper $\mathcal{X} - A$ group. If $G = \bigoplus_{i \in I} G_i$, then $G_k/G_k \cap G_k^- \cong G_k$. So again $G_k/G_k \cap G_k^-$ is an hyper $\mathcal{X} - A$ group. In either case 1.4.13 completes the proof. □

Proposition 1.4.15. [**residually g**] *Let $\mathcal{X}$ be any class of groups.*

(a) [**a**] *Suppose $\mathcal{X}$ is closed under quotients. Then hypercentral-by-$\mathcal{X}$ groups are hyper-$(\mathcal{X}, *)$ and nilpotent-by-$\mathcal{X}$ groups are poly-$(\mathcal{X}, *)$.*

(b) [**b**] *Hyper-$(\mathcal{X}, *)$ groups are hypercentral-by- $\mathbf{R}\mathcal{X}$). If $\mathcal{X}$ is closed under finite subdirect products then poly-$(\mathcal{X}, *)$-groups are nilpotent-by-$\mathcal{X}$.*

(c) [**c**] *If $\mathcal{X}$ is closed under quotients and finite subdirect products, then the nilpotent-by-$\mathcal{X}$-groups are exactly the finitely hyper-$(\mathcal{C}G, *)$ groups.*

Proof. (a) Let $H \trianglelefteq G$ such that H is hypercentral and $G/H \in \mathcal{X}$. Let $\mathcal{Z}$ be the hypercentral series for H. Then $\mathcal{Z}$ is G-invariant. If Z is a factor of $\mathcal{Z}$, then $[Z, H] = 1$ and so $G/C_G(Z)$ is a quotient of G/H. Thus $G/C_G(Z) \in \mathcal{X}$. Also $G/C_G(G/H)$ is a quotient of G/H and so $\mathcal{Z} \cup \{G\}$ is a hyper-$(\mathcal{X}, *)$ series for G. If H is nilpotent, $\mathcal{Z}$ is finite and (a) is proved.

(b) Let $\mathcal{M} = (M_\alpha)_\alpha$ be a hyper-$(\mathcal{X}, *)$-sequence for G and put

$$H = \bigcap \{C_G(E) \mid E \text{ a factor of } \mathcal{M}\}.$$

Since $G/C_G(E) \in \mathcal{X}$ for all factors E of $\mathcal{M}$, G/H is subdirect product of $\mathcal{X}$-groups and so an $\mathbf{R}\mathcal{X}$-group. Moreover $(M_\alpha \cap H)_\alpha$ is a hypercentral series for H and so H is hypercentral. If $\mathcal{M}$ is finite and $\mathcal{X}$ is closed under finite subdirect products, then $G/H \in \mathcal{X}$ and H polycentral, that is nilpotent. So (b) holds.

(c) Follows from (a) and (b). □

Proposition 1.4.16. [**hyper gw**] *Let $\mathcal{V}$ be a variety and W a set of words with $\mathcal{V} = \mathcal{V}(W)$. Let G be a group. Then the following are equivalent*

(a) [**a**] *G is hyper-$(\mathcal{V}), *)$ group.*

(b) [**b**] *G is hypercentral by $\mathcal{V}$.*

(c) [**c**] *$W(G)$ is a hypercentral group.*

Proof. (a) $\Longrightarrow$ (b): Suppose G is hyper-$(\mathcal{V}), *)$. Then by 1.4.15 G is hypercentral by $\mathbf{R}\mathcal{V}$. Since varieties are $\mathbf{R}$-closed, G is hypercentral by $\mathcal{V}$.

(b) $\Longrightarrow$ (c): Suppose M is a normal subgroup of G such that M is hypercentral and $G/M \in \mathcal{V}$. Then $W(G/M) = 1$ and so $W(G) \leq M$. Since subgroups of hypercentral groups are hypercentral, $W(G)$ is hypercentral.

(c) $\Longrightarrow$ (b): Note that $G/W(G) \in \mathcal{V}$. So if $W(G)$ is hypercentral G is hypercentral by $\mathcal{V}$.

(b) $\Longrightarrow$ (a): If G is hypercentral by $\mathcal{V}$, then by 1.4.15 G is hyper-$(\mathcal{V}, *)$. □

Definition 1.4.17. [**almost decreasing**] *Let $W = (W_i)_{i=1}^{\infty} \in \mathcal{P}(F)^{\infty}$ be a sequence of sets of words.*

(a) [**a**] *W is decreasing if $W_{i+1}(F) \leq W_i(F)$ for all i.*

(b) [**b**] *W is almost decreasing if for all $i, j \in \mathbb{Z}^+$ there exists $k \geq j$ with $W_k(F) \leq W_i(F)$.*

(c) [**c**] *$\mathcal{X}(W) = \bigcup_{i=1}^{\infty} \mathcal{V}(W_i)$.*

Lemma 1.4.18. [**trivial dec**] *Let G be group.*

(a) [**a**] *Let $V, W \in \mathcal{P}(W)$ with $V(F) \leq W(V)$. Then $V(G) \leq W(G)$.*

(b) [**b**] *Let $W \in \mathcal{P}(W)^{\infty}$ be almost decreasing. Then $(W_i(G))_{i=1}^{\infty}$ is almost decreasing, that is for $i, j \in \mathbb{Z}^+$ there exists $k \geq j$ with $W_k(G) \leq W_i(G)$.*

Proof. (a) Let $g \in V(G)$. Then $g \in V(H)$ for some finitely generated subgroup H of G. Let $\alpha : F \to H$ be an onto homomorphism. Then

$$g \in V(H) = V(\alpha(F)) = \alpha(V(F)) \leq \alpha(W(F))) = W(\alpha(F)) = W(H) \leq W(G)$$

and so $V(G) \leq W(G)$.

(b) follows from (a). □

Definition 1.4.19. [**def:outer**]

(a) [**a**] *For $i = 1, 2$ let w_i be a word and $m_i = m(w_i)$. Put*

$$\lceil w_1, w_2 \rceil := [\, w_1(\, (x_i)_{i=1}^{m_1}\,)\, ,\, w_2(\, (x_{m_1+i})_{i=1}^{m_2}\,)\,] \in F(m_1 + m_2)$$

$\lceil w_1, w_2 \rceil$ is called the outer commutator of w_1 and w_2.

(b) [**c**] *Let $w \in F^n$, $n \in \mathbb{N} \cup \{\infty\}$. Then $\check{w} \in F^{n+1}$ is inductively defined as follows: $\check{w}_1 = x_1$ and $\check{w}_{i+1} = \lceil \check{w}_i, w_i \rceil$.*

(c) [**d**] *Let $W \in \mathcal{P}(W)^n$, $n \in \mathbb{N} \cup \{\infty\}$. Then $\check{W} \in \mathcal{P}(W)^{n+1}$ is inductively defined as follows: $\check{W}_1 = \{x_1\}$ and $\check{W}_{i+1} = \{\lceil v, w \rceil \mid v \in \check{W}_i, w \in W_i\}$.*

For example, $\lceil x_1 x_2^3, x_1 x_2^2 \rceil = [x_1 x_2^3, x_3 x_4^2]$. Note that $m(\lceil w_1, w_2 \rceil) = m_1 + m_2$. Also $\check{W}_{i+1} = \{\check{w}_{i+1} \mid w \in \bigtimes_{j=1}^{i} W_j\}$. To improve readability we sometimes write $\check{w}$ for $\check{w}$.

Lemma 1.4.20. [**basic check**] *Let G be a group, $w \in F^\infty$, $g \in G^\infty$ and $i \in \mathbb{Z}^+$.*

(a) [**c**] *Put $n = m(\check{w}_i)$ and $m = m(w_i)$. Then*

$$\check{w}_{i+1}(g) = [\check{w}_i(g), w_i(\,(g_{n+j})_{j=1}^m\,)].$$

(b) [**b**] *Let $N \trianglelefteq G$. If $\check{w}_i(g) \in N$ then also $\check{w}_j(g) \in N$ for all $j \geq i$.*

(c) [**a**] *Let $W \in \mathcal{P}(W)^\infty$. Then $\check{W}_{i+1}(G) = \check{W}_i(G), W_i(G)] \leq \check{W}_i(G) \cap W_i(G)$.*

In particular, $\check{W}$ is decreasing.

Proof. (a) By definition $\check{w}_{i+1} = \lceil \check{w}_i, w_i \rceil$. So (a) follows from the definition of the outer commutator.

(b) and (c) follow from (a). □

Definition 1.4.21. [**def:h words**]

(a) [**a**] *Let $W \in \mathcal{P}(F)^\infty$. Then* Hyp$(W)$ *is the class of groups G such that for all $g \in G^\infty$ and all $w \in \bigtimes_{i=1}^{\infty} W_i$ there exists $n \in \mathbb{Z}^+$ with $\check{w}_n(g) = 1$.*

(b) [**b**] *Let $\mathcal{X}$ be a class of actions. Then* Hyp$\mathcal{X}$ *is the class of hyper-$\mathcal{X}D$-groups.* Poly$\mathcal{X}$ *is the class of Poly-$\mathcal{X}$-groups.*

Lemma 1.4.22. [**cX check**] *Let $W \in \mathcal{P}(F)^\infty$. Then for all $i \in \mathrm{Z}^+$, $\mathcal{V}(W_i) \leq \mathcal{V}(\check{W}_{i+1})$. In particular, $\mathcal{X}(W) \subseteq \mathcal{X}(\check{W})$.*

Proof. Let $G \in \mathcal{V}(W_i)$. Then $W_i(G) = 1$. Hence by **??**(**??**) $\check{W}_{i+1}(G) = [\check{W}_i(G), W_i(G)] = 1$ and so $G \in \mathcal{V}(\check{W}_{i=1})$. It follows

$$\mathcal{X}(W) = \bigcup_{i=1}^{\infty} \mathcal{V}(W_i) \subseteq \bigcup_{i=1}^{\infty} \mathcal{V}(\check{W}_{i+1}) \subseteq \mathcal{X}(\check{W})$$

□

Theorem 1.4.23. [**h and check**] *Let $W \in \mathcal{P}(F)^\infty$. Then*

(a) [**a**] *$\mathcal{X}(\check{W}) \subseteq$ Poly$(\mathcal{X}(W), *)$ with equality if W is almost decreasing.*

(b) [**b**] Hyp$(W) \subseteq$ Hyp$(\mathcal{X}(W), *)$ *with equality if W is almost decreasing.*

Proof. (a) Suppose $G \in \mathcal{X}(\check{W})$. Then $G \in \mathcal{V}(\check{W}_n)$ for some $n \in \mathrm{Z}^+$. Thus $\check{W}_n(G) = 1$. Then by 1.4.20(c) we obtain a finite series

$$(*) \qquad 1 = \check{W}_n(G) \leq \check{W}_{n-1}(G) \leq \ldots \leq \check{W}_2(G) \leq \check{W}_1(G) = G$$

there the last equality holds since $\check{(}W_1) = \{x_1\}$.

Observe that $[\check{W}_i(G), W_i(G)] \leq \check{W}_{i+1}(G)$ and so $W_i(G) \leq C_G(\check{W}_{i+1}(G)/\check{W}_i(G)$. Hence

$$G/C_G(\check{W}_{i+1}(G)/\check{W}_i(G) \in \mathcal{V}(W_i) \subseteq \mathcal{X}(W)$$

and (*) is a poly $(\mathcal{X}(W), *)$-series. Thus the first statement in (a) holds.

To prove the first statement in (b), let G be a group which is not hyper-$(\mathcal{X}(W), *)$. We will show that G is also not contained in $\mathrm{Hyp}(\check{W})$. Since every strongly hyper $(\mathcal{X}(W), *)$ group is hyper $(\mathcal{X}(W), *)$ (see **??**) we conclude that there there exists $N \lhd G$ such $N^*/N = 1$, whenever $N \leq N^* \trianglelefteq G$ with $(G/C_G(N^*/N), N^*/N) \in (\mathcal{X}(W), *)$. This implies

$$(*) \qquad \mathrm{C}_{G/N}(W_n(G)) = 1 \text{ for all } n \in \mathbb{Z}^+.$$

Let $g_1 \in G \setminus N$. Note that $x_1(g_1) = g_1 \notin N$. Suppose inductively that we already found $(g_i)_{i=1}^{n_k} \in G^{n_k}$ and $w_i \in W_i, 1 \leq i < k$ with $\check{w}_k(\,(g_i)_{i=1}^{n_k}\,) \notin N$, where $(\check{w}_i)_{i=1}^{k}) = (w_i)_{i=1}^{k-1}\check{}$. Then by (*) $[\check{w}_k(\,(g_i)_{i=1}^{n_k}\,), W_k(G)] \nleq N$ and there exist $w_k \in W_k$ and $(g_{n_k+j})_{j=1}^{m(w_k)} \in G^{m(w_k)}$ with $[\check{w}_k(g_i)_{i=1}^{n_k}, w_k(\,(g_{n_k+j})_{j=1}^{m(w_k)}\,)] \notin N$. Put $n_{k+1} = n_k + m(w_k)$. Then by 1.4.20(a),

$$\check{w}_{k+1}(\,(g_i)_{i=1}^{n_{k+1}}\,) \notin N.$$

where $w_{k+1} = \lceil \check{w}_k, w_k \rceil$. Put $g = (g_i)_{i=1}^{\infty}$ and $w = (w_i)_{i=1}^{\infty}$. Then $\check{w}_k(g) \neq 1$ for all k and so $G \notin \mathrm{Hyp}(W)$. Thus $\mathrm{Hyp}(W) \subseteq \mathrm{Hyp}(\mathcal{X}(W), *)$.

Suppose next that W is almost decreasing. We will prove the second assertions in (a) and (b) simultaneously. Let G be hyper-$(\mathcal{X}(W), *)$ and and let $(M_\alpha)_{\alpha \leq \rho}$ be any hyper-$(\mathcal{X}(W), *)$ sequence on G, with ρ finite in proof of (a). For the proof of (a) ρ let $V_i = W_i$ and $H_i = G$ for all $i \in \mathrm{Z}^+$. For the proof of (b) let $g \in G^\infty$, $w \in \bigtimes_{i=1}^{\infty} W_i$ infinite pick $w_i \in W_i$ and $g_i \in G$ and put $H_i = \{g_i\}$ and $V_i = \{w_i\}$

Let $g \in \bigtimes_{i=1}^{\infty} H_i$ and $w \in \bigtimes_{i=1}^{i} nftyV_i$. Then $\check{w}_1(g_1) = g_1 \in G = A_\rho$. So we can choose an ordinal α minimal such that there exists $n \in \mathbb{Z}^+$ with $\check{w}_n(g) \in G_\alpha$ for all $w \in \bigtimes_{i=1}^{\infty} V_i$ and $g \in \bigtimes_{i=1}^{\infty} H_i$.

We will show that $\alpha = 0$. Suppose for $\alpha = \beta+1$ for some ordinal β. Since $G/C_G(A_\alpha/A_\beta) \in \mathcal{X}(W)$, there exists $m \in \mathbb{Z}^+$ with $[M_\alpha, W_m(G)] \leq M_\beta$. Since W is almost decreasing we may assume $m \geq n$. Let $w \in \bigtimes_{i=1}^{\infty} V_i$. Then $\check{w}_n(g) \in M_\alpha$ and $m \geq n$. So by 1.4.20(b), $\check{w}_m(g) \in M_\alpha$. Hence

$$\check{w}_{m+1}(g) \in [\check{w}_m(g), W_m(G)] \leq [M_\alpha, W_m(G)] \leq A_\beta$$

for all $w \in \bigtimes_{i=1}^{\infty} V_i$ and $g \in \bigtimes_{i=1}^{\infty} H_i$, a contradiction to the minimal choice of α. Thus α is a limit ordinal.

Suppose that $\alpha \neq 0$. Then ρ is infinite and so by our choice of V_i, $|V_i| = 1$ and there exists a unique $w \in \bigtimes_{i=1}^{\infty} V_i$. Since $M_\alpha = \bigcup_{\beta<\alpha} M_\beta$ there exists $\beta < \alpha$ with $\breve{w}_n(g) \in A_\beta$, a contradiction to the choice of α.

Thus $\alpha = 0$ and so $\breve{w}_n(g) = 1$ for all $w \in \bigtimes_{i=1}^{\infty} V_i$.

If ρ is finite, $V_i = W_i$ and $H_i = G_i$. Thus $\check{W}_n(G) = 1$ and $G \in \mathcal{X}(\check{W})$. So (a) is proved.

In any case, $\breve{w}_n(g) = 1$ shows that $G \in \mathrm{Hyp}(W)$ and (b) holds. □

The following example shows that the inclusions in 1.4.23 may be proper if W is not almost decreasing:

Let $G = \mathrm{Sym}(3)$, $x = x_1$, $W_1 = \{x^2\}$ and $W_i = \{x\}$ for $i \geq 2$. Then $w = (x^2, x, x, x, \ldots)$ is the unique element in $\bigtimes_{i=1}^{\infty} W_i$. Also $1 \leq \mathrm{Alt}(3) \leq \mathrm{Sym}(3)$ is a finite hyper-$(\mathcal{X}(W), *)$ series. Thus $\mathrm{Sym}(3) \in \mathrm{Poly}(\mathcal{X}(W), *) \subseteq \mathrm{Hyp}(\mathcal{X}(W), *)$.

Put $g = ((12), (123), (12), (12), (12), \ldots)$. Then $\breve{w}_1(g) = g_1 = (12)$, $\breve{w}_2(g) = [(12), (123)^2] = (123)$, $\breve{w}_3(g) = [(123), (12)] = (123)$ and so for all $n \geq 2$, $\breve{w}_n(g) = (123)$. Thus $w_n(g) \neq 1$ for all n and $\mathrm{Sym}(3) \notin \mathrm{Hyp}(\check{W})$. Since $\mathcal{X}(\check{W}) \subseteq \mathrm{Hyp}(\check{W})$ we see that $\mathcal{X}(\check{W}) \neq \mathrm{Poly}(\mathcal{X}(W), *)$ and $\mathrm{Hyp}(\check{W}) \neq \mathrm{Hyp}(\mathcal{X}(W, *)$.

Lemma 1.4.24. [**char hyp**] *Let $W \in \mathcal{P}(F)^\infty$. Then there exists $V \in \mathcal{P}(F^\infty$ such that*

(a) [**a**] $\mathcal{X}(W) = \mathcal{X}(V)$.

(b) [**b**] *V is almost decreasing*

(c) [**c**] $\mathrm{Poly}(\mathcal{X}(W), *) = \mathcal{X}(\check{V})$.

(d) [**d**] $\mathrm{Hyp}(\mathcal{X}(W), *) = \mathrm{Hyp}(V)$.

Proof. Define

$$V = (W_1, W_1, W_2, W_1, W_2, W_3, W_1, W_2, W_3, W_4, W_1, \ldots).$$

Then clearly V is almost decreasing. For any W $\mathcal{X}(W)$ only depends on $\{W_i \mid i \in \mathbb{Z}^+\}$ and so $\mathcal{X}(W) = \mathcal{X}(V)$. Thus by 1.4.23

$$\mathcal{X}(\check{V}) = \mathrm{Poly}(\mathcal{X}(W), *) \text{ and } \mathrm{Hyp}(V) = \mathrm{Hyp}(\mathcal{X}(W), *).$$

□

Next we will give an example of a sequence $W \in \mathcal{P}(F)^\infty$, a group $G \in \mathrm{Hyp}(\mathcal{X}(W), *)$, $g \in G^\infty$ and $v \in \bigtimes_{i=1}^{\infty} \check{W}_i$ such that $v_n(g) \neq 1$ for all $n \in \mathrm{Z}^+$. (Note that this does not contradict **??** since our v will not be of the form $v = \breve{w}$ for some $w \in \bigtimes_{i=1}^{\infty} W_i$.

Put $W_1 = \{x_i \mid i \in \mathrm{Z}^+$ and for $i \geq 2$ put $W_i = \{x_1\}$. The for all $i \in \mathrm{Z}^+$, $\mathcal{V}(W)i) = \mathcal{T}$, the class of trivial groups. Hence also $\mathcal{X}(W) = \mathcal{T}$ and $\mathrm{Hyp}(\mathcal{X}(W), *)$ is the class of hypercentral groups. Put $G = \mathrm{D}_{22^\infty} = \mathrm{C}_{2^\infty}\langle\tau\rangle$. As seen before G is hypercentral group. Let $h_i \in \mathrm{C}_{2^\infty}$ with $|h_i| = 2^i$ and put $g_i = h_i\tau$.

Note that $\check{W}_1 = \{x_1\}$, $\check{W}_2 = \{\lceil x_1, x_i \rceil \mid i \in Z^+\} = \{[x_1, x_k] \mid 2 \le k \in Z^+\}$ and for any $i \ge 2$,

$$\check{W}_i = \{[x_1, x_k, x_{k+1}, \dots x_{k+i-2}] \mid 2 \le k \in Z^+\}$$

Define $v_1 := x_1$ and for $i \ge 0$:

$$v_i := [x_1, x_{2i}, x_{2i+1}, \dots, x_{3i-2}]$$

and $v_i \in \check{W}_i$ for all $i \in Z^+$.

Define $g_{i,0} := [g_1, g_i]$ and inductively $g_{i,j} := [g_{i,j-1}, g_{i+j}]$. Then $v_i(g) = g_{i,i-2}$. We will show by induction in j, that $g_{i,j}$ has order 2^{i-j-1}.

For $j = 0$,

$$g_{i,0} = [g_1, g_i] = g_1^{-1} g_i^{-1} g_1 g_i = \tau^{-1} h_1^{-1} \tau^{-1} h_i^{-1} h_1 \tau h_{2i} \tau = h_1 h_i^{-1} h_1 h_i^{-} 1 = h_i^{-2}$$

and $g_{i,0}$ has oder 2^{i-1}. Suppose inductively that $g_{i,j}$ has order 2^{i-j-1} and $g_{i,j} \in C_{2^\infty}$. Then g_{i+j+1} inverts $g_{i,j}$ via conjugation and so

$$g_{i,j+1} = [g_{i,j}, g_{i+j+1}] = g_{i,j}^{-1} g_{i,j}^{-1} = g_{i,j}^{-2}$$

Thus $g_{i,j+1} \in C_{2^\infty}$ and $g_{i,j+1}$ has order $2^{i-j-2} = 2^{i-(j+1)-1}$.

In particular $v_i(g) = g_{i,i-2}$ has order $2^{i-(i-2)-1} = 2$. Thus $v_i(g) \ne 1$ for all $i \ge 2$. Also $v_1(g) = g_1 = \tau h_1 \ne 1$ and so $v_i(g) \ne 1$ all $i \in Z^+$.

Definition 1.4.25. [**def:phi**]

(a) [**a**] *$\tau(0) = (x_1)_{i=1}^\infty$ and inductively $\tau(i+1) = \tau(i)\check{}$.*

(b) [**d**] *ϕ is the unique sequence of words with $\phi = \check{\phi}$. So $\phi_1 = x_1$ and inductively $\phi_{i+1} = \lceil \phi_i, \phi_i \rceil$.*

It might be worthwhile to list the first few terms of the above sequence of words:

$\tau(0)$:	x_1	x_1	x_1	x_1
$\tau(1)$:	x_1	$[x_1, x_2]$	$[[x_1, x_2], x_3]$	$[[[x_1, x_2], x_3], x_4]$
$\tau(2)$:	x_1	$[x_1, x_2]$	$[[x_1, x_2], [x_3, x_4]]$	$[[[x_1, x_2], [x_3, x_4]], [[x_5, x_6], x_7]]]$
ϕ :	x_1	$[x_1, x_2]$	$[[x_1, x_2], [x_3, x_4]]$	$[[[x_1, x_2], [x_3, x_4]], [[x_5, x_6], [x_7, x_8]]]$

Lemma 1.4.26. [**gw**]

(a) [**a**] *Let $\mathcal{T}(0)$ be the class of trivial groups and inductively let $\mathcal{T}(n+1)$ be the class of nilpotent-by-$\mathcal{T}(n)$ groups. Then $\mathcal{X}(\tau(n)) = \mathcal{N}(n)$. In particular, $\mathcal{X}(\tau(1))$ the class of nilpotent groups.*

(b) [**b**] *$\mathcal{V}(\phi_i)$ the class of solvable groups of derived length less than i. $\mathcal{X}(\phi)$ is the class of solvable groups.*

(c) [**c**] *$\mathrm{Hyp}(\tau(i))$ is the class of hyper $(\mathcal{T}(i),*)$-groups. In particular, $\mathrm{Hyp}(\tau(0))$ is the class of hypercentral groups, and $\mathcal{T}(1)$ is the class of hyper-(nilpotent,*) groups.*

(d) [**d**] *$\mathrm{Hyp}(\phi)$ is the class of hyper (solvable,*) groups.*

Proof. (a) Let $w \in F^{\infty}$ be almost decreasing. By 1.4.23(a), $\mathcal{X}(\breve{w}) = \mathrm{Poly}(\mathcal{X}(w),*)$ and so by 1.4.15(c):

$$(*) \qquad \mathcal{X}(\breve{w}) \text{ is the class of nilpotent-by-}\mathcal{X}(w)\text{groups.}$$

Clearly $\mathcal{X}(\tau(0)))$ is the class of trivial groups. Since $\tau(1) = \tau(0)\breve{}$, (*) says that $\mathcal{X}(\tau(1))$ is the class of nilpotent-by-trivial groups and $\mathcal{X}(\tau(1)) = \mathcal{T}(1)$. Inductively suppose that $\mathcal{X}(\tau(n)) = \mathcal{T}(n)$. Then (*) implies that $\mathcal{X}(\tau(n{+}1))$ is the class of nilpotent-by-$\mathcal{T}(n)$ groups. Thus $\mathcal{X}(\tau(n+1)) = \mathcal{T}(n+1)$ and (a) holds.

(b) We have $G = x_1(G) = \phi(G) = {}^{G}0$ and so inductively

$$\phi_{i+1}(G) = [\phi_i(G), \phi_i(G)] = [{}^{G}i-1, {}^{G}i-1] = {}^{G}i.$$

Hence $\mathcal{X}(\phi_i)$ is the class of solvable groups of derived length less than i and (b) holds.

By **??**(**??**), $\mathrm{Hyp}(\tau(n)) = \mathrm{Hyp}(\mathcal{X}(\tau(n),*)$. So rf c follows from (a).

By **??**(**??**), $\mathrm{Hyp}(\phi) = \mathrm{Hyp}(\mathcal{X}(\phi),*)$. So rf d follows from (b). □

We will now construct various examples of groups which are hyper-$(\mathcal{X},*)$ for some class of groups $\mathcal{X}$. By 1.4.15 we know that any such group is hypercentral-by-(residually $\mathcal{X}$). The next proposition gives a partial converse:

Example 1.4.27. [**main construction**] *Let $\mathcal{X}$ be a class of groups, $(H_i, i \in I)$ a family of $\mathcal{X}$-groups and H a subdirect product of $(H_i, i \in I)$. For $i \in I$ let A_i be an H_i-group. Suppose that*

(i) [**a**] *H is hyper-$(\mathcal{X},*)$.*

(ii) [**b**] *For each $i \in I$, A_i is abelian and H_i acts faithfully on A_i.*

(iii) [**c**] *For each $1 \neq N \trianglelefteq H$, there exists $i \in I$ such that N does not act hypercentrally on A_i.*

Put $A = \bigoplus A_i$. Note that H acts on A_i via its projection onto H_i and so also acts on A.Let $G = AH$ be the semidirect product of A and G Then G is hyper-$(\mathcal{X},*)$-group. Moreover, any hypercentral normal subgroup of G is contained in A.

Proof. Since $G/\mathrm{C}_G(A_i) \cong H_i \in \mathcal{X}$, G acts hyper-$(\mathcal{X},*)$ on A_i. So by 1.4.13, G is acts hyper-$(\mathcal{X},*)$ on A. Also $G/A \cong H$ is hyper-$(\mathcal{X},*)$ group and hence by 1.4.13 G is a hyper-$(\mathcal{X},*)$ group.

Let $M \trianglelefteq G$ with $M \nleq A$. Then $AM = AN$ for some $1 \neq N \trianglelefteq H$. By (iii) there exists $i \in I$ such that N does not act hypercentrally on A_i. So N also does not act hypercentrally on $[A_i, N]$. Since A is abelian, $[A_i, N] = [A_i, M] \leq M$ and M does not act hypercentrally on $[A_i, M]$. Thus M is not hypercentral. □

Lemma 1.4.28. [**hypercentral extension**] *Let $\mathcal{X}$ be a class of groups and H a group. Suppose H is a residually $\mathcal{X}$-group and a hyper-$(\mathcal{X}, *)$-group. Then there exists a hyper-$(\mathcal{X}, *)$ group G and an abelian normal subgroup A of G such that $G/A \cong H$ and such that every hypercentral normal subgroup of G is contained in A.*

Proof. Put $\mathcal{M} = \{M \trianglelefteq H \mid G/M \in \mathcal{X}\}$. Since H is residually-$\mathcal{X}$, $\bigcap \mathcal{M} = 1$. In particular, H is a subdirect product of $(G/M)_{M \in \mathcal{M}}$. For $M \in \mathcal{M}$ put $A_M = \mathbb{Z}[G/M]$. Then A_M is an abelian group with G/M acting faithfully on A_M by right multiplication. Let $1 \neq N \trianglelefteq H$ and choose $M \in \mathcal{M}M$ with $N \nleq M$. Then N does not act hypercentrally on A_M (indeed if NM/M is infinite, $\mathrm{C}_{A_M}(N) = 0$ and if NM/M is finite, choose a prime p with $p \nmid |NM/M|$ and observe that N does not act hypercentrally on A_M/pA_M.)

So 1.4.27 completes the proof. □

Corollary 1.4.29. [**not hypercentral x**] *Let $\mathcal{X}$ be a class of groups which is closed under homomorphic images but not under direct sums. Then there exists a hyper $(\mathcal{X}, *)$ groups which is not hypercentral by $\mathcal{X}$.*

Proof. Let $(H_i, i \in I$ be a family of $\mathcal{X}$ groups such that $H = \bigoplus_{i=1}^{\infty} H_i$ is not an $\mathcal{X}$-group. Then H is a subdirect product of $\mathcal{X}$ groups and so a residually $\mathcal{X}$-group. Each H_i is a $\mathcal{X}$-groups it also is a hyper $(\mathcal{X}, *)$ group. Hence by 1.4.14, H is hyper $(\mathcal{X}, , *)$. By **??** there exists a hyper $(\mathcal{X}, *)$-group G and an abelian normal subgroup A of G with $G/A \cong H$ and such that every hypercentral normal subgroup of G is contained in A. Suppose for a contradiction that G is hypercentral by $\mathcal{X}$ and let M be a hypercentral normal subgroup of G such that $G/M \in \mathcal{X}$. Then $M \leq A$ and $H \cong G/A \cong G/M\big/A/M$. Since $\mathcal{X}$ is **H**-closed, we conclude that $H \in \mathcal{X}$, a contradiction. □

Corollary 1.4.30. [**more hypercental x**] *Let $W \in \mathcal{P}(F)^{\infty}$ and suppose $\mathcal{X}(W) \neq \mathcal{V}(W_i)$ for all $i \in \mathrm{Z}^+$. Then there exists a hyper $(\mathcal{X}(W), *)$-group which is not hypercentral by $\mathcal{X}(W)$.*

Proof. For $i \in \mathrm{Z}^+$ pick $H_i \in \mathcal{X}(W) \setminus \mathcal{V}(W_i)$ and put $\oplus_{i \in I} H_i$. Since $W_i(H_i) \neq 1$ we have $W_i(H) \neq 1$. Thus $H \notin \mathcal{X}(W)$. H is a direct sum of $\mathcal{X}(W)$-group and so a residual $\mathcal{X}(W)$-group. Since H_i is a $\mathcal{X}(W)$-group and so a $(\mathcal{X}(W), *)$-group we conclude that from 1.4.14 that H is hyper $(\mathcal{X}(W), *)$. The corollary now follows from 1.4.29 □

Since there are solvable groups of arbitrary derived length and nilpotent groups of arbitrary class, the preceding corollary shows that there exists hyper (solvable,*) groups which are not hypercentral by solvable and hyper (nilpotent, *) groups which are not hypercentral by nilpotent.

Definition 1.4.31. [**def:locally cx**] *Let $\mathcal{X}$ be a class of groups and G a group. We say that G is locally $\mathcal{X}$, if for each finite subset I of G there exists $H \leq H$ with $I \subseteq H$ and $H \in \mathcal{X}$. The class of all locally $\mathcal{X}$ groups is denoted by $\mathbf{L}\mathcal{X}$.*

Observe that if $\mathcal{X}$ is closed under subgroups, then G is locally $\mathcal{X}$ if and only every finitely generated subgroup of G is an $\mathcal{X}$-group.

Proposition 1.4.32. [**schreier-reidemeister**] *Let G be finite generated subgroup and H a subgroup of finite index in G. Then H is finitely generated.*

Proof. Let X be a finite generating set for G with $x^{-1} \in X$ for all $x \in X$. For $T \in G/H$ pick $r_T \in T$ such that $r_H = 1$. Then $T = Hr_T$. Let $T \in G/H$ and $x \in X$. Then $r_T x \in (Hr_T)x = Tx = Hr_{Tx}$ and so there exists $h(T,x) \in H$ by

$$r_T x = h(T,x) r_{Tx}$$

Define $K = \langle h(T,x) \mid T \in G/H, x \in X \rangle$. We claim that

$$(*) \qquad g \in K r_{Hg} \text{ for all } g \in G$$

For this let $g = x_1 x_2 \dots x_n$ with $x_i \in X$ and $n \in \mathrm{N}$. If $n = 0$, then $g = 1$ and so $g \in K = K1 = K_{r_{H1}}$.

Suppose $n > 0$ and let $d = x_1 x_2 \dots x_{n-1}$. Then $g = dx_n$ and by induction on n, $d \in K r_{Hd}$.

Thus

$$g = dx_n \in K r_{Hd} x_n = K h(Hd, x_n) r_{Hdx_n} = K r_{Hg}$$

So (*) holds. If $g \in H$ we conclude $g \in K r_{Hg} = K r_H = K1 = K$. So $H \leq K$. Since $K \leq H$, this gives $K = H$ and so H is finitely generated. □

Let n be minimal number of generators of G and $i = |G/H|$. The preceding proof shows that H can be generated by $2ni$ elements. It can be shown that G is generated by $(n-1)i+1$ elements (Reidemeister-Schreier Theorem).

Corollary 1.4.33. [**lf by lf**] *The class $\mathbf{L}\mathcal{F}$ of locally finite groups is closed under subgroups, quotients and extensions.*

Proof. The first two assertions are obvious. Let G be a group and M a normal subgroup of G such that M and G/M are locally finite. Let S be a finite subset of G and $F = \langle S \rangle$. Then $FM/M = \langle sM \mid s \in S \rangle$ is finite generated and since G/M is finite, FM/M is finite. Hence also $F/F \cap M$ is finite and 1.4.32 implies that $F \cap M$ is finitely generated. Since M is locally finite, $F \cap M$ is finite. Hence F is finite and M is locally finite. □

Definition 1.4.34. [**def:p-group**] *Let G be a group and p a prime. Then G is called a p-group, if all elements of G have order a power of p.*

Note that by Cauchy's Theorem, a finite group if a p-group if and only if it has order a power of p.

Lemma 1.4.35. [**rg**] *Let R be a non-zero ring, G a group and H a non-trivial subgroup of G. Let $R[G]$ be the group of G over R and note that G acts on the abelian group $R[G]$ via $(\sum_{k\in G} r_k k)g = \sum_{k\in G} r_g kg$. Put $R_0[G] = \{\sum_{g\in G} r_g g \in R[G] \mid \sum_{g\in G} r_g = 0\}/$*

(a) [**a**] *Suppose H is infinite. Then $\mathrm{C}_{R[G]}(H) = 0$. In particular, H does not act hypercentrally on $R[G]$.*

(b) [**b**] *Suppose that $|H|r \neq 0$ for all $0 \neq r \in R^\sharp$. Then $\mathrm{C}_{R_0[H]}(H) = 0$. In particular, H does not act hypercentrally on $R[G]$.*

Proof. Let $a = \sum r_g g \in \mathrm{C}_{R[G]}(H)$. Then $r_g = r_{gh}$ for all $g \in G, h \in H$.

(a) If H is infinite, we get that conclude that $r_g = r_k$ for infinitely many $k \in G$. Since $r_g = 0$ for all but finitely many g, this implies $r_g = 0$ and so $a = 0$.

(b) Suppose H is finite and $|H|r \neq 0$ for all $r \in R_0[H]$. Let $a = \sum r_h h \in \mathrm{C}_{R_0[H]}(H)$ Then $r_h = r_1$ for all $h \in H$. Since $r \in R_0[H]$ this gives $0 = \sum_{h\in H} r_h = |H|r_1$ and so $r_1 = 0$. Hence $a = 0$. □

Lemma 1.4.36. [**easy zp=1**] *Let p be a prime and P a p-group with $\mathrm{Z}(P) = 1$. Then P has no non-trivial, finite normal subgroup. In particular, if $P \neq 1$, P is infinite.*

Proof. Suppose M is a non-trivial finite subgroup of P. Then $P/C_P(M)$ is also finite and acts on P. Since both $P/C_P(M)$ and M are p-groups, this gives $C_P(M) \neq 1$, a contradiction to $\mathrm{Z}(M) = 1$. □

Example 1.4.37. [**zp=1**] *Let p be a prime and k an integer with $k > 1$. Then there exists a locally finite, solvable p-group of derived length k with trivial center.*

Proof. If $k = 2$ let B be any infinite abelian p-group (for example $\bigoplus_{i\in \mathrm{N}} \mathrm{C}_p$. If $k > 2$ let B be any infinite, locally finite, solvable p-group of derived length $k-1$, which exists by induction (since by 1.4.36 a non-trivial p-group with trivial center is necessarily infinite). Put $A = \mathbb{F}_p[B]$. Then A is elementary abelian p group and B acts faithfully on A be right multiplication. Put $G = AB$, the semidirect product. Since B acts faithfully on A, $\mathrm{C}_G(A) = A$ and so $\mathrm{Z}(G) = C_A(G) = C_A(B)$. Since B is infinite, 1.4.35(a) gives $C_A(B) = 1$ and so $\mathrm{Z}(G) = 1$. Since $B^{(k-1)} = 1$ we have $G^{(k-1)} \leq A$ and so $G(k) \leq A' = 1$.

Suppose that $G^{(k-1)} = 1$. Since $B^{(k-2)} \leq G^{(k-2)}$ and $G^{(k-2)}$ is a normal subgroup of G, we have $[A, B^{(k-2)}]B^{(k-2)} \leq G^{(k-2)}$. Thus $[A, B^{(k-2)}, B^{(k-2)}] \leq G^{(k-1)} = 1$ and $B^{(k-2)}$ acts hyper-centrally on A. But by 1.4.36, $B^{(k-2)}$ is infinite, and so 1.4.35(a) gives a contradiction.

Thus $G^{(k-1)} \neq 1$ and G is solvable of derived length k.

Since both A and $B \cong G/A$ are locally finite p-groups, (**??**) implies that G is a locally finite p-group. □

Example 1.4.38. [**example**] *For each prime p there exists a locally finite, hyper (solvable,*) p-group which is not hypercentral-by-solvable.*

Proof. For $1 < k \in \mathbb{N}$ let H_k be a solvable p-group of derived length k with $\mathrm{Z}(H_k) = 1$ (see 1.4.37). Let $A_k = \mathbb{F}_p H_k$ and $H = \bigoplus_{k=2}^{\infty} H_k$. Let $1 \neq N \trianglelefteq H$ and choose k such that the projection N_k of N in H_k is not trivial. By **??** N_k is infinite. Hence by 1.4.35(a), N does not act hypercentrally on A_k. Put $A = \bigoplus A_k$ and $G = AH$. 1.4.27 now completes the proof. □

1.5 Radical Classes

Definition 1.5.1. [**def:delta asc**] *Let δ be a well ordered class, G a group and H a subgroup of G. We say that H is δ-ascending in G if the exists $\beta \in \delta$ and an ascending sequence $(H_\beta)_{\beta \leq \delta}$ from H to G. If H is an* Ord*-ascending subgroup of G, we write $H \mathrm{asc} G$ and say that H is an ascending subgroup of G. H is an ω-ascending subgroup of G, we write $H \trianglelefteq\trianglelefteq G$ and say that H is an subnormal subgroup of G.*

Definition 1.5.2. [**def:radical**] *Let $\mathcal{X}$ be a class of groups and G a group.*

(a) [**a**] *$\rho_{\mathcal{X}}(G)$ is group generated by all the normal $\mathcal{X}$-subgroups of G.*

(b) [**b**] *$\mathcal{X}$ is called $\mathbf{N}_0$ closed if any group generated by finitely many normal $\mathcal{X}$-subgroups is a $\mathcal{X}$ subgroup.*

(c) [**c**] *$\mathcal{X}$ is called $\mathbf{N}$ closed if any group generated by normal $\mathcal{X}$-subgroups is a $\mathcal{X}$ subgroup.*

(d) [**d**] *$\mathcal{X}$ is called $\acute{\mathbf{N}}$ closed if any group generated by ascending $\mathcal{X}$-subgroups is a $\mathcal{X}$ subgroup.*

(e) [**e**] *$\mathcal{X}$ is called $\mathbf{S}_n$-closed if every normal subgroup of an $\mathcal{X}$-group is a $\mathcal{X}$-group.*

Observe that $\mathcal{X}$ is $\mathbf{N}$-closed if and only if $\rho_{\mathcal{X}}(G)$ is $\mathcal{X}$-group for all groups G.

Lemma 1.5.3. [**asc and rho**] *Let $\mathcal{X}$ be an $\mathbf{N}$-closed class of groups, δ a well-ordered class and G a group. Suppose that whenever $\beta \in \delta$ is a limit ordinal, $K \mathrm{asc} L \mathrm{asc} G$ and $(M_\alpha)_{\alpha \leq \delta}$ is an ascending sequence from K to L such that $M_\alpha \in \mathcal{X}$ for all $\alpha < \delta$, then $L \in \mathcal{X}$. Then $\rho_{\mathcal{X}}(G)$ contains all δ-ascending $\mathcal{X}$-subgroups of G. In particular, if in addition, $\delta > 1$, then $\rho_{\mathcal{X}}(G)$ is the group generated by all the δ-ascending subgroups of G.*

Proof. Let H be an δ ascending subgroup of G and let $(H_\alpha)_{\alpha \leq \beta}$, $\beta \in \delta$ be an ascending sequence from H to G. For $\alpha \leq \beta$, define $\overline{H}_\alpha = \langle H^{H_\alpha} \}$.

We claim that $(\overline{H}_\alpha)_{\alpha \leq \beta}$ is a ascending series from H to $\langle H^G \rangle$. Since $H \leq H_\alpha \trianglelefteq H_{\alpha+1}$, $\overline{H}_{\alpha+1} \leq H_\alpha$. So $\overline{H}_\alpha \trianglelefteq \overline{H}_{\alpha+1}$. Also if α is a limit ordinal, then

$$\overline{H}_\alpha = \langle H^{H_\alpha} \rangle = \langle H^{\bigcup_{\gamma<\alpha} H_\alpha} \rangle = \bigcup_{\gamma<\alpha} \langle H^{H_\gamma} \rangle = \bigcup_{\gamma<\alpha} \overline{H}_\gamma$$

So $(\overline{H}_\alpha)_{\alpha\leq\beta}$ is a ascending series from $H = \langle H^H\rangle$ to $\langle H^G\rangle$.

Next we will use induction on α to show that $\overline{H}_\alpha \in \mathcal{X}$ for all $\alpha \leq \delta$.

Suppose first that $\alpha = 0$, then $\overline{H}_\alpha = H \in \mathcal{X}$.

Suppose next that $\alpha = \gamma + 1$ for some ordinal γ, then by induction, $\overline{H}_\gamma$ is a normal $\mathcal{X}$ subgroup of H_γ. Let $g \in H_\alpha$. Then g normalizes H_γ and so $\overline{H}_\gamma^g$ is a normal $\mathcal{X}$-subgroup of H_γ. Thus

$$\overline{H}_\alpha = \langle H^{H_\alpha}\rangle = \langle \overline{H}_\gamma^{H_\alpha}\rangle = \langle \overline{H}_\gamma^g \mid g \in H_\alpha\rangle$$

is generated by normal $\mathcal{X}$-subgroups. Since $\mathcal{X}$ is $\mathbf{N}$-closed, $\overline{H}_\alpha \in \mathcal{X}$.

Suppose that α is a limit ordinal. Then $(\overline{H})_{\gamma\leq\alpha}$ is an ascending sequence from H to $\overline{H}_\alpha$. By induction $\overline{H}_\gamma$ is an $\mathcal{X}$ groups for all $\gamma < \alpha$ and so by the assumption of the lemma, $\overline{H}_\alpha \in \mathcal{X}$.

We proved that $\overline{H}_\alpha \in \mathcal{X}$ for all $\alpha \leq \beta$. In particular, $\langle H^G\rangle = \overline{H}_\beta \in \mathcal{X}$. Thus $\langle H^G\rangle$ is a normal $\mathcal{X}$ subgroups of G and so $\langle H^G\rangle \leq \rho_{\mathcal{X}}(G)$. Hence also $H \leq \rho_{\mathcal{X}}(G)$. □

Corollary 1.5.4. [**rho and subnormal**]*Let $\mathcal{X}$ be an* $\mathbf{N}$ *closed class of groups. Then $\rho_{\mathcal{X}}(G)$ is the group generated by all the subnormal $\mathcal{X}$-subgroups of G.*

Proof. Note that ω does not contain a limit ordinal. So the condition in 1.5.3 holds vacuously for $\delta = \omega$. □

Corollary 1.5.5. [**ncx**] *Let $\mathcal{X}$ be class of groups, and let $\mathbf{N}\mathcal{X}$ be the class of groups which are generated by subnormal $\mathcal{X}$ groups. Then $\mathbf{N}\mathcal{X}$ is the smallest $\mathcal{N}$-closed class of groups containing $\mathcal{X}$, that is $\mathbf{N}\mathcal{X}$ is $\mathbf{N}$-closed and every $\mathbf{N}$-closed class of groups containing $\mathcal{X}$ also contains $\mathbf{N}\mathcal{X}$.*

Proof. Let G be a group generated by a family $\mathcal{M}$ of normal $\mathbf{N}\mathcal{X}$-groups. Then each $M \in \mathcal{M}$ is generated by a family $\mathcal{N}_M$ of subnormal $\mathcal{X}$-subgroups of M. Note that each $N \in \mathcal{N}_M$ is subnormal in G and so $\bigcup_{M\in\mathcal{M}} \mathcal{N}_M$ is a family of subnormal subgroups of G generating G. Thus $G \in \mathbf{N}\mathcal{X}$ and $\mathbf{N}\mathcal{X}$ is $\mathbf{N}$-closed.

Now let $\mathcal{Y}$ be any $\mathbf{N}$-closed class of groups with $\mathcal{X} \subseteq \mathcal{Y}$. Let $G \in \mathbf{N}\mathcal{Y}$. Then G is generate by subnormal $\mathcal{X}$ groups, and so also by subnormal $\mathcal{Y}$-subgroups. Thus 1.5.4, $G \leq \rho_{\mathcal{Y}}(G)$. Hence $G = \rho_Y(G)$ and so $G \in \mathcal{Y}$. □

Corollary 1.5.6. [**cap subnormal**] *Let $\mathcal{X}$ be an* $\mathbf{N}$*- and* $\mathbf{S}_n$*-closed class of groups. Let G be a group and $H \trianglelefteq\trianglelefteq G$. Then*

$\rho_{\mathcal{X}}(H) = \rho_{\mathcal{X}}(G) \cap H$.

Proof. Note that $\rho_{\mathcal{X}}(G) \cap H$ is subnormal subgroup of the $\mathcal{X}$ group $\rho_{\mathcal{X}}(G)$. Since $\mathcal{X}$ is $\mathbf{S}_n$-closed, $\rho_{\mathcal{X}}(G) \cap H$ is an $\mathcal{X}$ group. Since $\rho_{\mathcal{X}}(G) \cap H$ is normal in H this gives $\rho_{\mathcal{X}}(G) \cap H \leq \rho_{\mathcal{X}}(H)$.

Conversely, $\rho_{\mathcal{X}}(H)$ is a subnormal $\mathcal{X}$ subgroup of G and so by 1.5.4 $\rho_{\mathcal{X}}(H) \leq \rho_{\mathcal{X}}(G)$. Thus $\rho_{\mathcal{X}}(H) \leq \rho_{\mathcal{X}}(G) \cap H$ and the corollary holds. □

Definition 1.5.7. [**def:radical class**] *A class $\mathcal{X}$ of groups is called radical if it is* **N** *and* **H** *closed, and if for every group G*

$$\rho_{\mathcal{X}}(G/\rho_{\mathcal{X}}(G)) = 1$$

Lemma 1.5.8. [**char radical**] *A class of group is radical if and only if its* **N**, **H** *and* **P** *closed.*

Proof. Let $\mathcal{X}$ be class of groups which is **N** and **H**-closed.

Suppose first that $\mathcal{X}$ is radical and let G be a group which is $\mathcal{X}$-by-$\mathcal{X}$. Then there exists $M \trianglelefteq G$ such that M and G/M are $\mathcal{X}$-group. Then $M \in \rho_{\mathcal{X}}(G)$ and

$$G/\rho_{\mathcal{X}}(G) \cong G/M/\rho_{\mathcal{X}}(G)/M$$

Since G/M is an $\mathcal{X}$-group and $\mathcal{X}$ is **H**-closed we conclude that $G/\rho_{cX}(G)$ is an $\mathcal{X}$ groups. Thus

$$G/\rho_{\mathcal{X}}(G) \leq \rho_{\mathcal{X}}(G/\rho_{\mathcal{X}}(G)) = 1$$

and so $G = \rho_{cX}(G) \in \mathcal{X}$. Thus $\mathcal{X}$ is closed under extension, that is **P**-closed.

Suppose next that $\mathcal{X}$ is closed under extensions and let G be any group. Let M be the inverse image of $\rho_{\mathcal{X}}(G/\rho_{\mathcal{X}}(G))$ in G. Then M is a normal subgroups of G and both $\rho_{\mathcal{X}}(G)$ and $M/\rho_{\mathcal{X}}(G)$ are $\mathcal{X}$ groups. Thus M is a normal $\mathcal{X}$ subgroup of G and so $M \leq \rho_{\mathcal{X}}(G)$. Thus $M = \rho_{\mathcal{X}}(G)$ and $\rho_{\mathcal{X}}(G/\rho_{\mathcal{X}}(G)) = M/\rho_{\mathcal{X}}(G) = 1$. Thus $\mathcal{X}$ is a radical class. □

Definition 1.5.9. [**def rad cx**] *Let $\mathcal{X}$ be a class of groups. Then* $\mathrm{rad}\mathcal{X} = \mathrm{Hyp}(\mathbf{H}\mathcal{X}))$. *So* $\mathrm{rad}\mathcal{X}$ *is the class of all groups with ascending normal series all of whose factors are homomorphic images of an $\mathcal{X}$ group.*

Lemma 1.5.10. [**char rad cx**] *Let $\mathcal{X}$ be a class of groups. Then* $\mathrm{rad}\mathcal{X}$ *is the smallest radical class containing $\mathcal{X}$, that is* $\mathrm{rad}\mathcal{X}$ *is a radical class and contains all radical classes containing $\mathcal{X}$.*

Proof. By **??**(**??**), $\mathrm{rad}\mathcal{X}$ is **H**-closed. By 1.4.14, $\mathrm{rad}\mathcal{X}$ is **N**-closed and by 1.4.13, $\mathrm{rad}\mathcal{X}$ is **P** closed. So by 1.5.8, $\mathrm{rad}\mathcal{X}$ is a radical class.

Now let $\mathcal{Y}$ be radical class with $\mathcal{X} \subseteq \mathcal{Y}$. Let $G \in \mathrm{rad}\mathcal{X}$ and choose a hyper-$(*, \mathbf{H}\mathcal{X})$-sequence $(G_\alpha)_{\alpha \leq \beta}$ for G.So each We will show by induction that $G_\alpha \in \mathcal{Y}$ for all ordinals $\alpha \leq \beta$. If $\alpha = 0$, this is obvious. Suppose $\alpha = \delta + 1$ is a successor. Then by induction $G_\delta \in \mathcal{Y}$. Since $\mathcal{X} \subseteq \mathcal{Y}$ and $\mathcal{Y}$ is **H**-closed, $\mathbf{H}\mathcal{H} \subseteq \mathcal{Y}$. Thus $G_\alpha/G_\delta \in \mathcal{Y}$. Since $\mathcal{Y}$ is **P** closed this gives $G_\alpha \in \mathcal{Y}$.

Suppose α is limit ordinal. Then $G_\alpha = \bigcup_{\delta < \alpha} G_\delta = \langle G_\delta \mid \delta < \alpha \rangle$. By induction $G_\delta \in \mathcal{Y}$ and since $\mathcal{Y}$ is **N**-closed, $G_\alpha \in \mathcal{Y}$.

We proved that each $G_\alpha \in \mathcal{Y}$. In particular $G = G_\beta \in \mathcal{Y}$ and so $\mathrm{rad}\mathcal{X} \subseteq \mathcal{Y}$. □

Definition 1.5.11. [**def:central extension**] *Let G be a group and H be group. We say that G is a central extension of H if there exists $Z \leq \mathrm{Z}(G)$ with $G/Z \cong H$. If $\mathcal{X}$ is a class of groups, then $\mathbf{C}\mathcal{X}$ is class of central extensions of $\mathcal{X}$-groups.*

Proposition 1.5.12. [**cgrho**] *Let $\mathcal{X}$ be a $\mathbf{H}$-, $\mathbf{S}_n$- and $\mathbf{C}$-closed class of groups. Let $G \in \mathrm{rad}\mathcal{X}$ and put $H = \rho_{\mathcal{X}}(G)$. Then $\mathrm{C}_G(H) \leq H$.*

Proof. Since $\mathcal{X}$ is $\mathbf{H}$-closed and $G \in \mathrm{rad}\mathcal{X}$, there exists a hyper $\mathcal{X}$-sequence $(G_\alpha)_{\alpha\leq\beta}$ for G. We claim that $\mathrm{C}_G(H) \cap G_\alpha \leq H$ for all $\alpha \leq \beta$. This is obvious for $\alpha = 0$. So suppose $\alpha > 0$ and $\mathrm{C}_G(H) \cap G_\delta \in \mathcal{X}$ for all $\delta < \alpha$. If α is limit ordinal, then

$$\mathrm{C}_G(H) \cap G_\alpha = \mathrm{C}_G(H) \cap \bigcap_{\delta<\alpha} = \bigcap_{\delta<\alpha} (\mathrm{C}_G(H) \cap G_\delta) \leq H$$

So suppose $\alpha = \delta + 1$ for some ordinal delta. Put $D = \mathrm{C}_G(H) \cap G_\alpha = \mathrm{C}_G(H) \cap G_{\delta+1}$. Then DG_δ/G_δ is an normal subgroup of the $\mathcal{X}$-group $G_{\delta+1}/G_\delta$. Since $\mathcal{X}$ is $\mathbf{S}_n$-closed, DG_δ/G_δ is $\mathcal{X}$ group. Hence also $D/D_\cap G_\delta$ is an $\mathcal{X}$-group. Note the

$$[D, D_\cap G_\delta] \leq [\mathrm{C}_G(H), \mathrm{C}_G(H) \cap G_\delta] \leq [\mathrm{C}_G(H), H] = 1$$

and so $D \cap G_\delta \leq \mathrm{Z}(D)$. Thus D is a central extension of an $\mathcal{X}$ group. Since $\mathcal{X}$ is a $\mathcal{C}$-closed, $D \in \mathcal{X}$. Thus D is a normal $\mathcal{X}$ subgroup of G and so $D \leq H$.

Thus the claim holds. In particular, $\mathrm{C}_G(H) = \mathrm{C}_G(H) \cap G = \mathrm{C}_G(H) \cap G_\beta \leq H$. □

1.6 Finitely generated groups

Definition 1.6.1. [**def:rang**] *Let G be an A-group.*

(a) [**a**] *Let c be a cardinal. Then G is $c - A$-generated if the exists a subset I of G with $G = \langle I^A \rangle$ and $|I| \leq c$. We will also say that G is an c-generated A-group. Such an I is called $c - A$-generating set for G.*

(b) [**b**] *$r^A(G)$ is the least cardinal c such that G is $c - A$-generated.*

(c) [**c**] *If G is called finitely A-generated $r^A(G) \in \mathrm{N}$.*

(d) [**d**] $\mathrm{rank}^A(H) = \sup\{r^A(H) \mid H \leq G, r^A(G) \in \mathrm{N}\}$.

(e) [**e**] *If $A = 1$, we drop A in the previous notations.*

Lemma 1.6.2. [**factor and r**] *Let G be an A-group, H an A-subgroups and M a normal A-subgroup of G with HM.*

(a) [**a**] *There exists an $r^A(G)$-generated A-subgroup K of G with $G = \langle H, K \rangle$.*

(b) [**b**] $r^{HA}(M) \leq r^A(G) + r^{HA}(H \cap M)$.

Proof. (a): Let $I \subseteq G$ with $|I| = r^A(G)$ and $G = \langle I^A \rangle$. For $i \in I$ pick $h_i \in H$ and $m_i \in M$ with $i = h_i m_i$. Put $K = \langle m_i^A \mid i \in I \rangle$. Then K is an $r^A(G)$-generated A-subgroup of M. Also

$$G = \langle I^A \rangle = \langle h_i m_i^A \mid i \in I \rangle \leq \langle H, m_i^A \mid i \in I => = \langle H, K \rangle \leq G$$

and so (a) holds.

(b): Let K be as in (a). Then $G = \langle H, K \rangle = H \langle K^H \rangle$. Since $\langle K^H \rangle \leq M$ this gives $M = (H \cap M) \langle K^H \rangle$. Observe that $\langle K^H \rangle$ is an $r^A(G)$-generated HA-group and so M is an $r^A(G) + r^{HA}((H \cap M)$ generated HA-group. □

Lemma 1.6.3. [**simple rank**] *Let A be a group, G an A-group and H an A-subgroup of G.*

(a) [**a**] $\operatorname{rank}^A(H) \leq \operatorname{rank}^A(G)$.

(b) [**b**] *If H is normal in G then* $\operatorname{rank}^A(G/H) \leq \operatorname{rank}^A(G)$.

(c) [**c**] *If H is normal in G then* $\operatorname{rank}^A(G) \leq \operatorname{rank}^A(H) + \operatorname{rank}^A(G/H)$.

Proof. (a) and (b) are obvious. For (c) let L be a finitely A-generated A-subgroup of G. LH/H is an $\operatorname{rank}^A(G/H) - A$-subgroup of G/H and so there exists a finite subset I of L with $LH/H = \langle I^A \rangle H/H$ and $|I| \leq \operatorname{rank}^A(G/H)$. Then $L = \langle I^A \rangle (L \cap H)$. By 1.6.2(a), there exists a $|I| - A$-generated subgroup K of $L \cap H$ with $L = \langle I^A, K \rangle$. Since $K \leq H$, K is $\operatorname{rank}^A(H)$-generated and so $r^A(L) \leq \operatorname{rank}^A(G/H) + \operatorname{rank}^A(H)$. □

Definition 1.6.4. [**presentation**] *Let G be a group and c a cardinal.*

(a) [**a**] *A presentation of rank c for G is an onto homomorphism $\phi : F \to G$, where F is a free group of rank c.*

(b) [**b**] *A presentation $\phi : F \to G$ is called finite F has finite rank and* $\ker \phi$ *is finitely F generated.*

(c) [**c**] *A group is called finitely presented if its has a finite presentation.*

Example 1.6.5. [**finite groups are finitely presented**]

Proof. $G \cong \langle x_g \mid x_h x_h = x_{gh}, g, h \in G \rangle$. □

Lemma 1.6.6. [**finitely presented quotient**] *Let H be a finitely generated group and $M \trianglelefteq H$. if H/M is finitely presented, then M is finitely M generated.*

Proof. Put $G = H/M$ and define $\beta : H \to M, h \to hM$. Also let $\alpha : F \to G$ be a finite presentation of G. Let $(x_i, i \in I$ be basis for F and pick $h_i \in I$ with $\beta(h_i) = \alpha(x_i)$. Then there exists a unique homomorphism $\gamma : F \to H$ with $\gamma(x_i) = h_i$. then $\beta(\gamma(x_i)) = \beta(h_i) = \alpha(x_i)$ and so $\alpha = \beta \circ \gamma$. Note that $M = \ker \beta$ and $K = \operatorname{Im} \gamma$. Since $\beta(K) = \beta(\gamma(H)) = \alpha(H) = G$ we have $H = KM$. We compute

$$K \cap M = \{\gamma(f) \mid f \in F, \beta(\gamma(f)) = 1\} = \{\gamma(f) \mid f \in F \mid \alpha(f) = 1\} = \beta(\ker \alpha)$$

Since α is a finite presentation , $\ker \alpha$ is finitely H generated and so $K \cap M$ is finitely K-generated. Also H is finitely generated and so by 1.6.2(b), M is finitely H-generated. □

Proposition 1.6.7. [**all presentation finite**] *Let G be a finitely presented group. Then all presentation of finite rank for G are finite.*

Proof. Let $\beta : H \to G$ be a finite presentation and put $M = \ker \beta$. Then H is finite generated and $H/M \cong G$ is finitely presented. By 1.6.6, M is finitely H generated and so β is a finite presentation. □

Proposition 1.6.8. [**extensions of finitely presented groups**] *The class of finitely presented groups is closed under extensions.*

Proof. Let G be a group and N a normal subgroups of G such that both G/N and N are finitely presented. Let $\alpha : F \to G/N$ and $\beta : H \to N$ be finite presentation of G/N and N, respectively. Let I be a basis for F, J a basis for H, K a finite F-generating set for $\ker \alpha$ and L a finite H-generating set for $\ker \beta$. For $i \in I$ pick $g_i \in G$ with $\alpha(i) = g_i N$.Since F is free there exists a homomorphism $\alpha^* : F \to G$ with $\alpha^*(i) = g_i$. Then $\alpha^*(f)N = \alpha(f)$ for all $f \in F$.In particular $\alpha(f) = 1$ if and only if $\alpha^*(f) \in N$. If $k \in K, i \in I$ and $l \in L$, then $\alpha^*(k)$, $\beta(l)^{g_i}$ and $\beta(l)^{g_i^{-1}}$ all are in N and so $\alpha^*(k) = \beta(h_k)$, $\beta(l)^{g_i} = \beta(h_{ki})$ and $\beta(l)^{g_i^{-1}} = \beta(\tilde{h}_{ki})$ for some $h_k, h_{ki}, \tilde{h}_{ki} \in H$. Let T be the free product of F and H, that is the free group with basis $I \uplus J$. Note that F and H are subgroups of T. Let M be the normal subgroup of T generated by the elements

$$\begin{array}{ll} l & l \in L \\ kh_k^{-1} & k \in K \\ j^i h_{ki}^{-1} & j \in Ji \in I \\ j^{i^{-1}} \tilde{h}_{ji}^{-1} & j \in J, i \in I \end{array}$$

Let $\gamma : T \to G$ be the homomorphism defined by $\gamma(i) = g_i = \alpha^*(i)$ for $i \in I$ and $\gamma(j) = \beta(j)$ for $j \in J$. We will show that γ is onto and $\ker \gamma = M$. Observe that this implies that γ is a finite presentation for G.

Note that $\gamma \mid F = \alpha^*$ and $\gamma \mid K = \beta$. Thus $N = \beta(K) = \gamma(K) \leq \operatorname{Im} \gamma$. Since α is onto, $\alpha^*(F)N = G$ and so $\gamma(F)N = G$ and $\operatorname{Im} \gamma = G$.

Also $\gamma(l) = \beta(l) = 1$ for all $l \in L$, $\gamma(kh_k^{-1}) = \alpha^*(k)\beta(h_k)^{-1} = 1$, $\gamma(j^i h_{ki}^{-1}) = \beta(j)^{g_i}\beta(h_{ji}^{-1} = 1$, $\gamma(j^{i^{-1}}\tilde{h}_{ji}^{-1}) = \beta(j)^{g_i^{-1}})\beta(\tilde{h}_{ji}^{-1} = 1$. So all the generators of M are in $\ker \gamma$ and so $M \leq \ker \gamma$.

Since $j^i M = h_{ji} M \in HM$ and $j^{i^{-1}} M = \tilde{h}_{ji} M \in HM$ for all $j \in I$ and $i \in M$ we see that HM is normalized by $\langle I, J \rangle = T$. It follows that $T = \langle F, H \rangle = FHM$. For $k \in K$ we have $k \in h_k M \in HM$ and so $\ker \alpha \leq HM$.

Let $t \in \ker \gamma$, then $t = fhm$ for some $f \in F, h \in H$ and $m \in M$. Then $1 = \gamma(t) = \gamma(f)\gamma(h)\gamma(m) = \alpha^*(f)\beta(h) \in \alpha^*(f)N$. Thus $\alpha^*(f) \in N$ and so $\alpha(f) = 1$ and $f \in \ker \alpha \in HM$. Hence $t = fhm \in HM$ and we may assume that $f = 1$. Thus $1 = \beta(h)$ and $h \in \ker \beta$. Since $l \in M$ for all $l \in L$ we see that $\ker \beta \leq M$ and thus $t = hm \in M$. □

Corollary 1.6.9. [**polycyclic are finitely presented**] *All polycyclic groups are finitely presented. More generally all poly-(cyclic or finite) groups are finitely presented.*

Proof. □

1.7 Locally $\mathcal{X}$-groups

Definition 1.7.1. [**def:directed set**]

(a) [**a**] *A partially ordered set $(I, <)$ is called direct if for all $i, j \in I$ there exists $k \in I$ with $i \leq k$ and $j \leq k$.*

(b) [**b**] *A local system for a group G is a set $\mathcal{L}$ of subgroups such that $G = \bigcup \mathcal{L}$ and $(\mathcal{L}, \subset)$ is directed.*

Note that a partially ordered set is directed if and only if every non-empty subset has an upper bound.

Lemma 1.7.2. [**local system**]

(a) [**a**] *Let G be a group with a local system $\mathcal{L}$. Then each finitely generated subgroup of G is contained in member of $\mathcal{L}$.*

(b) [**b**] *Let $\mathcal{X}$ be a class of groups. Then every group with a local system of $\mathcal{X}$-groups is a local $\mathcal{X}$-group. In particular a union of a chain of $\mathcal{X}$-groups is a local $\mathcal{X}$-group.*

(c) [**c**] *$\mathcal{L}$ is a closure operation.*

Proof. (a) Let S be a finite subset of G. Since $G = \bigcup \mathcal{L}$, for each $s \in S$ there exists $L_s \in \mathcal{L}$ with $s \in \mathcal{L}$. Since $\mathcal{L}$ is directed , there exists an upper bound L for $\{L_s \mid s \in S\}$ in $\mathcal{L}$. Thus $s \in L_s \subseteq L$ and $\langle S \rangle \leq L$.

(b) follows immediately from (a).

(c) Let $\mathcal{X}$ be a class of groups. Let G be a group which is locally $\mathcal{L}\mathcal{X}$. Let S be a finite subset of G. Then there exists a $\mathcal{L}\mathcal{X}$-subgroup H of G with $S \subseteq H$. Since H is locally $\mathcal{X}$, there exists a subgroup K of H with $S \subseteq K$. Thus $G \in \mathcal{L}\mathcal{X}$. □

Proposition 1.7.3. [**n and l**] *An* $\mathbf{L}$*-closed class of groups is* $\mathbf{N}_0$ *if and only if its is* $\acute{\mathbf{N}}$*-closed*

Proof. The one direction is obvious. So suppose $\mathcal{X}$ is an $\mathbf{L}$ and $\mathbf{N}_0$ closed class of group. We will first show that it is bN closed. For this let G be a group which is generated by normal $\mathcal{N}$ subgroups. Let $\mathcal{L}$ be the set of subgroups of G which are generated by finitely many normal $\mathcal{X}$-subgroups. Note that $\mathcal{L}$ is a local system for G. Since $\mathcal{X}$ is $\mathbf{N}_0$-closed, $\mathcal{L} \subseteq \mathcal{X}$. So by 1.7.2(b), G is locally $\mathcal{X}$. Since $\mathcal{X}$ is $\mathcal{L}$ closed, $G \in \mathcal{X}$ and so $\mathcal{X}$ is $\mathcal{B}$-closed.

Now let G be group which is generated by ascending $\mathcal{X}$-subgroups. By 1.7.2(b), the unions of any chain of $\mathcal{X}$ subgroups of G is $\mathbf{L}\mathcal{X}$-group and so an $\mathcal{X}$-group. Thus the assumptions of 1.5.3 are fulfilled for $\delta = \text{Ord}$. Hence all ascending $\mathcal{X}$-subgroups of G are contained in $\rho_{\mathcal{X}}(G)$. So $G = \rho_{\mathcal{X}}(G) \in \mathcal{X}$. □

Lemma 1.7.4. [**easy locally**] *Let $\mathcal{X}$ be an* **S***-closed class of groups and G a group.Then the following are equivalent.*

(a) [**a**] *G is locally $\mathcal{X}$.*

(b) [**b**] *Every finitely generated subgroup of G is an $\mathcal{X}$-group.*

(c) [**c**] *G is locally $\mathcal{X} \cap \mathcal{F}$ (recall here that $\mathcal{F}$ is the class of finitely generated groups.*

Proof. (a) $\Longrightarrow$ (b): Let $S \subseteq G$ be finite. Since G is locally $\mathcal{X}$, $S \leq H$ for some $\mathcal{X}$ subgroup of G. Since $\mathcal{X}$ is **S**-closed, $\langle S \rangle$ is an $\mathcal{X}$-group.

(b) $\Longrightarrow$ (c): and (c) $\Longrightarrow$ (**??**): are obvious. □

Chapter 2

Locally nilpotent and locally solvable groups

2.1 Commutators

Lemma 2.1.1. [**commutator formulas**] *Let G be a group and x, y, z in G. Then*

(a) [**a**] $[x, y] = x^{-1}x^y = y^{-x}y$

(b) [**b**] $[x, yz] = [x, z]^y[x, z]$

(c) [**c**] $[xy, z] = [x, z]^y[y, z]$

(d) [**d**] $[x, y]^{-1} = [y, x]$.

(e) [**e**] $[x^{-1}, y] = [x, y]^{-x^{-1}}$.

(f) [**f**] $[x, y^{-1}, z]^y[y, z^{-1}, x]^z[z, x^{-1}, y]^x$.

Proof. Readily verified. □

Definition 2.1.2. [**def:comm groups**] *Let G be a group.*

(a) [**a**] *Let $X, Y \subseteq G$. The $[X, Y] := \langle [x, y] \mid x \in X, y \in U \rangle$.*

(b) [**b**] *Let $X_1, X_2, \dots X_n$ be subsets of G inductively define,*

$$[X_1] = \langle X_1 \rangle \text{ and } [X_1, X_2, \dots, X_n] := [[X_1, X_2, \dots X_{n-1}], X_n]$$

Lemma 2.1.3. [**comm 1**] *Let X and Y be subsets of a groups G.*

(a) [**a**] *If $1 \in Y$, then $\langle X^Y \rangle = \langle X, [X, Y] \rangle$.*

(b) [**b**] *If Y is a subgroup of G, then $[X, Y]$ is Y-invariant.*

Proof. (a)

$$\langle X^Y\rangle = \langle x^y \mid x\in X, y\in Y\rangle = \langle x[x,y] \mid x\in X, y\in Y\rangle \leq \langle X, [X,Y]\rangle$$
$$= \langle z, [x,y] \mid z\in x, z\in X, y\in Y\rangle = \langle z, x^{-1}x^y \mid x,z\in X, y\in Y\rangle \leq \langle X^Y\rangle$$

where, in the last inequality we used that $X\subseteq \langle X^Y\rangle$ since $1\in Y$.

(b) Let $x\in X$ and $y,z\in Y$. Then

$$x, zy] = [x,y]^z[x,z]$$

and so

$$[x,y]^z = [x,zy][x,z]^{-1}\in [X,Y]$$

where in the last assertion we used that Y and $[X,Y]$ are subgroups of G. □

Lemma 2.1.4. [**comm 2**] *Let X and Y be subsets of a group G and put $H=\langle X\rangle$ and $K=\langle Y\rangle$. Then*

$$[H,Y] = \langle [X,Y]^H\rangle$$

and

$$[H,K] = \langle [X,Y]^{HK}\rangle$$

Proof. Put $L=\langle [X,Y]^H\rangle$. By **??**(**??**), $[H,Y]$ is H-invariant. Since $[X,Y]\leq [H,Y]$, this gives $L\leq [H,Y]$. Since L is H acts on the cosets of L in G by conjugation, indeed $(Lg)^h = Lg^h$. Also Lg is fixed-point of $h\in H$ iff $Lg = lg^h$ and iff $[h,g]=g^{-h}\in L$. So all elements of X fix all Ly, $y\in Y$. Hence also $H=\langle X\rangle$ fixes all $Ly, y\in Y$ and so $[h,y]\in L$ for all $h\in H, y\in Y$. Thus $[H,Y]\leq L$ and $L=[H,Y]$.

This proves the first statement.

For the second, we use the fist statement twice:

$$[H,K] = \langle [H,Y]^K\rangle = \langle\langle [X,Y]^H\rangle^K\rangle = \langle [X,Y]^{HK}\rangle$$

□

2.2 Locally nilpotent groups

Definition 2.2.1. [**L**]*et G be a group and α and ordinal. Define subgroups $\mathrm{Z}_\alpha(G)$ and $\gamma_\alpha(G)$ inductively a follows:*

$\mathrm{Z}_0(G)=1, \mathrm{Z}_\alpha(G)/\mathrm{Z}_{\alpha-1} = \mathrm{Z}(G/\mathrm{Z}_{\alpha-1}(G))$, *if α is a successor and* $\mathrm{Z}_\alpha(G) = \bigcup_{\beta<\alpha}\mathrm{Z}_\beta(G)$ *if α is a limit ordinal*

$\gamma_0(G) = G, \gamma_\alpha(G) = [\gamma_{\alpha-1}(G), G]$, *if* α *is a successor and* $\mathrm{Z}_\alpha(G) = \bigcap_{\beta<\alpha} \mathrm{Z}_\beta(G)$ *if* α *is a limit ordinal*

$(\mathrm{Z}_\alpha)_\alpha$ *is called the upper central series of* G *and* $(\gamma_\alpha(G))_\alpha)$ *the lower centrals series of* G*.*

Lemma 2.2.2. [**char nilpotent**] *Let* $n \in \mathrm{N}$ *and* G *a group. Then the following statements are equivalent:*

(a) [**a**] $G = \mathrm{Z}_n(G)$*.*

(b) [**b**] *There exists a finite ascending normal series*

$$1 = A_0 \leq A_1 \leq \ldots A_{n-1} \leq A_n = G$$

of G *with* $[A_i, G] \leq A_{i-1}$ *for all* $1 \leq i \leq n$*.*

(c) [**c**] $\gamma_n(G) = 1$*.*

Proof. (a) $\Longrightarrow$ (b): Just put $A_i = \mathrm{Z}_i(G)$.

(b) $\Longrightarrow$ (a): We claim that $A_i \leq \mathrm{Z}_i(G)$. This is clearly true for $i = 0$. Suppose that $A_i \leq \mathrm{Z}_i(G)$. Then $[A_{i+1}, G] \leq A_i \leq \mathrm{Z}_i(G)$ and so $A_{i+1} \leq \mathrm{Z}_{i+1}(G)$. This proves the claim and so $G = A_N \leq \mathrm{Z}_n(G)$.

(b) $\Longrightarrow$ (c): We claim that $\gamma_i(G) \leq A_{n-i}$. Indeed this is true for $i = 0$. Suppose $\gamma_i(G) \leq A_{n-i}$. Then

$$\gamma_{i+1}(G) = [\gamma_i(G), G] \leq [A_{n-i}, G] \leq A_{n-(i+1)}$$

Thus the claim holds and $\gamma_n(G) \leq A_0 = 1$

(c) $\Longrightarrow$ (b): Just put $A_i = \gamma_{n-i}(G)$. □

Definition 2.2.3. [**def:nilpotent**] *Let* G *be a group. Then* G *is called nilpotent if* $\gamma_n(G) = 1$ *for some* $n \in \mathrm{Z}_n(G)$*. The smallest such* n *is called the nilpotency class of* G*.* $\mathcal{N}\rangle\updownarrow$ *denotes the class of nilpotent groups.*

Lemma 2.2.4. [**nilpotent and no**] *Let* K *and* L *be nilpotent normal subgroups of a group* G *of nilpotency class* k *and* l*, respectively. Then* KL *is nilpotent of class at most* $k + l$*. In particular,* $\mathcal{N}\rangle\updownarrow$ *is* $\mathbf{N}_0$ *closed.*

Proof. If $k = 0$ or $l = 0$, then $K = 1$ or $L = 1$ and the lemma holds. Now suppose $k > 0$ and $l > 0$. Note that $K\mathrm{Z}(L)/\mathrm{Z}(L)$ has nilpotency class at most k and $L/\mathrm{Z}(L)$ has nilpotency class $l - 1$. So by induction $KL/Z(L)$ class at most $k + l - 1$. Thus $\gamma_{k+l-1}(KL) \leq \mathrm{Z}(L)$. By symmetry, $\gamma_{k+l-1}(KL) \leq \mathrm{Z}(K)$. Since $\mathrm{Z}(K) \cap \mathrm{Z}(L) \leq \mathrm{Z}(KL)$ we conclude that

$$[\gamma_{k+l}(KL), KL] \leq [\mathrm{Z}(KL), KL] = 1$$

□

Definition 2.2.5. [**c generated**] *Let c be a cardinality. Then a group G is called c-generated if there exists a subset T of G with $G = \langle T \rangle$ and $|T| \leq c$.*

Lemma 2.2.6. [**polycyclic**] *Let G be a group with an ascending sequence $(G_\alpha)_{\alpha \leq \beta}$ all of whose factors are cyclic. Then every subgroups of G can is $|\beta|$-generated. In particular, all polycyclic groups are finitely generated.*

Proof. For $\alpha < \beta$, $G_{\alpha+1}/G_\alpha$ is cyclic and so there exists g_α with $G_{\alpha+1} = \langle g_\alpha \rangle G_\alpha$. We claim that for all $\gamma \leq \alpha$, $G_\gamma = \langle g_\delta \mid \delta < \gamma \rangle$. This is obvious of $\gamma = 0$ Suppose the claim is true for all ordinal less than γ. $\gamma = \alpha + 1$, then

$$G_\gamma = \langle g_\alpha \rangle G_\alpha = \langle g_\alpha \rangle \langle g_\delta \mid \delta < \alpha \rangle = \langle g_\delta \mid \delta < \gamma \rangle$$

If γ is a limit ordinal, then

$$G_\gamma = \bigcup_{\alpha < \gamma} G_\alpha = \bigcup_{\alpha < \gamma} \langle g_\delta \mid \delta < \alpha \rangle = \langle g_\delta \mid \delta < \gamma \rangle$$

So the claim holds. In particular, $G = G_\beta$ is $|\beta$ generated. If $H \leq G$, then $(H \cap G_\alpha)_{\alpha \leq \beta}$ is an ascending series with cyclic factors and so also H is $|beta|$-generated. □

Proposition 2.2.7. [**fg and nil**] *Let G be a nilpotent n-generated group of class $d > 0$ and suppose G can be generated by n elements.Put $m := \sum_{i=1}^{d} n^d$. Then $\gamma_{d-1}(G)$ is n^d-generated and G is polycyclic of length m. In particular, every subgroup of G is m-generated.*

Proof. Suppose $d = 1$. Then G is abelian and so polycyclic of length at most n. Also $\gamma_{d-1}(G) = G$ and so can be generated by $n^d = n$ elements. Thus proposition holds in this case.

So suppose $d > 1$ and put $D = \gamma_{d-1}(G)$ and $E = \gamma_{g-2}(G)$. Then $D \leq \mathrm{Z}(G)$ and $D = [E, G]$. Moreover by induction, E/D is generated by n^{d-1} elements and G/D is polycylic of length at most $\sum_{i=1}^{d-1} n^i$. So there exists $S \subseteq E$ with $|S| \leq n^{d-1}$ and $E/D = \langle sD \mid s \in S \rangle$. Note that $E = \langle S \rangle D$. Let $T \subseteq G$ with $G = \langle T \rangle$ and $|T| = n$. Then

$$D = [E, G] = [\langle S \rangle D, \langle T \rangle] = [\langle S \rangle, \langle T \rangle] = \langle [S, T] \rangle^{\langle S \rangle \langle T \rangle} = [S, T]$$

where the last equality holds since $[S, T] \leq [E, G] \leq D \leq \mathrm{Z}(G)$. Thus D is generated by $|S||T| \leq n^{d-1} n$ elements. Since D is abelian, D is polycyclic of length n^d. Since G/D is polycyclic of length $\sum_{i=1}^{d-1} n^i$, G is polycyclic of length

$$n^d + \sum_{i=1}^{d-1} n^i = \sum_{i=1}^{d} n^d$$

The last statement now follows from 2.2.6. □

Theorem 2.2.8. [**hirsch-plotkin**] *Let $\mathcal{X}$ be a $\mathbf{S}$- and $\mathbf{N}_0$-closed class of finitely generated groups. Then $\mathbf{L}\mathcal{X}$ is $\acute{\mathbf{N}}$-closed. In particular, for all groups G, $\rho_{\mathbf{L}\mathcal{X}}(G)$ is locally $\mathcal{X}(G)$ and contains all ascending locally $\mathcal{X}$-subgroups of G.*

Proof. We will first show that $\mathbf{L}\mathcal{X}$ is $\mathbf{N}_0$-closed. For this let L and M be normal locally $\mathcal{X}$-subgroups of a group H. We need to show that LM is locally $\mathcal{X}$.

So let S be a finite subsets of LM and choose finite subsets X and Y of L and M respectively with $S \subseteq \langle H, K\rangle$, where $H = \langle X\rangle$ and $K = \langle Y\rangle$. Note that $[X, Y]$ is finitely generated and $[X, Y] \leq [H, K] \leq [L, M] \leq L \cap M$ and so $< [X, Y], H\rangle =< [X, Y], X\rangle$ is a finitely generated subgroup of L. Since L is locally $\mathcal{X}$ we conclude that $< [X, Y], H\rangle$ is an $\mathcal{X}$ group. Since $\mathcal{X}$ is $\mathbf{S}$-closed also $[H, Y] = \langle [X, Y]^H\rangle$ is an $\mathcal{X}$ group. In particular, $[H, Y]$ is finitely generated. Hence

$$\langle K^H\rangle = [H, K]K = \langle [H, Y]^K\rangle K = \langle [H, Y], Y\rangle$$

is a finitely generated subgroup of M. Thus $\langle K^H\rangle$ is $\mathcal{X}$-group. By symmetry also $\langle H^K\rangle$ is $\mathcal{X}$-group. Since $\mathcal{X}$ is $\mathbf{N}_0$-closed we conclude from $\langle H, K\rangle = \langle H^K\rangle < K^H\rangle$ that $\langle H, K\rangle$ is an $\mathcal{X}$ groups. Since $S \subseteq \langle H, K\rangle$ this completes the proof that LM is locally $\mathcal{X}$.

Hence $\mathbf{L}\mathcal{X}$ is $\mathbf{N}_0$-closed. Since $\mathbf{L}\mathcal{X}$ is $\mathbf{L}$-closed, 1.7.3 implies that $\mathbf{L}\mathcal{X}$ is also $\acute{\mathbf{N}}$-closed. □

Definition 2.2.9. [**def:fitting**] *let G be groups.*

(a) [**a**] $\mathrm{F}(G) = \rho_{\mathrm{Nil}}(G)$*. So* $\mathrm{F}(G)$ *is is the group generated by the all the nilpotent normal subgroups of G.* $\mathrm{F}(G)$ *is called the Fitting subgroups of G.*

(b) [**b**] $\mathrm{HP}(G) = \rho_{\mathbf{L}\mathrm{Nil}}(G)$*. So* $\mathrm{F}(G)$ *is is the group generated by the all the locally nilpotent normal subgroups of G.* $\mathrm{HP}(G)$ *is called the Hirsch-Plotkin radical of G.*

Corollary 2.2.10 (Hirsch-Plotkin)**.** [**hp**] *Let G be a group.* $\mathrm{HP}(G)$ *is the largest ascending locally nilpotent subgroups of G, that is* $\mathrm{HP}(G)$ *is locally nilpotent and contains all ascending, locally nilpotent subgroups of G.*

Proof. Let $\mathcal{X} = \mathrm{Nil} \cap \mathcal{F}$, the class of finitely generated subgroups. By 2.2.7 and since subgroups of nilpotent are nilpotent, $\mathcal{X}$ is $\mathbf{S}$-closed. Note that Nil and $\mathcal{F}$ are $\mathbf{N}_0$-closed and so also $\mathcal{X}$ is $\mathbf{N}_0$-closed. Thus the assumption of **??** are fulfilled and so $\rho_{\mathbf{L}\mathcal{X}}(G)$ is the largest ascending, locally $\mathcal{X}$ subgroup of G. By 1.7.4, $\mathbf{L}\mathcal{X} = \mathbf{L}\mathrm{Nil}$ and the Corollary is proved. □

Lemma 2.2.11. [**cghp**] *Let G be a group.*

(a) [**a**] *If G is hyper abelian, then* $\mathrm{C}_H(\mathrm{F}(G)) \leq \mathrm{F}(G)$*.*

(b) [**b**] *If G is hyper (locally-nilpotent), then* $\mathrm{C}_G(\mathrm{HG}(G) \leq \mathrm{HP}(G)$*.*

Proof. (a) Note that G is hyper abelian, if and only if G is hyper nilpotent and if and only if $G \in \mathrm{radNil}$. Let K be a group such that $K/\mathrm{Z}(K)$ is nilpotent. Then $\gamma_n(K) \leq \mathrm{Z}(K)$ and $\gamma n + 1(G) \leq [\mathrm{Z}(K), K] = 1$. Thus Nil is closed under central extension. Clearly Nil is $\mathbf{H}$ and $\mathbf{S}_n$-closed and so the lemma follows from 1.5.12.

(b) Observe that G is hyper (locally nilpotent) just means $G \in \mathrm{rad}\mathbf{L}\mathrm{Nil}$. Since Nil is closed under central extensions, also $\mathbf{L}$Nil is closed under extensions. Clearly $\mathbf{L}$Nil is $\mathbf{H}$ and $\mathbf{S}_n$-closed and so the lemma follows from 1.5.12. □

Let G be a finite group. Then G is locally nilpotent iff G is nilpotent. So $\mathrm{F}(G) = \mathrm{HP}(G)$ is the largest normal nilpotent subgroup of G. Also G is hyper abelian iff G is solvable and iff G is hyper (locally nilpotent). So for finite groups, both parts of the previous lemma say that $\mathrm{C}_G(\mathbb{F}(G)) \leq \mathbb{F}(G)$ for every finite solvable group.

2.3 The generalized Fitting Subgroup

Definition 2.3.1. [**def:f*g**] *Let G be group.*

(a) [**a**] *G is called quasisimple, if G is perfect and $G/Z(G)$ is simple.*

(b) [**b**] *A component of G is a quasi simple ascending subgroup of G.*

(c) [**c**] $\mathrm{E}(G)$ *is the subgroup of G generated by all the components of G.*

(d) [**d**] $\mathrm{F}^*(G) = \mathrm{HP}(G)\mathrm{E}(G)$. $\mathrm{F}^*(G)$ *is called the general Fitting subgroup of G.*

Lemma 2.3.2. [**basic quasimple**] *Let K be quasisimple group and $M \trianglelefteq K$.*

(a) [**a**] $M = K$ *or* $M \leq \mathrm{Z}(K)$.

(b) [**b**] *If $M \neq K$, then $\mathrm{Z}(K/M) = \mathrm{Z} * K)/M$ and K/M is quasisimple.*

Proof. (a) We may assume $M \not\leq \mathrm{Z}(K)$. Since $K/Z(K)$ is simple this gives $K/\mathrm{Z}(K) = M\mathrm{Z}(K)/\mathrm{Z}(K)$ and $K = M\mathrm{Z}(K)$. Since K is perfect $K = [K, K] = [M\mathrm{Z}(K), M\mathrm{Z}(K)] = [M, M] \leq M$ and so $K = M$. (b) Suppose $M \neq K$. Then by (a) $M \leq \mathrm{Z}(K)$. Let D be the inverse image of $\mathrm{Z}(K/M)$ in K. Then $\mathrm{Z}(K) \leq D$. Also $[D, K, K] \leq [M, K] = 1$ and so also $[K, D, K] = 1$. The Three Subgroups Lemma implies that $[K, K, D] = 1$. Since K is perfect we conclude $[D, K] = 1$, $D \leq \mathrm{Z}(K)$ and $D = \mathrm{Z}(K)$. Hence $K/Z(M)\big/Z(K/Z(M)) = K/Z(M)//\mathrm{Z}(K)/Z(M) \cong K/\mathrm{Z}(K)$. The latter group is simple and so $K/Z(M)$ is quasisimple. □

Lemma 2.3.3. [**f* and asc**] *Let G be a group and M an ascending subgroup of G.*

(a) [**a**] $\mathrm{HP}(M) = \mathrm{HP}(G) \cap M$.

(b) [**b**] *A subgroup of M is a component of M iff its a component of G. In particular,* $\mathrm{E}(M) \leq \mathrm{E}(G)$ *and* $\mathrm{F}^*(M) \leq \mathrm{F}^*(G)$.

Proof. (a) Since $\mathrm{HP}(M) \trianglelefteq M \mathrm{asc} G$ we conclude from 2.2.10 that $\mathrm{HP}(M)$ is an ascending locally nilpotent subgroup of G and $\mathrm{HP}(M) \leq \mathrm{HP}(G)$. Also $\mathrm{HP}(G) \cap M$ is locally nilpotent normal subgroup of M and so $\mathrm{HP}(G) \cap M \leq \mathrm{HP}(M)$.

(b) If K is a component of M, then K is a quasisimple ascending subgroup of M. Since $M \mathrm{asc} G$ we get $K \mathrm{asc} G$ and so K is a component of G. □

Lemma 2.3.4. [**easy cf***] *Let G be a group.*

(a) [**a**] $C_{\mathrm{F}^*(G)}(\mathrm{E}(G)) = \mathrm{HP}(G)$.

(b) [**b**] *If M is subnormal in G, then $F^*(M) = M \cap F^*(G)$.*

Proof. Put $F = \mathrm{F}^*(G)$. (a) By **??** $[\mathrm{HP}(G), \mathrm{E}(G)] = 1$. Since $F = \mathrm{HP}(G)\mathrm{E}(G)$ this gives $C_F(\mathrm{E}(G))) = \mathrm{HP}(G)C_{\mathrm{E}(G)}\mathrm{E}(G)) = \mathrm{HP}(G)\mathrm{Z}(\mathrm{E}(G))$. Since $\mathrm{Z}(\mathrm{E}(G))$ is an abelian normal subgroup of G, $\mathrm{Z}(\mathrm{E}(G)) \leq \mathrm{HP}(G)$ and (a) holds.

(b) Put $E = \mathrm{E}(M)$. By **??** $\mathrm{HP}(G)$ and all components of G which are not contained in M centralizes all the components of M. Thus $F = C_F(E)E$ and so $(F \cap M) = (C_F(E) \cap M)E$. Put $D = C_F(E) \cap M$. Let K be a component of G with $K \not\leq M$. Then by **??**, $[K, M] = 1$. Thus D centralizes all components of G and so by (a) $D \leq C_F(\mathrm{E}(G)) = \mathrm{HP}(G)$. Hence D is locally nilpotent and thus $D \leq \mathrm{HP}(M) \leq \mathrm{F}^*(H)$. So also $F \cap M = DE \leq \mathrm{F}^*(M)$. Since $\mathrm{F}^*(M) \leq F$, (b) holds. □

Lemma 2.3.5. [**f* and factors**] *Let G be a group.*

(a) [**a**] *If $M \trianglelefteq G$ then $\mathrm{F}^*(G)M/M \leq \mathrm{F}^*(G/M)$.*

(b) [**b**] *If $M \leq \mathrm{Z}(G)$. Then $\mathrm{F}^*(G)/M = \mathrm{F}^*(G/M)$.*

Proof. (a) $\mathrm{HP}(G)M/M$ is locally nilpotent normal subgroup of G/M and so $\mathrm{HP}(G)M/M \leq \mathrm{HP}(G/M)$. Let K be a component of G. If $K \leq M$, then definitely $KM/M \leq \mathrm{E}(G/M)$. $K \not\leq M$, $K \cap M < K$ and by 2.3.2, $KM/M \cong K/K \cap M$ is quasisimple. Thus KM/M is a component of K. Hence $\mathrm{E}(G)M/M \leq \mathrm{E}(G/M)$ and (a) holds.

(b) Let H be the inverse image of $\mathrm{HP}(G/M)$ in G. Since H/M is locally nilpotent and $M \leq \mathrm{Z}(H)$, H is locally nilpotent and so $H \leq \mathrm{HP}(G)$. Thus $H = \mathrm{HP}(G)$.

Now let L be the inverse image of a component of G/M in G and put $K = L'$. Since L/M is perfect, $L/M = KM/M$ and so $L = KM$. Thus $L' = K' = L$ and so K is perfect. Let $D/M = Z(L/M)$. Then $D \not\leq K$ and so using **??**, $D \cap K \leq Z(K) \leq Z(L) \cap K \leq D \cap K$. Hence $D \cap K = Z(K)$ and $K/Z(K) = K/K \cap D \cong KD/D = L/D \cong L/M/Z(L/M)$. Therefore $K/Z(K)$ is simple and K is a component of G. Since $M \leq \mathrm{HP}(G)$ we get $L = KM \leq \mathrm{F}^*(G)$. It follows that $\mathrm{F}^*(G/M) \leq \mathrm{F}^*(G)/M$. Together with (a) this gives (b). □

Theorem 2.3.6. [**cf*g**] *Let $\mathcal{F}^*$ be the class of all groups H which are a central product of quasi-simple and locally nilpotent groups. Let G be group,*

(a) [**a**] *$G \in \mathcal{F}^*$ if and only if $G = \mathrm{F}^*(G)$.*

(b) [**b**] *$\mathcal{F}^*$ is $\mathbf{S}_n$-, $\mathbf{H}$-, $\mathbf{C}$- and $\mathbf{N}$-closed.*

(c) [**c**] *$\rho_{\mathcal{F}^*}(G) = \mathcal{F}^*(G)$.*

(d) [**d**] *If $G \in \mathrm{rad}\mathcal{F}^*$, then $C_G(\mathcal{F}^*(G)) \leq \mathcal{F}^*(G)$.*

Proof. (a): If $G \in \mathcal{F}^*$ then clearly $G = \mathrm{F}^*(G)$. Conversely, by **??**, $\mathcal{F}^*(G)$ is the central product of $\mathrm{HP}(G)$ and the components of G, so (a) holds.

(b) and (c): By **??**(**??**), $\mathcal{F}^*$ is $\mathbf{S}_n$-closed. By 2.3.5, $\mathcal{F}^*$ is **H** and **C** closed. Also if $N \trianglelefteq G$ with $N = \mathrm{F}^*(N)$, then by **??**(**??**), $N = \mathrm{F}^*(N) \leq \mathbb{F}^*(G)$. This shows that $\rho_{\mathcal{F}^*}(G) = \mathbb{F}^*(G)$ and that $\mathcal{F}^*$ is **N**-closed.

(d) By (b) and 1.5.12, $C_G(\rho_{\mathcal{F}^*}(G)) \leq \rho_{\mathcal{F}^*}(G)$. Thus (d) follows from (c). □

Definition 2.3.7. [**def:min**] *We say that a group G fulfills MIN if every non-empty sets of subgroups of G has a minimal element.*

Corollary 2.3.8. [**cf***] *Let G be a group with MIN, then $G \in \mathrm{rad}\mathcal{F}^*$. In particular, $C_G(\mathcal{F}^*(G)) \leq \mathcal{F}^*(G)$.*

Proof. Let $M \trianglelefteq G$ with $G \neq M$. Then G/M fulfills min and so G/M has a minimal normal subgroup E. Then E is simple and so either $|E|$ is a prime or E is quasisimple. In the first case $E \leq \mathrm{HP}(G/M)$ and in the second $E \leq \mathrm{E}(G/M)$. In either case $\mathrm{F}^*(G/M) \neq 1$. So G is strongly hyper $\mathcal{F}^*$ and hence by **??**(**??**), G is a hyper $\mathcal{F}^*$-group. Thus $G \in \mathrm{rad}\mathcal{F}^*$. The second statement now follows from **??**. □

2.4 Chieffactors of locally solvable groups

Proposition 2.4.1. [**chieffactors in locally nilpotent**] *let G be group.*

(a) [**a**] *If G locally nilpotent group, then G centralizes all chief-factors of G.*

(b) [**b**] *If G locally solvable group, then G all chief-factors of G are abelian.*

Proof. Let T/B be a chieffactor of G. Replacing G be G/B we may assume that $B = 1$ and so T is minimal normal subgroup of G. Let $H = G$ in (a) and $H = T$ in (b). We need to show that $[T, H] = 1$. So suppose $[T, H] \neq 1$. Since T is a minimal normal subgroup of G, $T = [T, H]$. Pick $1 \neq t \in T$. Then $T = \langle t^G \rangle$ and so $t \in [T, H] = [t^G, H]$. Thus there exists $g_1, g_2, \ldots, g_n \in G$ and $h_1, h_2, \ldots h_m \in H$ with

$$t \in [t^{\langle g_1, \ldots g_n \rangle}, \langle h_1, h_2, \ldots, h_m \rangle]$$

(a) Suppose G is locally nilpotent and put $D = \langle g_1, \ldots g_n, h_1, h_2, \ldots, h_n \rangle$. Then $t \in [\langle t^D \rangle, D]$. Since G is locally nilpotent, D is nilpotent and we can choose k minimal with $t \in \mathrm{Z}_k(D)$. Then

$$t \in [\langle t^D \rangle, D] \leq [Z_k(D), D] \leq Z_{k-1}(D)$$

a contradiction to the minimal choice of k.

(b) Suppose G is locally solvable and so $H = T = \langle t^G \rangle$. We we can choose $g_{jk} \in G$ with $h_j \in \langle t^{\langle g_{jk}, \ldots, g_{jt_j} \rangle} \rangle$. Put $D = \langle g_i, g_{jk} \mid 1 \leq i \leq n, 1 \leq j \leq m, 1 \leq k \leq t_j \rangle$. Then

$$t \in [\langle t^D \rangle, \langle t^D \rangle] = \langle t^D \rangle'$$

Since G is locally solvable, D is solvable and we can choose k maximal with $t \in G^{(k)}$. Then

$$t \in \langle t^D \rangle' \leq (G^{(k)})' = G^{(k+1)}$$

a contradiction to the maximality of k. $\square$

2.5 Polycyclic groups

Definition 2.5.1. [**def:c-series**] *Let G be a group. A c-series for G is finite series for G each of whose factors are isomorphic to $\mathbb{Z}_p$ or $\mathbb{Z}$. A strong c-series for G is a c-series of minimal length. A supersolvable series is a finite normal series all whose factors are cyclic. A group is called supersolvable if its has a supersolvable series.*

Definition 2.5.2. [**def:isomorphic set of groups**] *Let $\mathcal{M}$ and $\mathcal{N}$ be sets of groups, we say that $\mathcal{M}$ is isomorphic to $\mathcal{N}$ if there exists a bijection $\phi : \mathrm{M} \to \mathcal{N}$ with $M \cong \phi(M)$ for all $M \in \mathcal{M}$. We say that two series of a group have isomorphic factors, if the sets of factors of the two series are isomorphic.*

Definition 2.5.3. [**def:refinement**] *Let $\mathcal{A}$ be a series for the group G. A refinement of $\mathcal{A}$ is a series $\mathcal{B}$ of G with $\mathcal{A} \subseteq \mathcal{B}$.*

Proposition 2.5.4. [**refinement**] *Let $\mathcal{A}$ and $\mathcal{B}$ be ascending series of the group G. Define $\mathcal{A}^* = \{(A \cap B)A^- \mid A \in \mathcal{A}, B \in \mathcal{B}\}$ and $\mathcal{B}^* = \{(B \cap A)B^- \mid B \in \mathcal{B}, A \in \mathcal{A}\}$. Then $\mathcal{A}^*$ is an ascending refinement of $\mathcal{A}$, $\mathcal{B}^*$ is an ascending refinement of $\mathcal{B}$ and $\mathcal{A}^*$ and $\mathcal{B}^*$ have isomorphic factors. Moreover, the sets of factors of both $\mathcal{A}^*$ and $\mathcal{B}^*$ are isomorphic to*

$$\{A \cap B/(A^- \cap B)(A \cap B^-) \mid A \in \mathcal{A}, B \in \mathcal{B}, A \cap B \neq (A^- \cap B)(A \cap B^-)\}$$

Proof. We will first show that $\mathcal{A}^*$ is totally ordered. Let $X_1, X_2 \in \mathcal{A}^*$ and pick $A_i \in \mathcal{A}, B_i \in \mathcal{B}$ with $X_i = (A_i \cap B_i)A_i^-$. Without loss $A_1 \leq A_2$. Note that $A_i^- \leq X_i \leq A_i$. So if $A_1 < A_2$, then $X_1 \leq A_1 \leq A_2^- \leq X_2$. So suppose $A_1 = A_2$ and without loss $B_1 \leq B_2$. Then $X_1 \leq X_2$ and so $\mathcal{A}^*$ is totally ordered.

Note that $A = (A \cap G)A^- \in \mathcal{A}^*$ for all $A \in \mathcal{A}$ and so $\mathcal{A}^*$.

Let $X = (A \cap B)A^- \in \mathcal{A}^*$. Since $\mathcal{B}$ is well ordered we may assume that B is minimal in $\mathcal{B}$ with $X = (A \cap B)A^-$. Since $\mathcal{B}$ is well ordered we may assume that B is minimal in $\mathcal{B}$ with We will compute $X^- = \bigcup\{D \in \mathcal{A}^* \mid D < A\}$. If $A = A^-$(in $\mathcal{A}$) then $X = A = \bigcup\{D \in \mathcal{A} \mid D < A\} \leq X^-$ and so $X = X^-$. Suppose next that $A \neq A^-$. Let $E \in \mathcal{B}$ with $E < B$. By the minimal choice of B, $(A \cap E)A^- < (A \cap B)A^-$ and so $(A \cap E)A^- \leq X^-$. It follows that $(A \cap B^-)A^- \leq X^-$. So if $B = B^-$, then $X = X^-$. So suppose $B \neq B^-$. Let $\tilde{A} \in \mathcal{A}$ and $\tilde{B} \in \mathcal{B}$ with $(\tilde{A} \cap \tilde{B})\tilde{A}^- \leq X$. Then either $\tilde{A} \leq A^-$ or $\tilde{A} = A$ and $\tilde{B} \leq B^-$. In either case $(\tilde{A} \cap \tilde{B})\tilde{A}^-) \leq (A \cap B^-)A^-$ and so $X^- = (A \cap B^-)A^-$. Since $A^- \trianglelefteq A$ and $B^- \trianglelefteq B$ we have $X^- = A \cap B^-)A^-(A \cap B)A^- = X$ and so $\mathcal{A}^*$.

Let $\mathcal{M}$ be a non-empty subset of $\mathcal{A}^*$. Choose $A \in \mathcal{A}$ minimal with $(A \cap E)A^- \in \mathcal{M}$ for some $E \in \mathcal{B}$ and then choose $B \in \mathcal{B}$ minimal with $(A \cap B)A^- \in \mathcal{M}$. Then $(A \cap B)B^-$ is the minimal element of $\mathcal{M}$. So $\mathcal{A}^*$ is well ordered and $\bigcap \mathcal{M} = (A \cap B)B^- \in \mathcal{A}^*$. If $G \in \mathcal{M}$, then $\bigcup \mathcal{M} = G \in \mathcal{A}^*$. If $G \notin \mathcal{M}$ pick X minimal in $\mathcal{A}^*$ with $M < X$, for all $M \in \mathcal{M}$. Then clearly $\bigcup \mathcal{M} = X^- \in \mathcal{A}^*$. Thus $\mathcal{A}^*$ is a series for G and so an ascending refinement of $\mathcal{A}$. Also the factors of $\mathcal{A}^*$ are exactly the groups $|(A \cap B)A^-/(A \cap B^-)A^-$ where $A \in \mathcal{A}$, $B \in \mathcal{B}$ with $A \neq A^-$, $B \neq B^-$ and $(A \cap E)A^- < (A \cap B)A$ for all $E \in \mathcal{B}$ with $E < B$. Observe that these are exactly the groups $|(A \cap B)A^-/(A \cap B^-)A^-$ where $A \in \mathcal{A}$, $B \in \mathcal{B}$ and $(A \cap B)A^- \neq (A \cap B^-)A^-$.

Now

$$\begin{aligned}(A \cap B)A^-/(A \cap B^-)A^- &= (A \cap B)(A \cap B^-)A^-/(A \cap B^-)A^- \\ &\cong (A \cap B)/\big((A \cap B) \cap (A \cap B^-)A^-\big) \\ &= (A \cap B)/\big((A \cap B^-)(A \cap B \cap B^-)\big) \\ &= (A \cap B)/\big((A \cap B^-)(A \cap B^-)\big)\end{aligned}$$

and so the set of factors of $\mathcal{A}^*$ is isomorphic to the set

$$\{A \cap B/(A^- \cap B)(A \cap B^-) \mid A \in \mathcal{A}, B \in \mathcal{B}, A \cap B \neq (A^- \cap B)(A \cap B^-)\}$$

Observe that the last set is symmetric in A and B and all parts of the propositions are proved. □

Lemma 2.5.5. [**same number of infinite factors**] *Any two c-series of a polycyclic group have the same number of infinite factors.*

Proof. Let $\mathcal{A}$ and $\mathcal{B}$ be the c-series of the group G. By 2.5.4 we may assume that $\mathcal{A} \subseteq \mathcal{B}$. Let (X, Y) be a jump of $\mathcal{A}$ and consider the series

$$X = X_0 < X_1 < \dots X_n = Y$$

where $X_0, \dots, X_n$ are the members of $\mathcal{B}$ with $X \leq X_i \leq Y$. If $|Y/X|$ is cyclic of prime order then $n = 1$ and $X_1/X_0 = Y/X$. If $Y/X \cong \mathbb{Z}$, then $X_1/X_0 \cong \mathrm{Z}$ while X_i/X_{i-1} is finite for $2 \leq i \leq n$. So each infinite factor of $\mathcal{A}$ gives rise to exactly one infinite factor of $\mathcal{B}$. □

Lemma 2.5.6. [**cag cap kag**] *Let G be a group acting on the abelian group A. Let $g \in G$ with finite order n. Then $\mathrm{C}_A(g) \cap [V, g]$ has exponent dividing n.*

Proof. Let $a \in C_A(g) \cap [V, g]$. Since A is abelian, $[A, g] = \{[a, g] \mid a \in A\}$ and so there exists $b \in V$ with $a = [b, g]$. We claim that $a^m = [b, g^m]$ for all $m \in Z^+$. By definition this is true for $m = 1$. Note that $a^m \in C_A(g)$ and so by 2.1.1(b)

$$[b, g^{m+1}] = [b, g^m g] = [b, g][b, g^m] = aa^m = a^{m+1}$$

It follows that $a^n = [b, g^n] = [b, 1] = 1$. □

Proposition 2.5.7. [**supersolvabe**] *Let G be supersolvable group. Then* *(a)* [**a**] *There exists a strong c series $1 = G_0 < G_1 < G_2 < G_n$ and $0 \leq l \leq n$ such that G_i/G_{i-1} is has odd prime order for all $1 \leq i \leq l$ and G_i/G_{i-1} has order 2 or $infty$ for all $l < i \leq n$.*

(b) [**b**] *G has a unique maximal finite subgroup of odd order.*

(c) [**c**] *Any two strong c-series have isomorphic factors.*

Proof. Let $^A: 1 = H_0 < H_1 < H_2 < H_n$ be a strong c series for G and choose a c-series

$$\mathcal{B} : 1 = G_0 < G_1 < G_2 < G_n$$

and $a \leq b \in \mathbb{N}$ such that:

(a) $\mathcal{A}$ and $\mathcal{B}$ have isomorphic factors.

(b) G_i/G_{i-1} has odd order for all $1 \leq i \leq a$.

(c) G_i/G_{i-1} has order 2 or ∞ for $a < i \leq b$.

(d) If $b \neq n$, then G_{b+1}/G_b has odd prime order.

(e) a is maximal and then b is minimal.

Suppose that $b \neq n$. Then by maximality of a, $a \neq b$. Put Put $B = \bigcap G_{b-1}^{G_{b+1}}$, $\overline{G_b + 1} = G_{b+1}/B$, $p = |G_{b+1}/G_{b-1}$ and $m = |G_b/G_{b11}$. Then $G_{b+1}/G_{b-1} \cong \mathrm{Z}_p$, G_b/G_{b-1} is cyclic of order m, p is an odd prime and $m \in \{2, \infty\}$. Note that $G_b' \leq G_{b-1}$ and since $G_b' \trianglelefteq G_{b+1}$, $G_b' \leq B$. Thus $\overline{G_b}$ is abelian.

If $m = 2$, then $G_b^m \leq G_{b-1}$ and so $G_b^m \leq B$ and $\overline{G_b}$ is an elementary abelian 2-group.

Suppose $m = \infty$ and let $x \in G_b \setminus B$. Then there exists $g \in G_{b+1}$ with $x \not\leq G_{b-1}^g$. Since $G_b/G_{b-1}^g \cong \mathrm{Z}$, xG_{b-1}^g has infinite order in G_b/G_{b-1}^g. Hence also $\overline{x}$ has infinite order. So for either possibility of m, any non-trivial elements of $\overline{G_b}$ has order m.

Suppose for a contradiction the $D := [G_b, G_{b+1}]B \neq B$. Let $S_0 \leq S_1 \leq \ldots S_m = G$ be supersolvable series for G and pick k minimal with $S_k \cap D \not\leq B$. Then $\overline{E} := (S_k \cap D)B/B \cong S_k \cap D/S_k \cap B$ and since $S_{k-1} \cap D = S_{k-1} \cap D$, $\overline{E}$ is a quotient of

$$S_k \cap D/S_{k-1} \cap D = S_k \cap D/(S_k \cap D) \cap S_{k-1} \cong (S_k \cap D)S_{k-1}/S_{k-1}$$

Thus $\overline{E}$ is isomorphic to a section of the cyclic group S_k/S_{k-1}. Hence $\overline{E}$ is non-trivial cyclic subgroup of $\overline{G}_b$. Since non-trivial elements of $\overline{G_b}$ have order m, $\overline{E}$ is cyclic of order m. It follows that $\mathrm{Aut}(\overline{E})$ has order at most two. Observe that G_{b+1} acts on $\overline{E}$. G_b centralizes $\overline{G}_b$ and so also $\overline{E}$ and $G_{b+1}/G_b \cong \mathrm{Z}_p$ has order coprime to 2. Thus G_{b+1} centralizes $\overline{E}$. So $\overline{E} \leq [\overline{G}_b, G_{b+1}] \cap C_{\overline{G}_b}(G_{b+1})$. Thus by **??** $\overline{E}$ has exponent dividing $p = |G_{b+1}/G_b|$ a contradiction since $\overline{E}$ is cyclic of order m.

We proved that $[G_b, G_{b+1}] \leq B \leq G_{b-1}$. So $G_{b-1} = B \trianglelefteq G_{b+1}$ and $\overline{G}_b \leq \mathrm{Z}(\overline{G}_{b+1}$. Since G_{b+1}/G_b is cyclic we conclude that $\overline{G_{b+1}}$ is abelian. If $\overline{G_{b+1}}$ is cyclic, then

$$G_0 \leq \dots G_{b-1} \leq G_{b+1} \leq \dots G_n$$

is a c-series for G, a contradiction since $\mathcal{A}$ and so also $\mathcal{B}$ is a c-series of minimal length. Thus $\overline{G_{b+1}}$ is not cyclic and there exist $\overline{K} \leq \overline{G_{b+1}}$ with

$$\overline{G_{b+1}} = \overline{G_b} \times \overline{K}$$

Let K be the inverse image of $\overline{K}$ in G_{b+1}. The $K \trianglelefteq G_{b+1}$, $K/G_{b-1} \cong \mathrm{Z}_p$ and G_{b+1}/K is cyclic of order m.

Consider the series

$$G_0 \leq \dots G_{b-1} \leq K \leq G_{b+1} \leq \dots \leq G_n$$

If $b-1 = a$, we get a contradiction to the maximality of a and if $a < b-1$, we get a contradiction to the minimality of b.

This show that $n = b$ and so (a) holds.

Note that $H := G_l$ is a subgroup of odd order. Let g be any non-trivial element of odd order in G and pick $1 \leq t \leq n$ minimal with $g \in G_t$. Then gG_{t-1} is non-trivial elements of odd order in G_t/G_{t-1}. So G_t/G_{t-1} cannot by cyclic of order 2 or ∞ and so $t \leq l$ and $g \in G_l = H$. Thus H is the unique maximal finite subgroup of odd order in G and (b) is proved.

For any odd prime p let s_p the number of factors of $\mathcal{A}$ isomorphic to Z_p. Then s_p is also the number of factors of $\mathcal{B}$ isomorphic to Z_p and so $|H| = \prod\{p^{s_p} \mid p \text{ an odd prime}\}$. Thus s_p is independent of the choice of the strong c-series $\mathcal{A}$. By 2.5.5 any two strong c-series also have the same number of factors isomorphic to Z. By defintition, any two strong c-series have the same number of total factors. It follows that they also have the same number of factors isomorphic to Z_2. So (c) holds. □

Chapter 3

Groups with MIN

3.1 Basic properties of groups with MIN

Recall that a group with MIN is a group such that every non-empty set of subgroups has a minimal element.

Lemma 3.1.1. [**basic min**] *Let G be a group with MIN.*

(a) [**a**] *Every section of G fulfills MIN.*

(b) [**b**] *G is periodic, that is every element in G has finite order.*

Proof. (a) Let $B \trianglelefteq A \leq G$ and $\mathcal{M}$ a non-empty set of subgroups of A/B. Let $D \leq G$ be minimal with $B \leq D \leq A$ and $D/B \in \mathcal{M}$. Then D/B is a minimal element of $\mathcal{M}$.

(b) Let $g \in G$. By (a) $\langle g \rangle$ fulfills MIN and so $\langle g \rangle \ncong \mathbb{Z}$. Thus $\langle g \rangle$ is finite.

□

Lemma 3.1.2. [**min and com**] *Let G be a group with MIN. Then every series for G is an ascending series.*

Proof. Just recall that by definition a series is ascending if every non-empty subset of the series has a minimal element. □

Definition 3.1.3. [**def:gcird**] *Let G be a group. Then G° is the intersection of all the subgroups of finite index in G.*

Lemma 3.1.4. [**gcirc and min**] *Let G be a group with MIN. Then G° is the unique minimal subgroups of finite index in G.*

Proof. Let A minimal subgroups of finite index in G and B an arbitrary subgroup of index in G. $|A/A\cap B| = |AB/B| \leq\leq |G/B|$, $G/A\cap B| \leq |G/A||G/B|$. So $A\cap B|$ has finite index in A and so by minimality of A and B. $A = A\cap B \leq B$. So A is the unique minimal subgroup of finite index and $A = G^\circ$ □

Lemma 3.1.5. [**basic gcirc**] *Let G be a group and $H \leq G$. Then $H^\circ \leq G^\circ$.*

Proof. Let $F \leq G$ with $|G/F|$ finite. Then $|H/H \cap F| = |HF/F| \leq |G/F|$ and so $H^\circ \leq H \cap F \leq F$. Since this holds for all such F, $H^\circ \leq G^\circ$. □

3.2 Locally solvable groups with MIN

Definition 3.2.1. [**def:divisible**] *A group A is called divisible of it is abelian and for all $a \in A$ and $n \in \mathbb{Z}^+$ where exists $b \in A$ with $b^n = a$.*

$\mathbb{Q}$ and C_{p^∞} are divisible. $\mathbb{Z}$ is not divisible and all non-trivial divisible groups are infinite.

Lemma 3.2.2. [**basis divisible**] *Let A be an abelian group and D a divisible subgroup of A. Then $A = D \oplus K$ for some $K \leq A$.*

Proof. By Zorn's lemma there exists a subgroup K of A maximal with respect to $D \cap A = 0$. Let $a \in A$ and let $m \in \mathbb{N}$. Then $a^m \in DK$ if and only of $a^m = dk$ for some $d \in D, k \in K$ and so iff $a^m K \cap D \neq \emptyset$ and iff $a^m D \cap K \neq \emptyset$. Let n be the order of aDK in A/DK. If $n = \infty$ we conclude that $a \notin K$ and $| < a\rangle K \cap D = 1$, a contradiction to the maximality of K. Thus $n \in \mathrm{Z}^+$. Then $a^n = dk$ for some $d \in D$ and $k \in K$. Since D is divisible, $d = b^n$ for some $b \in D$. Put $e = ab^{-1}$. If $e^m K \cap D \neq \emptyset$ we get $e^m D \cap K \neq \emptyset$ and since $aD = eD$, $a^m D \cap K \neq emptyset$ and $a^m \in DK$, $n \mid m$ and $m = nl$ for some $l \in \mathrm{Z}$. Thus $e^m = (ab{-}1)^(nl) = (a^n b^{-n})^l = (a^n d^{-1})^l = k^k \in K$. It follows that $e^m \leq D \cap K = 1$ and so $\langle e\rangle K \cap D = 1$. By maximality of K, this gives $e \in K$ and so $a = eb \in KD$. Thus $A = DK$ and $AD \oplus K$. □

3.3 Locally finite groups with finite involution centralizer

Proposition 3.3.1. [**brauer fowler**] *Let H be a finite group, t an involution in H. Then there exist a non-trivial normal subgroup N of $G$with $|G/C_G(N) \leq (2|C_H(t)|^2)!$ and $N \leq [t, G]$.*

Proof. Put $\mathcal{D} = \{(x, y) \mid x, y \in t^H \mid x \neq y\}$. Note that $xy \neq 1$ for all $(x, y) \in \mathcal{D}$. For $a \in H^\sharp$, but $\mathcal{D}(a) = \{x, y) \in \mathcal{D} \mid xy = a\}$ and $k = \{\max|\mathcal{D}(h) \mid a \in G^\sharp$. Put $h = |H|$. Then $|cC| = |H/C_H(t)| = \frac{h}{c}$ and

$$\frac{h}{c}(\frac{h}{c} - 1) = |\mathcal{C}|(|\mathcal{C}| - 1) = |\mathcal{D}| = \sum_{a \in H^\sharp} |\mathcal{D}(a)| \leq (h-1)k$$

and so

$$\frac{h^2}{c^2} \leq hk - k + \frac{h}{c} \leq h(1\frac{1}{c} \leq 2h$$

and so

$$\frac{h}{k} \leq 2c^2$$

Pick $a \in H^\sharp$ with $|\mathcal{D}(a)| = k$ If $(x, y) \in \mathcal{D}(a)$ then $y = x^{-1}a = xa$, so y uniquely determined by x. Moreover x inverts $a = xy$. So if $(\tilde{x}, \tilde{y})$ is another element of $\mathcal{D}(a)$, then $xy^{-1} \in C_G(a)$. Thus $|\mathcal{D}(a)| \leq |C_H(a)|$. . It follows that

$$|a^H| = |H/C_H(a)| \leq \frac{h}{k} \leq c^2$$

Since $H/C(H(a^H)$ is isomorphic to a subgroup $\mathrm{Sym}(a^H)$ we conclude that $H/C_H(a^H)| \leq (2c^2)!$. Put $N = \langle a^G \rangle$. Then $|H/C_H(N)| \leq (2c^2)!$. Let $x = t^r$ and $y = x^s$ for some $r, s \in K$. Then $a = xy = x^{-1}x^s = [x, s] = [t^r, s]$. Since $[t, K] \trianglelefteq K$ this gives and $N \leq [t, G]$ and the lemme is proved. □

Lemma 3.3.2. [**brian**] *Let K be a group, $M \trianglelefteq K$, $\overline{K} = K/M$ and $h \in K$. Then $|C_{\overline{K}}(\overline{h})| \leq |C_K(h)$. Moreover if $|C_{\overline{K}}(\overline{h})| = |C_K(h)|$, then $Mh \subseteq h^K$.*

Proof. Define $A \leq K$ by $M \leq A$ and $A/M = C_{\overline{K}}(\overline{h})$. Note that $C_K(h) \leq A$. Consider the map

$$\tau : A \to H, a \to h^a$$

Since $[\overline{h}^{\overline{a}} = \overline{h}$ for all $a \in A$ we have $h^a \in Ma$ and so $\mathrm{Im}\,\tau \subseteq Mh$.

Note that $\tau(a) = \tau(b)$ iff $h^a = h^b$ iff $h^{ba^{-1}} = h$ iff $ba^{-1} \in C_K(h)$ iff $b \in a^{-1}C_K(h)$. Thus $\tau^{-1}(d) = |C_K(h)|$ for all $d \in \mathrm{Im}\,\tau$ and

$$|A| = |C_K(h)||\,\mathrm{Im}\,\tau| \leq |C_K(h)||Mh|\|C_K(h)|M|$$

and so

$$|C_{\overline{K}}(\overline{h})| = |A/M| \leq |C_K(h)|$$

If $|C_{\overline{K}}(\overline{h})| = |C_K(h)|$ we conlcude that $Mh = \mathrm{Im}\,\tau = h^A \subseteq h^K$. Note tat □

Lemma 3.3.3. [**h1 bouned**] *Let H be group acting on an abelian group A. Then $A/C_A(G)$ is bounded in terms of $|G/C_G(A)$ and $[A, G]$.*

Proof. Without loss $C_G(A) = 1$. For $g \in G$ we have $A/C_A(g) \cong [A, g] \leq [A, G]$ and so $|A/C_A(g)| \leq [A, G]$. Since $G/C_A(G)$ embeds into $\bigtimes_{g \in G} A/C_A(G)$, the lemma is proved. □

Proposition 3.3.4. [**g mod zl**] *Let G be a finite group and $t \in G$ with $t^2 = 1$. Put $L = [t, G]$. Then $|G/\mathrm{Z}_{\mathrm{Ord}}(L)|$ is bounded in terms of $|C_G(t)|$*

Proof. The proof is by induction on $C_G(t)$. Replacing G be $G/\mathrm{Z}_{\mathrm{Ord}}(L)$ we may assume that $\mathrm{Z}(L) = 1$. By 3.3.1 there exiss a non-trvial normal subgroup N of G such that $N \leq L$ and $G/C_G(N)$ is $|C_G(t)|$-bounded. Without loss N is a mininal normal subgroup of G. If t inverts N, then L centralizes N and so $L \leq \mathrm{Z}(L) = 1$, a contadiction. Hence there exists $n \in N$ such that t does not invert n. Since $n = (nt)t$ we conclude that (nt) does not have order two. So $nt \notin t^G$. Put $\overline{G} = G/N$. Then 3.3.2 implies that $|C_{\overline{G}}(t)| < |C_G(t)$. Let $Z/N = \mathrm{Z}_{\mathrm{Ord}}(\overline{L}))$. Then by induction G/Z is bounded in terms of $|\mathrm{C}_{\overline{g}}(\overline{t})$. Put $D = C_Z(N)$. Since $|Z/D| \leq |G/C_G(N)$ we conclude that Z/D and so also G/D are bounded in terms of $|C_G(t)|$.

It remains to bound the order of D. So suppose that $D \neq 1$ and let M be any non-trivial normal subgroup of G contained in D. Suppose that $M \cap D = 1$. Then $M \cong MN/N \leq ZN/N = \mathrm{Z}_{\mathrm{Ord}}(\overline{L}))$ and so $C_M(L) = 1 \neq 1$, a contradiction to $Z(L) = 1$. Thus $M \cap N \neq N$. Since N is a mininal normal subgroup of G this gives $N \leq M$. Thus N is the uniuqe mininal normal subgroup of G contained in D. In particular $N \leq D$ and so N is abelian. Since t does not invert N there a prime p and an elemenst of n of order p in $C_N(t)$. By mimimlity of N, $N = \langle n^G \rangle$. It follows that N is an elementary abelian p group and $|N| \leq p^{|}G/C_G(N)| \leq |C_G(t)|^{|G/C_G(N)}$. Thus $|N|$ is $|C_G(t)|$-bounded. Since Z/N is nilpotent and $N \leq Z(D)$, D is nilpotent. Observe that $N \cap O_{p'}(D) = 1$ and so $O_{p'}(D) = 1$. Thus D is a p-group and we conlcude that $[D, O^p(L)] \leq N$. If $C_D(O^p(L)) \neq 1$, then also $C_D(L) = 1$, a contradiction. Thus $C_D(O^p(L)) = 1$. From $[O^p(L), D, D] \leq [D, N] = 1$ and the Three subgroup lemma we get $[D', O^p(L)] = 1$ and so D is abelian. Since $|G/D|$ is bounded, we conclude that $O^p(L)/C_{O^p(L)}(D)$ is bounded. **??** now shows that $|D| = |D/C_D(O^p(L))|$ is bounded. □

Lemma 3.3.5. [**nilpotent and maximal abelian**] *Let P be a hypercentral groups and A a maximal abelian normal subgroup of P. Then $\mathrm{C}_P(A) = A$.*

Proof. Let $h \in C_P(A)$ with $[h, P] \leq A$. Then $\langle h \rangle A$ is an abelian normal subgroup of P and so by maximality of A, $h \in A$. Since P is hypercental this implies $\mathrm{C}_P(A) = A$. □

Lemma 3.3.6. [**2-group with small centralizer**] *Let P be a locally finite 2-group and $t \in P$ $t^2 = 1$ and with $|C_P(t)$ finite. Then there exists a integer n such that t inverts P^n and n and P/P^n are bounded in terms of $|C_P(t)|$*

Proof. Without loss P is finite. Let A be a maximal normal abelian subgroup of P and put $m = |C_P(t)|$. Let $m = 2^k$. Since $A/C_A(t) \cong [A, t]$ we have $|A/[A, t]| = |C_A(t)|\big||C_P(t)|$ and so $A^m \leq [A, t]$. Note that t inverts $[A, t]$ and so also A^m and $[\Omega_1 A(t), t]$. Thus $[\Omega_1 A(t), t] \leq C_{\Omega_1(A)}(t)$ and $|\Omega_1(A)| = |[\Omega_1 A(t), t]||C_{\Omega_1(A)}(t)| \leq |C_P(t)|^2 = m^2 = 2^{2k}$.

If follows that A has rank at most $2k$. and so A/A^m has order at most $m^{2k} = 2^{2k^2}$. order. Hence also $P/C_P(A/A^m)$ has m-bouned order. Put $E = C_P(A^m) \cap C_P(A/A^m)$. By 3.3.2 $P/[P, t]$ has order at most m and since $[P, t]$ centralizes A^m, $P/C_P(A^m)$ has order at most m. Put $E = C_P(A^m) \cap C_P(A/A^m)$. Then P/E has m-bouned order. Let $a \in A$ and $e \in E$. Then $[a, e]^m = [a^m, e] = 1$ and so $[a, e] \leq \Omega_k(A)$. Since $|\Omega_k(A)$ and A/A^m have order at most 2^{2k^2} we conclude that $E/C_E(A)$ has order at most $24k^4$. Thus $P/A = P/C_P(A)$ has m-bounded order. Hence $P^l \leq A$ for some m-bounded integer k. Then $P^{lm} \leq A^m$ and t inverts P^{lm}. Since $A^{lm} \leq A$, $|A/P^{lm}$ has order at most $(lm)^k$ and so $|P/P^{lm}|$ is lm-bounded. □

Lemma 3.3.7. [**coprime action**] *Let p be a prime and G a finite group acting a finite p-group P.Define $O^p(G) = \langle x \in G \mid x$ is a p' element$\rangle$*

(a) [**a**] *$G/O^p(G)$ is a p-group and so $O^p(G)$ is the unique smallest normal subgroup of G whose quotient is a p-group.*

(b) [**c**] *$[P, O^p(G)] = [P, O^p(G); n]$ for all $n \in \mathbb{Z}^+$.*

(c) [**d**] *The exists $n \in \mathbb{Z}^+$ with $[P, G; n] = 0$ if and only if $[P.O^p(G)] = 1$ and if and only if $G/C_G(P)$ is a p-group.*

Proof. (a) Let $x \in G$, then $x = yz$ where y is a p element and z is p'-elemenst. Thus $xO^p(G) = yO^p(G)$ and so $G/O^p(G)$ is a p-group.

□

Lemma 3.3.8. [**more coprime**] *Let P be a p-group acting on a p'-group Q.*

(a) [**a**] *Let $R \trianglelefteq S \leq Q$ be P-invariant subgroups of Q. Then $C_{S/R}(P) = C_S(P)R/R$.*

(b) [**b**] *Let $1 = Q_0 \trianglelefteq Q_1 \leq Q_2 \trianglelefteq \ldots \trianglelefteq Q_n = Q$ be a P invariant subnormal series of Q. Then*

$$|C_Q(P)| = \prod_{i=1}^{n} |C_{Q_i/Q_{i-1}}(P)|$$

Proof. (a) Let $T/R = C_{S/R}(Q)$. Then $C_S(R)Q \leq T$ and $[T, P] \leq R$. By Homework 1, $T = C_T(P)[T, T] \leq C_S(P)Q \leq T$ and so $T = C_S(P)Q$.

(b). This clearly holds for $n = 1$. Suppose $n > 1$ and put $k = n - 1$. Then

$$\begin{aligned}
|C_Q(P)| &= |C_Q(P)/C_{Q_k}(R)||C_{Q_k}(R)| &&= |C_Q(R)/C_Q(R) \cap Q_k||C_{Q_k}(R)| \\
&= ||C_Q(R)Q_k/Q_k||C_{Q_k}(R)| &&= |C_{Q/Q_k}(R)||C_{Q_k}(R)| \\
&= |C_{Q/Q_k}(R)|| \textstyle\prod_{i=0}^{k} |C_{Q_i/Q_{i-1}}(P)| &&= \textstyle\prod_{i=1}^{n} |C_{Q_i/Q_{i-1}}(P)|
\end{aligned}$$

□

Proposition 3.3.9. [**nilpotent by finite**] *Let G be a locally finite group and $t \in G$ with $t^2 = 1$. Then there exists a postive integer n such that n and $|G/Z_n([G, t])|$ are bounded in terms of $|C_G(t)|$. In particular, G is nilpotent by finite.*

Proof. Put $L = [t, G]$ and $Z = Z_{\text{Ord}}(L)$.

Supppose first that G is finite let n be minimal with $Z_n(L) = Z$. By 3.3.4 $|G/Z|$ is bounded in terms of $C_G(t)$. So we just need to show that n is bounded. Let r and s be minimal with $O_2(Z) \leq \mathrm{Z}_s(L)$ and $O(Z) \leq Z_r(L)$. Then $n = \max(r, s)$. By 3.3.6 there exists an integer m such that $O_2(Z)^m$ has bounded index in $O_2(Z)$ and $O_2(Z)^m$ is inverted by t. Then L centralizies $O_2(Z)^m$ and s is bounded.

For $1 \leq j \leq s$ put $Z_i = \mathrm{Z}_i(L) \cap \bar{(}Z)$. Then $Z_i/Z_{i-1} = \mathrm{C}_{O(Z)/Z_[i-1}(L)$ and $1 = Z_0 < Z_1 < Z_2 < \ldots < Z_r = O(Z)$. Let $i \in Z^+$ with $2i \leq t$. Then L does not centralizes Z_{2i}/Z_{2i-2}, t does not inverts Z_{2i}/Z_{2i-2}, $C_{Z_{2i}/Z_{2i-2}}(t) \neq 0$ and by Homework 1, $C_{Z_{2i}}(t) \not\leq Z_{2i-1}$. Thus

$$0 < C_{Z_2}(t) < C_{Z_4}(t) < \ldots$$

and we conclude that s is bounded in terms of $|C_G(t)|$.

So the proposition holds for finite groups. In particular there exist bounded integers n and m such that $|H/Z_n([[H, t])| \leq m$ for all finite subgroups H of G. For a finite subgroup subgroup H of G define

$$k(H) = \sup\{|\ |H/H \cap Z_n([[K, t])| \big| H \leq K \leq G, K \text{ finite}, H \cap [t, G] = H \cap [t, K]\}$$

Observe that since $H \cap [t, G]$ is a finite subgroup, there exists a finite subgroup K of G with $H \leq K$ and $H \cap [t, G] \leq [t, K]$. Hence $H \cap [t, G] = H \cap [t, K]$ and $k(H)$ is well defined. Also

$$|H/H \cap Z_n([[K, t])| = |HZ_n([[K, t]/Z_n([[K, t])| \leq |K/Z_n([[K, t])| \leq m$$

and so $k(H) \leq m$ and there exists a finite subgroup H^* of G with $H \leq H^* \leq G$, $H \cap [t, G] = H \cap [t, H^*]$ and $|H/H \cap Z_n([[H^*, t]| = k(H)$.

Put $k = \max\{k(H) \mid H \leq G, H \text{ finite}\}$. Then also $k \leq M$. Put

$$\mathcal{L} = \{H \leq G \mid H \text{ finite } k(H) = k\}$$

and for $L \in \mathcal{L}$ define

$$\mathcal{F}(L^*) = \{H \leq G \mid L^* \leq H, H \text{ finite}$$

We will prove next

1°. [1] *Let $L \in \mathcal{L}$ and $H \in \mathcal{F}(L^*)$. Then $L \cap [G, t] = L \cap [H, t]$, $L \cap Z_n([[L^*, t]) = L \cap Z_n([H^*, t])$ and $|L/L \cap Z_n([H^*, t]) = k$*

Indeed we have

$$L \cap [G, t] = L \cap [L^*, t] \leq L \cap [H, t] \leq L \cap [H^*, t] \leq L \cap [G, t]$$

and so $L \cap [G, t] = L \cap [L^*, t] = L \cap [H, t] = L \cap [H^*, t]$

Thus $[L \cap Z_n([H^*, t]), L^*; n] \leq Z_n([H^*, t]), H^*; n] = 1$ and hence

$$L \cap Z_n([H^*, t]) \leq \mathbb{L} \cap Z_n([L^*, t]$$

Therefore,

$$k = k(L) = |L/L \cap Z_n([L^*, t])| \leq |L/L \cap Z_n(H^*, t]| \leq k(L)$$

and (1°) is proved.

2°. [2] *Let $L \in \mathcal{L}$ and $H \in \mathcal{F}(L^*)$. Then $k(H) = k$ and $H = L(H \cap Z_n([H^*, t]))$.*

By (1°) we have

$$\begin{array}{rcccc} k & = & |L/L \cap Z_n([[H^*, t]| & = & |LZ_n([[H^*, t])/Z_n([[H^*, t]| \\ & \leq & |HZ_n([[H^*, t]/Z_n([[H^*, t]) & = & k(H) \leq k \end{array}$$

Thus $k = k(H)$, and $HZ_n([[H^*, t]) = LZ_n([[H^*, t]))$. Thus $H = L(H \cap \mathrm{Z}_n([[H^*, t]))$ and (2°) holds.

3°. [3] *Put $Z = \bigcup_{L \in \mathcal{L}} L \cap Z_n([L^*, t])$. Then Z is a normal subgroup of G.*

Let $L_1, L_2 \in \mathcal{L}$ and put $H = \langle L_1^*, L_2^* \rangle$. Then by (2°), $H \in \mathcal{L}$ and by (**??**), $L_i \cap Z_n([L_i^*, t]) \leq H \cap Z_n([t, H]) \leq Z$. Thus

$$\langle L_1 \cap Z_n([L_1^*, t], L_2 \cap Z_n([L_2^*, t]) \rangle \leq Z$$

and so Z is subgroup of G. Since $\mathcal{L}$ is invariant under G, also Z is invariant under G.

4°. [4] *$G = LZ$ for all $L \in \mathcal{L}$ and $|G/Z| \leq k \leq m$.*

Let $g \in G$ and put $H = \langle L^*, g \rangle$. Then by (2°), $H \in \mathcal{L}$ and $g \in H = L(H \cap Z_n[H^*, t]) \leq LZ$. Thus $G = LZ$ and so $G/Z| = |L/L \cap Z| \leq |L/L \cap Z_n([L^*, t])| = k \leq m$.

5°. [5] *$Z \leq Z_n([G, t])$.*

Clearly $Z \leq [G, t]$ and so we only need to show that $[Z, [G, t]; n] = 1$. This holds if an only if $[z, F; n] = 1$ for all $z \in Z$ and all finite subgroups F of $[G, t]$. . Pick $L \in \mathcal{L}$ with $z \in L \cap Z_n([L^*, t])$ and then $H \leq G$ with H finite, $L^* \leq H$ and $F \leq [H, t]$. Then using (1°), $z \in L \cap Z_n([L^*, t]) = L \cap Z_n([H^*, t])$ and so $[z, F; n] \leq [Z_n([H^*, t]), [H^*, t]; n] = 1$. So (5°) hold.

By (4°) and (5°), $G/Z_n([G, t])| \leq m$ and the theorem is proved.

□

Corollary 3.3.10. [**infinite centralizer**] *Let H be an infinite locally finite simple group and t an involution in H. Then $C_H(t)$ is infinite.*

Proof. This follows immediately from 3.3.9 □

3.4 Locally finite groups with MIN

This section is entirely devoted the proof of the following Theorem

Theorem 3.4.1. [**lf with min**] *Every locally finite group which fulfills MIN is a cernikov group.*

Suppose the theorem is false.

Step 1. [**step 1**] *There exists an infinite locally finite simple groups G all of whose proper subgroups are Cernikoóvgroups.*

Proof. Let G_0 be a locally finite group with MIN which is not Cernikoóv. Let G_1 be a subgroup of G_0 minimal with respect to not being Cernikoóv. **??** implies that G_1 has a component K with $K/Z(K)$ infinite. Put $G = K/Z(K)$. By minimality of $G-1$, all proper subgroups of G_1 and so also of G are Cernikoóvgroups. □

Step 2. [**step 2**] *G is not a $2'$-group.*

Proof. Otherwise the Odd Order Theorem implies that all finite subgroups of G are solvable. But then G is locally solvable and all chief factor of G are abelian, a contradiction. □

Let $\mathcal{P}$ be the set of all positive primes, $\pi \subseteq \mathcal{P}$, $\mathcal{D}_\pi$ be the set of maximal divisible abelian π-subgroups of G and $\mathcal{D} = \mathcal{D}_\pi$.

Step 3. [**step 3**] *Let H be proper subgroup of G and put $H_\pi = \{x \in H^\circ \mid x$ is a $\pi-$element. Then H_π contains every divisible abelian π-subgroup of H and is contained in every maximal π-subgroup of H.*

Proof. Let D be a divisible abelian π-subgroup of H. Then $D = D^\circ \leq H^\circ$ and so $D \leq H_\pi$.

Let M be maximal π-subgroup of H. Since H_π is normal in H, $H_\pi M$ is π-subgroup of G and so $M = H_\pi M$ by maximality of M. □

Step 4. [**step 4**] *Let $1 \neq D \in \mathcal{D}_\pi$ and $D \leq H < G$. Then $D = H_\pi$ and $H \leq N_G(D)$. So $N_G(D)$ is the unique maximal subgroup of G containing D.*

Proof. We have $D \leq H_\pi$ and so by maximality of D, $D = H_\pi$. Since $H_\pi \trianglelefteq H$, $H \leq N_G(D)$. □

Step 5. [**step 5**] *Let $D \in D_\pi$ and E a divisible abelian π subgroup of G. Then $E \leq D$ or $E \cap D = 1$.*

Proof. Assume that $E \cap D \neq 1$. Then $D \neq 1$. Put $H = C_G(E \cap D)$. Since G is simple, $E \cap D \ntrianglelefteq G$ and so $H \neq G$. Note that $\langle E, D\rangle \leq H$ and by Step 4, $D = H_\pi$. Thus by Step 3, $E \leq D$. □

Step 6. [**step 6**] *Every every non-trivial divisible abelian subgroup A of G lies in a unique maximal divisible abelian subgroup $\overline{A}$ of G. If in addition A is a π-group, then $\overline{A}_\pi$ is the unique maximal divisible abelian π-subgroup of G containing A.*

Proof. Let $D, E \in \mathcal{D}$ with $A \leq D$ and $A \leq E$. Then $A \leq D \cap E$. By Step 5 $D = E$. Now suppose A and B are divisible by groups with $A \leq B$. Then $A \leq \overline{B}$ and so $\overline{B} = \overline{A}$ and $B \leq \overline{A}_\pi$. □

Step 7. [**step 7**] *Let D be non-trivial divisible abelian subgroup of G. Then $N_G(D) \leq N_G(\overline{D})$ and if $D \in \mathcal{D}_\pi$, then $N_G(D) = N_G(\overline{D})$.*

Proof. Let $g \in \mathrm{N}_G(D)$. Then $D \leq \overline{D}^g \in \mathcal{D}$ and so $\overline{D} = \overline{D}^g$ by the uniqueness of $\overline{D}$. So the first statement holds. For the second observe that $D = \overline{D}_\pi$ and so $\mathrm{N}_G(\overline{D}) \leq \mathrm{N}_G(D)$. □

Step 8. [**step 19**]

(a) [**a**] *Every maximal subgroup of G is infinite.*

(b) [**b**] *Every proper infinite subgroup R of G lies in a unique maximal subgroup $\tilde{R}$ of G, namely $\tilde{R} = B_G(\overline{R^\circ})$.*

(c) [**c**] *If M_1 and M_2 are maximal subgroups of G with $M_1 \cap M_2$ infinite, then $M_1 = M_2$.*

(d) [**d**] *Let M be a maximal subgroup of G and $H \leq G$ with $M \cap H$ infinite. Then $H \leq M$.*

Proof. (a) Suppose F be a finite subgroup of G and let $g \in G \setminus F$. Then $\langle F, g\rangle$ is finite, $F < \langle F, g\rangle < G$ and so F is not maximal.

(b) Let $R \leq M < G$. Then $R^\circ \leq M^\circ \leq \overline{R^\circ}$ and so $\overline{R^\circ} = \overline{M^\circ}$. Thus $M \leq N_G(\overline{R^\circ})$.

(c) By (b) $M_1 \cap M_2$ is contained in a unique maximal subgroup and so $M_1 = M_2$.

(d) By (b) H lies in a maximal subgroup $\tilde{M}$ of G. Then $H \cap M \leq M \cap \tilde{M}$ and so by (c), $M = \tilde{M}$. Thus $H \leq M$. □

Step 9. [**char max**] *Let $M < G$. Then following are equivalent.*

(a) [**a**] *M is a maximal subgroup of G.*

(b) [**c**] *$1 \neq M^\circ \in \mathcal{D}$ and $M = N_G(M^\circ)$.*

(c) [**b**] *$M = N_G(D)$ for some set of prime π and some $1 \neq D \in D_\pi$.*

Proof. (a) $\Longrightarrow$ (c): Suppose M is maximal in G. By Step 8(a), M is infinite and so $M^\circ \neq 1$. By Step 8(b), $M = N_G(\overline{M^\circ})$ and so $\overline{M^\circ} \leq M$ and thus $M^\circ = \overline{M^\circ} \in \mathcal{D}$.

(c) $\Longrightarrow$ (b): Just set $\pi = \mathcal{P}$ and $D = M^\circ$.

(b) $\Longrightarrow$ (a): See Step 4. □

Definition 3.4.2. [**omega**] *Let H be a group. Then $\Omega^m_n(H) = \langle x \in H \mid x^{m^n} = 1\}$. If H is a p group for some prime p, then $\Omega_m(H) = \Omega^p_m(H)$.*

Step 10. [**step 9**] *Let p be a prime and $1 \neq D \in \mathcal{D}_p$. Let T be p-subgroup of G with $\Omega_2(D) \leq T$. Then $T \leq \mathrm{N}_G(D)$ and $|T/T \cap D| \leq |N_G(D)/\overline{D}|_p$.*

Proof. Since $D \leq \mathrm{N}_G(\Omega_2(D))$, Step 4 implies $\mathrm{N}_G(\Omega_2(D)) \leq \mathrm{N}_G(D)$. Since T is a Cernikoóvp-group, $1 \neq \mathrm{Z}(T)$. Observe that $[\Omega_2(D), \mathrm{Z}(T)] = 1$ and $\mathrm{Z}(T) \leq \mathrm{N}_G(\Omega_2(D)) \leq \mathrm{N}_G(D)$. Thus by **??**, $[D, \mathrm{Z}(T)] = 1$. We have $D \leq \mathrm{C}_G(\mathrm{Z}(T)) < G$ and so usingStep 4, $T \leq \mathrm{C}_G(\mathrm{Z}(T)) \leq \mathrm{N}_G(D)$. Since $D = \overline{D}_p$, D/D_p is p'-group and so $T \cap D \leq D_p$. Thus $T/T \cap D = T/T \cap oD \cong T\overline{D}/\overline{D} \leq N_G(D)/\overline{D}$ and Step 13 is proved. □

Lemma 3.4.3. [**cernikov and sylow**] *Let H be a Cernikoóvgroup and p a prime, then H acts transitively on $\mathrm{Syl}_p(H)$.*

Proof. Note that $H_p \trianglelefteq H$ and H_p is a p-group. Let $T \in \mathrm{Syl}_p(H)$. Then H_pS is a p-group and so $H_p \leq S$. Since H°/H_p is a p'-group, $S \cap H^\circ = H_p$. Thus $|S/H_p| = |SH^\circ/H^\circ|$ and so S/H_p is finite. Note that S/H_p is a Sylow p-subgroup of H/H_p. We conclude from **??** that all Sylow p-subgroups of H/H_p are conjugate in H/H_p. Hence all Sylow p-subgroups of H are conjugate. □

Step 11. [**scirc**] *Let $S \in \mathrm{Syl}_p(G)$. then $S^\circ \in \mathcal{D}_p$ and $S^\circ = \overline{S^\circ}_p$*

Proof. Since S° is a divisible abelian p-goup, $S^\circ \leq \overline{S^\circ}_p$. Pick $D \in \mathcal{D}_p$ with $\overline{S^\circ}_p \leq D$. By Step 4, D is unique and so S normalizes D. Thus SD is p-group and so $D \leq S$ by maximality of S. Hence $D \leq S^\circ$ and so $S_p = \overline{S^\circ}_p = D$. □

Step 12. [**transitive on syl**] *Let $H \leq G$. Then H acts transitively on $\mathrm{Syl}_p(H)$.*

Proof. If $H \neq G$, then H is a Cernikoóvgroup and we are done by 3.4.3.

So suppose $G = H$ and let S_1 and S_2 be Sylow p-subgroups of G. If S_1 or S_2 is finite we are done by **??**. So we may assume that $S_i^\circ \neq 1$ for $i = 1$ and 2. Put $E_i = \Omega_2(S_i^\circ)$ and $L = \langle E_1, E_2 \rangle$. Then L is a finite group and so by Sylow's Theorem $\langle E_1, E_2^g \rangle$ is a p-group for some $g \in L$. Thus by Step 13 $E_2^g \leq N_G(S^\circ)$ and so E_2^g is contained in a Sylow p-subgroup of $\mathrm{N}_G(S_1^\circ)$. By the first paragraph of the proof $E_2^{gh} \leq S_1$ for some $h \in N_G(S_1^\circ)$. Hence by Step 13, $S_1 \leq \mathrm{N}_G(S_2^{\circ gh}$ and then by the first paragraph, $S_2^{ghk} = S_1$ for some $k \in \mathrm{N}_G(S_2^{\circ gh}$. □

Step 13. [**step 9**] *Let p be a prime. Then G acts transitively on $\mathcal{D}_p$.*

Proof. Let $D_1, D_2 \in \mathcal{D}_p$ and pick $S_i \in \mathrm{Syl}_p(G)$ with $D_i \leq S_i$. Then $S_1^g = S_2$ for some $g \in G$. Since $D_i = S_i^\circ$, this gives $D_1^g D_2$. □

Definition 3.4.4. [**def rank**] *Let H be a locally finite group and p a prime. Then $m_p(G) = \sup\{k \in \mathrm{N} \mid$ there exists$A \leq H$ with $A \cong C_p^k\}$.*

Step 14. [**step 12**] *Let p be prime. Then $m_p(G)$ is finite.*

Proof. Let $S \in \mathrm{Syl}_p(G)$. Every elementary abelian subgroup of G is contained in Sylow p-subgroup and so conjugate to a subgroup of S. Thus $m_p(G) = m_p(S)$. By **??**, $k := m_p(S^\circ)$ is finite. Put $|S/S^\circ| = p^l$ and let A be an elementary abelian subgroup of S. Then $|S^\circ \cap A| \leq p^k$ and $AS^\circ/S^\circ| \leq p^l$. Thus $|A| \leq p^{k+l}$ and so $m_p(S) \leq k + l$. □

Theorem 3.4.5. [**walter feit**] *Let H be a finite simple group and with dihedral Sylow 2 subgroups. Then $H \cong Alt(7)$ or $L_2(p^k)$, where p is an odd prime and $|p^k| > 3$.*

Lemma 3.4.6. [**l2p**] *Let $H \cong L_2(p^k)$, p an odd prime.*

(a) [**a**] *Let $T \in \mathrm{Syl}_p(H)$. Then T is elementary abelian p group of rank k and $|N_H(T)/C_H(T)| = \frac{p^k-1}{2}$.*

(b) [**b**] *Let A be an elementary abelian r subgroup of H, where r is an odd prime, $r \neq p$. Then $|N_H(T)/C_H(T)| \leq 2$.*

Proof. Readily verified. □

Step 15. [**s is not dihedral**] *S be a Sylow 2-subgroup of G. Then $S \not\cong D_{22^k}$ for $k \in \mathrm{Z}^+ \cup \infty$.*

Proof. Suppose $S \cong D_{22^k}$. If $|S| = 2$ let $R = S$ otherwise pick $R \leq S$ with $R \cong C_2 \times C_2$. choose $R \leq H_1 < H_2 < H_3 < \ldots H_n < \ldots$ with $(H_i, 1) \in \mathcal{K}$ and $|H_1| \geq 7!$. Let $S_i \in \mathrm{Syl}_2(H_i)$ with $R \leq S_i$. By Step 12 there exists $g \in G$ with $S_i \leq S^g$. It follows that $S - i$ is either a dihedral group or cyclic. Since $R \leq S_i$, S_i is a dihedral group. Thus by 3.4.5, $H_i \cong L_2(p_i^{k_i}$, p_i an odd prime or Alt(7). Since $|H_i| \geq |7!$, $H \not\cong$ Alt(7) and $H \not\cong L_2(5)$. So by 3.4.5 $H_i \cong L_2(p_i^{k_i}$, $p_i^{k_i} > 5$. Let $p = p_1$ and $A \in Syl_p(H_1)$. Then by **??**(**??**) $|N_{H_1}/C_{H_1}(A)| = \frac{p^{k_1}-1}{2} > \frac{5-1}{2} = 2$. Thus **??**(**??**) implies that $p = p_i$ for all i. Since $H_i < H_{i+1}$, $k_i < k_{i+1}$. Since $m_p(G) \geq m_p(H_i) = k_i$, this gives $m_p(G) = \infty$ a contradiction to **??** □

Definition 3.4.7. [**def:strongly p-embedded**] *Let H be a locally finite group, p a prime and M a subgroup of H. Then M is called strongly p-embedded if*

(i) [**i**] *M is not a p'-group.*

(ii) [**ii**] *$M \cap M^g$ is p'-group for all $g \in H \setminus M$.*

Theorem 3.4.8. [**bender**] *Let H be a finite group with a proper strongly 2-embedded subgroup. The one of the following holds:*

1. [**1**] *$[z, H]$ has odd order for all involutions z of H.*

2. [**2**] *$H/O(H)| \leq f(m_2(H))$ where $f : \mathrm{Z}^+ \to \mathrm{Z}^+$ is some function independent of H.*

Proof. Suppose first that $m_2(H) = 1$. Then H has a unique class of involution and $[x, z] \neq 1$ for all involutions x, z in H with $x \neq z$. Thus Glauberman's Z^* theorem shows that $[z, H]$ has odd order.

Suppose next that $m_2(H) \geq 2$. Then Bender's strongly embeded theorem shows that $H/O(H) \cong L_2(q)$, $Sz(q)$ or $U_3(q)$, where $q = 2^k$ for some $k \in \mathbb{Z}^+$. It follows that $m_2(H) = k$ and $|H/O(H)| \leq q^9 = 2^{9k} = 2^{m_2(H)}$. □

Step 16. [**step 13**] *G has no proper strongly 2-embedded subgroup.*

Definition 3.4.9. [**def:kegel cover**] *Let H be locally finite group. Then a Kegel cover $\mathcal{K}$ for H is a set of pairs of subgroup of H such that*

(i) [**1**] *If $(K, M) \in \mathcal{K}$ then $M \trianglelefteq K \leq H$, K is finite and K/M is simple.*

(ii) [**2**] *If F is a finite subgroup of H, then there exists $(K, M) \in \mathcal{K}$ with $F \leq K$ and $F \cap M = 1$.*

Theorem 3.4.10. [**kegel**] *Every locally finite simple group has a Kegel cover.*

Proof. Let H be a locally finite group. Define $\mathcal{K}$ to be the set of all pairs (K, M) such that $M \trianglelefteq K \leq H$, K is finite and K/M is simple. F be a non-trivial finite subgroup of H. Let $1 \neq f \in F$. Since H is simple $H = \langle f^H \rangle$ and so there exists a finite subset I_f of H with $F \leq \langle f^{I_F} \rangle$. But $F^* = \langle F, I_f \mid f \in F^\sharp \rangle$. Then $F \leq \langle f^{F^*} \rangle$ for all $f \in F^\sharp$. Put $K = \langle F^{F^{**}} \rangle$. Let N be the intersection of the maximal normal subgroups of K. Then N is characteristic subgroup of K and $N \neq K$. Since F^{**} normalizes K it also normalizes N. If $F \leq N$ we get $K = \langle F^{F^{**}} \rangle \leq N$, a contradiction. Thus $F \nleq N$ and there exists a maximal normal subgroup M of K with $F \nleq M$. Note that $(K, M) \in \mathcal{K}$ and $F \leq H$. Suppose that $F \cap M \neq 1$ and pick $f \in F^\sharp$. Then $f \in F^*$ and so $F^* \leq \langle f^{F^{**}} \rangle \leq K$. Hence $F \leq \langle f^{F^*} \rangle \leq \langle M^H \rangle = M$, a contradiction. Thus $F \cap M = 1$ and $\mathcal{K}$ is a Kegel cover. □

Step 17. [**step 14**] *There exists a finite subgroup Q of G such that $M = 1$ for all finite subgroups M of G with $Q \leq N_G(M)$ and $Q \cap M = 1$.*

Proof. Suppose not. Put $L_1 = M_1$ be a arbitrary non-trivial finite subgroup of G and assume inductively that we already define finite subgroups L_i, M_i, $1 \leq i \leq n$ in G. By assumption there exists non-trivial finite subgroup M_{n+1} of G with $L_n \leq N_G(M_{n+1})$ and $L_n \cap M_{n+1} = 1$. Put $L_{n+1} = L_n M_{n+1}$.

Define $H_n = \langle M_i \mid i \in \mathbb{Z}^+, i \geq n \rangle$. Then clearly

$$H_1 \geq H_2 \geq H_3 \geq \dots$$

Fix $n \geq 2$. We will now show that $L_{n-1} \cap H_n = 1$. Let $g \in L_{n-1} \cap H_n$. For $m \geq n$ define $R_m = \langle M_i \mid n \leq i \leq m \rangle$. Then $H_n = \bigcup_{m=n}^{\infty} R_m$ and so we can choose m minimal with $x \in R_m$. Suppose that $m \neq n$. Then $R_m = \langle R_{m-1}, M_m \rangle$. Note that $R_{m-1} \leq L_{m-1}$ and so R_{m-1} normalizes M_m and $R_m = R_{m-1} M_n$. Since $x \in L_{n-1} \leq L_{m-1}$ and $R_{m-1} \leq L_{m-1}$ we get

$$x \in L_{m-1} \cap R_{m-1}M_n = R_{m-1}(L_{m-1} \cap M_n) = R_{m-1}$$

a contradiction to the minimal choice of m. Thus $m = n$, $x \in R_n = M_n$ and $x \in L_{n-1} \cap M_n = 1$.

So $L_{n-1} \cap H_n = 1$ and so $H_{n-1} > H_n$, a contradiction since G fulfills MIN. □

Step 18. [**simple cover**] *Let F be a finite subgroup of G and $m \in \mathrm{Z}^+$. Then there exists a finite simple subgroup K of G with $F < K$ and $|K| \geq m$.*

Proof. Let Q be as in Step 17. Since G is infinite there exists $I \subseteq G$ with $|I| \geq m$ and $F \subseteq I$. Put $R = \langle I, Q\rangle$. Then R is finite and by 3.4.10 there exists a finite subgroup K of G and maximal normal subgroup M of G with $R \leq K$ and $R \cap M = 1$. Then $Q \leq K \leq N_G(M)$ and $Q \cap M = 1$. Thus by Step 17, $M = 1$. So K is simple. Since $F \subset I \subseteq R \leq K$, $F < K$. Since $|I| \geq m$, $|K| \geq m$ and so **??** □

Lemma 3.4.11. [**normalizer condition**]

(a) [**a**] *Let S be a nilpotent group and $T \leq S$. If $\mathrm{N}_S(T) = T$, then $T = S$.*

(b) [**b**] *Let S be a locally nilpotent group and T a finitely generated subgroup of S. If $\mathrm{N}_S(T) = T$, then $S = T$.*

Proof. (a) Let $Z_0 \leq Z_1 \leq \ldots \leq Z_n$ be the upper central series of S. Note that $Z_0 \leq T$. Assume inductively that $Z_i \leq T$. Then

$$[Z_{i+1}, T] \leq [Z_{i+1}, S] \leq Z_i \leq T$$

and so $Z_{i+1} \leq \mathrm{N}_S(T) = T$. Thus $S = Z_n \leq T$ and $T = S$.

(b) Let $s \in S$ and put $R = \langle T, s\rangle$. Then R is finitely generated and so R is nilpotent. Also $T \leq \mathrm{N}_R(T) \leq \mathrm{N}_S(T) = T$ and so by (a), $R = T$. Thus $s \in T$ and $S = T$. □

Proposition 3.4.12. [**char strongly p-embedded**] *Let H be a locally finite group, p a prime and $M \leq H$. Suppose that*

(a) [**i**] *M is not a p' group and $M \neg H$.*

(b) [**ii**] *If $x \in M$ has order P, then $C_G(x) \leq M$.*

(c) [**iii**] *Let S be a Sylow p-subgroup of G.*

1. [**1**] *If S is finite, then $N_G(S) \leq H$.*
2. [**2**] *If S is infinite, then each $h \in H \setminus M$, $M \cap M^h$ has finite Sylow p-subgroups.*

Then M is a strongly p-embedded subgroup of H.

Proof. Suppose not and let $h \in H \setminus M$ such that $M \cap M^h$ is not a p' group. Let $T \in \mathrm{Syl}_p(H \cap H^g)$ and $S \in \mathrm{Syl}_p(T)$. By (c:1), T is finite. Suppose that $S \neq T$. Then by **??**(**??**), $\mathrm{N}_S(T) \neq T$ and so there exists $T < P \leq N_S(T)$ with P finite. Thus there exists $1 \neq x \in C_T(P)$. Then by (b), $P \leq C_H(x) \leq M$ and thus $T < P \leq H \cap M^h$, a contradiction since P is p-groups and T is a Sylow p-subgroup of $H \cap H^\gamma$.

Thus $T = S$ and so $T \in \mathrm{Syl}_p(M^g)$. In particular, M has finite Sylow p-groups. It follows that M^g acts transitively on $\mathrm{Syl}_p(M^g)$. Since $T \leq M$, $T^h \leq M^g$ and $T^h \in \mathrm{Syl}_p(M^g)$. Thus $T^{hk} = T$ for some $k \in M^h$. Then $hk \in N_H(T)$ and so by (c:2), $hk \in M$. Thus $M = M^{hk} = (M^h)^k = M^h$ and so $k \in M$ and $h = (hk)k^{-1} \in M$, contrary to the choice of h. □

Lemma 3.4.13. [**dihedral**] *Let x and y be non-conjugate involution in a group H. Then $|xy|$ has even order, $\langle xy\rangle$ contains a unique involution u, and any involution in $< x, y\rangle$ is either equal to u or conjugate to x or to y.*

Proof. This follows easily from the fact that $\langle x, y\rangle$ is dihedral group. □

Step 19. [**step 20**] *Let $\mathcal{M}$ be a finite set of maximal subgroups of G and K a non empty G-invariant subset of $G^\sharp$. Then $K \setminus \bigcup \mathcal{M}$ is infinite.*

Proof. Suppose that $K \setminus \bigcup \mathcal{M}$ is finite. If K is finite, $\langle K\rangle$ would be a non-trivial finite normal subgroups of G, a contradiction, since G is infinite and simple. So K and $K \cap \bigcup \mathcal{M}$ are infinite. Since $\mathcal{M}$ is finite, there exists $M \in \mathcal{M}$ such that $K \cap M$ is infinite. Let $g \in G$. Then $(K \cap M)^g = K \cap M^g$ is infinite and so there exists $N \in \mathcal{M}$ with $K \cap M^g \cap N$ infinite. Hence by **??**(**??**), $M^g = M \in \mathcal{M}$. Thus M^G is finite. Then also $G/\mathrm{C}_G(M^G)$ is finite and $\mathrm{C}_G(M^G)$is a normal subgroup of finite index in G. Hence $\mathrm{C}_G(M^G) = G$ and $M \trianglelefteq G$, a contradiction □

For $z \in \mathcal{I}_\infty$ let H_x be the unique maximal subgroup of G containing $C_G(z)$.
p

Lemma 3.4.14. [**lemma 14**] *Let D be a divisible abelian group and $\alpha \in \mathrm{Aut}(D)$ with $\alpha^2 = \mathrm{id}_D$. If $C_D(\alpha)$ is finite, then α inverts D.*

Proof. Observe that the map $\tau : D \to D, d \to dd^\alpha$ is a homomorphism with $\mathrm{Im}\,\tau \leq \mathrm{C}_D(\alpha)$. Thus $D/\ker\alpha$ is finite. Since divisible groups of no proper subgroup of finite index, $D = \ker\tau$ and so $dd^\alpha = 1$ for all $d \in D$. Hence $d^\alpha = d^{-1}$. □

Step 20. [**step 15**] *Let $z \in \mathcal{I}$ and M a maximal subgroup of G with $z \in M \not\leq H_z$. Then z inverts M°.*

Proof. If C_{M° is finite, then by Step 17 z inverts M°. So suppose $C_{M^\circ}(z)$ is infinite. Since $C_{M^\circ}(z) \leq H_z \cap M$, **??**(**??**) gives $M = H_z$. □

Step 21. [**step 16**] *Let $A \leq G$ be a fours group (that is $A \cong C_2 \times C_2$) and M a maximal subgroup of G containing A. Then $M = H_x$ for some $x \in A^\sharp$. If $C_G(A)$ is infinite, then M is the unique maximal subgroup of G containing A.*

Proof. Let $A^\sharp = \{a, b, c\}$. If a does not inverts M°, then by (**??**), $M = H_a$. Similary if b does not inverts M°, then $M = H_\circ$. If a and b inverts M°, then $ab = c$ centralizes M° and so $M = H_c$.

Thus $M = H_x$ for some $1 \neq x \in A$. Suppose $C_G(A)$ is infinite. Then $C_G(A) \leq C_G(x) \leq H_x = M$ and so M is the unique maximal subgroup containing $C_G(A)$. □

Step 22. [**cga not in hz**] *Let $1 \neq z \in \Omega_1 \mathrm{Z}(S)$. There exists $a \in S$ with $|a| = 2$ and $H_a \neq H_z$.*

Proof. Suppose first that $N_G(S) \not\leq H_z$ and pick $g \in N_G(S) \setminus H_z$. Then $z^g \in S$ and $H_{z^g} = H_z^g \neq H_z$.

Suppose next that $N_G(S) \leq H_z$. Since H_z is not strongly 2-embedded there exists $b \in H_z$ with $\beta| = 2$ and $C_G(b) \leq H_z$. Then $H_b \neq H_z$. Also a is conjugate to an element a of S and so Step 22 holds. □

Step 23. [**rank less than 2**] $m_2(S^\circ) \leq 1$.

Proof. Let $D = \overline{S^circ}$ and $M = \mathrm{N}_G(D)$. Let y be any involution in M. Put $A = \Omega_1(D)$. Since $S^\circ \leq C_G(A)$, $C_G(A)$ is infinite. Since $m_2(S^\circ) > 1$, A contains a fours group. Thus A is contained in a unique maximal subgroup of G. We claim that $H_y = M$. If y does not invert M°, then by Step 20, $M = H_y$. If y inverts M°, then $A \leq C_G(y) \leq H_y$ and again $H_y = M$. Thus $C_G(y) \leq H_y \leq M$.

Let $g \in G \setminus M$. If $M \cap M^g$ is infinite then **??** implies that $M = M^g$ and $D = D^g$ and $g \in N_G(D) = M$. Thus $M \cap M^g$ is finite and so by **??** M is a strongly 2-embedded on G, a contradiction to Step 16. □

Lemma 3.4.15. [**transitive on coset**] *Let H be a group, A and abelian subgroup of G with $A = A^2$ and $y \in \mathrm{N}_G(A)$. If y inverts A, then A acts transitively in Ay.*

Proof. Note that also y^{-1} inverts A. Let $a \in A$. Since $A = A^2$, $a^{-1} = b^2$ for some $b \in A$. Then $y^b = b^{-1}yb = b^{-1}yby^{-1}y = b^{-1}b^{y^{-1}}y = b^{-1}b^{-1}y = (b^2)^{-1}y = ay$. □

Step 24. [**step 18**] *Suppose $m_2(S^\circ) \geq 1$. Then G acts transitively on $\{x \in I \mid D_x$ is a not$2'-$ group$\}$.*

Proof. Put $\mathcal{I}^* = \{x \in I \mid D_x$ is a not$2' -$ group. Since $m_2(S^\circ) = 1$, S° has a unique involution x.

Note that $S^\circ = (D_x)_2$ and so x is the unique involution in D_x and D_x is not a $2'$-group. Thus $x \in \mathcal{I}^*$ and $x \in Z(H_x)$.

Suppose that G does not act transitively on cI^* and pick an involution y in G. which is not conjugate to x. Since G is simple $G = \langle x^G \rangle$ and so $x^g \notin H_y$. Thus $x \notin H_y^{g^{-1}}$ and replacing y by $y^{g^{-1}}$ we may assume that $x \notin H_y$.

Since x and y are not conjugate there exists a unique involution $u \in \langle xy \rangle$. Then $u \in \mathrm{C}_G(y) \leq H_y$. By **??**, Since $(D_y)_2 \leq S^h$ for some $h \in G$. Since $y \in \mathcal{I}^*$, $(D_y)_2$ is a nontrivial divisible group. hence $(D_y)^2 = S^{\circ h}$. Thus $D_y \cap D_x^h \neq 1$, $D_y = D_x^h$ and x^h is the unique involution in D_y. Thus by u and y centralizes x^h. Put $A = \langle y, x^h \rangle$. Since $y \notin x^G$, A is a fours group. Since $C_G(y)$ is infinite, also $C_{D_y}(y)$ is infinite and so $C_G(A)$ is infinite. Thus by Step 21, A lies in a unique maximal subgroup of G. Note that $A \leq H_y$ and $A \leq C_G(u) \leq H_u$. Thus $H_y = H_u$ and $x \leq C_H(u) \leq H_u = H_y$, a contradiction. □

Step 25. [**s is finite**] *S is finite.*

Proof. Suppose S is infinite, then by Step 23 $m_2(S^\circ) = 1$. Let $x \in S^\circ$ with $|x| = 2$.

Suppose that $C_S(S^\circ) \neq S^\circ$ and pick $S^\circ \leq T \leq C_S(S^\circ)$ with $|T/S^\circ| = 2$. Then T is abelian and so by **??**, $T = S^\circ \times K$ for some $L \leq T$. $y \in K$ with $|x| = |y| = 2$. Since $S^\circ \leq D_x \cap D_y$ we have $D_x = D_y$. Hence D_y is not a $2'$-group and by Step 24 $y = x^g$ for some $g \in G$. Thus $D_x = D_y = D_x^g$. Since $x \in S^\circ = (D_x)_p$ this gives $y = x^g \in (D_x^g)_p = (D_x)_p = S^\circ$, a contradiction.

Hence $C_S(S^\circ) = S^\circ$. Put $S_0 = \{z \in S^\circ \mid z^4 = 1\}$. By **??**, $C_S(S_0) = C_S(S^\circ) = S^\circ$. Since $|S_0| = 4$ we conclude that $|S/S^\circ| \leq 2$.

Suppose that x is the only involution in S. Let y be any involution in H_x. Note Then $y^h \in S$ for some $h \in H_x$ and so $y^h = x$. Thus $C_G(y) = C_G(x^{h^{-1}}) \leq H_x$. Let $g \in G$ with $|H_x \cap H_x^g| = \infty$. Then by **??**, $D_x = D_x^g$ and so $g \in N_G(D_x) = H_x$. 3.4.12 now shows that H_x is a strongly 2-embedded subgroup, a contradiction to **??** □

Theorem 3.4.16. [**brauer**] *Let H be a finite simple group, T a Sylow 2-subgroup of G and $x_0, x_1, x_2 \in T$ with $|x_1| = |x_2| = 2$. Then one of the following holds:*

(a) [**1**] *For $0 \leq i \leq 2$, there exists $y_i \in S \cap x_i^G$ with $y_1 y_2 = y_0$ and $C_T(y_0) \in Syl_2(C_G(y_0))$.*

(b) [**2**] *$|H| \leq \alpha(s_0, s_1, s_2)$, where $s_i = |C_H(x_i)/O(C_H(x_i))$ and $\alpha : \mathbb{Z}^3 \to \mathbb{Z}^+$ is a function independent of H.*

Let $1 \neq z \in \Omega_1 \mathrm{Z}(S)$.

Step 26. [**brauer step**] *For all $1 \neq x_0 \in S$ there exists $y_1, y_2 \in S \cap z^G$ and $y_0 \in S \cap y_0^G$ with $y_1 y_2 = y_0$ and $C_S(y_0) \in \mathrm{Syl}_2(C_G(y_0))$.*

Proof. Put $x_i = z$ for $i = 1, 2$ and for $0 \leq i \leq 2$ define $t_i = \mathrm{C}_G(x_i)/\mathrm{C}_G(x_i)^\circ|$. Put $m = \max\{\alpha(s_0, s_1, s_2) \mid 1 \leq s_i \leq t_i\}$. Pick $T \in \mathrm{Syl}_2(C_G(x_0)$ and let H be finite simple subgroup of G with $\langle T, S \rangle \leq H$ and $|H| > m$. Put $s_i = |\mathrm{C}_H(x_i)/O(\mathrm{C}_H(x_i)$. Since S is finite, $C_G(x_i)^\circ$ is a $2'$ group and so $\mathrm{C}_H(x_i) \cap \mathrm{C}_G(x_i)^\circ \leq O(C_H(x_i)$. Hence

$$s_i = |\mathrm{C}_H(x_i)/\Omega(\mathrm{C}_H(x_i)| \leq |\mathrm{C}_H(x_i)/\mathrm{C}_H(x_i) \cap C_G(x_i)^\circ| \leq \mathrm{C}_H(x_i)\mathrm{C}_G(x_i)^\circ)/C_G(x_i)^\circ| \leq t_i$$

and so $|H| > m > \alpha(s_0, s_1, s_2)$. Thus by 3.4.16 there exists $y_i \in S \cap x_i^H$ such that $y_1 y_2 = y_0$ and $C_S(y_0) \in \mathrm{Syl}_2(C_H(y_0)$. Since $T \leq C_H(x_0)$ we get $C_S(y_0)| \geq |T|$ and so $C_S(y_0) \in \mathrm{Syl}_2(C_G(y_0)$. □

Step 27. [**2 central fours group**] *There exists a fours group $E \leq S$ in G with $z \in E$ and $E^\sharp \in z^G$.*

Proof. By Step 26 applied with $x_0 = z$, there exists $y_i \in z^G \cap S$ with $y_1 y_2 = y_0$. Put $F = \langle y_1, y_2 \rangle$. Then $F^\sharp \subseteq z^G$. Moreover, $y_1^g = z$ for some $g \in G$ and so $z \in F^g \leq C_G(z)$. Since S is a Sylow 2-subgroup of $C_G(z)$ and so by Step 12 there exists $h \in C_G(z)$ with $E := F^{gh} \leq S$. Also $z = z^h \in E$. □

Lemma 3.4.17. [**centralizer of hyper planes**] *Let B be finite elementary abelian p group acting on a locally finite abelian p'-group D. Then $D = \langle C_D(X) \mid X \leq B, |H/X| = p \rangle$.*

Proof. See MTH913 Homework 1. □

Step 28. [**step CGA**] *Let $A \leq S$ be a fours group and suppose that A is contained in more than one maximal subgroup of G. Then $\Omega_1^2(C_G(A)) = A$ and there exists $d \in z^G \cap S$ with $z \notin \mathrm{C}_S(A)$. In particular, $A \not\leq \mathrm{Z}(S)$.*

Proof. Suppose there exists an involution $b \in C_G(A) \backslash A$. Put $B = \langle A, b \rangle$. Then $B \cong C_2^3$. Let M_1 and M_2 be two distinct maximal subgroups of G containing A. By Step 21, $M_i = H_{a_i}$ for some $a_i \in A$. Thus $B \leq \mathrm{C}_G(a_i) \leq M_i$. By **??** $M_i^\circ = \langle \mathrm{C}_{M_i^\circ}(X) \mid X \leq B, |B/X| = 2 \rangle$. Thus there exists $B_i \leq B$ with $|B/B_i| = 2$ and $C_{M^\circ}(B_i)$ infinite. The B_i is a foursgroup and by Step 21, B_i is contained in a unique maximal subgroup of G, a contradiction to $B_i \leq M_1 \cap M_1$.

Thus $\Omega_1^2(C_G(A)) = A$. Suppose S is elementary abelian. Then $S \leq \Omega_1(C_S(A)) = A$ and so $S \cong D_4$, a contradiction. So there exists $x_0 \in S$ with $|x_0| > 2$. By Step 26 there exists involutions $y_1, y_2 \in S \cap z^G$ and $y_0 \in S \cap x_0^g$ with $y_1 y_2 = y_0$. Suppose y_1 and y_2 are in $C_S(A)$. Then $y_0 \in \langle y_1, y_2 \rangle \leq \Omega_1(C_S(A)) = A$ and so $y_0^2 = 1$, a contradiction. Thus one of y_1 and y_2 is not in $C_S(A)$. □

Step 29. [**s in a unique maximal**] *H_z is the unique maximal subgroup of G containing S.*

Proof. Suppose $S \leq M$ with $M \neq H_z$. If $|\Omega_1 \mathrm{Z}(S)| \geq 4$, we can choose $A \leq \Omega_1 \mathrm{Z}(S)$ with $|A| = 4$, a contradiction to Step 28. Thus $\Omega_1 \mathrm{Z}(S) = \langle z \rangle$. By Step 20, z inverts M°. Thus $\Omega_1 \mathrm{Z}(S) \cap C_S(M^\circ) = 1$. Since $C_S(M^\circ)$ is normal in S this implies $C_S(M^\circ) = 1$. Let E be as in Step 27 and let $E \backslash \langle z \rangle = \{a, b\}$. If a inverts M° we get $b = az \in C_S(M^\circ)$, a contradiction. Thus a does not invert M° and by Step 21, $M = H_a$. By symmetry, $M = H_b$. Thus a and b invert D_z and so $ab = z$ centralizes D_z. Since $a \in z^G$, a centralizes $D_a = M^\circ$, again a contradiction. □

Let $e \in S$ be an involution in S with $H_e \neq H_z$. If $H_e \in H_z^G$, put $x = a$. If $H_e \notin H_z^G$, then choose $g, h \in G$ with $e = z^g z^h$ and put $x = e^{g^{-1}}$. In either case put $A = \langle x, z\rangle$, $y = zx$ and $\mathcal{A} = \{a \in A \mid H_a \in H_z^G\}$. Let $T \in \mathrm{Syl}_2(H_x \cap H_y)$.

Step 30. [**basic a**] *A is a foursgroup, $A = \langle x, z\rangle$, $H_x \neq H_z$ and $|\mathcal{A}| \geq 2$.*

Proof. If $H_e \in H_z^G$, then $a = e$, $a \in \mathcal{A}$, $H_a = H_e \neq H_z$, $a \in S \leq C_G(z)$ and $A = \langle a, z\rangle$ is a fours group.

If $H_e \notin H_z^G$, then $x = e^{g^{-1}} = (z^g z^h)^{g^{-1}} = z z^{hg^{-1}}$ and so $y = zx = z^{hg^{-1}} \in z^G$. Thus zx has order two and A is fours group. Also $H_y = H_z^{hg^{-1}} \in H_z^G$ and so $y \in \mathcal{A}$. Since $H_x = H_e^{g^{-1}} \notin H_z^G$, $H_x \neq H_z$. □

For $a \in A^\sharp$ pick $S_a \in \mathrm{Syl}_2(H_a)$ with $T \cap H_a \leq S_a$ and define $T_a = N_{S_a}(C_{S_a}(A))$.

Step 31. [**omega t**] *Let*

(a) [**a**] $\mathcal{A} = A^\sharp \subseteq z^G$.

(b) [**b**] $A = \Omega_1 \mathrm{Z}(T) = \Omega_1(T)$ *and* $C_{S_a}(A) = T$

(c) [**c**] $\Omega_1 \mathrm{Z}(S_a) = \Omega_1 \mathrm{Z}(T_a) = \langle a\rangle$

(d) [**d**] $T_a = N_{S_a}(T) = N_{S_a}(A)$ *and* $|T_a/T| = 2$.

(e) [**e**] $N_G(T)/N_G(T) \cap C_G(A) \cong \mathrm{Sym}(A^\sharp)$

Proof. Let $a \in \mathcal{A}$. By definition of $\mathcal{A}$, H_a is conjugate to H_z and so contains a Sylow 2-subgroup of G. Thus S_a is Sylow 2 subgroup of G. By **??**$S_a \neq C_{S_a}(A)$ and $A = \Omega_1(C_{S_a}(A))$. Thus also $T_a \neq (C_{S_a})(A)$ and $A \trianglelefteq T_a$. It follows that $1 < C_A(T_a) < A$ and so there exists a unique $1 \neq a^* \in C_A(T_a)$. Note that both $\Omega_1 \mathrm{Z}(S_a)$ and $\Omega_1 \mathrm{Z}(T_a)$ are contained in $\Omega_1(C_{S_a}(A))$ and so also in $C_A(T_a)$. Thus $\Omega_1 \mathrm{Z}(S_a) = \Omega_1 \mathrm{Z}(T_a) = \langle a^*\rangle$ Then $S_a \leq C_G(a^*)$ and so by **??** $H_{a^*} = H_a$. If $a \neq a^*$ we get $A^\sharp = \{a^*, a, a^t\}$, where $t \in T_a \setminus C_{S_a}(A)$. Since $t \in H_a$ this gives $H_a^t = H_a = H_{a^*}$ and Step 21 implies that H_a is the unique maximal subgroup of G containing A, a contradiction, since $A \leq H_x \cap H_y$. Thus $a = a^*$.

Since $|\mathcal{A}| \geq 2$, we can choose $b \in \mathcal{A}$ with $b \neq a$. Note that T_a acts as the two cycle with fix-point a on $A^\sharp$ and T_b as the 2 cycle with fix point b. Thus $\langle T_a, T_b\rangle$ acts as $\mathrm{Sym}(\mathcal{A}^\sharp)$ on $A^\sharp$. So all elements in $A^\sharp$ are conjugate in G and $\mathcal{A} = A^\sharp \subseteq z^G$.

Suppose now that $a \in \mathcal{A}$ with $T \leq H_a$. Note that $C_{S_a}(A) \leq H_x \cap H_z$ and $\langle T, C_{S_a}(A)\rangle \leq S_a$. Since T is a Sylow 2 subgroup of $H_x \cap H_z$ we conclude that $C_{S_a}(A) = C_T(A)$. Also $|N_{S_a}(A)/C_{S_a}(A)| \leq 2$ and so $N_{S_a}(A) = T_a C_{S_a}(A) = T_a$.

If $A \not\leq Z(T)$, then $N_T(C_T(A)) \neq C_T(A)$ and since $|T_a/C_{S_a}(A)| = 2$, $T_a = N_T(C_T(A)$. This hold for $a = z$ and x and so $T_x = T_z$ centralizes $\langle x, z\rangle = A$, a contradiction.

Thus $A \leq Z(T)$, $C_{S_a}(A) = C_T(A) = T$ and $T_a = N_{S_a}(T)$. Hence $\langle T_a, T_b\rangle \leq N_G(T)$ and $\Omega_1 \mathrm{Z}(T) \leq \Omega_1(T) \leq \Omega_1^2(C_G(A)) = A \leq \Omega_1 \mathrm{Z}(T)$. So $N_G(T)$ acts transitively on $A^\sharp$ and thus $T \leq H_a$ for all $a \in A^\sharp$. □

Definition 3.4.18. [**def:quasidihedral**] *Let n be positive integer. Then $QD_{8n} := \langle s, t \mid s^2 = 1, (ss^t)^{2n} = 1, t^2 = (ss^t)^n \rangle$. QD_{8n} is called the quasidihedral group of order $8n$.*

Lemma 3.4.19. [**char quasidihedral**] *Let P be a finite 2-group and A a fours group in P with $C_P(A) = A$. Then P is a dihedral or quasidihedral group.*

Proof. Observe that $Z(P) \leq C_P(A) \leq A$. If $A \leq Z(P)$, then $P \leq C_P(A) \leq S$ and we are done. So suppose $A \not\leq Z(P)$ and pick $1 \neq a \in A \setminus Z(P)$ and $1 \neq z \in Z(P)$. Then $C_P(a) = C_P(\langle a, z \rangle) = C_P(A) + A$. Let $D \leq P$ such that D is dihedral group maximal with respect to $A \leq D$. If $D = P$ we are done. So suppose $D \neq P$.

Let $Q = N_P(D)$. Then $D < Q$. Let $\mathcal{A} = \{t \in D \setminus Z(P) \mid t^2 = 1\}$. Put $|D| = 4n$. Then $|^A| = 2n$. Note that Q acts on $\mathcal{A}$ and so

$$2n = |cA| \geq |a^Q| = |Q/C_Q(a)| = |Q/A| = |Q/D||D/A| \geq 24n4 = 2n$$

It follows that $\mathcal{A} = a^Q$ and $|Q/D| = 2$. Let $b \in \mathcal{A}$ with $\langle a, b \rangle = D$. Then there exists $t \in Q$ with $a^t = b$. Put $x = ab$. Then either $|D| = 4$ and $x = z$ or $|D| > 4$ and $\langle x \rangle$ is the unique cylcic subgroup of order $2n$ in D. In either case $X \trianglelefteq Q$. So also $Y = \langle x^2 \rangle \trianglelefteq Q$. Consider $\overline{Q} = Q/Y$. Then $\bar{t}^2 \in C_{\overline{D}}(t) = \overline{X}$ and replacing t by at if necessary we may assume that $\bar{t}$ has order 2. Thus $t^2 \in Y$ and so $t^2 = x^l$ for some even integer with $0 \leq l < 2n$. Thus $b^t = a^{t^2} = x^{-l}ax^l = aa^{-1}x^{-l}ax^l = ax^lx^l = ax^{2l}$ and so $x^t = (ab)^t = bax^{2l} = x^{-1}x^{2l} = x^{2l-1}$. Since t centalizes $t^2 = x^l$ this means $x^l = (x^l)^t = x^{l(2l-1)}$ and so $x^{l(2l-2)} = 1$. Since x has order m we conclude $2n \mid l(2l-2) = 2l(l-1)$. Since m is power of 2 and l is even , we infer $2n \mid 2l$ and so $n \mid l$. As $0 \leq l < 2n$ we have $l = 0$ or $l = n$. If $t^2 = 1$ and in the second case $t^2 = x^n$. In either case $b^t = ax^{2n} = a$. Observer that $Q = D\langle t \rangle = \langle a, b, t \rangle = \langle a, t \rangle$. So if $t^2 = 1$ then Q is a dihedral group, a contradiction to the maximality of D. Hence $t^2 = x^n$ and Q is a quasi dihidral group or order $8n$. Sine $l = n$ and l is even, Q has order at least 16. group.

Put $E = \langle D^{\mathrm{N}_P(Q)} \rangle$. Then $D \leq E \leq Q$ and E is generated by involutions. By Homework 1, Q is not generated by involutions. Since $|Q/D| \leq 2$ this gives $E = D$ and so $D \trianglelefteq N_P(Q)$, $N_P(Q) = Q$ and $Q = P$. □

Theorem 3.4.20. [**semidihedral**] *If H is a finite simple group with quasidihedral Sylow 2-subgroup of order at least 16, then $H \cong M_{11}$, $L_3(p^k)$ or $U_3(p^k)$, where p is an odd prime.*

Proof. □

Lemma 3.4.21. [**basic semidihedral**] *Let $H \cong L_3(q$ or $U_3(q)$, q a power of an odd prime. and $t \in H$ with $|t| = 2$. $C_H(t)$ has a normal subgroup isomorphic to $SL_2(q)$. Moreover, $|H| \leq q^{18}$.*

Proof. Put $\mathbb{K} = \mathbb{F}_q$ and define $GL_n^+(\mathbb{K}) = GL_n(\mathbb{K})$ and $GL_n^-(\mathbb{K}) = GU_n(\mathbb{K})$. Put $\tilde{H} = GL^\epsilon(\mathbb{K})$ and $V = \mathbb{F}^3$, where $\mathbb{F} = \mathbb{K}$ in the $L_3(q)$ case and $\mathbb{F} = \mathbb{K}_{q^2}$ in the $U_3(\mathbb{K})$. Then $\tilde{H}/Z(\tilde{H})$. Note that $|H| \leq |GL_3(q^2)| = (q^6 - 1)(q^6 - q^2)(q^6 - q^4) \leq q^{18}$. Since $\mathrm{Z}(SL_3^\epsilon(\mathbb{K}))$ has order dividing 3, there exists a unique element of order two $\tilde{t}$ in $\mathrm{Z}(SL_3^\epsilon(\mathbb{K})$ which maps which maps to t. Since $|\tilde{t}| = 2$ and $\det \tilde{t} = 1$ and $\operatorname{char} \mathbb{K} \neq 2$ we have $V = [V, \tilde{t}] \oplus C_V(\tilde{t})$ with $\dim[V, \tilde{t}] = 2$ and $\dim C_V(\tilde{2}) = 1$. 2-dimensional. In the $GU_3(\mathbb{K})$ case, $[V, \tilde{t}] \perp C_V(t)$ and so this direct sum is an orthogonal sum. It follows that $C_{\tilde{H}}(\tilde{t}) = GL^\epsilon([V, \tilde{t})] \times GL^\epsilon(C_V(\tilde{t}) \cong GL_2^\epsilon(\mathbb{K}) \times GL_1^\epsilon(\mathbb{K})$. It follows that $C_{\tilde{H}}(\tilde{t})$ has a normal subgroup K isomorphic to $SL_2^\epsilon(\mathbb{K})$. K centralizes $C_V(\tilde{t})$, and since the elements of $Z(\tilde{H})$ acts by scalar multiplication on V, and $K \cap Z(\tilde{H})$. Thus $K \cong KZ(\tilde{H})/Z(\tilde{H})$ and so $C_H(t)$ has a subgroup isomorphic to $\mathrm{SL}_2^\epsilon(\mathbb{K})$. Since $SU_2(\mathbb{K}) \cong SL_2(\mathbb{K})$, the lemma is proved. □

Step 32. [**step semidihedral**] *S is not a quasidihedral group.*

Proof. Suppose S is a quasidihedral group. By **??** S is not a dihedral group and so $|S| \geq 16$. Pick a finite simple subgroup H of G with $|H| > (|C_G(z)/D_z|)^{18}$. and $S \leq H$. Since $|M_{11}| = 11 \cdot 10 \cdot 9 \cdot 8 \leq 2^1 8 < |H|$, we conclude from 3.4.20 that $H \cong L_3^\epsilon(q)$, q a power of an odd prime and $q > |C_G(z)/D_z|$. Let $K \leq C_H(z)$ with $K \cong SL_2(q)$. Then $\mathrm{Z}(K)$ has order two, and $\mathrm{Z}(K)$ is the unique minimal normal subgroup of K. Since D_z is $2'$-group, $Z(K) \nleq D_z$ and so $K \cap D_z = 1$. Hence $|KD_z/D_z| \geq |K| > q > |C_G(z)/D_z|$, a contradiction. □

.

Step 33. [**t not a**] *$T \neq A$.*

Proof. Otherwise $C_{S_a}(A) = T = A$ and by **??**, S_a is a dihedral or quasidihedral group, a contradiction to **??** and **??** □

Step 34. [**z centralizes hz**] *Let $a, b \in A^\sharp$ with $a \neq b$.*

(a) [**a**] *$H_a \neq H_b$.*

(b) [**b**] *z centralizes D_z.*

(c) [**c**] *Let $C_G^*(D_z)$ be the set of elements in G which centralize or inverts D_z. Then $t \in C_G^*(D_z)$ and $[H_z, t] \leq C_G(D_z)$ for all $t \in z^G \cap H_z$*

(d) [**d**] *$C_G(D_a) \cap C_G(D_b) = 1$.*

Proof. (a) By Step 31 there exists $g \in N_G(T)$ with $x^g = a$ and $z^g = b$. Since $H_x \neq H_z$, $H_a \neq H_b$.

(b) From (a) and Step 20 both x and xz invert D_z and so $z = x(xz)$ centralizes D_z.

(c) If $H_z = H_t$ then by (b), t centralizes $D_t = D_z$. And if $H_t \neq H_z$, then by Step 20 t inverts D_z. So $t \in C^*_{H_z}(D_z)$.

Since $C_G^*(D_z)$ is a normal subgroup of H_z and $C_G^*(D_z)/C_G(D_z)| \leq 2$ we have $[C_G^*(D_z), G] \leq C_G(D_z)$. and so (c) holds.

(d) Suppose that $X := C_G(D_a) \cap C_G(D_b) \neq 1$. Then $\langle D_a, D_b \rangle \leq C_G(X)$ and so $D_a = X^\circ = D_b$. Hence also $H_a = N_G(D_a) = H_b$, contradiction.

□

Step 35. [**ngt**] *For each $a \in A^\sharp$ there exist $t_a \in z^\cap T_a \setminus T$ such that if $S_a \neq T_a$, then $[T, t_a] \leq \langle a \rangle$. For any such $t_a' s$ and any $a, b \in A^\sharp$ with $a \neq b$:*

(a) [**b**] *Put $k := t_a t_b$. Then $a^k = c$, $c^k = b$, $b^k = a$, $k^3 = 1$ and $C_T(k)$=1.*

(b) [**c**] $T = [T, t_a][T, t_b]$.

Proof. We first show that existence of t_a. Suppose first that $S_a \neq T_a$. Pick $s_a \in N_{S_a}(T_a) \setminus T_a$.If $A^{s_a} \leq T$, then $A^{s_a} \leq \Omega_a(T) = A$. Thus $A = A^{s_a}$ and $s_a \in N_{S_a}(A)$. So by Step 31 $s_a \in T_a$, a contradiction. Thus $A^{s_a} \neq T$ and $\langle a \rangle \leq T \cap A^{s_a}$. Since $A \trianglelefteq T_a$ also $A^{s_a} \trianglelefteq T_a$ and so $[T, A^{t_a}] \leq T \cap A^{t_a} = \langle a \rangle$.

If $S_a = T_a$ the existence of t_a follows from Step 28.

Since t_a acts as the cycle (b, c) and t_b as the cycle (a, c) in $A^\sharp$, k acts as $(b, c)(a, c) = (a, c, b)$ on $A^\sharp$. Thus $k^3 \in C_G(A) \leq H_a$. By (**??**) Step 34(c), $k^6 = [k^3, t_a] \in C_G(D_a)$. By symmetry, $k^6 in C_G(D_b)$ and so by Step 34(d), $k^6 = 1$. Thus $k^3 \in \Omega_1^2(C_G(A)) = A$. Since $C_A(k) = 1$ this implies $k^3 = 1$. Since $\Omega_1(T) = A$ and $C_A(k) = 1$, $C_T(k)$ contains no element of order 2 and so $C_T(k) = 1$

(b) By Homework 1, since $|k|$ is coprime to $|T|$, $T = C_T(k)[T, k] =][T, k]$. Thus
$T = [T, k] \leq [T, \langle t_a, t_b \rangle] = [T, t_a][T, t_b] \leq T$ and (b) holds. □

Step 36. [**t normal in s**] *$T \trianglelefteq S_a$ for all $1 \neq a \in A$.*

Proof. By Step 35, $T = [T, t_a][T, t_b] \leq A$ and so $T = A$, a contradiction to Step 33 □

Step 37. [**step c**] *For $a \in A^\sharp$ define $C_a = C_T(D_a)$ and Then $C_a = [T, t_a]$, $T = C_a \times C_b$ and T is abelian.*

Proof. By Step 34(**??**) $[T, t_a] \leq C_G(D_a)$ and since t_a normalizes C_a, $[T, t_a] \leq C_a$. Thus by Step 35(**??**), $T = C_a C_b$. By Step 34(d), $C_a \cap C_b = 1$. Since both C_a and C_b are normal in T this implies $[C_a, C_b] = 1$ and $T = C_a \times C_b$. Moreover, C_c is centralized by C_a and C_b and so $C_c \leq Z(T)$. The same holds for C_a and C_b and so $T = C_a \times C_b$ is abelian. □

Step 38. [**sz**] *$Z(S)$ has order two.*

Proof. Let $x_0 \in \mathrm{Z}(S)$. Then $S \leq C_G(x_0)$. By Step 26, there exists $y_1, y_2 \in z^G \cap S$ and $y_0 \in x_0^G$ with $x_0 = y_1 y_2$ and $C_S(y_0) \in \mathrm{Syl}_2(C_G(y_0)$. Since $C_G(x_0)$ and so also $C_G(y_0)$ contains a Sylow 2-subgroup of G, we conclude that $C_S(y_0) = S$. Thus $[y_0, y_1] = 1$. Since $y_0 = y_1 y_2$, y_1 inverts y_0 and so y_0 has order two. Hence $x_0 \in \Omega_1 \mathrm{Z}(S) = \langle z \rangle$. □

Step 39. [**step contradiction**] *The final contradiction.*

Proof. Let $d \in C_b$. Then dd^{t_a} is centralizes by $C\langle t_a \rangle = T\langle t_a >= S_a$ and so $dd^t \in \mathrm{Z}(S)$. Thus dd^t has order at most two. Since $C = C_b \times C_b^{t_a}$, $|d| = |d^t|$. Thus $d^2 = 1$. So $d \in C_b$ and $C_b \leq A$. By symmetry, $C_a \leq A$ and so $T = C_a \times C_b = A$, a contradiction to Step 33 □

3.5 J_1

In this section we prove:

Theorem 3.5.1 (Janko)**.** [**j1**] *Let G be a finite group of even order and $t \in G$ with $|t| = 2$. Suppose that all involutions in G are conjugate and $C_G(t) \cong C_2 \times \mathrm{Alt}(5)$. Then $|G| = 2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19 = 11(11+1)(11^3 - 1) = 175,560$. Moreover such a group exits and is unique up to isomorphism.*

Before we start the proof we will prove need to prove a few lemmas from finite group theory.

Lemma 3.5.2. [**even more coprime action**] *Let A be a finite abelian p-group acting on an finite p' group Q.*

(a) [**a**] $Q = \langle C_Q(B) \mid B \leq A, A/B\,\textit{cyclic}\rangle$.

(b) [**b**] *If $A \cong C_p \times C_p$, then*

$$|Q| = \frac{\prod\{|C_Q(B)| \mid B \leq A, |B| = p\}}{|C_Q(A)|^p}$$

Proof. Let $H = QA$ be the semidirect product of A and Q. Let q be a prime dividing the order of Q and $S \in \mathrm{Syl}_q(Q)$. Then by the Frattini argument, $H = QN_H(S)$. Then $|A|$ divides $N_H(S)$ and so $N_H(S)$ contains a Sylow p-subgroup, $\tilde{A}$ of H. Choose $h \in H$ with $\tilde{A}^h = A$. Then A normalizes S^h. So if (a) and (b) holds whenever Q is a q-group for some prime $q \neq p$, then it also for any arbitray p' group. Thus we may and do assume that Q is a q-group.

(a) Put $\overline{Q} = Q/Q'$. Then $\overline{Q}$ is abelian and so by , Since $\overline{Q}$ is a p'-group, $\overline{Q}^{p^m} = \overline{Q}$ for all $m \in \mathbb{Z}^+$. Hence by Homework 1

$$\overline{Q} = \langle C_{\overline{Q}}(B) \mid B \leq A, A/B\text{cyclic}\rangle$$

By 3.3.8, $C_{\overline{B}} = \overline{C_Q(B)}$ and thus

$$Q = \langle C_Q(B)| \mid B \leq A, A/B\text{cyclic}\rangle Q'$$

By the induction on —Q—,

$$Q' = \langle C_{Q'}(B)| \mid B \leq A, A/B\text{cyclic}\}$$

and so (a) holds.

(b) Let M a maximal A invariant normal subgroup of Q and define $\overline{Q} = Q/M$ and $\mathcal{B} = \{B \leq A \mid A/B\text{is cyclic}C_{\overline{Q}}(B) \neq 1$.

By (a) $\overline{Q} = \langle C_{\overline{Q}}(B) \mid B \in \mathcal{B}\rangle$ and so $|\mathcal{B}| \geq 1$. Since $\overline{Q}'$ is a proper A invariant normal subgroup of $\overline{Q}$, the maximality of M implies that $\overline{Q}' = 1$ and so $\overline{Q}$ is abelian. Let $B \in \mathcal{B}$, then $C_{\overline{Q}}(B)$ is a non-trivial A-invariant normal subgroup of $\overline{Q}$. Thus $C_{\overline{Q}}(B) = C_{\overline{Q}}(B)$. We claim that (b) holds for $\overline{Q}$ in place of Q. Suppose first that $|\mathcal{B}| = 1$. Then $|C_{\overline{Q}}(B)| = |\overline{Q}|$ while $|C_{\overline{Q}}(C)| = 1$ for each of subgroup C of A with $|C| = p$ and $C \neq B$. In particular, $|C_{\overline{Q}}(A)| = 1$ and so

$$\frac{\prod\{|C_{\overline{Q}}(D)| \mid D \leq A, |D| = p\}}{|C_{\overline{Q}}(A)|^p}\} = \frac{|\overline{Q}|1^p}{1^p} = |\overline{Q}|$$

and the claim holds in this case.

Suppose next that $|\mathcal{B}| \geq 2$ and let $B_1, B_2 \in \mathcal{B}$ with $B_1 \neq B_2$. Then $A = B_1B_2$ and since B_1 and B_2 centralize $\overline{Q}$, A centralizes $\overline{Q}$. Thus $|C_{\overline{Q}}(B)| = |\overline{Q}|$ for each of the $p+1$ subgroups of order p in A. Also $C_{\overline{Q}}(A)| = |\overline{Q}|$ and thus

$$\frac{\prod\{|C_{\overline{Q}}(D)| \mid D \leq A, |D| = p\}}{|C_{\overline{Q}}(A)|^p} = \frac{|\overline{Q}|^{p+1|}}{|\overline{Q}|^p} = |\overline{Q}|$$

and again the claim holds.

By induction on $|Q|$ we also have

$$\frac{\prod\{|C_M(D)| \mid D \leq A, |D| = p\}}{|C_M(A)|^p}\}$$

Since $|Q| = |M||\overline{M}|$ and $|C_Q(X)| = |C_M(X)||C_{\overline{Q}}(X)$ for any $X \leq A$ we conclude that (b) holds. □

Definition 3.5.3. [**def:weakly closed**]

(a) [**a**] *Let G be a group, and $A \leq H \leq G$. Then A is called weakly closed in H with respect to G if $A^g = A$ for all $g \in G$ with $A^g \leq H$. (That is if A is the only conjugate of A in G contained in H.*

(b) [**b**] *Let p a prime, and A a p subgroup of finite group G. Then A is called a weakly closed subgroup of G if there exists a Sylow p-subgroup S of G with $A \leq S$ such that A is weakly closed in S with respect to G.*

Lemma 3.5.4. [**char weakly closed**] *Let p be a prime, G a finite group and A a p-subgroup of G. Then the following are equivalent.*

(a) [**a**] *A is a weakly closed subgroup of G.*

(b) [**b**] *Each Sylow p subgroup of G contains exactly one conjugate of A in G*

(c) [**c**] *Each p-subgroup of G contains at most one conjugate of A in G*

Proof. Suppose (a) holds. Then there exists some Sylow p subgroup S of G such that $A \leq S$ and A is weakly closed in S with respect to G. So S contains a unique G-conjugate of A (namely A). Since any two Sylow subgroups are conjugate in G we see that (a) holds.

Suppose (b) holds and let T be a p subgroup of G. Then $T \leq S$ for some Sylow p-subgroup of G. By (a) , S contains a unique conjugate of A in G and so T contains at most one conjugate of A in G. Thus (c) holds.

Suppose (c) holds and let S be a Sylow p-subgroup of G with $A \leq S$. Then by (c), A is weakly closed in S with respect to G and so (c) holds. □

Lemma 3.5.5. [**weakly closed and conjugate**] *Let A be a weakly closed p-subgroup of a finite group G and $A \leq H \leq G$. If $g \in G$ with $A^g \leq H$. Then $A^g = A^h$ for some $h \in H$.*

Proof. Let $A \leq S \in \mathrm{Syl}_p(H)$ and $A^g \leq T \in \mathrm{Syl}_p(H)$. By Sylow's Theorem, $S^h = T$ for some $h \in H$ and so both A^h and A^g are G-conjugates of A in T. Thus by 3.5.4, $A^h = A^g$. □

Lemma 3.5.6. [**control fusion**] *Let A be a weakly closed p-subgroup of a finite group G and X and Y A-invariant subsets of A. If $X^g = Y$ for some $g \in G$, then $X^h = Y$ for some $h \in N_G(A)$.*

Proof. Observe $A \leq N_G(X)$ and $A \leq N_G(Y)$. Hence also $A^g \leq N_G(X^g) = N_G(Y)$. So be 3.5.5, $A^{gl} = A$ for some $l \in N_G(Y)$. Hence $gl \in N_G(A)$ and $X^{gl} = Y^l = Y$. □

Corollary 3.5.7. [**fusion for abelian**] *Let G be a finite group and $S \in \mathrm{Syl}_2(G)$. Suppose S is abelian and $x^g \in S$ for some $g \in G$ and $x \in S$. Then $x^g = x^h$ for some $h \in N_G(S)$.*

Proof. Just observe that S is weakly closed an, since S is abelian, $\{x\}$ and $\{x^g\}$ are S invariant subsets of S. So we can apply 3.5.6 □

Lemma 3.5.8. [**tompson transfer**] *Let G be a finite group, $S \in \mathrm{Syl}_2(G)$, $T \leq S$ with $|S/T| = 2$ and $x \in S$. Then one of the following holds:*

1. [**a**] *$x^g \in T$ for some $g \in G$.*
2. [**b**] *$y^g \in S \setminus T$ for some $y \in \langle x^2 >$ and some $g \in G$.*
3. [**c**] *G has a subgroup H with $|G/H| = 2$ and $x \notin H$.*

Proof. We assume without loss that neither (1) nor (2) holds. Consider the action of G on G/T by right multiplication. We will show that x induces an odd permutation on G/T. Then (3) hold with H consisting of all the elements in G which induces an even permutation on G/T.

Define $\Phi : G/T \to G/S, Tg \to Sg$. Since $Sg = STg$, this is well defined. Observe that for all $g, h \in G$,

$$\Phi((Tg)h) = \Phi(T(gh)) = S(gh) = (Sg)h = \Phi(Tg)h$$

and so Φ is G equivariant.

Put $X = \langle x \rangle$. Let A be an orbit for X on G/S of size m and put $m = \Phi^{-1}(A)$. Since Φ is G-equivariant, B is X-invariant. Since $|S/T| = 2$, $|\Phi^{-1}(\alpha)| = 2$ for all $\alpha \in G/S$ and so $|B| = 2m$. Pick $\beta = Tg \in B$ and put $\alpha = \Phi(\beta) = Sg$. Observe that $C_X(\alpha) = X \cap S^g$ and $C_X(\beta) = X \cap T^g$. We will show

1°. [**1**] *One of the following holds:*

I [**I**] $X^{g^{-1}} \cap S = X^{g^{-1}} \cap T$ *and* X *has two orbits of length* m *on* B*.*

II [**II**] $X^{g^{-1}} \cap S \neq X^{g^{-1}} \cap T$ *and* X *has an orbits of length* $2m$ *on* B*.*

Suppose first that $X^{g^{-1}} \cap S = X^{g^{-1}} \cap T$. Then also $X \cap S^g = X \cap T^g$, $C_X(\alpha) = C_X(\beta)$ and

$$|\beta X| = |X/C_X(\beta)| = |X/C_X(\alpha)| = |\alpha X| = |A| = m$$

Suppose next that $X^{g^{-1}} \cap S \neq X^{g^{-1}} \cap T$. Then also $X \cap S^g \neq X \cap T^g$, $|S^g/\cap X/T^g \cap X| = 2$ and $|C_X(\alpha)/C_X(\beta)| = 2$. Thus

$$|\beta X| = |X/C_X(\beta)| = 2|X/C_X(\alpha)| = 2|\alpha X| = 2|A| = 2m$$

So (1) holds.

This allows us the determine the orbits of X on G/T in terms of the orbits X on G/T:

Suppose that $|A| > 1$. Then $X \neq X \cap S^{g^{-1}}$ and so $X^{g^{-1}} \cap S \neq X$ and $X^{g^{-1}} \cap S \leq \langle x^2 \rangle$. Since by assumption (2) fails, we conclude that $X^{g^{-1}} \cap S \leq X^{g^{-1}} \cap T$. Hence by (1°), X has two orbits of length m on B. Thus x is an even permutation on B. Since this holds for all non-trivial orbits for X on G/S, x is an even permutation on $\Phi^{-1}(\mathrm{Supp}_{G/S}(X))$.

Suppose next that $|A| = 1$. Then $X \leq S^g$ and so $x^{g^{-1}} \in S$. Since (1) fails, we get $x^{g^{-1}} \notin T$ and so $X^{g^{-1}} \cap S = X^{g^{-1}} \neq X^{g^{-1}} \cap T$. Thus by (1°), X has an orbits of length 2 on B. Since this holds for each trivial orbit on A in G/S, X has $|\mathrm{Fix}_{G/S}(X)$ orbits of length 2 on $\Phi^{-1}(\mathrm{Fix}_{G/S}(X)$. Observe that $|G/S|$ is odd, while $|\mathrm{Supp}_{G/S}(X)|$ is even. Hence $|\mathrm{Fix}_{G/S}(X)|$ is odd and so X has an odd number of orbits of length two on $\Phi^{-1}(\mathrm{Fix}_{G/S}(X)$. It follows that X is an odd permutation on $\Phi^{-1}(\mathrm{Fix}_{G/S}(X)$ and so also on G/S. □

Lemma 3.5.9. [**burnside**] *Let* G *be finite group and* $S \in \mathrm{Syl}_2(G)$*. Suppose that* $S \leq Z(N_G(S))$*. Then* $G = O(G)S$*.*

Proof. Since $S \leq N_G(S)$ we have $S \leq Z(S)$ and so S is abelian.

We will first show:

1°. [**1**] *If* $a \in S$ *and* $g \in G$ *with* $a^g \in S$*, then* $a^g = a$*.*

By **??**, $a^g = a^h$ for some $h \in N_G(S)$. Since $S \leq Z(N_G(S))$ this gives $a^g = a$. So (1°) is proved.

If $S = 1$, then $G = O(G)$ and the lemma holds. So suppose $S \neq 1$ and pick $T \leq S$ with $|S/T| = 2$ and $x \in S \setminus T$.

Let $g \in G$ with $x^g \in S$. Then by (1°), $x^g = x \notin T$ and **??**thompson transfer]a does not hold.

Let $y \in \langle x^2 \rangle$ and $g \in G$ with $y^g \in S$. Then by (1°), $y^g = y$. Since $|S/T| = 2$, $x^2 \in T$ and so $y^g = y \in T$. So also **??**thompson transfer]b does not hold

Thus **??**thompson transfer]c must hold and there exist a subgroup H of G with $|G/H| = 2$. Then $G = HS$, $H \trianglelefteq G$ and $H \cap S$ is a Sylow 2-subgroup of H. We claim that $H \cap S \leq Z(N_G(H \cap S))$. For this let $a \in H \cap S$ and $g \in N_G(H \cap S)$. Then $a^g \in H \cap S \leq S$ and so by (1°), $a^g = a$. Thus indeed $H \cap S \leq Z(N_G(H \cap S)$. By induction on $|G|$ we conclude that $H = O(H)(H \cap S)$. Since $H \trianglelefteq G$, $O(H) \leq O(G)$ and so $G = HS = O(H)(H \cap S)S = O(G)S$. □

We now start the proof of Janko's Theorem. So let G be a finite group of even order with a unique conjugacy class of involutions and $z \in G$ with $z^2 = 1$ and $C_G(z) \cong C_2 \times \mathrm{Alt}(5)$. Let $S \in \mathrm{Syl}_2(C_G(z))$. For $t \in G$ with $t^2 = 1$, define $G_t = C_G(t)$ and $K_t = G_t' \cong \mathrm{Alt}(5)$. So $K_t \cong \mathrm{Alt}(5)$ and $G_t = \langle t \rangle \times K_t$.

Step 1. [**j1-1**]

(a) [**a**] $S \cong C_2 \times C_2 \times C_2$.

(b) [**b**] $S \in \mathrm{Syl}_2(G)$.

(c) [**c**] $C_G(B) = S$ *for all* $B \leq S$ *with* $|B| \geq 4$.

(d) [**d**] $|N_G(S)| = 2^3 \cdot 3 \cdot 7$.

Proof. (a) Just observe that $\langle (12)(34), (14)(23) \rangle$ is a Sylow 2 subgroup of $\mathrm{Alt}(5)$ and is isomorphic to $C_2 \times C_2$.

(b) Let $T \in \mathrm{Syl}_2(G)$ with $S \leq T$ and pick $1 \neq t \in \Omega_1 \mathrm{Z}(T)$. Then $T \leq C_G(t)$ and $C_G(t) \cong C_2 \times \mathrm{Alt}(5)$. Thus $|T| \leq 8$ and $S = T$.

(c) Without loss $|B| = 4$. Pick $1 \neq b \in B$. Then $C_G(B) = C_{G_b}(B)$. Since $G_b = \langle b \rangle \times K_b$ we have $B = \langle b \rangle \times (B \cap K_t)$ and $C_{G_b}(B) = \langle b \rangle \times C_{K_b}(B \cap K_b)$. $\mathrm{Alt}(5)$ has a unique class of involutions and $C_{\mathrm{Alt}(5)}((12)(34)) = \langle (12)(34), (13)(24) \rangle$ has order 4. This $C_G(B)$ has order eight and $C_G(B) = S$.

(d) Let $s \in S^\sharp$. Then $|s| = 2$ and so there exists $g \in G$ with $z^g = s$. By **??**, $z^h = s$ for some $h \in N_G(S)$. Thus $N_G(S)$ acts transitively on $S^\sharp$ and so $|N_G(S)/N_G(S) \cap G_z| = |S^\sharp = 7$. Also $N_G(S) \cap G_z = \langle z \rangle \times N_{K_z}(S \cap K_z)$. Since $N_{\mathrm{Alt}(5)}(\langle (12)(34), (13)(24) \rangle = \mathrm{Alt}(4)$ we conclude that $N_G(S) \cap G_z \cong C_2 \times \mathrm{Alt}(4)$ has order $2^3 \cdot 7$. Thus $N_G(S)$ has order $2^3 \cdot 3 \cdot 7$. □

For $x \in G$ let $G_t = N_G(\langle x \rangle)$ and $0_t = O(G_t)$. In order to count the involutions in G we need to compute G_d where d is an element of order 3 in G_z. For this we have to investigate subgroup L of G such that $O(L) \neq 1$ and $4 \big| |L|$. Let L be such a group, $Y = O(L)$, $A \in \mathrm{Syl}_2(L)$ and for $a \in A^\sharp$ put $Y_a = C_Y(a)$.

Step 2. [**j1-2**]

(a) [**a**] *For* $a \in A\sharp$, Y_a *has order* $1, 3$ *or* 5.

(b) [**b**] $|A| = 4$.

(c) [**c**] $|Y| = \prod_{a \in A^\sharp} |Y_a| = 3^x 5^y$ *for some* $x, y \in \mathbb{N}$ *with* $x + y \leq 3$.

Proof. (a) Observe that Y_a is a subgroup of odd order in G_a. Thus $Y_a \leq K_a \cong \mathrm{Alt}(5))$. By Lagrange's Y_a has order $1, 3, 5, 15$. Since Alt(5) is simple it has no subgroup of index 4 and so $|Y_a| \neq 15$.

(b) Suppose that $|A| = 8$ and let $B \leq A$ such that $|A/B|$ is cyclic. Then B has order at least 4 and so by Step 1, $C_G(B)$ has order eight. Thus $C_Y(B) = 1$. Hence

$$Y = \langle C_Y(B) \mid B \leq A, A/B \text{ is cylic } \rangle = 1$$

a contradiction.

(c) By 3.5.2

$$|Y| = \prod(|C_Y(B)| \mid B \leq A, |B| = 2) = \prod_{a \in A^\sharp} |Y_a|$$

Together with (a) this gives (c). □

Step 3. [**j1-3**] *One of the following holds:*

1. [**a**] *$L = YA$ and $N_L(A) = A$.*
2. [**b**] *Y is elementary abelian of order p^3 for some $p \in \{3, 5\}$, Y is a minimal normal subgroup of L and $N_L(A) \cong \mathrm{Alt}(4)$.*

Proof. Since $|C_G(A)| = 8$ and A is a Sylow 2 subgroup of L, $C_L(A) = A$. Moreover $N_L(A)/C_L(A)$ is isomorphic to subgroup of odd order of $\mathrm{Aut}(A) \cong \mathrm{Sym}(3)$ and so $N_L(A) = C_L(A) = A$ or $N_L(A)/A \cong C_3/$

Suppose first that $N_L(A) = A$. Then $A \leq Z(N_L(A))$ and by 3.5.9, $L = O(L)A = YA$. So (1) holds.

Suppose next that $N_L(A)/A \cong C_3$. Then $N_L(A) \cong \mathrm{Alt}(4)$ and $N_L(A)$ acts transitively on $A^\sharp$. Let $1 \neq a \in A$ and put $p = |Y_a|$. Then $p \in \{1, 3, 5\}$ and $|Y_b| = p$ for all $b \in A^\sharp$. Hence $|Y| = p^3$ and $p \in \{3, 5\}$. So Y is a p-group. Let D be a minimal normal subgroup of L contained in Y. Since $D = \langle C_D(a) \mid a \in \mathbb{A}^\sharp \rangle$ we get $C_D(a) \neq 1$ for some $a \in A^\sharp$. Since $|Y_a| = p$ this gives $Y_a \leq D$ and since $N_L(A)$ acts transitively on $A^\sharp$, $Y_a \leq D$ for all $a \in A^\sharp$. Thus $|D| = p^3$ and $Y = D$. In particular, $Y = \Omega_1 \mathrm{Z}(Y)$ and so Y is elementary abelian. □

Step 4. [**j1-4**] *Let D be a non-trivial A-invariant subgroup of G of odd order.*

(a) [**a**] *If $D \leq L$, the $D \leq Y$.*

(b) [**b**] *If D is not elementary abelian or 3^3 or 5^3, then $N_G(D) = O(N_G(D))A$ and every subgroup of odd order normalizing D is contained in $O(N_G(D))$.*

Proof. (a) If $L = YA$, this is obvious. So suppose $L \neq YA$. Then $|Y| = p^3$. Since $DY \leq O(DYA)$ we conclude from Step 2 that applied to DYA in place of L, that $Y = O(DYA)$ and so $D \leq Y$.

(b) Put $\tilde{L} = N_G(D)$. Then D is a non-trivial normal subgroup of $\tilde{L}$ contained in $O(\tilde{L})$. Thus Step 3 applied to $\tilde{L}$ shows that $\tilde{L} = O(\tilde{L})A$ and so (b) holds. □

Step 5. [**j1-4.3**] *Let $D \leq Y$ with $|D| = p^2$, $p \in \{3, 5\}$. Then $D \trianglelefteq Y$ and if $|Y| \neq p^3$, then $D \trianglelefteq L$.*

Proof. If $D = Y$, this is obvious. So suppose $D \neq Y$. If $|D| = p^3$, then $D < N_Y(D) \leq Y$ and so $D \trianglelefteq Y$. If $|Y| \neq p^3$ the by Step 2, $|D| = p^2 q$ where $q \in \{3, 5\}$ with $p \neq q$. Thus D is a Sylow p-subgroup of Y and the number of Sylow p-subgroup of Y divides q and is equal to 1 (mod p). Since $3 \not\equiv 1$ (mod 5) and $5 \not\equiv 1$ (mod 3) we conclude that D is the unique Sylow p subgroup of Y. Thus $D \trianglelefteq L$. □

Step 6. [**j1-4.6**] *Let $p \in \{2, 3\}$ and for $i = 1, 2$ let $D_i \leq G$ with $|D_i|$ and $|C_G(D_i)|$ even. Let $t_i \in C_G(D_i)$ with $|t_i| = 1$. Then there exists $g \in G$ with $t_1^g = t_2$ and $D_1^g = D_2$. In particular, D_1 and D_2 are conjugate in G.*

Proof. Since all involutions in G are conjugate, there exists $h \in G$ with $t_1^h = t_2$. Then both D_2 and D_1^h are contained in $C_G(t_2)$. Since $C_G(t_2) \cong C_2 \times \mathrm{Alt}(5)$, the Sylow p subgroups of G have order p. Thus D_2 and D_1^h are Sylow p-subgroups of $C_G(t_2)$ and so there exists $l \in C_G(t_2)$ with $D_1^{hl} = D_2$. Also $t_1^{hl} = t_2^l = t_2$ and so the lemma holds with $g = hl$. □

Step 7. [**j1-5**] *Suppose $|Y|$ does not divide 15 and put $Y^* = C_G(Y)$ and $L^* = N_G(L^*)$. Then $L \leq L^*$, $Y \leq Y^*$, $Y^* = O(L^*)$ and $L^* \neq Y^* A$.*

Proof. Since $|Y|$ does not divide 15 and $|Y| = 3^x 5^y$ with $x + y \leq 3$ there exists $p \in \{3, 5\}$ with $p^2 \mid |Y|$. Let D be a Sylow p-subgroup of Y. If $|Y| \neq p^3$, then $|D| = p^2$ and so by Step 5, $D \trianglelefteq L$. If $|Y| = p^3$, then $D = Y$ and again $D \trianglelefteq L$. Since D is a p-group, $\Omega_1 \mathrm{Z}(D) \neq 1$ and so there exists $a \in A^\sharp$ with $C_{\Omega_1 \mathrm{Z}(D)}(a) \neq 1$ and so $Y_a \leq \Omega_1 \mathrm{Z}(D)$. Since $|D| \geq p^2$ there exists $b \in A^\sharp$ with $C_D(b) \not\leq Y_a$. Then $b \neq a$. Put $E = Y_a Y_b$. Since $Y_a \leq Z(D)$, $E \cong C_p \times C_p$. By **??** $Y \leq N_G(E)$ and so by Step 4, $Y \leq F := O(N_G(E))$. By Step 6 there exists $g \in G$ with $a^g = b$ and $Y_a^g = Y_b$. Let $e \in \{a, b\}$. Then E is a subgroup of odd order in G_e and so by Step 4, $E \leq O_e := O(N_G(Y_e))$. So by Step 6, $E \trianglelefteq O_e$. Thus another application of Step 4 shows that $O_e \leq F$. Observe that F/E has order $1, 3$ or 5, $E \leq O_a \cap O_b$ and $|O_a| = |O_b|$. Thus either $E = O_a = Q_b$ or $F = O_a = O_b$. In any case $O_a = Q_b$ and so $g \in \tilde{L} := N_G(O_a)$. Put $\tilde{Y} = O(\tilde{L})$. Since $a^g = b$, $\tilde{L} \neq \tilde{Y} A$. Hence by Step 3, $\tilde{Y}$ is elementary abelian of order p^3 and $\tilde{Y} = O_a = O_b$. Since $Y Q_a \leq F$, this gives $\tilde{Y} = F$ and $Y \leq \tilde{Y}$. Since Y has order at least p^2, $C_G(Y)$ has odd order. Since $\tilde{Y} \leq C_G(Y)$ we conclude from Step 2, that $\tilde{Y} = C_G(Y) = O(N_G(Y))$. In particular, $L \leq N_G(\tilde{Y})$ and the lemma is proved. □

Step 8. [**j1-6**] *$|Y|$ divides 15.*

Proof. Suppose not. Then we can apply Step 7 and replacing L by L^* we may assume that $|Y| = p^3$, $L = N_G(Y)$ and $L \neq YA$. Let $a \in A^\sharp$. Then $|Y_a| = p$. By Step 4, $N_G(Y_a) = O(N_G(Y_a))A$ and it follows that $Y = O(N_G(Y_A)$ and $N_G(Y_a) = YA$. By Step 3 $N_L(A) \cong \mathrm{Alt}(4)$ and so there exists $d \in N_L(A)$ with $|d| = 3$. Put $b = a^d$ and $c = b^d$. Then $A^\sharp = \{a, b, c\}$ and $Y = Y_a \times Y_b \times Y_c$. Let $1 \neq y_a \in Y_a$ and put $y_b = y_a^d$, $y_c = y_b^c$ and $y = y_a y_b y_c$. Since d has order three, $y \in C_Y(d)$. Also $y_e \in Y_e$, $y \neq 1$ and $|y = p$. Since $Y\langle d\rangle \leq C_G(y)$, $C_G(y)$ has order divisible by $3p^3$ and so $\langle y\rangle$ is not conjugate to Y_a. Put

$\tilde{S} = C_G(A)$. Then $|\tilde{S}| = 8$ and d normalizes $\tilde{S}$. Thus d centralizes an element $\tilde{a}$ of order 2 in $\tilde{S}^\sharp$. In $G_{\tilde{a}}$ we see that there exists a subgroup $\tilde{A}$ of order 4 inverting d. Thus $\tilde{L} = N_G(\langle d \rangle$ is divisible by 4. From Step 4 we conclude that $y \in \tilde{Y} := O(N_G(\langle d \rangle))$.

Suppose that $p = 5$. Then 15 divides $\tilde{Y}$ and by Step 7 we conclude that $|\tilde{Y}| = 15$. Thus $\langle y >$ is the unique subgroup of order 5 in $\tilde{Y}$, $\tilde{A}$ normalizes $\langle y \rangle$ and so $[y, \tilde{b}] = 1$ for some $\tilde{b} \in \tilde{A}^\sharp$. But then $\langle y \rangle$ is conjugate to Y_a, a contradiction.

Thus $p = 3$. We will show that $L = YN_L(A)$. For this we investigate the action of L on the set $\mathcal{P}$ of subgroups of order 3 of Y. Note that $|\mathcal{P}| = 13$. $N_L(A)$ has three orbits $\mathcal{P}_3$, $\mathcal{P}_4$ and $\mathcal{P}_6$ on $\mathcal{P}$ of size 3, 4 and 6 respectively. Indeed $\mathcal{P}_2 = \{Y_e \mid e \in A^\sharp\}$, $\mathcal{P}_4 = \langle y \rangle^{N_L(A)}$ and $\mathcal{P}_6 = \langle y_a y_b \rangle^{N_L(A)}\}$. Since $\langle y \rangle$ is not conjugate to Y_a in G there are three possibilities for the orbits of L on $\mathcal{P}$:

(a) $\mathcal{P}_3$, $\mathcal{P}_4$ and $\mathcal{P}_6$.

(b) $\mathcal{P}_3 \cup \mathcal{P}_6$ and $\mathcal{P}_4$.

(c) $\mathcal{P}_3$ and $\mathcal{P}_4 \cup \mathcal{P}_6$.

In any case there exists $i \in \{3, 4\}$ such that $\mathcal{P}_i$ is an orbit for L on $\mathcal{P}$. Put $Q = C_L(\mathcal{P}_i)$. Then L/Q is isomorphic to a subgroup of $\mathrm{Sym}(i)$ and $N_L(A)Q/Q \cong \mathrm{Alt}(i)$. Thus $|L/N_L(A)Q| \leq 2$. Since A is a Sylow 2 subgroup of L we get $L = N_L(A)Q$. Note that $|Q/C_Q(U)| \leq 2$ for all $U \in \mathcal{P}_i$ and so $Q/C_Q(Y)|$ is a 2-group. Since $Y = C_G(Y)$ this gives $Q = C_Q(Y)(Q \cap A) \leq YA$ and $L = N_L(A)YA = N_L(A)Y$.

Note that this implies that $\mathcal{P}_3$ is an orbit for L on $\mathcal{P}$. Let $g \in G$ with $Y_a \leq Y$. Then by Step 4, $Y \leq O(N_G(Y_a)^g$ and $Y = O(N_G(Y_a)) = Y^g$. So $g \in N_G(Y) = L$ and $Y_a^g \in \mathcal{P}_3$. So Y contains exactly three G conjugates of Y_a and these three conjugate generate Y. Since $\langle d \rangle$ is conjugate to Y_a the same is true for $\tilde{Y}$.

Put $R = C_Y(d)\langle d \rangle = \langle y, d \rangle$. Then $R \leq \tilde{Y}$ and Then $R < N_{YR}(R) = N_Y(R)R$. So $N_Y(R) \neq C_Y(d)$ and $|N_Y(R)/C_Y(d)| = 3$. Also $[N_Y(R) \cap N_Y(\langle d \rangle), \langle d \rangle] \leq Y \cap \langle d \rangle = 1$ and so $|\langle d >^{N_Y(R)}| \geq 3$. Hence R contains at least G-three conjugate of Y_A. But the R contains all G conjugates of Y_A in $\tilde{Y}$ and so $R = \tilde{Y}$, a contradiction. □

Step 9. [**j1-7**] *$L \cong D_{12}, D_{20}$ or $D_6 \times D_{10}$.*

Proof. By Step 8, $|Y| = 3, 5$ or 15 and so by Step 3, $L = YA$. So L has order 12, 20 or 60 and the lemma follows. □

Step 10. [**j1-8**] *For $p = 3, 5$ let S_p be a Sylow p subgroups of $C_G(z)$. The one of the following holds.*

1. [**a**] *$N_G(S_3) \cong D_{12}$ and $N_G(S_5) \cong D_{20}$.*

2. [**b**] *$N_G(S_3) \cong D_6 \times D_10 \cong N_G(S_5)$.*

Proof. Let $p \in \{2,3\}$. Then by Step 9, $N_G(S_p) \cong D_{4p}$ or $D_6 \times D_{10}$. So either (2) holds or $N_G(S_p) \cong D_6 \times D_{10}$. Suppose the latter and let $\{p,q\} = \{3,5\}$. Then $N_G(S_p)$ as a normal Sylow q subgroup T_q. Moreover $N_G(S_p) \cap C_G(T_q)$ contains an involution and so T_q is conjugate to S_q. Thus also $N_G(S_q) \cong D_6 \times D_{10}$ and (1) holds. □

Proposition 3.5.10. [**bender counting**] *Let G be a finite group of even order and $\mathcal{J}$ the set of involutions in G and $\mathcal{I} = \{t \in \mathcal{J} \mid H \cap H^t \neq 1\}$. Let H be a subgroup of G. Let $j_n = |\{U \in G/H \mid U \neq H, |U \cap \mathcal{J}| = n\}|$ and $i_n = |\{U \in G/H \mid U \neq H, |U \cap \mathcal{I}| = n\}|$. For $\mathcal{K} = \{\mathcal{I}, \mathcal{J}\}$ put $\mathcal{K}_n = \{t \in \mathcal{K} \mid t \notin H, |Ht \cap \mathcal{I}| = n\}$. Let m be the number of orbits of H on $\mathcal{J}_1 \setminus \mathcal{I}_1$. Put $c = \frac{|G|}{|\mathcal{I}|}$ and $h = |H|$. Then*

(a) [**a**] *For all $t \in \mathcal{J} \setminus H$, $Ht \cap \mathcal{I} = \{ht \mid h \in H \cap H^t, h^t = h^{-1}\}$. In particular $\mathcal{I}_n = \mathcal{J}_n$ for all $n \geq 2$.*

(b) [**b**] *Let $U = Hg \in G/H$ with $U \neq H$ and put $l = |U \cap \mathcal{J}|$. Then $U \cap \mathcal{I} \subseteq \mathcal{J}_l$. Moreover, either $H \cap H^g \neq 1$ and $U \cap cI \subseteq \mathcal{I}_l$ or $H \cap H^g = 1$, $l \leq 1$ and $U \cap \mathcal{I} \subseteq \mathcal{J}_l \setminus \mathcal{I}_l$.*

(c) [**c**] *For all $n \in \mathrm{Z}^+$, $|\mathcal{J}_n| = nj_n$ and $\mathcal{I}_n = |ni_n|$. In particular $i_n = j_n$ for all $n \geq 2$.*

(d) [**d**] *$j_1 = i_1 + mh$ and $|\mathcal{J}| = |\mathcal{I}| + mh$.*

(e) [**e**] *$|\mathcal{J}| = |\mathcal{J} \cap H| + \sum_{n=1}^{\infty} nj_n = |\mathcal{J} \cap H| + |mh + \sum_{n=1}^{\infty} ni_n$*

(f) [**f**] *$|G/H| = 1 + \sum_{n=0}^{n} j_n = 1 + j_0 + mh + \sum_{n=1}^{n} i_n$*

(g) [**g**] *$h\big((h-c)m + j_0\big) = |\mathcal{J} \cap H|c - h + \sum_{n=1}^{\infty}(nc - h)i_n$*

Proof. (a) Let $h \in H$. Since $ht \notin H$, $ht \neq 1$ and so $|ht| = 2|$ iff $(ht)^2 = 1$. Since $(ht)^2 = htht = hh^t$, we have $(ht)^2 = 1$ if and only if $h^t = h^{-1}$. Observe that $h^t = h^{-1}$ implies $h \in H \cap H^t$. So if $t \in \mathcal{J}_n$ for some $n \geq 2$, then $H \cap H^t$ contains at least two elements inverted by t and so $H \cap H^t \neq 1$ and $t \in \mathcal{I}$. Thus $Ht \cap \mathcal{J} = H \cap cI$ and $t \in \mathcal{I}_n$.

(b) Observe that $U = Ht$ for all $t \in U \cap \mathcal{J}$. Thus $|Ht \cap \mathcal{J}| = |U \cap \mathcal{J}| = l$ and so $U \cap \mathcal{J} \subseteq \mathcal{J}_l$. Observe also that $H \cap H^t = H \cap H^g$. So if $H \cap H^g \neq 1$, then $U \cap \mathcal{J} \subseteq \mathcal{I}_n$ and if $H \cap H^g = 1$, then $U \cap \mathcal{J} \subseteq \mathcal{J}_n \setminus \mathcal{I}_n$. In the latter case, (a) implies $n \leq 1$.

(c) Obvious.

(d) Let $t \in \mathcal{J}_1 \setminus \mathcal{I}_1$. Then $C_H(t) \leq H \cap H^t = 1$ and so all orbits of $|H|$ on $\mathcal{J}_1 \setminus \mathcal{I}_1$ have length $h = |H|$. Hence $|\mathcal{J}_1 \setminus \setminus \mathcal{I}_1| = mh$ and so $|\mathcal{J}_1| = |\mathcal{I}_1| + |\mathcal{J}_1 \setminus \mathcal{I}_1| = i_1 + mh$. Since $\mathcal{J}_n = \mathcal{I}_n$ for all $n \geq 2$ this implies

$$|\mathcal{J} \setminus H| = \sum_{n=1} |\mathcal{J}_n| = mh + \sum_{n=1} |\mathcal{I}_n| = mh + |\mathcal{I}|$$

(e) This follows from (c) and (d).

(f) This follows from (c) and (d).

(g) Note that $c|\mathcal{J}| = |G| = h|G/H|$. So by (e) and (f):

$$c\left(|\mathcal{J}\cap H| + mh + \sum_{n=1}^{\infty} n i_n\right) = h\left(1 + j_0 + mh + \sum_{n=1}^{n} i_n\right)$$

and so (g) holds. □

Lemma 3.5.11. [**computing in**] *Retain the assumption and notation from 3.5.10. For $g \in G$ and $K \leq H$ with $K^g = K$ define $g_K \in \mathrm{Aut}(K)$ by $k^{g_K} = k^g$. Define*

$$\Xi = \{(K,s) \mid 1 \neq K \leq H, s \in \mathrm{Aut}(K), s^2 = 1\}.$$

Note the H acts on Ξ via $(K,s)^g = (K^g, s^g)$, where $s^g \in \mathrm{Aut}(K^g)$ is defined by $l^(s^g) = (l^{g^{-1}})^s)^g$. Let Λ be the set of orbits for H on Ξ and $\lambda, \mu \in \Lambda$ Let $(K,s) \in \lambda$ and define

$$\begin{aligned}
a_\lambda &= |\{(L,t) \in \mathcal{I} \setminus H \mid 1 \neq L \leq H, t \in J \setminus H, L^t = L, (L, t_L) \in \lambda\}| \\
b_\lambda &= |\{t \in \mathcal{I} \setminus H \mid (H \cap H^t, t_{H\cap H^t}) \in \lambda\}| \\
n_\lambda &= |\{k \in K \mid k^s = k^{-1}\}| \\
r_{\mu\lambda} &= |\{L \leq K \mid L^s = L, (L, s_L) \in \mu\}|
\end{aligned}$$

Then

(a) [**a**] *Let $(K,s) \in \lambda$. Then $a_\lambda = |H/N_H(K)| \cdot |\{t \in N_G(K) \setminus H \mid (K, t_K) \in \lambda\}|$.*

(b) [**b**] *Let $\mu \in \Lambda$. Then $b_\mu = a_\mu - \sum_{\mu \neq \lambda \in \Lambda} r_{\mu\lambda} b_\lambda$.*

(c) [**c**] *$i_n = \frac{1}{n} \sum (b_\lambda \mid \lambda \in \Lambda, n_\lambda = n)$.*

Proof. Define

$$\begin{aligned}
A_\lambda &= \{(L,t) \in \mathcal{I} \setminus H \mid 1 \neq L \leq H, t \in J \setminus H, L^t = L, (L, t_L) \in \lambda\} \\
B_\lambda &= \{t \in \mathcal{I} \setminus H \mid (H \cap H^t, t_{H\cap H^t}) \in \lambda\}
\end{aligned}$$

□

Appendix A

Set Theory

A.1 The basic language of sets theory

A simple term is a set or a variable. A formula is any expression which can be obtained in finite steps according to the following rules:

(a) [**a**]

$$x = y \text{ and } x \in y$$

are formulas, where x and y are simple terms.

(b) [**b**] If ϕ and ψ are formulas and x a variable, then

$$(\neg\phi)$$
$$(\phi \to \psi)$$
$$(\phi \vee \psi)$$
$$(\exists x \phi)$$

are formulas.

These formulas are pronounced as follows:
$x = y$: x is equal to y.
$x \in y$: x is an element of y.
$(\neg\phi)$: not ϕ
$(\phi \to \psi)$: ϕ is equivalent to ψ.
$(\phi \vee \psi)$: ϕ or ψ.
$(\exists x \phi)$: there exists x such that ϕ.

We use following abbreviations:
$(\forall x \phi)$ means $(\neg(\exists x(\neg\phi)))$

$(\phi \wedge \psi)$ means $(\neg(\exists x((\neg\phi)) \vee (\neg\psi))))$
$(\phi \to \psi)$ means $((\neg\phi) \vee \psi)$
$\exists! x(\phi)$ means $(\exists y(\forall x(x = y \leftrightarrow \phi)))$, where y is any variable not appearing in ϕ.
$(\exists(x \in y)\phi)$ means $(\exists x(x \in y \wedge \phi))$.
$(\forall(x \in y)\phi)$ means $(\forall x(x \in y \to \phi))$.

Let ϕ be a formula and v a variable. We inductively define the terminologies, 'v is free variable of ϕ' and 'free appearance of "x" in ϕ If ϕ is $x = y$ or $x \in y$, then any x or y equal to v is called a free appearance of x in ϕ. Any variable is called free variable of ϕ.

If ϕ is $\neq \psi$ then a free variable of ϕ is free variable of ψ. A free appearance of v in ψ is free appearance of v in ψ.

If ϕ is $(\psi \leftrightarrow \tau$ or $(\psi \vee \tau$, then a free variable of ϕ is a free variable of ψ or of τ. A free appearance of v in ϕ is free appearance of v in ψ or in τ.

If $\phi \equiv (\exists x\psi)$, then v is a free variable of ϕ if $v \neq x$ and v is a free variable of ψ. If $v \neq x$, then any free appearance of v in ψ is a free appearance of v in ϕ.

A variable which is not free variable of ϕ is called a bound variable of ϕ.

Now let ϕ a formula, v a variable. ϕ and t a simple term. Then $\phi(v \searrow t)$ is the formula obtained to replacing all free appearances of v by t. More formally $\phi(v \searrow t)$ is inductively defined

Let r, s be simple terms distinct v and let $\diamond$is one of $=, \in$, Then

If $\phi \equiv r \diamond s$ then $\phi(v \searrow t) \equiv r \diamond s$. If $\phi \equiv v \diamond s$ then $\phi(v \searrow t) \equiv t \diamond s$. If $\phi \equiv r \diamond v$ then $\phi(v \searrow t) \equiv r \diamond v$. If $\phi \equiv v \diamond v$ then $\phi(v \searrow t) \equiv t \diamond t$. If $\phi \equiv (\neq \psi)$, then $\phi(v \searrow t) \equiv (\neq \psi(v \searrow t))$.

Let $\diamond$ is one of $\to$ or $\vee$. If $\phi \equiv (\psi \diamond \tau)$, then $\phi(v \searrow t) \equiv (\psi[v \searrow t] \diamond \tau[v \searrow t)$

If $\phi \equiv (\exists x\psi)$ and x is a variable different from v, then $\phi(v \searrow t) \equiv (\exists s\psi(v \to t)$. If $\phi \equiv (\exists v\psi)$ then $\phi(v \searrow t) \equiv (\exists v\psi)$.

We will often use the following more convenient notion: We use the symbol $\phi(v)$ in place of ϕ and from then on $\phi(t)$ denotes the formula $\phi(v \searrow t)$. So $\phi(v)$ is a formulas with a distinguished variable v.

A class A is just a formula $\phi(v)$ with a free distinguished variable v. But we think about A as the collection of all sets which fulfill ϕ and write

$$A = \{x \mid \phi(x)$$

Any set s can be viewed as the class

$$\{x \mid x \in s\}$$

The class $V := \{x \mid x = x\}$ is called the universe. Every set is a member of the universe.

The class $\emptyset := \{x \mid x \neq x\}$ is called the empty class. The empty class has no members.

We introduce an extended language: A simple class term is a variable, a set or a class. Now a class formula is defined in the save way as a formula: just replace 'simple term' by 'simple class term'.

Any class formula Φ has a corresponding set formula $\tilde{\Phi}$ inductively defined as follows: Let A and B be simple class terms, and s a simple set term. If A is a set or variable, let

$\phi(v)$ be the formula $v \in A$, where v is a variable distinct from A. If A is a class, let $\phi(v)$ be the formula used to define A. Also u is a variable different from s and not involved in ϕ and ψ.

If $\Phi \equiv A = B$, then $\tilde{\Phi} = \forall u(\phi(u) \leftrightarrow$
$psi(u)$. If $\Phi \equiv s \in B$, where s is a set term, then $\tilde{\Phi} \equiv \psi(s)$. If $\Phi \equiv A \in B$ and A is a class, then $\tilde{\Phi} \equiv (\exists u(u = A \wedge u \in B)$, If $\Phi \equiv \Psi \leftrightarrow \Sigma$, then $\tilde{\Phi} \equiv \tilde{\Psi} \leftrightarrow \tilde{\Sigma}$. If $\Phi \equiv \Psi \vee \Sigma$, then $\tilde{\Phi} \equiv \tilde{\Psi} \vee \tilde{\Sigma}$. If $\Phi \equiv (\neg\Psi)$, then $\tilde{\Phi} \equiv (\neg\tilde{\Psi})$. If $\Phi \equiv (\exists x\Psi)$, then $\tilde{\Phi} \equiv (\exists s\tilde{\Psi})$.

$\tilde{\Phi}$ is called the translation of Φ. Note that if s and t are sets terms then $s = t$ is translated into $\forall u(u \in s \leftrightarrow u \in t)$. This is justified be the following Axioms of Set Theory

Set Axiom 1 $\forall x \forall y(x = y \leftrightarrow (\forall z(z \in x \leftrightarrow z \in y))$

Definition A.1.1. [**def:int**]

(a) [**a**] *Let $\Phi(x)$ a class formula. Then $\{x \mid \Phi(x)$ denotes the class $\{x \mid \tilde{\Phi}(x)\}$ defined by the translated formula $\tilde{\Phi}(x)$.*

(b) [**b**] *Let A be class. Then $\bigcap A :\equiv \{x \mid (\forall a \in A) x \in a\}$.*

(c) [**c**] *Let A be a class. Then $\bigcup A :\equiv \{x \mid (\exists a \in A) x \in a\}$*

If $A = \{x \mid \phi(x)\}$, then

$$\bigcap A \equiv \{x \mid (\forall a \in A) x \in a\} = \{x \mid \forall a(a \in A \to x \in A\} = \{x \mid \forall a(\phi(a) \to x) \in a\}$$

and

$$\bigcup A \equiv \{x \mid (\exists a \in A) x \in a\} = \{x \mid \exists a(x \in A\} = \{x \mid \exists a(\phi(a) \wedge x \in a\}$$

A.2 The Axioms of Set Theory

To continue we need

Set Axiom 2 $\forall x \forall y \exists z \forall w(w \in z \leftrightarrow (w = x \vee w = y))$

Note that this just says that for any sets x and y, there exists a set z whose elements are exactly x and y. We denote this set by $\{x, y\}$. The special case $x = y$, show that there exists a set $\{x\}$ whose only element is x.

Definition A.2.1. [**def:ordered pair**] *Let a, b be sets. Then (x, y) denotes the set $\{\{x\}, \{x, y\}\}$. (x, y) is called the ordered pair x and y.*

Lemma A.2.2. [**ordered**] *Let a, b, c, d be sets. Then $(a, b) = (c, d)$ if and only if $a = b$ and $c = d$.*

Proof. See Homework 2 □

Definition A.2.3. [**def:relation**]

(a) [**a**] *A relation is a class R such that all members of R are ordered pairs. If x and y are sets then xRy means $(x, y) \in R$.* $\mathrm{Dom}(R) := \{a \mid aRb \text{ for some} b\}$ *and* $\mathrm{Ran}(R) := \{b \mid aRb \text{ for some } a\}$.

(b) [**b**] *A function is a relation F such that $b = c$ for all sets a, b, c such that $(a, b) \in F$ and (a, c) is in F. $F(a) = b$ means that $(a, b) \in F$. Also if F is a function and A a class then $\{F[A] := \{b \mid a \in A, b = F[a]\}$. $F[A]$ is called the image of A under F. $FA|_{:}= \{(a, b) \mid a \in A, b = F(a)\}$.*

Lemma A.2.4. [**int class**] *Let A be a class.*

(a) [**a**] *If $A = \emptyset$, then $\bigcap \emptyset = V$.*

(b) [**b**] *If $A \neq \emptyset$, then $\bigcap A$ is a set.*

Proof. (a) If $\bigcap \emptyset = \{x \mid x \in y \text{ for all } y \in \emptyset\} = \{x|\} = V$.
(b) Let $a \in A$. Then $\bigcap A \subseteq a$.Since $\bigcap A$ is a class, A.2.5 implies that $\bigcap A$ is a set. □

If A and B are classes we define $A \subseteq B$ to mean $(\forall x(x \in A \to x \in B)$.
We are able to state all the Axioms of Set Theory :

Set Axiom 1 [**1**] $\forall x \forall y(x = y \leftrightarrow (\forall z(z \in x \leftrightarrow z \in y))$, that is two sets are equal if and only if they have the same elements.

Set Axiom 2 [**2**] $\forall x \forall y \exists z \forall w(w \in z \leftrightarrow (w = x \vee w = y))$ (That is for all sets x and y there exists a set z with exactly x and y as elements.

Set Axiom 3 [**3**] For all sets x, $\{y \mid y \subseteq x\}$ is a set.

Set Axiom 4 [**4**] For all sets x, $\bigcup x$ is a set.

Set Axiom 5 [**5**] For all functions F and all sets x, $F[x]$ is a set.

Set Axiom 6 [**6**] There exists a set z such that $\emptyset \in z$ and for all $x \in z$ also $x \cup \{x\} \in z$.

Set Axiom 7 [**7**] For all non-empty classes A, there exists $x \in A$ such that $y \notin A$ for all $y \in x$.

(6) includes the statement that the empty class is a set. Indeed $\emptyset \in z$, means that there exists a set x with $x = \emptyset$ and $x \in z$. Henceforth we will call the empty class, the empty set.

Lemma A.2.5. [**subclass**]

(a) [**a**] *If x is a set and A a class, then $x \cap A$ is a class.*

(b) [**b**] *If x is a class and A a set with $A \subseteq x$, then A is class.*

(c) [**c**] *A function is a set if and only if* Dom$f$ *is a set.*

Proof. See Homework 2. □

Lemma A.2.6. [**compatible**] *Let A be a class of compatible functions, that is A is class, if $f \in A$, then f is a function and a set, and if $f, g \in A$, then $f(x) = g(x)$ for all $x \in \mathrm{Dom} f \cap \mathrm{Dom} g$. Then $\bigcup A$ is a function.*

Proof. Let $a \in \bigcup A$. Then $a \in f$ for some $f \in A$ and so a is an ordered pair. Now let a, b, c be sets with $(a, b) \in \bigcup A$ and $(a, c) \in \bigcup A$. The $(a, b) \in f$ and $(a, c) \in g$ for some $f, g \in A$. Thus $a \in \mathrm{Dom} f \cap \mathrm{Dom} g$ and so

$$b = f(a) = g(a) = c$$

So $\bigcap A$ is a function. □

A.3 Well ordered sets and the Recursion Theorem

Definition A.3.1. [**def:relation**] *Let R be a relation and A a class*

(a) [**a**] *aRb means $(a, b) \in R$ and $a \not R B$ mean $(a, b) \notin R$.*

(b) [**b**] *R is called irreflexive on A if $a \not R a$ for all $a \in A$.*

(c) [**c**] *R is transitive of A aRc for all $a, b, c \in A$ with aRb and bRc.*

(d) [**d**] *T partially orders A if R is irreflexive and transitive on A.*

(e) [**d**] *R totally orders A if R is partially orders A and for all $a, \in A$ one of aRb, $a = b$ and bRA holds.*

(f) [**e**] *An R-minimal element of A is an element $m \in A$ such that for all $a \in A$, $m = a$ or mRa.*

(g) [**e**] *If x is any object that $A^R_x := \{a \in A \mid bRx\}$.*

Lemma A.3.2. [**trivial total orders**]*Suppose the relations R totally orders the class A. Then for all a, b in R exactly one of aRb, $a = b$ and bRa holds,*

Proof. By definition of a total ordering, at least one of aRb, $a = b$ and bR holds. Id $a = b$, then $a \not R b$ and $b \not R a$ since R is irreflexive on A. If aRb and bRA, then aRa since R is transitive, a contradiction since R is irreflexive. □

Definition A.3.3. [**def:well orders**] *Let R be a relation and A a class. We say that R well-orders A if*

(i) [**i**] *R totally orders A.*

(ii) [**ii**] *Every non-empty subset x of A has a RR-minimal element.*

(iii) [**iii**] *For all $a \in A$, A_a^R is a set.*

Lemma A.3.4. [**minimal for class**] *If the relation R well orders the class A, then every non-empty subclass of A has a R-minimal element.*

Proof. Let B be a subclass of $b \in B$. If b is a minimal element of B we are done. So suppose b is not a minimal element. Then there exists $a \in B$ such that neither $a = b$ nor bRa. So aRb and thus B_b^R is not empty. not-empty. By definition of a well-ordering A_b^R is a set and so also $B_b^R = B \cap A_b^R$, since the intersection of a class with a set is a class. Since B_b^R is a set, the definition of a well ordering implies that B_b^R has a minimal element m. Since $m \in B_b^R$, we have mRb. Let $y \in B$. If yRb, then $y \in B_R^b$ and so $y = m$ or mRy. If $y = b$ then mRy. If bRy then mRy since R is transitive on A. Thus m is a minimal element of B. □

Definition A.3.5. [**def:segment**] *Let R be a relation, A a class and B a subclass of A.*

(a) [**a**] *B is called in initial R-segment of A if $a \in B$ for all $b \in B$ and $a \in A$ with aRb.*

(b) [**b**] *B is called an R-section of A if $B = A_a^R$ for some $a \in A$.*

With this definition the last condition on a well-ordered class says that every section is a se

Lemma A.3.6. [**union of segments**] *Let R be a relation, A a class and T a non-empty class of initial R-segments of A. Then $\bigcup T$ and $\bigcap T$ are initial R-segment of A.*

Proof. Observe first that $\bigcup T$ is a subclass of A. Let $b \in \bigcup T$ and $a \in A$ with aRb. Then $b \in B$ for some $B \in T$. Thus $a \in B$ since B is an initial R-segment of A. Hence $a \in \bigcup T$ and so $\bigcup T$ is an initial R-segment of A.

A similar proof shows that $\bigcap T$ is an initial R-segment of A. □

Lemma A.3.7. [**segments**] *Let R be relation which well orders the class A and let B be an initial R-segment of A. Then $B = A$ or B is an R-section of A. In particular, $B = A$ or B is a set.*

Proof. Suppose $B \neq A$. Then $A \setminus B$ is a non-empty subclass of A and so has a R-minimal element m. Let $a \in A$. We claim that aRm if and only if $a \in B$. If aRm, then $a \notin A \setminus B$, since m is the minimal element of $A \setminus B$. Thus $a \in B$. If $a = m$, then $a \notin B$ since $m \in A \setminus B$. Suppose mRa and $a \in B$. Since B is an initial segment this gives $m \in B$, a contradiction. Thus proves the claim and so $B = A_m^R$ and B is an R-section of A. □

Theorem A.3.8 (Recursion Theorem)**.** [**recursion**] *Let R be a relation which well-orders the class A. Let τ be a function with domain the universe V. Then there exists a unique function F with domain A such that for all $a \in A$*

$$(*) \qquad F(a) = \tau(F \mid_{A_a^R})$$

Proof. Recall that two functions F and G are called compatible if $F(x) = G(x)$ for all $x \in \mathrm{Dom}(F) \cap \mathrm{Dom}(G)$. Just in this proof we will call a function F recursive if its domains is an initial segment of A and $F(a) = \tau(F\mid_{A_a^R})$ for all $a \in \mathrm{Dom}(F)$.

1°. [**1**] *Any two recursive functions are compatible.*

Let F_1 and F_2 be recursive functions and $x \in \mathrm{Dom}(F_1) \cap \mathrm{Dom}(F_2)$. By induction we may assume that $F_1(y) = F_2(y)$ for all $y \in \mathrm{Dom}(F_1) \cap \mathrm{Dom}(F_2)$ with yRx. Since $\mathrm{Dom}(F_i)$ is an initial segment we have $A_x^R \subseteq \mathrm{Dom}(F_1) \cap \mathrm{Dom}(F_2)$. So the induction assumptions shows that $F_1\mid_{A_x^R} = F_2\mid_{A_x^R}$. Thus

$$F_1(x) = \tau(F_1\mid_{A_x^R}) = \tau(F_2\mid_{A_a^R}) = F_2(x)$$

So F_1 and F_2 are indeed compatible.

Observe that (1°) implies the uniqueness statement of the Theorem. To prove the existence

Let T be the class of all recursive functions whose domains are sets. Put $F = \bigcup T$.

2°. [**2**] *F is a recursive function.*

By (1°) and A.2.6 F is a function. Observe that $\mathrm{Dom}(F) = \bigcup\{\mathrm{Dom}(G) \mid G \in T\}$. Since the unions of a class of initial segment is an initial segment, $\mathrm{Dom}(F)$ is an initial segment. Now let $x \in \mathrm{Dom}(F)$ and $G \in T$ with $x \in \mathrm{Dom}(G)$. Then $A_x^R \subseteq \mathrm{Dom}(G)$ and so

$$F(x) = G(x) = \tau(G\mid_{A_x^R}) = \tau(F\mid_{A_x^R})$$

and so F is indeed a recursive function.

3°. [**3**] $\mathrm{Dom}(F) = A$.

Suppose not. Then by A.3.7 $\mathrm{Dom}F = A_R^x$ for some $x \in A$. Let $G = F \cup \{(x, \tau(F)\}$. Since $x \notin A_x^R = \mathrm{Dom}(F)$ we see that G is a function. Let $y \in \mathrm{Dom}(G)$. Then either $y \in \mathrm{Dom}(F)$ or $y = x$. In the first case $A_y^R \subseteq \mathrm{Dom}(F) \subseteq \mathrm{Dom}(G)$ and $G(y) = F(y) = \tau(F\mid_{A_y^R}) = \tau(G\mid_{A_y^R})$. Also $A_x^R = \mathrm{Dom}(F) \subseteq \mathrm{Dom}(G)$ and $G(x) = \tau(F) = \tau(G\mid_{A_x^R})$. Hence in either case $A_y^R \subseteq \mathrm{Dom}(G)$ and $G(y) = \tau(G\mid_{A_y^R})$. Thus $\mathrm{Dom}(G)$ is an initial segment of A and G is a recursive function. By definition of a well-ordered class, $A_R(x)$ is a set and so also $\mathrm{Dom}(G) = A_x^R \cup \{x\}$ is a set. Thus $G \in T$. But then $x \in \mathrm{Dom}(G) \subseteq \bigcup\{\mathrm{Dom}H \mid H \in T\} = \mathrm{Dom}(F) = A_x^R$, a contradiction. Thus (3°) holds.

By (2°) and (3°) F fulfills the conclusion of the theorem. □

A.4 Ordinals

Definition A.4.1. [**def:ordinal**] *An ordinal is a set α such that every elements of α is a subset of α and '$\in$' well-orders α.* Ord *is the class of all ordinals.*

For example $\emptyset$, $\{\emptyset\}$, $\{\emptyset, \{\emptyset\}\}$ all are ordinals.

Lemma A.4.2. [**basic ord**] *Let α be an ordinal.*

(a) [**a**] $\beta \notin \beta$ *for* $\beta \in \alpha$.

(b) [**b**] $\alpha \notin \alpha$.

(c) [**c**] *Every elements of α is an ordinal.*

(d) [**d**] $\alpha \cup \{\alpha\}$ *is an ordinal.*

Proof. (a) This holds since $'\in'$ is a well-ordering and so irreflexive on α. (b) If $\alpha \in \alpha$, (b) gives $\alpha \notin \alpha$.

(c) Let α be an ordinal and $\gamma \in \beta \in \alpha$. Since β is a subset of α, γ is an element of α and so a subset of α. Let $\delta \in \gamma$. Then $\delta \in \alpha$. Since $\gamma \in \beta$ and $\in$ is transitive on α, $\delta \in \beta$ and so γ is a subset of β. A restriction of a well ordering to a subset is a well ordering and β is an ordinal.

(d) Since $\beta \in \alpha$ for all $\beta \in \alpha$, α is a maximal element of $\alpha \cup \{\alpha\}$ with respect to $\in$. This easily implies that $\in$ well orders $\alpha \cup \{\alpha\}$. If $\beta \in \alpha \cup \{\alpha\}$ the either $\beta \in \alpha$ or $\beta = \alpha$. In either case β is a subset of α and so also of $\alpha \cup \{\alpha\}$. □

Notation A.4.3. [**alpha+1**] *If α is an ordinal, we denote the ordinal $\alpha \cup \{\alpha\}$ by $\alpha + 1$. We also denote $\emptyset$ by 0, $0 + 1$ be 1, $1 + 1$ by 2 and so on.*

Theorem A.4.4. [**ord well-ordered**] $'\in'$ *well-orders* Ord.

Proof. Let α, β and γ be ordinals. By A.4.2(a), $\alpha \notin \alpha$ and so $\in$ is irreflexive on Ord. If $\alpha \in \beta$ and $\beta \in \gamma$, then β is a subset of γ and so $\alpha \in \beta$ and so $\in$ is transitive on Ord.

To show that one of $\alpha \in \beta, \alpha = \beta$ and $\beta \in \gamma$ holds, put $\delta = \alpha \cup \beta$. We will show that δ is a initial segment of α. So let $\epsilon \in \alpha$ and $\gamma \in \delta$ with $\epsilon \in \gamma$. Note that $\gamma \in \beta$ and so $\epsilon \in \beta$ since γ is a subset of β. Hence $\epsilon \in \alpha \cap \beta = \delta$. So δ is indeed and initial segment of α. **??** choose that either $\delta = \alpha$ or there exists $\rho \in \alpha$ with

$$\delta = \alpha_\rho = \{x \in \alpha \mid x \in \rho\} = \rho$$

We proved that $\delta = \alpha$ or $\delta \in \alpha$. By symmetry, $\delta = \beta$ or $\delta \in \beta$.

Suppose that $\delta = \alpha$. Then $\alpha = \beta$ or $\delta \in \beta$ and we are done with this part of the proof. So we may assume $\delta \in \alpha$ and by symmetry also $\delta \in \beta$. But then $\delta \in \alpha \cap \beta = \delta$, a contradiction to $\delta \in \alpha$ and **??**(**??**).

Now let x be any non-empty subset of Ord. Pick $\alpha \in x$. Suppose α is not a minimal elements of x. Then $\{\beta \in x \mid x \in \alpha\}$ is a non-empty subclass of α and so has a minimal element γ. But then γ is also an minimal element of Ord. Hence any case x has minimal element.

For any $\alpha \in \text{Ord}$, $\text{Ord}_\alpha = \{\beta \in \text{Ord} \mid \beta \in \alpha\} = \alpha$ and so Ord_α is a set. We verified all the defining properties of a well-ordered class and the Theorem is proved. □

Corollary A.4.5. [**intersect ordinals**] *Let A be non-empty class of ordinals. Then $\bigcap A$ is the minimal element of A with respect to $\in$.*

Proof. Since Ord is well ordered with respect to $\in$, **??** shows that A has a minimal elements α. Let $\gamma \in A$. Then $\alpha = \gamma$ or $\alpha \in \gamma$. In any case $\alpha \subseteq \gamma$ and so $\alpha \subseteq \bigcap A$. Since $\bigcap A \subseteq \alpha$, this gives $\bigcap A = \alpha$. □

Lemma A.4.6. [**unions of ordinals**] *Let A be a class of ordinals.*

(a) [**a**] *If $\bigcup A$ is a set, then $\bigcap A$ is an ordinal. In particular, if A is a set, then $\bigcup A$ an ordinal.*

(b) [**b**] *If $\bigcup A$ is not a set, then $\bigcup A = \mathrm{Ord}$.*

Proof.

1°. [**1**] $\bigcup A \subseteq \mathrm{Ord}$

Thus holds since every element of ordinal is an ordinal.

2°. [**2**] *$\in$ well-order $\bigcup A$.*

Since $\in$ well -orders Ord, this follows from (1°).

3°. [**3**] *Every element of $\bigcup A$ is a subset of $\bigcup A$.*

Let $x \in \bigcup A$. Then $x \in \alpha$ for some $\alpha \in A$. Thus $x \subseteq \alpha$. Since $\alpha \subseteq A$ thus gives $x \subseteq A$

(a) If $\bigcup A$ is a set, then (2°) and (3°) shows that $\bigcup A$ is a ordinal.

(b) Suppose now that $\bigcup A$ is not a set and let δ be ordinal. Since δ is a set, and subclasses of sets are sets, we get $\bigcup A \nsubseteq \delta$. Thus there exists $\alpha \in A$ with $\alpha \nsubseteq \delta$. Note that $\alpha = \delta$ or $\alpha \in \delta$ imply $\alpha \subseteq \delta$, a contradiction. Since $\in$ is a totally ordering on Ord we conclude that $\delta \in \alpha$ and so $\delta \in \bigcup A$. Since this holds for all ordinals, $\mathrm{Ord} \subseteq \bigcup A$. So (1°) implies (b). □

A.5 The natural numbers

Definition A.5.1. [**ordering**] *Let α and β be ordinals. We will write $\alpha < \beta$ if $\alpha \in \beta$ and $\alpha \leq \beta$ if $\alpha = \beta$ or $\alpha \in \beta$.*

Lemma A.5.2. [**in and sub**] *Let α and β be ordinals.*

(a) [**a**] *$\alpha \in \beta$ iff $\alpha < \beta$ and iff $\alpha \subset \beta$.*

(b) [**b**] *($\alpha \in \beta$ or $\alpha = \beta$) iff $\alpha \leq \beta$ iff $\alpha \subseteq \beta$.*

(c) [**c**] *If $\alpha < \beta$, then $\alpha + 1 \leq \beta$. So $\alpha + 1$ is the least ordinal larger than α.*

Proof. (a) The first statement is just the definition of $\alpha < \beta$. If $\alpha \in \beta$, then the definition of and ordinal implies $\alpha \subseteq \beta$. Since $\in$ is irreflexive on Ord, $\alpha \neq \beta$ and so $\alpha \subset \beta$. Suppose now that $\alpha \subseteq \beta$. Since $\in$ is total ordering $\alpha \in \beta, \alpha = \beta$ or $\beta \in \alpha$. The last two statements imply that $\beta \subseteq \alpha$, a contradiction to $\alpha \subseteq \beta$. Hence $\alpha \in \beta$.

(b) follows immediately from (a).

(c) Otherwise (b) gives $\beta \in \alpha + 1 = \alpha \cup \{\alpha\}$. So
beta $\in \alpha$ or $\beta = \alpha$, a contradiction to $\alpha \in \beta$. □

Definition A.5.3. [**limit ordinals**] *Let α be an ordinal.*

(a) [**a**] *We say that α is an successor if $\alpha = \beta + 1$ for some ordinal β.In this case β is denoted by $\alpha - 1$.*

(b) [**b**] *We say that α is a limit ordinal, if α is neither zero, nor an ordinal.*

(c) [**c**] *We say that α is a natural number of $\alpha + 1$ contains no limit ordinal.*

(d) [**d**] *$\mathbb{N}$ is the class of natural numbers.*

Note that first $\alpha + 1$ contains no limit ordinal iff neither α nor any element of α is a limit ordinal. α is a natural numbers if and only if either $\alpha = 0$; or α is an successor and each non-zero ordinal β with $\beta \in \alpha$ is successor.

Lemma A.5.4. [**natural numbers**]

(a) [**a**] *Let α and β be ordinal with $\alpha \in \beta$. If β is a natural numbers, so is α.*

(b) [**b**] *Let n be a natural numbers. Then $n + 1$ is a natural number.*

(c) [**c**] *Let n be a non-zero natural number. Then $n - 1$ is a natural number.*

Proof. (a) Observe that $\alpha + 1 \subseteq \beta + 1$. Since $\beta + 1$ contains no limit ordinal, $\alpha + 1$ contains no limit ordinal.

(b) If $x \in n + 1$, then $x \in n$ or $x = n + 1$. In neither case x is limit ordinal.

(c) Observe first that is neither 0 nor a limit. Hence $n - 1$ is defined. Since $n - 1 \in n$, (c) follows from (a). □

Lemma A.5.5. [**induction on n**] *Let A be a class. If $0 \in A$ and $a \cup \{a\} \in A$ for all $a \in A$, then $\mathbb{N} \subseteq A$.*

Proof. Note that $B := \mathbb{N} \setminus A$ is subclass of $\mathbb{N}$. Suppose $B \neq \emptyset$ and let n be the minimal element of B. Then $n \neq 0$. By minimality of n, $n - 1 \in A$ and so also $n = (n-1) + 1 = (n-1) \cup \{n-1\} \in A$, a contradiction. □

Lemma A.5.6. [**n a set**]

(a) [**a**] *$\mathbb{N}$ is a set.*

(b) [**b**] *$\mathbb{N}$ is an ordinal, in fact $\mathbb{N}$ is the smallest limit ordinal.*

Proof. (a) By Set Axiom 6, there exists a set z such that $0 \in z$ and $z \cup \{z\} \in Z$. So by A.5.5, $\mathbb{N} \subseteq z$. Since subclasses of subsets are sets, $\mathbb{N}$ is a set.

(b) Since $\mathbb{N}$ is a subclass of the well-ordered class Ord, $\in$ is a well ordering in $\mathbb{N}$. Let $n \in \mathbb{N}$ and $\alpha \in n$. Then by A.5.4(a), $\alpha \in \mathbb{N}$. So n is a subset of $\mathbb{N}$. Thus $\mathbb{N}$ is an ordinal. Let δ be any limit ordinal. Then $0 \in \delta$ and if $\gamma \in \delta$, then $\gamma + 1 \leq \delta$ and since δ is not a successor. Thus $\gamma + 1 \in \delta$. So A.5.5 implies that $\mathrm{N} \subseteq \delta$, and so $\mathrm{N} \leq \delta$. □

Definition A.5.7. [**def:sum of ordinals**] *Let α and β be ordinals, then the ordinal $\alpha+\beta$ is inductively defined by*

$$\alpha+\beta := \begin{cases} \alpha & \text{if } \beta = 0 \\ (\alpha+\delta)+1 & \text{if } \beta = \delta+1 \\ \bigcup_{\gamma<\beta} \alpha+\gamma & \text{if } \beta \text{ is a limit ordinal} \end{cases}$$

Since $1 = 0+1$ is an ordinal we now have two definitions of $\alpha+1$. But since $\alpha+(0+1) = (\alpha+0)+1 = \alpha+1$, these two definitions agree.

Lemma A.5.8. [**sum of ordinals**] *Let α, β be ordinals and $n, m \in \mathrm{N}$. Then*

(a) [**a**] $(\alpha+\beta)+n = \alpha+(\beta+n)$.

(b) [**b**] $n+m = m+n$ *and* $n+m$ *is a natural number.*

Proof. (a) If $n = 0$, thus is obvious. So suppose (a) is true for n, then

$$(\alpha+\beta)+(n+1) = ((\alpha+\beta)+n)+1 = (\alpha+(\beta+n))+1 = \alpha+((\beta+n)+1) = \alpha+(\beta+(n+1))$$

and so (a) also holds for $n+1$.

(b) If $n = m = 0$, then both sides are zero. Suppose next $0+m = m+0$. Then

$$0+(m+1) = (0+m)+1 = (m+0)+1 = m+1 = (m+1)+0$$

So (**??**) holds whenever $n = 0$. By symmetry it also holds whenever $m = 0$.

Suppose $1+m = m+1$. Then

$$1+(m+1) = (1+m)+1 = (m+1)+1$$

and so (b) holds whenever $n = 1$.

Suppose (b) holds for some $n \in \mathbb{N}$ and all $m \in \mathbb{N}$

$$m+(n+1) = (m+n)+1 = (n+m)+1 = n+(m+1) = n+(1+m) = (n+1)+m$$

and so (b) holds for $n+1$ and for all $m \in \mathbb{N}$. □

Lemma A.5.9. [**decompose ordinals**] *Let α be an ordinal then there exists a non-successor β and a natural numbers n with $\alpha = \beta+n$.*

Proof. Note that $\alpha = \alpha+0$ and so there exists a least ordinal β such that $\alpha = \beta+n$ for some natural numbers n. Suppose that β is a successor and let $\delta = \beta-1$. Then

$$\alpha = \beta+n = (\delta+1)+n = \delta+(1+n) = \delta+(n+1)$$

Since $n+1$ is natural number we get a contradiction to the minimal choice of β. □

A.6 Cardinals

Definition A.6.1. [**def:cardinals**] *Two sets a and b are called isomorphic, if there exits a bijection from a to b. The cardinal $|a|$ of a set a is the least ordinal isomorphic to a.*

Lemma A.6.2. [**injective**] *Let a and b be sets, then there exists a injection from a to b if and only if $|a| \leq |b|$.*

Proof. Let $F : a \to |a|$ and $G : b \to |b|$ be bijection.

Suppose first that $|a| \leq |b|$. Then $|a| \subseteq |b|$. Thus $G^{-1} \circ F$ is an injection from a to b.

Suppose next that $H : a \to b$ is a injection. Then $I = G \circ H \circ F^{-1}$ is an injection from $|a|$ to $|b|$. Put $d = I(|a|$. Then $d \subseteq |b|$. Define $\Phi : d \to \mathrm{Ord}$ inductively by $\Phi(e)$ is the least elements of $\mathrm{Ord} \setminus \{\Phi(c) \mid c \in d, c < e$. We claim that $\Phi(e) \leq e$ for all $e \in d$. Indeed if $c < e$, then by induction $\Phi(e) \leq e$ and so $\Phi(e) \neq e$. Thus $\Phi(e) \leq e$ by defintion of $\Phi(b)$.

Since $\Phi(e) \leq e$ and $|b|$ is an initial segment of Ord, $\Phi(e) \in |b|$. We claim that $\Phi[d]$ is an initial segment of $|b|$. Indeed of $\alpha < \Phi(e)$, then $\alpha = \Phi(c)$ for some $c \in d$ with $c < e$. Thus $\Phi(d)$ is an ordinal, also $\Phi(d) \leq |b|$ and $\Phi(d)$ isomorphic to a. Thus $|a| \leq |\Phi(d) \leq |b|$. □

Corollary A.6.3. [**sb**] *Let a and b sets. If the exits an injection from a to b and an injection from b to a, then a and b are isomorphic.*

Proof. By A.6.2 $|a| \leq |b|$ and $|b| \leq |a|$. Thus $|a| = |b|$ and a and b are both isomorphic to $|a|$. □

Appendix B

Homework

B.1 Homework 3 from MTH912

Let $\mathbb{K}$ be a division ring and V_1, V_2 and V_3 a left $\mathbb{K}$ space. A function $f : V_1 \to V_2, v \to vf$ is called $\mathbb{K}$-linear if $v + \tilde{v})f = vf + \tilde{v}f$ and $kv.fk, vf$ for all $v \in V$ and $k \in \mathbb{K}$. If $f : V_1 \to V_2$ and $g : V_2 \to V_3$ are $\mathbb{K}$-linear, then fg is the $\mathbb{K}$-linear function from $V_1 \to V_2$ defined by $v.fg = vf.g$. $\mathrm{Hom}_{\mathbb{K}}(V_1, V_2)$ denotes the set of all $\mathbb{K}$-linear map from $V_1 \to V_2$. $\mathrm{End}_{\mathbb{K}}(V) = \mathrm{Hom}_{\mathbb{K}}(V, V)$. Note that $\mathrm{End}_{\mathbb{K}}(V)$ is a ring.

Similarly let W_1, W_2 and W_3 a left $\mathbb{K}$ space. A function $f : W_1 \to W_2, w \to fw$ is called $\mathbb{K}$-linear if $f(w, \tilde{w}) = fw + f\tilde{w}$ and $fw.k = f.w$ for all $w, \tilde{w} \in V$ and $k \in \mathbb{K}$. If $f : W_1 \to W_2$ and $g : W_2 \to W_3$ are $\mathbb{K}$-linear, then gf is the $\mathbb{K}$-linear function from $W_1 \to W_2$ defined by $fg.w = f.gw$. $\mathrm{Hom}_{\mathbb{K}}(W_1, W_2)$ denotes the set of all $\mathbb{K}$-linear map from $W_1 \to W_2$.

So we view function on a left vectors space to be acting from the right. while functions on a right vector space act from the left.

Let V be left- and W a right $\mathbb{K}$-space. Let $s : V \times W \to \mathbb{K}$ be a $\mathbb{K}$-bilinear function. So for all $v, \tilde{v} \in V$, $w, \tilde{w} \in W$ and $k \in \mathbb{K}$, $(v + \tilde{v})w = vl + \tilde{v}w$, $v(w + \tilde{w}) = vw + v\tilde{w}$, $kv.w = k.vw$ and $vw.k = vw.k$. Noe that just means taht for each $v \in V$, the map $s_v : W \to W, w \to vw$ is $\mathbb{K}$-linear and for each $w \in W$, the map $s_w : v \to vw$ is $\mathbb{K}$-linear.

Put $E := \mathrm{End}^s_{\mathbb{K}}(V, W)$ be the set of all $(\alpha, \beta) \in \mathrm{End}_{\mathbb{K}}(V) \times \mathrm{End}_{\mathbb{K}}(W)$ such that $v\alpha.w = v.\beta w$. for all $v \in V, w \in W$. Note that V is a right E-module via $v(\alpha, \beta)v\alpha$ and W is a left E-module via $(\alpha, \beta)w = \beta w$. So if $\delta = (\alpha, \beta) \in E$ the $v\delta.w = v.\delta w$ for all $v \in V$, $w \in W$. Observer that E is a subring of $\mathrm{End}_{\mathbb{K}}(V) \times \mathrm{End}_{\mathbb{K}}(W)$.

Define $wv \in \mathrm{End}_{\mathbb{K}}(V_{\times}\mathrm{End}_{\mathbb{K}}(W)$ by $\tilde{v}.wv = \tilde{v}w.v$ and $wv.\tilde{w} = w.v\tilde{w}$ for all $\tilde{v} \in V, \tilde{w} \in W$. We claim that $wv \in E$. Indeed

$$
\begin{aligned}
& (\tilde{v}(wv))\tilde{w} \\
= \; & ((\tilde{v}w)v)\tilde{w} && \text{definition of } wv \\
= \; & (\tilde{v}w)(v\tilde{w}) && s_{\tilde{w}} \text{ is linear} \\
= \; & \tilde{v}(w(v\tilde{w}) && s_{\tilde{v}} \text{ is linear} \\
= \; & \tilde{v}((wv)\tilde{w}) && \text{definition of } wv
\end{aligned}
$$

So $wv \in E$.

Observe that we now have binary operation, $\mathbb{K} \times \mathbb{K} \to \mathbb{K}$, $\mathbb{K} \times V \to V$, $W \times \mathbb{K} \to W$, $V \times E \to V$, $E \times W \to W$ and $E \times E \to E$.

We say that $\mathbb{K}$ has type $(0,0)$, V has type $(0,1)$, W has type $(1,0)$ and E has type $(1,1)$. If X has type (i,j), Y has type (k,l) and Z has type (m,n), then we have a binary operation $X \times Y \to Z$ if and only if $j = k$ and $(m,n) = (i,l)$. In particular, if $x, y, z \in \mathbb{K} \cup V \cup W \cup E$, then $xy.z$ is defined if and only if $xy.z$ is defined.

We will now show if $xy.z$ is defined, then $xy.z = x.yz$. Indeed, almost all of theses equations follows immediately from the definitions, except for $wv.\alpha = w.v\alpha$ and $\alpha w.v = alpha.wv$, there $v \in V, w \in W$ and $\alpha \in E$.

Note that $wv \in E$ and so $wv.\alpha \in E$. So to show that $wv.\alpha = w.v\alpha$ we need to show that they act the same way on V and W. So let $\tilde{V} \in V$ and $\tilde{W} \in W$. Then

$$
\begin{aligned}
& \tilde{v}((wv)\alpha)) = \\
= \; & (\tilde{v}(wv))\alpha && \text{definition of mult. in } E \\
= \; & (\tilde{v}w)v))\alpha && \text{definition of } wv \\
= \; & (\tilde{v}w)(v\alpha) && \alpha \text{ is linear} \\
= \; & \tilde{v}(w(v\alpha)) && \text{definition of } w(v\alpha)
\end{aligned}
$$

B.2 Homework 4 from MTH912

Homework B.2.1. **[t in m']** *Let $\mathbb{F}$ be a division ring, V a left $\mathbb{F}$ space, W a right $\mathbb{F}$ space, $s : V \times W \to \mathbb{F}$ a bilinear form and $\mathcal{N}$ a series of closed $\mathbb{F}$-subspace of V. Let $M = M^s_{\mathcal{N}}(V, W)$ be the corresponding McLain group and let $v \in V^\sharp$ and $w \in W^\sharp$ with $t(v,w) \in M'$. Then $T_v < T_w$. Here $T_v = \bigcap\{E \in \mathcal{N} \mid v \in E\}$ and $T_w = \bigcup\{E \in \mathcal{N} w \in E^\perp$.*

Proof. Since $t(v,w) \in M$ we have $T_v \leq T_w$. Let $B_v = \{bigcupD \mid v \notin D\}$. Then $v \notin B_v$. Since B_v is closed, $v \notin B_v^{\perp\perp}$ and so $B_v^\perp \not\leq v^\perp$. Thus $[t(v,w), B_v^\perp] \neq 0$ and so $w \in w\mathbb{F} = [t(v,w), B_v^\perp]$. On the other hand (B_v, T_v) is a jump of $\mathcal{N}$ and by **??**

$$M' = \{g \in M \mid [B^\perp, g] \leq (T^\perp)^- \text{ for all jumps}(B, T) \text{ of } \mathcal{N}\}$$

Thus $w \in [t(v,w), B_v^\perp] \leq (T_v^\perp)^-$. Since $(T_v^\perp)^- = \bigcup\{D^\perp \mid T_v < D \in \mathcal{N}\}$ we conclude that $w \in D^\perp$ for some $D \in \mathcal{N}$ with $T_v < D$. Then $D \leq T_w$ and so $T_v < T_w$. □

Definition B.2.2. [**def:component**]

(a) [**a**] *If H is an ascending subgroup of G. the $\delta_G(H)$ is the mimial length of an ascending sequence from H to G.*

(b) [**b**] *A component of a group is a quasisimple ascending subgroup of G.*

Homework B.2.3. [**basic components**] *Let K and L be components of a group G and M a subnormal subgroup of G.*

(a) [**a**] *$K = L$ or $[K, L] = 1$.*

(b) [**b**] *$K \leq M$ or $[K, M] = 1$.*

Proof. Let K be a components of G

1°. [**1**] *Let $M \trianglelefteq\trianglelefteq G$. If $K \trianglelefteq \langle K^H\rangle$, then $K \leq M$ or $[K, M] = 1$.*

Suppose first that M is normal in G, that is $\delta_G(M) \leq 1$. Put $H = \langle K^G\rangle$ and assume that $K \leq M$. Then $K \cap M \trianglelefteq K$ and since $K \cap M \neq K$ we get $K \cap M \leq \mathrm{Z}(K)$. Since $H \cap M$ normalize K we have $[H \cap M, K] \leq K \cap M \leq \mathrm{Z}(M)$ and thus $][H \cap M, K, K] = 1$. Hence also $[K, H \cap M, K] = 1$ and the Three Subgroup Lemma implies that $[K, K, H \cap M] = 1$. Since K is perfect, $[H \cap M, K] = 1$. Since H and M are normal in G and $K \leq H$, $[M, K] \leq [M, H] \leq H \cap M$ and so $[M, K, K] = 1$. Another application of the three subgroups lemma shows that $[M, K] = 1$.

Suppose nest tat $\delta_G(M) \geq 2$. The there exists $M \mathrm{asc} M^* \trianglelefteq G$ with $\delta_{M^*}(M) = \delta_G(M) - 1$. If $K \neq M^*$, then by the previous paragraph, $[K, M^*] = 1$ and so also $[K, M] = 1$. If $K \leq M^*$, then by induction on $\delta_G(K)$ we have $K \leq M$ or $[K, M] = 1$. Thus (1°) is proved.

2°. [**1.5**] *Let K and L be components of G with $K \trianglelefteq \langle K^G\rangle$ and $L \trianglelefteq \langle L^G\rangle$. Then $K = L$ or $[K, L] = 1$.*

Since $L \trianglelefteq \langle L^G\rangle$, $L \trianglelefteq\trianglelefteq G$. Thus by (1°), $K \leq L$ or $[K, L] = 1$. By symmetry $L \leq K$ or $[L, K] = 1$ and so (2°) is proved.

Let $(G_\alpha)_{\alpha \leq \delta_G(K)}$ be an ascending sequence from K to G.

3°. [**2**] *Suppose that $K = L$ or $[K, L] = 1$ for all $\beta < \delta$ and all components L of G_β with $\delta_{G_\beta}(K) = \delta_{G_\beta}(K)$. Then $K = K^g$ or $[K, K^g] = 1$ for all $g \in G$ and so $K \trianglelefteq \langle K^G\rangle$.*

If $\gamma \leq \delta$ be minimal with $g \in G_\gamma$. Note that $\gamma = 0$, γ is a limit ordinal or $\gamma = \beta + 1$ for some ordinal β. In the first case $g \in K$ and so $K = K^g$. If the second case, $g \notin \bigcup_{\alpha<\gamma} G_\alpha = G_\gamma$, a contradiction. In the third case g normalizes G_β and so $\delta_{G_\beta}(K) = \delta_{G_\beta}(K^g)$ and K^g is a component of G_β. Hence assumption of (3°) imply that $K = K^g$ or $[K, K^g] = 1$.

4°. [**3**] *$K = L$ or $[K, L] = 1$ for all components K and L of G with $\delta_G(K) = \delta_G(L)$.*

Suppose inductively that $K^* = L^*$ or $[K^*, L^*] = 1$ whenever K^*, L^* are components of a group G^* and $\delta_{G^*}(K^*) = \delta_{G^*}(L^*) < \delta_G(K)$. Then the assumptions of (3°) are fulfilled. Thus $K \trianglelefteq \langle K^G \rangle$. By symmetry, $L \trianglelefteq \langle L^G \rangle$ and so (4°) follows from (2°).

5°. [**4**] *Let $g \in G$. Then $K = K^g$ or $[K, K^g] = 1$. In particular, $K \trianglelefteq \langle K^G \rangle$.*

This follows immediately from (4°).

(a) follows from (5°) and (2°). (b) follows from (5°) and (1°). □

Homework B.2.4. [**component and hp**] *Let K be a component of G. Then $[K, \mathrm{HP}(G)] =$.*

Proof. By **??** $K \leq \mathrm{HP}(G)$ or $[K, \mathrm{HP}(G)] = 1$. In the first case K would be locally nilpotent and so all chief-factors of K would be abelian. But $K/\mathrm{Z}(K)$ is a non-abelian chief-factor of K. □

Definition B.2.5. [**def:invert**] *Let H be a group acting on a abelian group A and I a subset of H and $h \in H$. We say that h inverts A of $a^h = a^{-1}$ for all $a \in A$. We say that I inverts A if each elements of I either centralizes A or inverts A.*

Homework B.2.6. [**basic invert**] *Let H be a group acting on an abelian group A.*

(a) [**a**] *If $I \subseteq H$ with $H = \langle I \rangle$, then H inverts A if and only of I inverts I.*

(b) [**b**] *Let $h \in H$ with $h^2 = 1$. Put $\mathrm{I}_A(h) = \{a \in A \mid a^h = a^{-1}\}$ and $\mathrm{I}^*_h = \{aa^h \mid a \in A\}$.*

- *(a)* [**a**] *$A/ \cong \mathrm{I}_A(h) \cong \mathrm{I}^*_H(h)$ and $A/C_A(h) \cong [A, h]$ as*
- *(b)* [**b**] *$\mathrm{I}_H(a)$ is largest subgroup of A inverted by h and $\mathrm{I}^*(h)$ is the smallest subgroup of A whose quotient is inverted by h.*
- *(c)* [**c**] *$[A, h] \leq \overline{I}_H(a)$ and $\overline{I}^*_H a \leq \mathrm{C}_A(h)$.*

(c) [**c**] *Suppose H is an finite elementary abelian 2-group. Then there exists a finite series*

$$1 = A_0 \leq A_1 \leq \dots A_m = A$$

of H-invariant subgroups of A all of whose factors are inverted by A.

Proof. (a) Let $i, j \in I$. If i and j centralizes A, or i and j inverts A, then ij centralize A. If one of i and j centralizes A and the other inverts A, then ij inverts A. So the set of elements of A which centralizes or inverts A forms a subgroup of H.

(b:a) Consider the homomorphisms $A \to A, a \to aa^h$ and $A \to A, a \to a^{-1}a^h$. The first has $\mathrm{I}_A(h)$ as kernel and $\mathrm{I}_A(h)$ as image. The second has $\mathrm{C}_A(h)$ as kernel and $[A, h]$ a image.

(b:b) Readily verified.

(b:c) $(a^{-1}a^h)^h = (a^{-1})^h a^{h^2} = (a^h)^{-1}a = (a^{-1}a^h)^{-1}$ and $(aa^h)^h = (a^h a^{h^2}) = a^h a = aa^h$.

(c) Let $H = \langle h_1, h_2, \dots h_n \rangle$ for some $h_i \in H$ and put $H_0 = \langle h_1, \dots h_{n-1}$. By (b) h_n inverts $[A, h_n]$ and centralizes $A/[A, h_n]$. Since H is abelian, $[A, h_n]$ is H_0 invariant and so H_0 acts on $[A, h_n]$ and $A/[A, h_n]$. By induction on n there exitss H_0 invariant subgroups,

$$1 = A_0 \leq A_1 \leq \ldots A_t = [A, h_n] \leq A_{t+1} \leq \ldots A_m = A$$

such that H_0 inverts each of the factors. Note h_n inverts each of the factors A_i/A_{i-1} for $1 \leq i \leq t$ and centralizes each the factors A_i/A_{i-1}, $t < i \leq m$. Thus by (b), H each of the factors. □

Homework B.2.7. [**char subsolvable**] *Let G be a group with no non-trivial finite normal subgroup of odd order. Then G is super-solvable if and only if GG is finitely generated and G^2 is nilpotent.*

Proof. Suppose first that G is super solvable. Then G is polycyclic and so finitely generated. Moreover, there exists a strong composition series

$$1 = G_0 \leq G_1 \leq \ldots \leq G_k \leq G_{k+1} \leq G_n = G$$

such that for $1 \leq i \leq k$, G_k/G_{k-1} has odd prime order and for $k < i \leq n$, G_k/G_{k-1} is cyclic of order 2 or ∞. Then G_k is the unique maximal subgroup of odd order. So G_k is normal in G and so by assumption, $G_k = 1$ and thus $k = 0$. It follows that for all $1 \leq i \leq n$, $\mathrm{Aut}(G_i/G_{i-1})$ has order at most 2. Thus G^2 centralizes G_i/G_{i-1}. Hence

$$1 = G_0 \cap G^2 \leq G_1 \cap G^2 \leq \ldots G_n \cap G^2 = G^2$$

is a finite normal series for G^2 all of whose factor are centralized by G^2. Thus G^2 is nilpotent.

Suppose next that G is finitely generated and G^2 is nilpotent. Note that G/G^2 is a finitely generated elementary abelian 2 group and so finite. Since subgroups of finite index in finitely generated group are finitely generated, G^2 is a finitely generated nilpotent groups. Thus every section of G^2 is finitely generated. Let

$$1 = Z_0 \leq Z_1 \leq Z_m = G^2$$

be the upper central series for G^2. But $Z_{m+1} = G$. Then each Z_i is G invariant and Z_i/Z_{i-1} an finitely generated abelian group centralized by G^2. So we can apply B.2.6 with $H = G/G^2$ and $A = Z_i/Z_{i-1}$ to obtain a G invariant series of subgroup

$$Z_{i-1} = Z_{i,0} \leq Z_{i,1} \leq \ldots Z_{i,j_i} = Z_i$$

all of whose factors are inverted by G. Since $Z_{i,j}/Z_{i,j-1}$ is finitely generated there exists a finite series

$$Z_{i,j-1} = Z_{i,j,0} \leq Z_{i,j,1} \leq Z_{i,j,k_{ij}} = Z_{i,j}$$

of subgroups of $Z_{i,j}$ all of whose factors are cyclic. Since G^2 inverts $Z_{i,j}/Z_{i,j-1}$ each of $Z_{i,j,k}$ are G invariant. Thus the $Z_{i,j,k}$ from a supersolvable series for G and G is supersolvable. □

Homework B.2.8. [**char series for supersolvable**] *Let G be a supersolvable group and $p_1 > p_2 > \ldots > p_k$ the order of the strong chief-factors of odd order of G. Then there exists series*

$$1 \leq S_1 \leq S_2 \leq \ldots S_k \leq S_\infty \leq G$$

of characteristic subgroups of G such that G/S_∞ is a finite 2-group, S_∞/S_k is a torsion free nilpotent group, and for $1 \leq i \leq k$, S_i/S_{i-1} is a finite p_i-group.

Proof. Let H be the unique maximal subgroup of odd order of G. Let

$$H_0 \leq H_1 \leq \ldots \leq H_k$$

be chief-series series such that $(|H_1/H_0|, |H_2/H_1|, \ldots, |H_k/H_{k-1}|)$ is maximal in lexiographic order. Suppose that $p := |H_i/H_{i-1}| < q := |H_{i+1}/H_{i-1}$ for some $1 \leq i < k$. Then H_{i+1}/H_{i-1} is a group of order pq. By Sylow's Theorem H_{i+1}/H_{i-1} has a unique Sylow q-subgroups H_i^*/H_{i-1}. But then

$$H_0 \leq H_1 \leq H_{i-1} \leq H_i^* \leq H_i \ldots \leq H_k$$

is a chief-series of G of higher lexiographic order, a contradiction.

Thus $||H_i/H_{i-1}| \leq |H_{i+1}/H_{i-1}$. For $1 \leq j \leq k$ let i_j be maximal with $|H_{i_j}/H_{i_j-1}| = p_j$. Put $S_j = H_{i_j}$, $S_0 = 1$ and $i_0 = 0$ Then

$$S_{j-1} = H_{i_{j-1}} \leq H_{i_{j-1}+1} \leq \ldots H_{i_j} = S_j$$

is a series all of whose factors have order p_j and so S_j/S_{j-1} is a finite p_j-group. Hence S_j is finite $\{p_1, \ldots, p_j\}$ group. Let x be $\{p_1, \ldots, p_j\}$ element in H and pick l minimal with $x \in S_l$. Then xS_{j-1} is a non-trivial $\{p_1, \ldots, p_j\}$ element in the p_l-group S_l/S_{l-1} and so $l \leq j$. Thus S_j is unique maximal subgroup $\{p_1, \ldots, p_j\}$-subgroup of H. Hence S_j is a characteristic subgroup of H and G. Note that $S_k = H$.

Replacing G by G/H we may assume from now on that G has no non-trivial normal finite subgroups of odd order. Choose a supersolvable series

$$1 = G_0 \leq G_1 \leq \ldots \leq G_a \leq \ldots G_b \leq \ldots G_n = G$$

such that

(i) [**i**] $|G_i/G_{i-1}| = \infty$ $1 \leq i \leq a$.

(ii) [**ii**] $|G_i/G_{i-1}| = 2$ for $1 \leq i \leq a$. equals 2 for

(iii) [**iii**] $|G_{b+1}/G_b| = 2$ if $b < n$.

(iv) [**iv**] a is maximal with respect to (i)-(iii).

(v) [**v**] b is minimal with respect to (i)-(iv).

We claim that $b = n$. Suppose not. If $a = b$ then (i)–(iii) are fulfilled with $b+1$ in place of a, contradicting the maximality of a. So $a < b$. Put $\overline{G_{b+1}} = G_{b+1}/G_{b-1}$. Then $\overline{G_b}$ has order 2 and $\overline{G_{b+1}}/\overline{G_b}$ is cyclic of infinite order. Pick $x \in G_b \setminus G_{b-1}$ and $y \in G_{b+1}$ with $\langle y \rangle G_b = G_{b+1}$. Suppose that $\overline{x} \in \langle \overline{y} \rangle$. Then $\overline{G}_{b+1}$ is cyclic and the series

$$G_0 \leq \ldots G_a \leq \ldots \leq G_{b-1} \leq G_{b+1} \leq G_n = G$$

contradiction the maximality of a (if $a = b-1$) and the minimality of b if $a \neq b-1$.

Thus $\overline{x} \notin \langle \overline{y} \rangle$ and $\overline{G}_b = \langle \overline{x} \rangle \times \langle \overline{y} \rangle$. Thus $\overline{G_b} = \langle oy^2 \rangle$. Put $A = G_{b-1}\langle y^2 \rangle$. Then $\overline{A} = \overline{G_{b+1}}^2$ is a characteristic subgroup of $\overline{G_{b+1}}$ and so A is normal in G. Note that A/G_{b-1} is cyclic of infinite order, while AG_b/A and G_{b+1}/AG_b both have order 2. Thus

$$1 = G_0 \leq G_1 \leq \ldots \leq G_a \leq \ldots \leq G_{b-1} \leq A \leq AG_b \leq G_{b+1} \ldots G_n = G$$

contradiction the maximality of a (if $a = b-1$) and the minimality of b if $a \neq b-1$.

So $b = n$ and G/G_a is a finite of order 2^{n-a}. . Let $g \in G$ be a nontrivial element of finite order and let i be minimal with $g \in G_i$. Then gG_{i-1} is an element of finite order in G_i/G_{i-1} and so $i > a$. Thus G_a is torsion free. Put $m = \max n-a, 1$ and $S_\infty = G^{2^m}$. Then S is a characteristic subgroup of G and $S_\infty \leq G_a \cap G^2$. By **??** G^2 is nilpotent and so S_∞ is torsion free and nilpotent. It remains the show that S/S_∞ has finite order. For $1 \leq i \leq a$, G_i/G_{i-1} is cyclic of infinite order. Thus $G_i/G_i^{2^m}G_{i-1}$ has order 2^m and so $G_i/(G_i \cap S_\infty)G_{i-1}$ has order at most 2^m. Thus $G_a/G_a \cap S_\infty$ has order at most 2^{ma} and G/S_∞ has order at most $2^{ma+(n-a)}$. □

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