MTH 481
Lecture Notes
Based on
Harris,Hirst and Mossinghoff,
Combinatorics and Graph Theory

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Chapter 1

Combinatorics

1.1 Some Essential Problems

Theorem 1.1.1 (Sum Rule)**.** *Suppose that $S_1, S_2, \ldots S_m$ are pairwise disjoint finite sets. For $0 \leq i \leq$ let $n_i := |S_i|$ be the size of the set S_i. Then the number of ways to select one element from one of the sets $S_1, S_2, \ldots S_n$ is the sum*

$$n_1 + n_2 + n_3 + \ldots + n_m$$

Example 1.1.2. Count the number of two digits positive integers n which have the following three properties:

(i) the first digit of n is odd.

(ii) n is divisible by 3.

(iii) the second digit is less than the first.

We break the problems into cases bases on the first digit: The first digit is odd and so has to be one of five numbers

$$1, 3, 5, 7, \text{ and } 9$$

For each case we look the consider the possible second digit (which has less than the first) and decide whether the resulting integer is divisible by 3

first digit	second digit less than the first	divisible by 3	number of integers in this case
1	–	–	0
3	$30, 31, 32$	30	1
5	$50, 51, \ldots 54$	$51, 54$	2
7	$70, 71, \ldots 76$	$72, 75$	2
9	$90, 91, \ldots 98$	$90, 93, 96$	3

So according to the sum rule, the total number of such integers is

$$0 + 1 + 2 + 2 + 3 = 8$$

Theorem 1.1.3 (Product Rule)**.** *Suppose $S_1, S_2, \dots S_m$ are finite sets. For $1 \le i \le m$ let $n_i := |S_i|$ be the size of the set S_i. Then the number of ways to select one element for S_1, then select one element from S_2, and so on, ending by selecting one element from S_m, is the product*

$$n_1 n_2 n_3 \dots n_m$$

Example 1.1.4. Count the number of valid US phone numbers.

A US phone number consists of a three digit area code, followed by a three digit exchange code, and then four digit station station code and has to follow these rules:

(i) The area code cannot begin with 0 or 1.

(ii) The second digit of the area code cannot be 9.[1]

(iii) The exchange code cannot begin with 0 or 1.

So there are 8 choices for the first digit of the area code, 9 choices for the second digit of the area code, 10 for the last digit of the area code, 8 for the first digit of the exchange code, 10 for the second and third digit of exchange and 10 for each of the four digits of the station code. So according to the Product Rule the total number of valid phone numbers is

$$(8 \cdot 9 \cdot 10) \cdot (8 \cdot 10^2) \cdot 10^4 = 576 \cdot 10^6 = 5,760,000,000$$

Definition 1.1.5. (a) *$\mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \dots\}$ is the set of integers.*

(b) *$\mathbb{N} = \{0, 1, 2, 3, \dots\}$ is the set of non-negative integers. We will also use the term natural number for a non-negative integer.*

(c) *$\mathbb{Z}^+ = \{1, 2, 3, \dots\}$ is the set of positive integers.*

(d) *$\mathbb{R}$ is the set or real numbers.*

Definition 1.1.6. *Let $n \in \mathbb{N}$, $k \in \mathbb{Z}$ and $x \in \mathbb{R}$.*

(a) *A k-set is a set with exactly k elements.*

(b) *$\binom{n}{k}$ is the number of k-subsets of an n-sets. The expression $\binom{n}{k}$ is called a binomial coefficient and $\binom{n}{k}$ is pronounced n choose k.*

[1] Area codes of the form X9Y are reserved for possible future expansion of the phone number system

(c) $x^{\underline{k}}$ *is inductively defined by* $x^{\underline{0}} := 1$ *and if* $k > 0$ *then* $x^{\underline{k}} := x^{\underline{k-1}}(x-(k-1))$*. So*

$$x^{\underline{k}} = x(x-1)(x-1)\dots(x-(k-1))$$

$x^{\underline{k}}$ *is pronounced* x *to the* k *falling.*

(d) $x^{\overline{k}}$ *is inductively defined by* $x^{\overline{0}} := 1$ *and if* $k > 0$ *then* $x^{\overline{k}} := x^{\overline{k-1}}\big(x+(k-1)\big)$*. So*

$$x^{\overline{k}} = x(x-1)(x-1)\dots(x+(k-1))$$

$x^{\overline{k}}$ *is pronounced* x *to the* k *rising.*

(e) $k! := k^{\underline{k}} = k(k-1)(k-2)\dots 2\cdot 1$. $k!$ *is pronounced* k *factorial.*

Remark 1.1.7. *Let* $k, n \in \mathbb{N}$ *with* $k \le n$*. Then*

$$n^{\underline{k}} = \frac{n!}{(n-k)!}$$

Proof.

$$\begin{aligned} n^{\underline{k}} &= n(n-1)\dots(n-(k-1)) \\ &= \frac{n\cdot(n-1)\cdot(n-2)\dots(n-k+2)\cdot(n-k+1) \quad\cdot\quad (n-k)\cdot(n-k-1)\cdot\ldots\cdot 2\cdot 1}{(n-k)\cdot(n-k-1)\cdot\ldots\cdot 2\cdot 1} \\ &= \frac{n!}{(n-k)!} \end{aligned}$$

□

Example 1.1.8. How many ways are there to order a collection of n objects.

The first object (in any ordering) can be any of the n elements, so there are n choice for the first object.

The second object must be different than the first, so there are $n-1$ choices for the second object.

The third object must be different than the first two, so there are $n-2$ choices for the third object.

⋮

The second to last object must be different than the first $n-2$, so there are 2 choices for the second to last object.

The last object must be different than the first $n-1$, so there is 1 choice for the last object. So according the the product rule there are

$$n! = n\cdot(n-1)\cdot(n-2)\cdot\ldots 2\cdot 1$$

ways to order a collection of n objects.

Example 1.1.9. How many ways are there to choose an ordered list of k-distinct element from a set of n elements?

As in the previous example:

There are n choices for the first element, $n-1$ choices for the second elements, $n-2$-choices fro the third element, ..., $n-\big((k-1)-1\big)$ choices for the $k-1^{\text{th}}$-element and $n-(k-1)$ choices for the k^{th} element. So the total number of choices is

$$n \cdot (n-1) \cdot (n-2) \ldots (n-(k-2)) \cdot (n-k+1) = n^{\underline{k}}$$

Example 1.1.10. Let $k, n \in \mathbb{N}$ with $k \leq n$. How many ways are there to choose an unordered list of k distinct element from a collection of n elements?

Let m be the number of unordered list of k-distinct elements from a collection of n-elements. According to 1.1.8 each of these m list can be ordered in $k!$ ways. Thus the number of ordered list of k-distinct elements from a collection of n elements is

$$m \cdot k!$$

On the other hand by 1.1.9 this number is

$$n^{\underline{k}}$$

Hence $m \cdot k = n^{\underline{k}}$ and so

$$m = \frac{n^{\underline{k}}}{k!} = \frac{n!}{k!(n-k)!}$$

Theorem 1.1.11. *Let $k \in \mathbb{Z}$ and $n \in \mathbb{N}$. Then*

$$\binom{n}{k} = \begin{cases} \frac{n!}{k!(n-k)!} = \frac{n^{\underline{k}}}{k!} & \text{if } 0 \leq k \leq n \\ 0 & \text{if } k < 0 \text{ or } k > n \end{cases}$$

Proof. Suppose first that $0 \leq k \leq n$. By defintion $\binom{n}{k}$ is the number of k-sets in an n-sets, but that is just the number of unordered list of k distinct elements from a collection of n-objection. By 1.1.10 that number is $\frac{n^{\underline{k}}}{k!} = \frac{n!}{k!(n-k)!}$. So the theorem holds in this case.

Suppose $k < 0$. There are no sets of negative cardinality, so there are also no k-subsets of an n-subset. Thus $\binom{n}{k} = 0$.

Suppose $k > n$. No set has a subset of larger size, so so there are also no k-subsets of an n-subset. Thus $\binom{n}{k} = 0$. □

Example 1.1.12. standard deck of card consists of four suits (clubs, spades, hearts and diamond). Each suits consist of thirteens card with distinct face-value (2,3,4,5,7,7,8,9,10, jack, queen, king,ace). A poker hand is set of five cards from a standard card deck.

(a) Count the number of poker hands.

Since there are 52 cards and five cards in each poker hand the number of poker hands is

$$\binom{52}{5} = \frac{52^{\underline{5}}}{5} = 2,598,960$$

(b) How many poker hands have exactly three cards of the same face value?

We can choose such a hand as follows:

First choose one of the 13 face-values: $\binom{13}{1}$ choices.

Nest choose three of the four cards of the choosen face-value: $\binom{4}{3}$ choices.

Finally choose 2 of 48 card of different face-value: $\binom{48}{2}$.

Hence, by the product rule, the total number of such poker hands is

$$\binom{13}{1}\binom{4}{3}\binom{48}{2} = 13 \cdot 4 \cdot \frac{48 \cdot 47}{2} = 58,656.$$

1.2 Binomial Coefficients

Theorem 1.2.1 (Symmetry). *Let $n \in \mathbb{N}$ and $k \in \mathbb{Z}$. Then*

$$\binom{n}{k} = \binom{n}{n-k}.$$

Proof. If $k < 0$ or $k > n$ then also $n - k > n$ or $n - k < 0$. So both sides of the equation are 0.

So suppose $0 \leq k \leq n$. Then also $0 \leq n - k \leq n$.

We will first give an algebraic proof:

$$\binom{n}{k} \overset{1.1.11}{=} \frac{n!}{k!(n-k)!} = \frac{n!}{\big(n-(n-k)\big)!(n-k)!} = \binom{n}{n-k}$$

Next we give a combinatorial proof.

Let A be any set of size n Let B be k-subset of A. Recall that

$$A \smallsetminus B = \{a \in A \mid a \notin B\}$$

and that (for example by the Sum Rule) $A \smallsetminus B$ is $n-k$-subset of A. Also if D is a subset of size $n-k$ of A then $A \smallsetminus D$ is a k-subset of A. Since $A \smallsetminus (A \smallsetminus B) = B$, we conclude that the function

$$B \quad \mapsto \quad A \smallsetminus B$$

is a bijection from the k-subsets of A to the $n-k$-subsets of A with inverse

$$D \quad \mapsto \quad A \smallsetminus D$$

Hence the number of such subsets must be equal, that is

$$\binom{n}{k} = \binom{n}{n-k}$$

□

Theorem 1.2.2 (Addition)**.** *Let $n \in \mathbb{Z}^+$ and $k \in \mathbb{Z}$. Then*

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

Proof. If $k < 0$ or $k > n$, all three terms are equal to zero. If $k = n$, then the first two terms are equal to 1 and the last terms is equal to 0. If $k = 0$, the first and the last terms is equal to 1, while the middle term is 0. In any of the three cases the equation holds. So suppose $1 \leq k \leq n-1$.

Again we will first give an algebraic proof:

$$\begin{aligned}
\binom{n-1}{k-1} + \binom{n-1}{k} &= \frac{(n-1)!}{(k-1)!\big((n-1)-(k-1)\big)!} + \frac{(n-1)!}{k!(n-1-k)!} && \mid \quad 1.1.11 \\
&= \frac{(n-1)!}{(k-1)!(n-k-1)!(n-k)} + \frac{(n-1)!}{k(k-1)!(n-k-1)!} \\
&= \frac{(n-1)!}{(k-1)!(n-k-1)!}\left(\frac{1}{n-k} + \frac{1}{k}\right) \\
&= \frac{(n-1)!}{(k-1)!(n-k-1)!}\left(\frac{k+(n-k)}{k(n-k)}\right) \\
&= \frac{(n-1)!}{(k-1)!(n-k-1)!}\frac{n}{k(n-k)} \\
&= \frac{n!}{k!} && \mid \quad 1.1.11
\end{aligned}$$

Next we give a combinatorial proof: Let $A = \{1, 2, \ldots, n\}$. By definition $\binom{n}{k}$ is the number of k-subsets of A. We now count such subsets B by consider the two cases $n \in B$ and $n \notin B$.

If $n \in B$, then $B \setminus \{n\}$ (the remaining elements of B) form a $k-1$-subset of $1, \ldots, n-1$. So there are $\binom{n-1}{k-1}$ choices of $B \setminus \{b\}$, and hence there are also $\binom{n-1}{k-1}$ k-subsets of A containing n.

If $n \notin B$, then B is a k-subset of the $n-1$-set $\{1, \ldots, n-1\}$. Thus there are $\binom{n-1}{k}$ k-subsets of A which do not contain n. The Sum Rule now shows that

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

□

The addition rules allows as to easily compute the all the binomial coefficients using what is called the Pascal's triangle: Write $\binom{n}{k}$'s $0 \leq k \leq n$ in form of a triangle:

$$
\begin{array}{c}
\binom{0}{0} \\[1em]
\binom{1}{0} \quad \binom{1}{1} \\[1em]
\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2} \\[1em]
\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3} \\[1em]
\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4} \\[1em]
\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5} \\[1em]
\vdots \qquad\qquad\qquad\qquad \vdots \\[1em]
\binom{n-1}{0} \; \binom{n-1}{1} \quad \cdots \quad \binom{n-1}{k-1} \; \binom{n-1}{k} \quad \cdots \quad \binom{n-1}{n-2} \; \binom{n-1}{n-1} \\[1em]
\binom{n}{0} \quad \binom{n}{1} \quad \cdots \quad \cdots \quad \binom{n}{k} \quad \cdots \quad \cdots \quad \binom{n}{n-1} \quad \binom{n}{n}
\end{array}
$$

According to the Addition rule any entry (except the two outer one in each row) is the sum of the two entries immediately above it. Note also the $\binom{n}{0} = 1$ and $\binom{n}{n} = 1$. This leads to

$$
\begin{array}{c}
1 \\[1em]
1 \quad 1 \\[1em]
1 \quad 2 \quad 1 \\[1em]
1 \quad 3 \quad 3 \quad 1 \\[1em]
1 \quad 4 \quad 6 \quad 4 \quad 1 \\[1em]
1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1 \\[1em]
1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1 \\[1em]
1 \quad 7 \quad 21 \quad 35 \quad 35 \quad 21 \quad 7 \quad 1 \\[1em]
1 \quad 8 \quad 28 \quad 56 \quad 70 \quad 56 \quad 28 \quad 8 \quad 1
\end{array}
$$

Theorem 1.2.3. *Let $n, m \in \mathbb{N}$. Then*

$$\binom{n+1}{m+1} = \sum_{k=0}^{n} \binom{k}{m}$$

Proof. We will first give an algebraic proof using induction on n. For this we first consider the case $n = 0$. The left side is equal

$$\binom{1}{m+1} = \begin{cases} \binom{1}{1} = 1 & \text{if } m = 0 \\ 0 & \text{if } m \geq 1 \end{cases}$$

The right side has just one summand and is equal to

$$\binom{0}{m} = \begin{cases} \binom{0}{0} = 1 & \text{if } m = 0 \\ 0 & \text{if } m \geq 1 \end{cases}$$

So the equation holds for $n = 0$. Suppose it holds for n. Then

$$\begin{aligned} \sum_{k=0}^{n+1} \binom{n}{k} &= \binom{n+1}{m} + \sum_{k=0}^{n} \binom{k}{n} \\ &= \binom{n+1}{m} + \binom{n+1}{m+1} && \text{– Induction Assumption} \\ &= \binom{(n+1)+1}{m+1} && \text{– Addition Rule 1.2.2} \end{aligned}$$

So the equation also holds for $n + 1$.

Next we give a combinatorial proof by counting the number of $m + 1$ sets in the $n + 1$ set $A = \{0, 1, \ldots, n\}$. We choose an arbitrary $m + 1$-subset B of A as follows:

First we choose the largest element k of B. Note that k can be any of the numbers $0, 1, 2, \ldots n$.

Next for a give k we choose the remaining m elements of B. They all are smaller than k and so form an m-subset of the k-set $\{0, 1, 2, \ldots, k-1\}$. So there are $\binom{k}{m}$ choices for $B \setminus \{k\}$. Thus, by the Sum Rule, the total number of $m + 1$ sets in a $n + 1$ set is $\sum_{k=0}^{n} \binom{k}{m}$ and so indeed

$$\binom{n+1}{m+1} = \sum_{k=0}^{n} \binom{k}{n}$$

□

Example 1.2.4. Compute $(x+4)^4$.

$$
\begin{aligned}
& (x+y)^4 \\
= \; & (x+y)(x+y)(x+y)(x+y) \\
= \; & yyyy && \text{1 term: } \binom{4}{0} \text{ ways to choose the 0 positions of } x \\
& + xyyy + yyxy + yxyy + yyyx && \text{4 terms: } \binom{4}{1} \text{ ways to choose the 1 position of } x \\
& + xxyy + xyxy + xyyx + yxxy + yxyx + yyxx && \text{6 terms: } \binom{4}{2} \text{ ways to choose the 2 positions of } x \\
& + xxxy + xxyx + xyxx + yxxx4 && \text{4 terms: } \binom{4}{3} \text{ ways to choose the 3 positions of } x \\
& + xxxx && \text{1 term: } \binom{4}{4} \text{ ways to choose the 4 positions of } x \\
= \; & y^4 + 4xy^3 + 6x^2y^2 + 4x^3y + y^4 \\
= \; & \sum_{k=0}^{4} \binom{4}{k} x^k y^{4-k}
\end{aligned}
$$

Theorem 1.2.5 (Binomial Theorem). *Let $x, y \in \mathbb{R}$ and $n \in \mathbb{N}$. Then*

$$(x+y)^n = \sum_{k\in\mathbb{Z}} \binom{n}{k} x^k y^{n-k} = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

Proof. We first give a proof by induction on n. For $n = 0$ both sides are equal to 1 and so the equation holds. Suppose it holds for n. Then

$$
\begin{aligned}
(x+y)^{n+1} &= (x+y)(x+y)^n \\
&= (x+y) \sum_{k\in\mathbb{Z}} \binom{n}{k} x^k y^{n-k} && \text{– induction assumption} \\
&= x \cdot \sum_{k\in\mathbb{Z}} \binom{n}{k} x^k y^{n-k} + y \cdot \sum_{k\in\mathbb{Z}} \binom{n}{k} x^k y^{n-k} && \text{– Distributive Law} \\
&= \sum_{k\in\mathbb{Z}} \binom{n}{k} x^{k+1} y^{n-k} + \sum_{k\in\mathbb{Z}} \binom{n}{k} x^k y^{n-k+1} && \text{– Distributive Law} \\
&= \sum_{k\in\mathbb{Z}} \binom{n}{k-1} x^k y^{n-(k-1)} + \sum_{k\in\mathbb{Z}} \binom{n}{k} x^k y^{(n+1)-k} && \text{– substitution: } k \to k-1 \text{in the left summand} \\
&= \sum_{k\in\mathbb{Z}} \left[\binom{n}{k-1} + \binom{n}{k} \right] x^k y^{(n+1)-k} && \text{– Distributive Law} \\
&= \sum_{k\in\mathbb{Z}} \binom{n+1}{k} x^k y^{(n+1)-k} && \text{– Addition Rule 1.2.2}
\end{aligned}
$$

So the Binomial Theorem also holds for $n+1$, and this for all $n \in \mathbb{N}$.

Let $A = \{x, y\}$.

$$\begin{aligned}(x+y)^n &= \underbrace{(x+y)(x+y)\dots(x+y)}_{n\text{-times}} \\ &= \sum_{z_1\in A}\sum_{z_2\in A}\dots\sum_{z_n\in A} z_1z_2\dots z_n\end{aligned}$$

To each monomial $z_1z_2\dots z_n$ we can associate a subset I of $\{1,\dots,n\}$, namely

$$I = \{i \in \{1,\dots,n\} \mid z_i = x\}$$

Let $k = |I|$. Then k of the z_i's are equal to x and $n-k$ of the z_i's are equal to y. Thus

$$z_1z_2\dots z_n = x^ky^{n-k}$$

There are $\binom{n}{k}$ subsets of size k, hence exactly $\binom{n}{k}$ of the monomials $z_1\dots z_n$ are equal to x^ky^{n-k}. This gives

$$\sum_{z_1\in A}\sum_{z_2\in A}\dots\sum_{z_n\in A} z_1z_2\dots z_n = \sum_{k=0}^{n}\binom{n}{k}x^ky^{n-k}$$

and so the binomial theorem holds. □

Example 1.2.6. A lotto game consists of selecting four different numbers from 1 to 10 to match a randomly drawn set of four such numbers (the winning numbers). For each $0 \le k \le 4$ determine the number of selections which match exactly k of the winning numbers.

$k = 0$: All four numbers are from the 6 loosing numbers: $\binom{6}{4} = \binom{6}{2} = 15$ selections.
$k = 1$: Choose one of the four winning numbers and then 3 of 6 losing numbers: $\binom{4}{1}\cdot\binom{6}{3} = 4\cdot 20 = 80$.
$k = 2$: Choose two of the four winning numbers and then 3 of 6 losing numbers: $\binom{4}{2}\cdot\binom{6}{2} = 6\cdot 15 = 90$.
$k = 3$: Choose three of the four winning numbers and then 1 of 6 losing numbers: $\binom{4}{3}\cdot\binom{6}{1} = 6\cdot 6 = 24$.
$k = 4$: All four number are winning numbers: $\binom{4}{4} = 1$
Total number of selections:

$$15 + 80 + 90 + 24 + 1 = 210$$

Of course the total number of selection is $\binom{10}{4} = \frac{10\cdot 9\cdot 8\cdot 7}{4\cdot 3\cdot 2} = 10\cdot 3\cdot 7 = 210$. We derived the equation

$$\sum_{k=0}^{4}\binom{4}{k}\binom{6}{4-k} = \binom{4+6}{4}$$

This is referred to as the Convolution rule:

Theorem 1.2.7 (Convolution)**.** *Let $m, n \in \mathbb{N}$ and $l, p \in \mathbb{Z}$. Then*

(a) $\binom{n+m}{l} = \sum_{k\in\mathbb{Z}} \binom{n}{k}\binom{m}{l-k}$.

(b) $\binom{n+m}{p+l} = \sum_{k\in\mathbb{Z}} \binom{n}{p+k}\binom{m}{l-k}$.

Proof. (a): Let A and B be disjoint sets of size m and n, respectively. Then $D := A \cup B$ is a set of size $n+b$. We will choose an arbitrary l-subsets E of D has follows:

We first choose how many elements of E are contained in A. Call this number k. k can be any integer. Note that E will contain exactly $l-k$ elements of B.

Next we choose the k-elements of A which are contained E: $\binom{m}{k}$ choices. Finally we choose the $l-k$-element of B which are contained in E: $\binom{n}{l-k}$ choices. Hence the total number of choices is

$$\sum_{k\in\mathbb{Z}} \binom{m}{k}\binom{n}{l-k}$$

Of course the total number of l-subsets of the $m+n$-set D is $\binom{m+n}{l}$ and so (a) holds.

(b) follows from (a) via the substitution $l \to l+p$ and $k \to k+p$. Note here that $(p+l)-(p+k) = l-k$. □

Remark 1.2.8. *The Addition Rule 1.2.2 is the special case $m = 1$ in the Convolution rule 1.2.7.*

Proof. For $n = 1$ 1.2.7 says

$$\binom{1+n}{l} = \sum_{k\in\mathbb{Z}} \binom{1}{k}\binom{n}{l-k}$$

Note that $\binom{1}{k} = 1$ for $k = 0$ and $k = 1$ and $\binom{1}{k} = 0$ otherwise. So the right side becomes

$$1 \cdot \binom{n}{l-0} + 1 \cdot \binom{n}{l-1}$$

and thus

$$\binom{n+1}{l} = \binom{n}{l} + \binom{n}{l-1}.$$

□

1.3 Multinomial Coefficients

Example 1.3.1 (Placements)**.** Consider 10 objects and 3 boxes. How many are there to place three of the 10 objects in the first box, 2 objects into the second box, and four objects into the third box? Here we assume that the order in the which elements are placed in one of the boxes is irrelevant, but the order of the boxes is relevant. Also each objects is only placed in one box.

There are $\binom{10}{3}$ to choose the three objects to be placed in Box 1. Then there are $\binom{7}{2}$ ways the choose two objects from the remaining 7 objects to be placed in Box 2. Finally there are $\binom{5}{4}$ ways to choose four objects of the remaining 5 objects to be placed in Box 3.

$$\binom{10}{3}\cdot\binom{7}{2}\cdot\binom{5}{4} = \frac{10!}{3!7!}\cdot\frac{7!}{2!5!}\cdot\frac{5!}{4!}1! = \frac{10!}{3!2!4!1!} = \frac{10\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5}{3\cdot 2\cdot 2} = 10\cdot 9\cdot 7\cdot 4\cdot 5 = 90\cdot 140 = 12,600.$$

Definition 1.3.2. *Let $m, n \in \mathbb{N}$ and $k_1, k_2 \ldots, k_m \in \mathbb{Z}$. Given n objects and m boxes labeled Box 1 to Box m. Then*

$$\binom{n}{k_1, k_2, \ldots, k_m}$$

is the number of ways to place k_1 of the n-objects in Box 1, then place k_2 objects in Box 2 of the, ... and k_m objects in Box m. (If $m = 0$, we take the view that there is a unique way to do nothing and define $\binom{n}{}$ to be 1.)

The expression $\binom{n}{k_1,k_2,\ldots,k_m}$ is called a multinomial coefficient.

Remark 1.3.3. *Let $n \in \mathbb{N}$ and $k \in \mathbb{Z}$. Then we have given two definitions of $\binom{n}{n}$. Namely the binomial $\binom{n}{k}$ which counts the number of ways to choose k elements from a set of n, and the multinomial $\binom{n}{k}$ which counts the number of ways to place k of the n objects into one box. But observe that both of the number are the same.*

Theorem 1.3.4. *Let $n, m, l \in \mathbb{N}$ and $k_1, k_2 \ldots, k_m \in \mathbb{Z}$ with $l \le m$.*

$$\binom{n}{k_1, k_2, \ldots, k_m} = \binom{n}{k_1, \ldots, k_l}\binom{n - k_1 - \ldots k_l}{k_{l+1}, \ldots, k_m}$$

Proof. To place k_i of the n-objects into Box i (for $1 \le i \le m$, we can first place k_i of the m-objects into Box i (for $1 \le i \le l$) and then place k_j of the remaining n_0 objects into Box j for $l+1 \le j \le m$. The product rule now implies the Theorem. □

Theorem 1.3.5. *Let $n, m \in \mathbb{N}$ and and $k_1, k_2 \ldots, k_m \in \mathbb{Z}$. Put $k_0 := n - \sum_{i=1}^m k_i$.*

$$\binom{n}{k_1, k_2, \ldots, k_m} = \begin{cases} \frac{n!}{k_0!k_1!\ldots k_m!} & \text{if } k_i \ge 0 \text{ for all } 0 \le i \le m \\ 0 & \text{if } k_i < 0 \text{ for some } 0 \le i \le m \end{cases}$$

Proof. If $k_i < 0$ for some $1 \le i \le n$, there is no way to place k_i-object the equation into a box, so both sides of the equation are 0. If $k_0 < 0$, then $n < \sum_{i=1}^n k_i$ and again there is now the to place k_i of the n-objects into a box for $1 \le i \le m$. So we now assume that $k_i \ge 0$ for all $0 \le i \le m$ and proceed by induction on m.

Suppose that $m = 0$. Then $\binom{n}{} = 1$. Also $k_0 = n - \sum_{i=1}^m k_i = n - 0 = n$ and so $\frac{n!}{k_0!} = \frac{n!}{n!} = 1$. So gain the equation holds.

Suppose next $m > 0$ and that the theorem holds for $m-1$. Put $n_0 := n - \sum_{i=0}^{m-1} k_i$ and note that $n_0 - k_m = k_0$.

$$\begin{aligned}
\binom{n}{k_1,\dots,k_m} &= \binom{n}{k_1,\dots,k_{m-1}} \cdot \binom{n_0}{k_m} && \Big|\ \text{1.3.4 with } l = m-1 \\
&= \frac{n!}{n_0!k_1!\dots k_{m-1}!}\frac{n_0!}{k_m!(n_0-k_m)!} && \Big|\ \text{Induction Hypothesis} \\
&= \frac{n!}{k_1!\dots k_{m-1}!k_m!k_0!} \\
&= \frac{n!}{k_0!k_1\dots k_m!}
\end{aligned}$$

So the Theorem holds for m. □

Example 1.3.6. Given 10 objects and three boxes. How many ways to place 4 objects in Box 1, 3 objects in Box 2 and 2 objects in Box 3?

This is the same as the number of ways to place 3 objects in Box 1, 2 objects in Box 2 and 4 objects in Box 3. Hence the answer is the same as in Example 1.3.1, that is there are 12,600 ways. Algebraically, this says

$$\binom{10}{3,4,2} = \frac{10!}{1!4!3!2} = \frac{10!}{1!3!2!4!} = \binom{10}{3,2,4}$$

In general the same argument gives:

Theorem 1.3.7 (Symmetry)**.** *Let $n \in \mathbb{N}, m \in \mathbb{Z}^+$ and $k_1,\dots,k_m \in \mathbb{Z}$. Then for any permutation* [2] *π from $\{1,\dots,m\}$:*

$$\binom{n}{k_{\pi(1)},\dots,k_{\pi(m)}} = \binom{m}{k_1,\dots,k_m}$$

Theorem 1.3.8 (Addition)**.** *Let $n \in \mathbb{Z}^+$, $m \in \mathbb{N}$ and $k_1,\dots,k_m \in \mathbb{Z}$ with $n = \sum_{i=1}^m k_i$. Then*

$$\binom{n}{k_1,k_2,\dots,k_m} = \binom{n-1}{k_1-1,k_2,\dots,k_m} + \binom{n-1}{k_1,k_2-1,\dots,k_m} + \dots + \binom{n-1}{k_1,k_2,\dots,k_m-1}.$$

Proof. Consider the problem to place k_i of the n-objects into Box i (for $1 \le i \le m$). Since $n = \sum_{i=1}^n k_i$ each object will be placed into a box. Fix one of the objects, say Object A. Then A will be placed into Box j for some $1 \le j \le m$. For a fixed j, we still need to place $k_j - 1$ objects of the remaining $n-1$ objects into Box j, and for $1 \le i \le m$ with $i \ne j$ we need to place k_i objects into Box i. So there are

$$\binom{n-1}{k_1,\dots,k_{j-1},k_j-1,k_{j+1},\dots,k_m}$$

possible placements. This holds for each $1 \le j \le m$ and the Addition Rule shows the total number of placements is

[2] A permutation of a set I is bijection from I to I

$$\binom{n-1}{k-1, k_2, \dots, k_m} + \binom{n-1}{k_1, k_2 - 1, \dots, k_m} + \dots + \binom{n-1}{k_1, k_2, \dots, k_m - 1}$$

On the other hand the total number of placements is $\binom{n}{k_1, k_2, \dots, k_m}$ and the Addition Theorem is proved. □

Remark 1.3.9. *The addition rule for binomial coefficients follows from the addition rule for multinomial coefficients.*

Proof. Let $n \in \mathbb{Z}^+$ and $k \in \mathbb{N}$. Then

$$\binom{n}{k} = \binom{n}{k, n-k} = \binom{n-1}{k-1, n-k} + \binom{n-1}{k, n-k-1} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

□

Notation 1.3.10. *Let $n \in \mathbb{N}$ and $a_1, \dots, a_n$ objects. Then*

(a) $\{a_1, \dots, a_n\}$ *denotes the set with elements $a_1, \dots, a_n$. So $\{a_1, \dots, a_n\}$ and for any object b*

$$b \in \{a_1, \dots, a_n\} \quad \Longleftrightarrow \quad a = a_j \text{ for some } \leq j \leq n$$

To sets are equal if and only if the have the same elements, so the order in which the elements are listed does not matter, and also repeating an element does not change the set. For example

$$\{1,2,3,4\} = \{2,3,1,4\} = \{1,2,3,2,1,4\}$$

(b) $(a_1, \dots, a_n)$ *denotes the sequence formed by $a_1, \dots, a_n$. Sequences $(a_1, \dots, a_n)$ and $(b_1, \dots, b_m)$ are equal if and only of $n = m$ and $a_i = b_i$ for all $1 \leq i \leq n$. So order and repeated elements matter. For example*

$$(1,2,3,4) \neq (1,3,2,4) \neq (1,3,1,2,4)$$

(c) $[a_1, a_2, \dots, a_n]$ *denotes the multiset formed by $a_1, \dots, a_n$. Multisets $[a_1, \dots, a_n]$ and $[b_1, \dots, b_m]$ are equal if and only if there exists a bijection $\pi : \{1, \dots, n\} \to \{1, \dots m\}$ with $b_i = a_{\pi(i)}$ for all $1 \leq i \leq n$. So order does not matter, but repeats do. For example*

$$[1,2,3,4] = [1,3,2,4] \neq [1,3,2,4,1]$$

(d) *For convenience we will use the notation $a_1 a_2 \dots a_n$ to denote both sequences and multisets. We will only do this if, in the given context, it does not matter whether $a_1 a_2 \dots a_n$ is a sequence or multiset, or if we clearly pointed out that we meant a sequence or meant a multiset.*

(e) *We will also use the terms 'ordered lists' and 'string' for a sequence and the term 'unordered list' for a multiset.*

Example 1.3.11. How many ways to order the unordered list

$$[a, b, a, a, c, c, a, c, b, c, c]$$

We first count how often each element appears in the list:

$$a: \ 4, \quad b: 2, \quad c: \ 5$$

The length of the list is 11. So if the elements would be distinct there would exists 11! different ways to order the list. But permuting the four position of the a's has no effect on our ordered list. The same holds for two positions of the b"s and the 5 positions of c. Hence there are $4! \cdot 2! \cdot 5!$ ways to permute to positions without effected the list. So the number of ways to order the list is

$$\frac{11!}{4!2!5!}$$

and so number of ways to order the list is

$$\binom{11}{4, 2, 5}$$

We will know explain why this the same as placing 4 of 11 objects in Box 1, 2 into Box 2 and 5 into Box 2. Label the three boxes with a, b, c. Given an ordering, say

$$(a, b, c, a, a, c, a, c, c, b, c)$$

We will place the numbers from 1 to 11 into boxes as follows. a appears in the first position, so we place 1 into the Box a (the box labelled a). b is in the second position,, so the place 2 into Box a and so on

number:	1	2	3	4	5	6	7	8	9	10	11
box:	a	b	c	a	a	c	a	c	c	b	c

So Box a contains $1, 4, 5$ and 7. Box b contains 2 and 10, and Box c contains 3,6,8,9 and 11.

If we know which elements are in Box a, we know exactly in which position the a's appear. So the ordering is completely determined by the box placement. Hence we found a bijection between the ordered lists and the box placements. This show that the number of ways to order the multiset is indeed $\binom{11}{4,2,5}$.

Definition 1.3.12. *Let $A = a_1 a_2 \dots a_n$ be an ordered or unordered list of objects.*

(a) *n is called the length of A.*

(b) *We say that A is from the set B if $a_i \in B$ for all $1 \leq i \leq n$.*

(c) *A is called non-repeating if $a_i \neq a_j$ for all $1 \leq i < j \leq n$.*

(d) *Let b be an object and $k := |\{1 \le j \le n \mid b = a_i\}$. k is called the multiplicity of b in A and is denoted by* $\mathrm{mult}_b(A)$. *We will also say that b appears exactly k-times in A.*

(e) *We say that A has multiplicities $(k_1, k_2, \dots k_m)$ with respect to the non-repeating sequence $(b_1, \dots, b_m)$ if $\{a_1, \dots, a_n\} \subseteq \{b_1, \dots b_m\}$ and $k_i = \mathrm{mult}_{b_i}(A)$ for all $1 \le i \le m$. In this case we will also say that A has multiplicities $[k_1, \dots, k_m]$.*

(f) *If A is an unordered list with multiplicity $(k_1, \dots, k_n)$ with respect to $(b_1, \dots, b_n)$ we will denoted A by $[b_1^{*k_1}, \dots, b_n^* k_n]$ and also by $b_1^{*k_1} \dots b_n^{*k_n}$.*

Example 1.3.13. Consider the sequence or multiset $A = 122133323422232$. Then A has length 15 and

$$\mathrm{mult}_1(A) = 2, \quad \mathrm{mult}_2(A) = 6, \quad \mathrm{mult}_3(A) = 5, \quad \mathrm{mult}_4(A) = 1, \mathrm{mult}_5(A) = 0$$

Hence the multiplicities of A with respect to $(1, 2, 3, 4, 5)$ are $(2, 6, 5, 1, 0)$.

With this notation we can now state Example 1.3.11 as a general theorem.

Theorem 1.3.14. *An unordered list of length n and multiplicities $[k_1, \dots, k_m]$ can be ordered in $\binom{n}{k_1, \dots, k_m}$ ways.*

Example 1.3.15. Compute the coefficient of $x^{12}y^9z^{21}$ in $(x + y + z)^{42}$.

Expanding $(x + y + z)^{42}$ using the Distributive Law we see that $(x + y + z)^{42}$ is a sum of terms $z_1 \dots z_{42}$ there each z_i is one of x, y, z. Using the Commutative Law this product is equal to $x^k y^l z^m$ with $k, l, m \in \mathbb{N}$ and $k + l + m = 42$. Observe that product $z_1 \dots z_{42}$ is equal to the product $x^{12}y^9z^{21}$ if and only if ordered list $z_1 \dots z_{42}$ is an ordering of the unordered list $x^{12}y^9z^{21}$. By 1.3.14 the number of such ordering is $\binom{42}{12,9,21}$.

The general version of this argument shows:

Theorem 1.3.16 (Multinomial Theorem)**.** *Let $n, m \in \mathbb{N}$. Then*

$$(x_1 + x_2 + \dots + x_m)^n = \sum_{\substack{k_1, k_2, \dots k_m \in \mathbb{Z} \\ k_1 + k_2 + \dots + k_m = n}} \binom{n}{k_1, \dots k_m} x_1^{k_1} x_2^{k_2} \dots x_n^{k_m}$$

Proof. In addition to the above combinatorial argument we will give an algebraic proof by induction on n.

If $m = 0$, then both sides are equal to 0 and if $m = 1$ both sides are equal to x_1^m. Suppose now that $m \ge 2$ and the Theorem holds for $m - 1$ and all n. Then

$$
\begin{aligned}
&(x_1 + x_2 + \dots x_m)^n \\
=&\big(x_1 + (x_2 + \dots x_m)\big)^m && \text{Associative Law} \\
=&\sum_{k_1 \in \mathbb{Z}} \binom{n}{k_1} x_1^{k_1} (x_2 + \dots x_m)^{n-k_1} && \text{Binomial Theorem} \\
=&\sum_{k_1 \in \mathbb{Z}} \binom{n}{k_1} x_1^{k_1} \sum_{\substack{k_2,\dots k_m \in \mathbb{Z} \\ k_2 + \dots + k_m = n - k_1}} \binom{n}{k_2, \dots k_m} x_2^{k_2} \dots x_m^{k_m} && \text{Induction Hypothesis} \\
=&\sum_{k_1 \in \mathbb{Z}} \sum_{\substack{k_2,\dots k_m \in \mathbb{Z} \\ k_2 + \dots + k_m = n - k_1}} \binom{n}{k_1} \binom{n - k_1}{k_2, \dots k_m} x_1^{k_1} x_2^{k_2} \dots x_m^{k_m} && \text{Distributive Law} \\
=&\sum_{k_1 \in \mathbb{Z}} \sum_{\substack{k_2,\dots k_m \in \mathbb{Z} \\ k_2 + \dots + k_m = n - k_1}} \binom{n}{k_1, k_2, \dots k_m} x_1^{k_1} x_2^{k_2} \dots x_m^{k_m} && \text{Theorem 1.3.4} \\
=&\sum_{\substack{k_1, k_2,\dots k_m \in \mathbb{Z} \\ k_1 + k_2 + \dots + k_m = n}} \binom{n}{k_1, k_2, \dots k_m} x_1^{k_1} x_2^{k_2} \dots x_m^{k_m}
\end{aligned}
$$

□

Example 1.3.17. Use the Multinomial Theorem to compute $(x + y + z)^3$.

$$
\begin{aligned}
(x + y + z)^3 = &\binom{3}{3,0,0} x^3 y^0 z^0 + \binom{3}{0,3,0} x^0 y^3 z^0 + \binom{3}{0,0,3} x^0 y^0 z^3 \\
&+ \binom{3}{2,1,0} x^2 y^1 z^0 + \binom{3}{2,0,1} x^2 y^0 z^1 + \binom{3}{1,2,0} x^1 y^2 z^0 \\
&+ \binom{3}{0,2,1} x^0 y^2 z^1 + \binom{3}{1,0,2} x^1 y^0 z^2 + \binom{3}{0,1,2} x^0 y^1 z^2 \\
&+ \binom{3}{1,1,1} x^1 y^1 z^1 \\
= &\ x^3 + y^3 + z^3 + 3(x^2 y + x^2 z + x y^2 + y^2 z + x z^2 + y z^2) + 6xyz
\end{aligned}
$$

1.4 The Pigeon Hole Principal

Theorem 1.4.1 (Pigeon Hole Principal)**.** *Let $n, k \in \mathbb{Z}^+$ with $k > n$. If k objects are placed in n boxes, then at least one box contains at least two objects.*

Proof. For $1 \le i \le n$ let m_i be the number of objects in box i. Then $k = \sum_{i=1}^{n} k_i$. Suppose for a contradiction that no box contains at least two elements. Then $m_i \le 1$ for all $1 \le i \le n$. Thus

$$k = \sum_{i=1}^{n} m_i \le \sum_{i=1}^{n} 1 = n$$

contrary to the assumption that $k > n$. □

Example 1.4.2. (1) If 400 students are enrolled in a class than at least two have there birthday on the same day of a year.

(2) A human has no more than 250,000 hair on there head. The City of Phoenix as more that 1.5 Million residents. So at least people in Phoenix have the same number of hair on their head. Actually there are at least six people in Phoenix have the same number of hair.

Theorem 1.4.3 (Pigeon Hole Principal)**.** *Let $k, n, m \in \mathbb{Z}^+$ with $k > nm$. If objects k objects are placed into n boxed then at least one box contains at least $m+1$ objects*

Proof. For $1 \leq i \leq n$ let m_i be the number of objects in box i. Then $k = \sum_{i=1}^n k_i$. Suppose for a contradiction that no box contains at least $m+1$ elements. Then $m_i \leq m$ for all $1 \leq i \leq n$. Thus

$$k = \sum_{i=1}^n m_i \leq \sum_{i=1}^n m = nm$$

contrary to the assumption that $k > nm$. □

Theorem 1.4.4. *Let $n \in \mathbb{Z}^+$ and $a_1, \ldots, a_n \in \mathbb{R}$. Let $\mu = \frac{\sum_{i=1}^n a_n}{n}$ be the average of $(a_1, \ldots, a_n)$ The there exist $1 \leq i \leq n$ and $1 \leq j \leq n$ with $a_i \leq \mu$ and $a_j \geq \mu$. In other words:*

$$\min(a_1, \ldots a_n) \leq \mathrm{av}(a_1, \ldots, a_n) \leq \max(a_1, \ldots, a_n)$$

Proof. Suppose that $a_i > \mu$ for all $1 \leq i \leq n$. Then

$$\mu = \frac{\sum_{i=1}^n a_i}{n} > \frac{\sum_{i=1}^n \mu}{n} = \frac{n\mu}{n} = \mu,$$

a contradiction. Thus there exists $1 \leq i \leq n$ with $a_i \leq \mu$. Similarly (or by applying this results with each a_i replaced by $-a_i$) we get $a_j \geq \mu$ for some $1 \leq j \leq n$. □

Definition 1.4.5. *Let $s = a_1 a_2 \ldots a_n$ be a sequence of objects and $1 \leq i_1 < i_2 < \ldots < i_m \leq n$. Then*

$$a_{i_1} a_{i_2} \ldots a_{i_m}$$

is called the subsequence of s at the positions $(i_1, \ldots, i_m)$.

Definition 1.4.6. *Let $s = a_1 a_2 \ldots a_n$ be a sequence of real numbers.*

(a) *s is called increasing if*

$$a_1 \leq a_2 \leq a_3 \leq \ldots \leq a_n$$

and is called strictly increasing if

$$a_1 < a_2 < a_3 < \ldots < a_n$$

(b) *s is called decreasing if*

$$a_1 \geq a_2 \geq a_3 \geq \ldots \geq a_n$$

and is called strictly decreasing if

$$a_1 > a_2 > a_3 > \ldots > a_n$$

Example 1.4.7. Consider the sequence

$$s = (4, 8, 12, 3, 7, 11, 2, 6, 10, 1, 5, 9)$$

What is the length of the longest increasing subsequence? What is the length of the longest decreasing subsequence?

There many increasing subsequence of length three, for example

$$(4, 8, 12), \quad (4, 8, 9), \quad (4, 7, 11, \quad (3, 7, 11), \quad (3, 7, 10), \quad (2, 6, 9), \quad (1, 5, 9)$$

but no increasing subsequence of length four. To see the latter, arrange the elements in a array

$$\begin{matrix} 4 & 8 & 12 \\ 3 & 7 & 11 \\ 2 & 6 & 10 \\ 1 & 5 & 9 \end{matrix}$$

Let ab be subsequence of length two of s such that a and b appear in the same column of the table. Since subsequence are in the original order, a must appear in a higher row than b. The entries in any columns are strictly decreasing from top to bottom, so $a > b$. Thus ab cannot be part of an increasing subsequence. Hence any increasing subsequence of s can have at most one entry from each column. Since there are three columns, an increasing subsequence can have length at most 3. So the length of a longest decreasing subsequence is 3.

Next let ab be subsequence of length two of s such that a and b appear in the same row of the table. Since subsequence are in the original order, a must appear on the left of b. Note that the entries in any row are strictly increasing from left to right, so $a < b$. Thus ab cannot be part of decreasing sequence. Thus any decreasing subsequence of s. can have at most one entry from each row. There are four rows, so a decreasing subsequence can have length at most 4. So the length of a longest decreasing subsequence is 4.

Observe that the length of s is 12 and $12 = 3 \cdot 4$. Our next theorem shows that a sequence of length 13 would have to have a increasing sequence of length 4 or a decreasing subsequence of length 4.

Theorem 1.4.8. *Let $k, m, n \in \mathbb{Z}^+$ with $k > mn$. Then any sequence of real number has an increasing subsequence of length $m + 1$ or a strictly decreasing subsequence of length $n + 1$ (or both).*

Proof. Let $s = s_1 \dots s_k$ be a sequence of real number of length k. For $1 \leq i \leq k$ let d_i be the length of a longest strictly decreasing subsequence of s starting at the position i of s. Similarly let e_i be the length of longest increasing subsequence of s starting the position i of s. We will first show

($*$) *Let $1 \leq i < j \leq n$. If $s_i < s_j$, then $d_i > d_j$ and if $s_i \geq s_j$, then $e_i > e_j$. In particular, $(d_i, e_i) \neq (d_j, e_j)$.*

By definition of d_j there exists a strictly decreasing subsequence a of s of length d_j starting at position j of s. So $a = a_1 \dots a_{d_j}$, $a_1 = s_j$ and $a_1 > a_2 > \dots > a_{d_j}$.

Similarly there there exists a increasing subsequence b of s of length e_j starting at the position j of s. So the $b = b_1 \dots a_{b_j}$, $b_1 = s_j$ and $b_1 \leq b_2 > \dots \geq b_{e_j}$.

Suppose first that $s_i > s_j$. Since $a_1 = s_j$, we get

$$s_i > a_1 > a_2 > \dots a_{d_j},$$

and so $s_i a_1 \dots a_{d_j}$ is a strictly decreasing subsequence of s starting at position i with length $d_j + 1$. Hence $d_i > d_j$.

Suppose next that that $s_i \leq s_j$. Since $b_1 = s_j$, we get

$$s_i \leq b_1 \leq b_2 > \leq b_{e_j},$$

and so $s_i b_1 \dots b_{e_j}$ is a increasing subsequence of s starting at position i with length $e_j + 1$. Hence $e_i > e_j$. This completes the proof of ($*$)..

By ($*$) we obtain k distinct pairs (d_i, e_i). Suppose for a contradiction that $d_i \leq n$ and $e_j \leq m$ for each $1 \leq i \leq k$. Then $1 \leq d_i \leq n$, $1 \leq e_i \leq m$ and so there are n choices for d_i and m choices for e_i and nm choices for the pair (d_i, e_i). But then $k \leq nm$, contrary the hypothesis of the theorem. □

Theorem 1.4.9 (Direchlet's Approximation Theorem). *Let $a \in \mathbb{R}$ and $Q \in \mathbb{Z}$ with $Q \geq 2$. Then there exist $p, q \in \mathbb{Z}$ with $1 \leq q < Q$ and*

$$\left| a - \frac{p}{q} \right| \leq \frac{1}{qQ}$$

Proof. Divide the real interval $[0,1]$ into Q subintervals of equal length $\frac{1}{Q}$:

$$\left[0, \frac{1}{Q}\right), \quad \left[\frac{1}{Q}, \frac{2}{Q}\right), \quad \dots, \quad \left[\frac{Q-2}{Q}, \frac{Q-1}{Q}\right), \quad \left[\frac{Q-1}{Q}, 1\right]$$

For $0 \leq i < Q$ let $ia = s_i + d_i$ where $s_i \in \mathbb{Z}$ and $d_i \in \mathbb{R}$ with $0 \leq d_i < 1$. Put $d_Q = 1$. Then $d_i \in [0,1]$ for all $0 \leq i \leq Q$ so each of the d_i lies in one of the Q subintervals. There are $Q+1$ choices for i, so by the pigeon hole principal there exists $0 \leq i \leq j \leq Q$ such that $i \neq j$ and d_i and d_j lie in the same subinterval. Since the subintervals have length $\frac{1}{Q}$ we get

$$|d_j - d_i| \leq \frac{1}{Q}$$

Note that $d_k = r_k a - s_k$ with $r_k, s_k \in \mathbb{Z}$. Indeed for $k < Q$, choose s_k as above and $r_k = k$. For $k = Q$, choose $r_Q = 0$ and $s_Q = -1$. As $Q > 1$,

$$|d_Q - d_0| = 1 - 0 > \frac{1}{Q}$$

and so $\{i, j\} \neq \{0, Q\}$. It follows that $r_i \neq r_j$ and $|r_j - r_i| < Q$. Without loss $r_i < r_j$. Put $p = r_j - r_i$ and $q = s_j - s_i$. Then $p, q \in \mathbb{Z}$ and $1 \leq q < Q$. Also

$$\frac{1}{Q} \geq |d_j - d_i| = |(r_j a - s_i) - (r_j a - s_j)| = |(r_j - r_i)a - (s_j - s_i)| = |qa - p|$$

Dividing by q now gives the Theorem. □

1.5 The Principal of inclusion and exclusion

Suppose A_1 and A_2 are subsets of a set A. Count the number of elements of A which are are neither in A_1 nor A_2:

At most $|A|$, but we need to exclude the elements from $|A|_1$ and $|A|_2$. giving $|A| - (|A_1| + |A_2|)$. But we exclude the elements of $|A_1 \cap A + 2$ twice so the correct answer is

$$|A| - (|A_1| + |A_2|) + |A_1 \cap A_2|$$

With three sets consider

$$|A| - (|A_1| + |A_2| + |A_3|) + (|A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3|)$$

The elements of $A_1 \cap A_2 \cap A_3$ are counted once in the first summand, excluded 3 times in he second summand and included again three times the third term. Hence they need to be excluded one more time. So the correct answer is

$$|A| - (|A_1| + |A_2| + |A_3|) + (|A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3|) - |A_1 \cap A_2 \cap A_3|$$

Theorem 1.5.1 (Inclusion-Exclusion Principal)**.** *Let A and I be finite set and for each $i \in I$ let A_i be a set. For $J \subseteq I$ define*

$$A_J := \{a \in A \mid a \in A_j \text{ for all } j \in J\}.$$

Then

$$\Big|A \setminus \bigcup_{i \in I} A_i\Big| = \sum_{J \subseteq I} (-1)^{|J|} \left|A_J\right|$$

Proof. Observe first that

$$(*) \qquad \sum_{J \subseteq I} (-1)^{|J|} \left|A_J\right| = \sum_{J \subseteq I} \sum_{a \in A_J} (-1)^{|J|} = \sum_{a \in A} \sum_{\substack{J \subseteq I \\ a \in J}} (-1)^{|J|}.$$

Fix $a \in A$. Define $K := \{i \in I | a \in A_i\}$ and $n := |K|$. Then $a \in A_i$ if and only if $i \in K$. Let $J \subseteq I$. Observe that

$$(**) \qquad a \in A_J \iff a \in A_j \text{ for all } j \in J \iff j \in K \text{ for all } j \in J \iff J \subseteq K$$

Put $B := A \setminus \bigcup_{i \in I} A_i$. Observe that $n = 0$ if and only if $a \notin A_i$ for all $i \in I$ and if and only if $a \in B$. Define $\chi(a) = 1$ of $a \in B$ and $\chi(a) = 0$ if $a \notin B$. Then

$$\chi(a) = 1 \iff a \in B \iff n = 0 \quad \text{and} \quad \chi(a) = 0 \iff a \notin B \iff n > 0$$

We compute

$$(***) \qquad \sum_{\substack{J \subseteq I \\ a \in A_J}} (-1)^{|J|} \overset{(**)}{=} \sum_{J \subseteq K} (-1)^{|J|} = \sum_{i \in \mathbb{Z}} \sum_{\substack{J \subseteq K \\ |J| = i}} (-1)^i = \sum_{i \in \mathbb{Z}} (-1)^i \binom{n}{i} = (-1+1)^n = \begin{cases} 0 & \text{if } n > 0 \\ 1 & \text{if } n = 0 \end{cases} = \chi(a)$$

Substituting $(***)$ into $(*)$ gives

$$\sum_{J \subseteq I} (-1)^{|J|} \left| A_J \right| = \sum_{a \in A} \chi(a) = \sum_{a \in B} 1 + \sum_{a \in A \setminus B} 0 = |B|$$

□

Before applying the Principal of Inclusion and Exclusion, here is an equivalent formulation:

Theorem 1.5.2 (Inclusion-Exclusion Principal). *Let A be a set of size N, let $r \in \mathbb{N}$ and let $b_1 \ldots, b_r$ be a list of properties. For $k \in \mathbb{N}$ and $1 \le i_1 < i_2 < \ldots < i_k \le r$, let $N(b_{i_1} \ldots b_{i_k})$ be the number of objects in the set A which have each of the properties $b_{i_1}, b_{i_2}, \ldots b_{i_k}$. Let N_0 be the number of objects in the set A which do not have any of the properties. Then*

$$N_0 = N - \sum_i N(b_i) + \sum_{i<j} N(b_i b_j) - \sum_{i<j<k} N(b_i b_j b_k) + \ldots + (-1)^k \sum_{i_1 < \ldots < i_k} N(b_{i_1} \ldots b_{i_k}) + \ldots + (-1)^r N(b_1 \ldots b_r)$$

Definition 1.5.3. (a) *Let $m, n \in \mathbb{Z}$, not both 0. Then $\gcd(m,n)$ is the largest integer d such that d divides both m and n in $\mathbb{Z}$.*

(b) *Let $n \in \mathbb{Z}^+$. Then $\phi(n)$ is the number of integers m such that $1 \le m \le n$ and $\gcd(m,n) = 1$.*

Example 1.5.4. Let $1 \le m \le 24$. The only prime divisors of 24 are 2 and 3. Hence $\gcd(m, 24) = 1$ if and only if neither 2 nor 3 divides m. So we can find all such m by listing all elements from 1 to 24 and cross out all multiples of 2 and all multiplies of 3.

1, ~~2~~, ~~3~~, ~~4~~, 5, ~~6~~, 7, ~~8~~, ~~9~~, ~~10~~, 11, ~~12~~, 13, ~~14~~, ~~15~~, ~~16~~, 17, ~~18~~, 19, ~~20~~, ~~21~~, ~~22~~, 23, ~~24~~

There are 8 elements left and thus $\phi(24) = 8$. Rather than making the list we could have used the Inclusion-Exclusion Principal. There are 24 integers between 1 and 24, $\frac{12}{2}$ are divisible by 2, $\frac{24}{3}$ are divisible by 3 and $\frac{24}{6}$ are divisible by 2 and 3. Thus

$$\phi(8) = 24 - 12 - 8 + 4 = 8$$

We now use this method the compute $\phi(n)$ in general.

Theorem 1.5.5. *Let $n \in \mathbb{Z}^+$ and I the set of positive prime integers dividing n. Then*

$$\phi(n) = n \prod_{i \in I} \left(1 - \frac{1}{i}\right)$$

Proof. Let $A = \{1 \ldots, n\}$ and for $i \in I$ define $A_i = \left\{a \in A \,\middle|\, i \,|\, a\right\}$. Let $m \in A$. Then $\gcd(n, m) = 1$ if and only if $i \nmid m$ for all primes i dividing n. Thus

$$(*) \qquad \phi(n) = \left|A \setminus \bigcup_{i \in I} A_i\right|.$$

Note that

$$(**) \qquad |A_J| = |\{a \in A \,|\, j \,|\, a \text{ for all } j \in J\}| = |\{a \in A \,|\, \prod_{j \in J} j \,|\, a\}| = \frac{n}{\prod_{j \in J} j}$$

and so

$$\begin{aligned}
\phi(n) &= \left|A \setminus \bigcup_{i \in I} A_i\right| && - (*) \\
&= \sum_{J \subseteq I} (-1)^{|J|} |A_J| && - \text{Inclusion-Exclusion Principal} \\
&= \sum_{J \subseteq I} (-1)^{|J|} \frac{n}{\prod_{j \in J}} j && - (**) \\
&= n \sum_{J \subseteq I} (-1)^{|J|} \prod_{j \in J} \frac{1}{j} \\
&= n \prod_{j \in J} \left(1 - \frac{1}{j}\right) && - \text{See Homework 4}
\end{aligned}$$

□

Example 1.5.6. $\phi(24) = 24 \cdot (1 - \frac{1}{2})(1 - \frac{1}{3}) = 24 \cdot \frac{1}{2} \cdot \frac{2}{3} = 8$.

Example 1.5.7. Let n be a positive integer. How many primes are there between 1 and n? Let $1 \le m \le m$. If m is not a prime, then $m = ab$ with $1 < a \le b < m$. Let p be a prime dividing a. Then $p \le a \le \sqrt{ab} = \sqrt{m} \le \sqrt{m}$. So we can count the number if primes between 1 and n by excluding all the numbers which are divisible by a prime between 1 and $\sqrt{n}$. But note that thus also exclude all those primes and does include 1, so we will have to adjust for that.

Let $A = \{1, 2, 3, \ldots n\}$, $I = \{p \in \mathbb{Z} \mid 1 \le p \le \sqrt{n}, p \text{ is a prime}\}$. For $J \subseteq I$ define

$$A_J = \{a \in A \mid j \,|\, a \text{ for all } j \in J\}|$$

Then

$$(*) \qquad |A_J| = |\{a \in A \mid \prod_{j \in J} j \,|\, a\}| = \left\lfloor \frac{n}{\prod_{j\in J} j} \right\rfloor$$

Hence the number of primes between 1 and b is

$$\left(\sum_{J \subseteq I} (-1)^{|J|} |A_J| \right) - |J| + 1 = \left(\sum_{J \subseteq I} (-1)^{|J|} \left\lfloor \frac{n}{\prod_{j\in J} j} \right\rfloor \right) + |J| - 1$$

Consider $n = 48$. Then $6 \leq \sqrt{48} < 7$, so the primes between 1 and $\sqrt{48}$ are $2, 3, 5$. We compute $\left\lfloor \frac{n}{\prod_{j\in J} j} \right\rfloor$ for each subset J of $2, 3, 5$:

$$\left\lfloor \frac{48}{1} \right\rfloor = 48, \left\lfloor \frac{48}{2} \right\rfloor = 24, \left\lfloor \frac{48}{3} \right\rfloor = 16, \left\lfloor \frac{48}{5} \right\rfloor = 9, \left\lfloor \frac{48}{2\cdot 3} \right\rfloor = 8, \left\lfloor \frac{48}{2\cdot 5} \right\rfloor = 4, \left\lfloor \frac{48}{3\cdot 5} \right\rfloor = 3, \left\lfloor \frac{48}{2\cdot 3\cdot 5} \right\rfloor = 1.$$

Thus the number of primes between 1 and 48 is

$$[48 - (24 + 16 + 9) + (8 + 4 + 3) - 1] + 3 - 1 = 48 - 49 + 15 + 1 = 15$$

To confirm here is the list

$$2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47$$

Definition 1.5.8. *Let I be a set.*

(a) *Let I be a set. A permutation of I is a bijection from I to I.*

(b) *A derangement of I is a permutation π of I such that $\pi(i) \neq i$ for all $i \in I$.*

(c) *Let $n \in \mathbb{N}$. Then $n¡$ is the number of derangements of a set of size n.*

Example 1.5.9. Let $n \in \mathbb{N}$. Compute $n¡$.

Let I be a set of size n. We start with all the permutation and the exclude the permutation with a fixed-point. Let A be the set of permutation of I and for $i \in I$ and $J \in I$ let $A_i = \{\pi \in A \mid \pi(i) = i\}$ and

$$A_J = \{\pi \in A \mid \pi \in A_j \text{ for all } j \in J\} = \{\pi \in A \setminus \pi(j) = j \text{for all } j \in J\}$$

Note that

$$|A| = n!$$
$$|A_i| = (n-1)!$$

and if $k = |J|$

$$|A_J| = (n-k)!$$

Thus

$$\begin{aligned}
n\text{¡} &= \left|\{A \setminus \bigcup_{i\in I} A_i\right| \\
&= \sum_{J\subseteq I} (-1)^{|J|} |A_J| \\
&= \sum_{k=0}^{n} \sum_{\substack{J\subseteq I \\ |J|=k}} (-1)^k (n-k)! \\
&= \sum_{k=0}^{n} \binom{n}{k} (-1)^k (n-k)! \\
&= \sum_{k=0}^{n} (-1)^k \frac{n}{k!(n-k))!} (n-k)! \\
&= n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}
\end{aligned}$$

It follows that

$$\lim_{n\to\infty} \frac{n\text{¡}}{n!} = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{(-1)^k}{k!} = \sum_{n=0}^{\infty} \frac{(-1)^k}{k!} = e^{-1} = \frac{1}{e}$$

Note that $\sum_{n=0}^{\infty} \frac{(-1)^k}{k!}$ converges quickly, so $\frac{n\text{¡}}{n!}$ is effectively independent of n. Indeed

$$\begin{aligned}
\frac{6\text{¡}}{6!} &= 0.36805555555\ldots \\
\frac{8\text{¡}}{8!} &= 0.36788194444\ldots \\
\frac{10\text{¡}}{10!} &= 0.36787946428\ldots \\
\frac{13\text{¡}}{13!} &= 0.36787944116\ldots \\
\frac{1}{e} &= 0.36787944117\ldots
\end{aligned}$$

Observe that the quotient $\frac{n}{n!}$ can be viewed as probability. For example the probability that, after thoroughly shuffling a 52-card deck of cards, one of cards is in the same positions as before shuffling is $\frac{52\text{¡}}{52!} \approx \frac{1}{e} \approx 36.8\%$.

1.6 Generating function

Definition 1.6.1. *Let $I \subseteq \mathbb{N}$ and let $a = (a_i)_{i\in I}$ be a sequence of real number then*

$$G_a(x) := \sum_{i \in I} a_i x^i$$

is called the generated function of a.

Example 1.6.2. Let $n \in \mathbb{N}$. Then the generating function of the sequence $\left(\binom{n}{k}\right)_{k=0}^{n}$ is

$$\sum_{k=0}^{n} \binom{n}{k} x^k,$$

which by the Binomial theorem is equal to

$$(x+1)^n$$

1.6.1 Generating Functions of multisets

Definition 1.6.3. *Let $k, n \in \mathbb{N}$ and let a be a sequence, multiset or set.*

(a) *$\mathcal{P}(a)$ is the set of all subsequences, sub-multisets and subsets of a, respectively.*

(b) *$\mathcal{P}_k(a)$ is the set of all $b \in \mathcal{P}(a)$ such that b has length k (Here we define length of a set be the size of a set.*

Example 1.6.4. (1) Consider the sequence $a = (1, 2, 1, 3)$. Then

$$\mathcal{P}_3(a) = \{(1,2,1), (1,2,3), (1,1,3), (2,1,3)\}$$

(2) Consider the multiset $b = [1, 2, 1, 3]$. Then

$$\mathcal{P}_3(b) = \{[1,2,1], [1,2,3], [1,1,3]\}$$

Note here that $[2, 1, 3] = [1, 2, 3]$.

(3) Consider the multiset $c = \{1, 2, 1, 3\}$ Then $c = \{1, 2, 3\}$ and so

$$\mathcal{P}_3(b) = \{\{1,2,3\}\}$$

Definition 1.6.5. *Let $a = [a_1, a_2, \ldots, a_n]$ and $b = [b_1, \ldots, b_m]$ be multiset with multiplicities $(k_1, \ldots, k_m)$ and $(l_1, \ldots, l_m)$ with respect to the sequence $(d_1, d_2, \ldots d_m)$. Then ab, $a \cup b$ and $a \cap b$ are the multisets whose multiplicities with respect to $(d_1, \ldots, d_m)$ are*

$$(k_1 + l_1, k_2 + l_2, \ldots, k_m + l_m)$$
$$(\max(k_1, l_1), \max(k_2, l_2), \ldots, \max(k_m, l_m)$$

and

$$(\min(k_1, l_1), \min(k_2, l_2), \ldots, \min(k_m, l_m),$$

respectively.

a and b are called disjoint if $a \cap b = [\,]$.

Example 1.6.6. Consider the multisets $a = [1,2,2,2,2,4,4,4,4]$ and $b = [1,1,2,3,3,5,5]$. Then

$$\begin{aligned} ab &= [1,2,2,2,2,4,4,4,4,1,1,2,3,3,5,5] = [1,1,1,2,2,2,2,2,3,3,4,4,4,4,5,5] \\ a \cup b &= [1,1,2,2,2,2,,3,3,4,4,4,4,5,5] \end{aligned}$$

and

$$a \cap b = [1,2]$$

Definition 1.6.7. *Let A be the sequence, multiset and set. Then $G^A(x)$ is the generating function of the sequence $\left(|\mathcal{P}_k(A)|\right)_{k\in\mathbb{N}}$. So*

$$G^A(x) = \sum_{k\in\mathbb{N}} |\mathcal{P}_k(A)| x^k$$

Example 1.6.8. (1) Compute $G^A(x)$ if A is a set of size n.

Let $k \in \mathbb{N}$. Then $\mathcal{P}_k(A)$ is the set of k-subsets of the n-set A. So

$$|\mathcal{P}_k(A)| = \binom{n}{k}$$

and

$$G_A(x) = \sum_{k\in\mathbb{N}} |\mathcal{P}_k(A)| x^k = \sum_{k\in\mathbb{N}} \binom{n}{k} x^k = (x+1)^n$$

(2) Compute $G^A(x)$ if A is the multiset a^{*n} where $n \in \mathbb{N}$ and a is object.

Let $k \in \mathbb{N}$. If $k \le n$ then a^{*n} has a unique sub-multiset of length k, namely a^{*k} if $k \ge n$, the a^{*n} has no multi-subset of length n. Thus

$$|\mathcal{P}_k(A)| = \begin{cases} 1 & \text{if } k \le n \\ 0 & \text{if } k < n \end{cases}$$

and so

$$G_A(x) = \sum_{k\in\mathbb{N}} |\mathcal{P}_k(A)| x^l = \sum_{k=0}^{n} 1x^k + \sum_{k=n+1}^{\infty} 0 \cdot x^k = \sum_{k=0}^{n} x^k = \frac{x^n - 1}{x - 1}$$

(3) Compute $G^A(x)$ if A is infinite multiset with just one object a.

This time $|\mathcal{P}_k(A)| = 1$ for all $k \in \mathbb{N}$. So

$$G_A(x) = \sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$$

Example 1.6.9. Find all sublist of the multiset $aaabb$ and compute $G^{aaabb}(x)$.

Consider the formal calculation

$$\begin{aligned}
([\,] + a + aa + aaa) \cdot ([\,] + b + bb) = \; & [\,] && \mid 1 \text{ sublist of length } 0 \\
& + a + b && \mid 2 \text{ sublists of length } 1 \\
& + aa + ab + bb && \mid 3 \text{ sublists of length } 2 \\
& + aaa + aab + abb && \mid 3 \text{ sublists of length } 3 \\
& + aaab + aabb && \mid 4 \text{ sublists of length } 4 \\
& + aaabb && \mid 1 \text{ sublists of length } 5
\end{aligned}$$

Hence $G^{aaabb}(x) = 1 + 2x + 3x^2 + 3x^3 + 4x^4 + x^5$

Substitute x for a and b. Then each sublist of length k is replaced by x^k and the coefficient of x^k is the number of sublist of length k. Then

$$(1 + x + x^2 + x^3) \cdot (1 + x + x^2) = 1 + 2x + 3x^2 + 3x^3 + 4x^4 + x^5$$

that is

$$G^{aaa}(x) \cdot G^{bb}(x) = G^{aaabb}(x).$$

Theorem 1.6.10. *Let C be a multisets from the set S. Then $G^C(x)$ is obtained from $\sum_{A \in \mathcal{P}(C)} C$ by replacing each $s \in S$ by x and $[\,]$ by 1.*

Proof. Let $A \in \mathcal{P}(C)$ be of length k. Then $a = s_1 \dots s_k$ with $s_i \in S$. So a is replaced by x^k. Since $\sum_{A \in \mathcal{P}(C)} C = \sum_{k \in \mathbb{N}} \sum_{A \in \mathcal{P}_k(C)} C$ we conclude that $\sum_{A \in \mathcal{P}(C)} C$ is replaced by

$$\sum_{k \in \mathbb{N}} \sum_{A \in \mathcal{P}_k(C)} x^k = \sum_{k \in \mathbb{N}} |\mathcal{P}_k(C)| \, x^k = G^C(x).$$

□

Theorem 1.6.11. *Let A and B be disjoint multisets. Then*

$$G^A(x) \cdot G^B(x) = G^{AB}(x)$$

Proof. Let E be sublist of AB. Since A and B are disjoint, $E = CD$ for unique sublist C of A and D of B. We conclude

$$\sum_{C \in \mathcal{P}(A)} C \cdot \sum_{D \in \mathcal{P}(B)} D = \sum_{C \in \mathcal{P}(A), D \in \mathcal{P}(B)} CD = \sum_{E \in \mathcal{P}(AB)} E$$

Let S be the union of the sets obtained from A and B. According to 1.6.10 replacing each $s \in S$ by x we obtain

$$G^A(x) \cdot G^B(x) = G^{AB}(x)$$

□

Corollary 1.6.12. (a) *Let A be a multiset with multiplicities $[n_1, n_2, \dots, n_m]$. Then*

$$G^A(x) = \prod_{i=1}^{m} \sum_{k=0}^{n_i} x^k = \prod_{i=1}^{m} \frac{x^{n_i+1} - 1}{x - 1}$$

(b) *Let A be a multiset with multiplicities $[\infty^{*m}]$, so A has m distinct objects and each objects appears infinitely often. Then*

$$G^A(x) = \prod_{i=1}^{m} \sum_{k \in \mathbb{N}} x^k = \frac{1}{(1-x)^m}$$

Example 1.6.13. Let $m, n \in \mathbb{N}$ and let A be a multiset with multiplicities $[\infty^{*m}]$. Compute the number of sublists of length n of A

Solution 1:

Put $f(x) = G^A(x)$. We need to compute the coefficient of x^n in $f(x)$. By (1.6.12)(b) $f(x) = \frac{1}{(1-x)^m}$. By Taylor's Theorem

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(0) x^n$$

We compute

$$\begin{aligned} f^{(1)}(x) &= (-m)\frac{1}{(1-x)^{m+1}}(-1) &&= m\frac{1}{(1-x)^{m+1}} \\ \frac{1}{2} f^{(1)}(x) &= \frac{1}{2} m(m+1)\frac{1}{(1-x)^{m+2}} &&= \binom{m+1}{2}\frac{1}{(1-x)^{m+2}} \end{aligned}$$

Suppose inductively that $\frac{1}{n!} f^{(n)}(x) = \binom{n+m-1}{n} \frac{1}{(1-x)^{m+n}}$.
Then

$$\begin{aligned} \frac{1}{(n+1)!} f^{(n+1)}(x) &= \frac{1}{n+1}\left(\frac{1}{n!} f^{(n)}(x)\right)' \\ &= \frac{1}{n+1}\left(\binom{m+n-1}{n}\frac{1}{(1-x)^{n+m}}\right)' \\ &= \frac{1}{n+1}(n+m)\binom{m+n-1}{n}\frac{1}{(1-x)^{n+m+1}} \\ &= \frac{m+n}{n+1}\binom{n+m-1}{n}\frac{1}{(1-x)^{n+m+1}} \\ &= \binom{m+n}{n+1}\frac{1}{(1-x)^{n+m+1}} \end{aligned}$$

Thus the formula also holds for $m+1$

We conclude that the number of sublist of length n of A is

$$\frac{1}{n!}f^{(n)}(0) = \binom{n+m-1}{n}\frac{1}{(1-0)^{m+n}} = \binom{n+m-1}{n}$$

and so also

$$G^A(x) = \frac{1}{(1-x)^m} = \sum_{n\in\mathbb{Z}}\binom{n+m-1}{n}x^n$$

Solution 2 Let $a_1, a_2 \dots, a_m$ be the distinct elements of A. Then any sublist of length m can be written as

$$\underbrace{a_1a_1\dots a_1}_{k_1\text{-times}}\underbrace{a_2a_2\dots a_2}_{k_2\text{-times}}\dots\underbrace{a_ma_m\dots a_m}_{k_m\text{-times}}$$

where $k_i \in \mathbb{N}$ with $k_1 + k_2 + \dots + k_m = n$. Insert a "divider' $|$ in between any two of the consecutive maximal constant subsequences and replace each a_i by $\bullet$. We obtain a sequence of length $n+m-1$ with n dots and $m-1$ dividers.

$$\underbrace{\bullet\bullet\dots\bullet}_{k_1\text{-times}}|\underbrace{\bullet\bullet\dots\bullet}_{k_2\text{-times}}|\dots|\underbrace{\bullet\bullet\dots\bullet}_{k_m\text{-times}}$$

Observe the sublist is uniquely determined by the sequence of dots and dividers and any sequence such sequence of dots and dividers comes from a sublist. Hence the number of sublist of length of A is equal the number sequences of length $n+m-1$ with n dots and $m-1$ dividers. Any such sequence is determined by the choosing the n positions for the dots among the $n+m-1$ available positions. So the number such sequences of

$$\binom{n+m-1}{n}$$

Remark 1.6.14. *Let $n, m \in \mathbb{N}$ and $(k_1, \dots, k_m)$ a sequence of cardinalities. Then the following numbers are equal*

(a) *The number of sublist of length n of a multiset A, where A has with multiplicities $[k_1, \dots, k_m]$.*

(b) *The number of ways to choose n objects from a set of m objects with repetition, if objects i can be chosen at most k_i time, assuming that order does not matter.*

(c) *The number of sequence $(l_1, l_2, \dots l_m)$ of natural numbers with $l_i \leq k_i$ and $l_1 + l_2 + \dots l_m = n$.*

(d) *The number of ways to place n identical objects into m boxes if box i can contain at most k_i objects.*

Example 1.6.15. How many ways are there to choose 10 objects from a set of 5 objects with arbitrary repetition, assuming that order does not matter?

$$\binom{5+10-1}{10} = \binom{14}{10}$$

Example 1.6.16. Let $n \in \mathbb{N}$. Compute $G^A(x)$ if A is a multiset with multiplicities $[2^{*n}]$.

We have $A = a_1a_1a_2a_2\dots a_ma_m$ for distinct objects $a_1,\dots,a_m$. By (1.6.8)(2) $G^{a_ia_i} = 1 + x + x^2$ and so by 1.6.11

$$\begin{aligned}
G^A(x) &= \prod_{i=1}^{n} G^{a_ia_i}(x) \\
&= (1+x+x^2)^n \\
&= \sum_{k\in\mathbb{Z}} \binom{n}{k}(x+x^2)^k && \text{– Binomial Theorem} \\
&= \sum_{k\in\mathbb{Z}} \binom{n}{k} x^k(1+x)^k \\
&= \sum_{k\in\mathbb{Z}} \binom{n}{k} x^k \sum_{j\in\mathbb{Z}} \binom{k}{j} x^j && \text{– Binomial Theorem} \\
&= \sum_{k\in\mathbb{Z}} x^k \sum_{j\in\mathbb{Z}} \binom{n}{k}\binom{k}{j} x^{k+j} && \text{– Binomial Theorem} \\
&= \sum_{m\in\mathbb{Z}} \left(\sum_{j\in\mathbb{Z}} \binom{n}{m-j}\binom{m-j}{j} \right) x^m && - m := k+j, k = m-j
\end{aligned}$$

So the number of ways to choose m objects from a set of n objects, if each objects can be choose at most twice, is

$$\sum_{j\in\mathbb{Z}} \binom{n}{m-j}\binom{m-j}{j}$$

So the number of five cards poker hands from a double standard deck of cards is

$$\binom{52}{5}\binom{5}{0} + \binom{52}{4}\binom{4}{1} + \binom{52}{3}\binom{3}{2} = 3,748,160$$

The first card summand is the number of ways to choose the five cards without repetition. The second summand is the number of ways to choose the fives cards with exactly one card repeated: There are $\binom{52}{4}$ to choose the four distinct card and then $\binom{4}{1}$ to choose the card among the four cards which is repeated. The last summand is the number of ways to choose five cards with two cards being repeated: There are $\binom{52}{3}$ to choose the three distinct card and then $\binom{3}{3}$ to choose the two cards among the three cards which are repeated.

1.6.2 Fibonacci Numbers

The sequence

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots$$

is called the Fibonacci sequence. It its define recursively be

$$F_0 = 0, \quad F_1 = 1, \quad \text{and} \quad F_n = F_{n-2} + F_{n-1} \text{ for all } n \in \mathbb{N} \text{ with } n \geq 2$$

Let $F(x) = \sum_{n\in\mathbb{N}} F_i x^n$ be the corresponding generating function. We will first find an explicit formula for $F(x)$, and then use this formula to compute the Fibonacci numbers F_n.

$$\begin{aligned}
F(x) &= \sum_{n=0}^{\infty} F_n x^n \\
&= F_0 + F_1 x + \sum_{n=2}^{\infty} F_n x^n \\
&= x + \sum_{n=2}^{\infty} (F_{n-2} + F_{n-1}) x^n && - \quad -F_0 = 0, F_1 = 1, \quad F_n = F_{n-2} + F_{n-1} \\
&= x + \sum_{n=2}^{\infty} F_{n-2} x^n + \sum_{n=2}^{\infty} F_{n-1} x^n \\
&= x + x^2 \sum_{n=2}^{\infty} F_{n-2} x^{n-2} + x \sum_{n=2}^{\infty} F_{n-1} x^{n-1} \\
&x + x^2 \sum_{n=0}^{\infty} F_n x^n + x \sum_{n=1}^{\infty} F_n x^n && - \text{shift } n \text{ by 1 and 2 respectively} \\
&x + x^2 \sum_{n=0}^{\infty} F_n x^n + x \sum_{n=0}^{\infty} F_n x^n && - F_0 = 0 \\
&= x + x^2 F(x) + x F(x)
\end{aligned}$$

We proved that $F(x) = x + x^2 F(x) + xF(x)$. So $-x = (x^2 + x - 1)F(x)$ and

$$F(x) = \frac{-x}{x^2 + x - 1}$$

We will now use partial fractions to compute the power series of $\frac{-x}{x^2+x-1}$.

$$x^2 + x - 1 = \left(x + \frac{1}{2}\right)^2 - \frac{5}{4} = \left(x + \frac{1}{2} + \frac{\sqrt{5}}{2}\right)\left(x + \frac{1}{2} - \frac{\sqrt{5}}{2}\right) = (x + \alpha)(a + \beta)$$

where

$$\alpha = \frac{1}{2} + \frac{\sqrt{5}}{2} \qquad \text{and} \quad \beta = \frac{1}{2} - \frac{\sqrt{5}}{2}.$$

We need to find $A, B \in \mathbb{R}$ with

$$-\frac{x}{x^2+x-1} = \frac{A}{x+\alpha} + \frac{B}{x+\beta}$$

Multiplying with $(x+\alpha)(x+\beta)$ gives

$$-x = A(x+\beta) + B(x+\alpha) = (A+B)x + A\beta + B\alpha$$

and so

$$A+B=-1 \qquad \text{and} \qquad A\beta + B\alpha = 0$$

Multiplying the first equation with α and subtracting the second gives

$$A(\alpha-\beta) = -\alpha \quad \text{and} \quad B = -\frac{\beta}{\alpha}A$$

Note that $\alpha - \beta = \sqrt{5}$. so

$$A = -\frac{\alpha}{\sqrt{5}} \quad \text{and} \quad B = -\frac{\beta}{\alpha} \cdot -\frac{\alpha}{\sqrt{5}} = \frac{\beta}{\sqrt{5}}.$$

Thus

$$\begin{aligned}
-\frac{x}{x^2+x-1} &= \frac{1}{\sqrt{5}}\left(\frac{-\alpha}{x+\alpha} + \frac{\beta}{x+\beta}\right) \\
&= \frac{1}{\sqrt{5}}\left(\frac{-\alpha}{-\alpha\beta x+\alpha} + \frac{\beta}{-\alpha\beta x+\beta}\right) && -\alpha\beta = -1 \text{ since} (x+\alpha)(x+\beta) = (x^2+x-1) \\
&= \frac{1}{\sqrt{5}}\left(\frac{-1}{-\beta x+1} + \frac{1}{-\alpha x+1}\right) \\
&= \frac{1}{\sqrt{5}}\left(\frac{1}{1-\alpha x} - \frac{1}{1-\beta x}\right) \\
&= \sum_{n=0}^{\infty} \frac{1}{\sqrt{5}}(\alpha^n - \beta^n) && \text{geometric series}
\end{aligned}$$

It follows that, for all $n \in \mathbb{N}$,

$$F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}}$$

Observe that $\sqrt{5} \approx 2.2$ and so $\alpha = \frac{1}{2}(1+\sqrt{5}) \approx \frac{1+2.2}{2} = 1.6$ and $\beta = \frac{1}{2}(1-\sqrt{5}) \approx \frac{1-2.2}{2} = -0.6$. In particular, $|\alpha| > 1$ and $|\beta| < 1$ So for large n:

$$F_n \approx \frac{\alpha^n}{\sqrt{5}}$$

1.6.3 The Generalized Binomial Theorem

Let α be any real number and $n \in \mathbb{N}$. What is the power series of $(1+x)^\alpha$? We compute

$$\begin{aligned}
f(x) &= (1+x)^\alpha \\
\frac{1}{1!}f^{(1)}(x) &= \alpha(1+x)^{\alpha-1} \\
\frac{1}{2!}f^{(2)}(x) &= \frac{\alpha(\alpha-1)}{2}(1+x)^{\alpha-2} \\
\vdots &= \vdots \\
\frac{1}{n!}f^{(n)}(x) &= \frac{\alpha(\alpha-1)\dots(\alpha-n+1)}{n!}(1+x)^\alpha \\
&= \frac{\alpha^{\underline{n}}}{n!}(1+x)^\alpha
\end{aligned}$$

Thus

$$\frac{1}{n!}f^{(n)}(0) = \frac{\alpha^{\underline{n}}}{n!}(1+0)^\alpha = \frac{\alpha^{\underline{n}}}{n!}.$$

We define

Definition 1.6.17. *Let $\alpha \in \mathbb{R}$ and $n \in \mathbb{Z}$ Then*

$$\binom{\alpha}{k} := \begin{cases} \frac{\alpha^{\underline{n}}}{n!} & \text{if } n \geq 0 \\ 0 & \text{if } n < 0 \end{cases}$$

$\binom{\alpha}{k}$ is called a generalized binomial coefficient.

We proved

Theorem 1.6.18 (Generalized Binomial Theorem)**.** *Let $\alpha \in \mathbb{R}$. Then*

$$(1+x)^\alpha = \sum_{n\in\mathbb{Z}} \binom{\alpha}{n} x^n$$

1.6.4 Catalan numbers

For $n \in \mathbb{N}$ let C_n be the number of many way to calculate a product of a sequence $x_0, x_1, \dots, x_n$ of length $n = 1$ of real numbers, or matrices or any set with a binary operation? Consider the case $n = 3$. Then $C_3 = 5$:

$$\big((x_0x_1)x_2\big)x_3 \quad \big(x_0(x_1x_2)\big)x_3, \quad (x_0x_1)(x_2x_3), \quad x_0\big((x_1x_2)x_3\big), \quad x_0\big(x_1(x_2x_3)\big)$$

Note that sequences are ordered, so we do not allows products like $(x_3x_2)(x_1x_0)$.

C_n is called the n'th Catalan number. The C_n's can be computed recursively: Consider the very last multiplication done to compute the product:

$$(x_0x_1\dots x_i)(x_{i+1}x_{i+2}\dots x_n)$$

Note that i can be any integer with $0 \le i \le n-1$. The sequence $(x_0,\dots,x_i)$ has length $i+1$ and so there are C_i ways to compute $x_0x_1\dots x_i$. The sequence $x_{i+1}\dots x_n$ has length $n-i$ and so there are C_{n-i-1} to compute the product $x_{i+1}\dots x_n$. Thus

$$(*) \qquad C_n = \sum_{i=0}^{n-1} C_iC_{n-i-1} \;=\; C_0C_{n-1} + C_1C_{n-2} + \dots C_{n-2}C_1 + C_{n-1}C_0.$$

There is only one way to compute a product of one element. So

$$\begin{aligned}
&C_0 && &&= 1\\
&C_1 = C_0C_0 &&= 1\cdot 1 &&= 1\\
&C_2 = C_0C_1 + C_1C_0 &&= 1\cdot 1 + 1\cdot 1 &&= 2\\
&C_3 = C_0C_2 + C_1C_1 + C_2C_1 &&= 1\cdot 2 + 1\cdot 1 + 2\cdot 1 &&= 5\\
&C_4 = C_0C_3 + C_1C_2 + C_2C_1 + C_3C_0 &&= 1\cdot 5 + 1\cdot 2 + 2\cdot 1 + 5\cdot 1 &&= 14\\
&C_5 = C_0C_4 + C_1C_3 + C_2C_2 + C_3C_1 + C_4C_0 &&= 1\cdot 14 + 1\cdot 5 + 2\cdot 2 + 5\cdot 1 + 14\cdot 1 &&= 42
\end{aligned}$$

Let $C(x) = \sum_{n=0}^{\infty} C_nx^n$ be the generating function of the Catalan sequence. Let $n \in \mathbb{Z}^+$. Then coefficient of x^{n-1} in $C^2(x) = C(x)C(x)$ is

$$\sum_{i=0}^{n-1} C_iC_{n-i}$$

Hence this is also the coefficient of x^n in $xC^2(x)$. From $(*)$ we conclude that for $n \ge 1$ $C(x)$ and $xC^2(x)$ have the some coefficients. Note that the coefficient of x^0 in $C(x)$ is $C_0 = 1$ and in $xC^2(x)$ is 0, so $C(x) = 1 + xC^2(x)$. Thus

$$C(x) = \frac{1 \pm \sqrt{1-4x}}{2x}$$

Since $C_0 = 1$ we need the function on the right side to have removable discontinuity at 0. In particular, the numerator needs to be 0 at $x = 0$. Thus

$$C(x) = \frac{1 - \sqrt{1-4x}}{2x}$$

We now use the Generalized Binomial Theorem to compute the power series for $\sqrt{1-4x}$:

$$\begin{aligned}
\sqrt{1-4x} &= (1-4x)^{\frac{1}{2}} \\
&= \sum_{k=0}^{\infty} \binom{\frac{1}{2}}{k} (-4x)^k \\
&= 1 - 4x \sum_{k=1}^{\infty} \binom{\frac{1}{2}}{k} (-4x)^{k-1} \\
&= 1 - 4x \sum_{k=0} \binom{\frac{1}{2}}{k+1} (-4)^k x^k
\end{aligned}$$

and so

$$C(x) = \frac{1-\sqrt{1-4x}}{2} = 2 \sum_{k=0} \binom{\frac{1}{2}}{k+1} (-4)^k x^k$$

Hence

$$\begin{aligned}
C_k &= 2(-4)^k \binom{\frac{1}{2}}{k+1} && = (-1)^k 2^{1+2k} \frac{\left(\frac{1}{2}\right)^{\underline{k+1}}}{(k+1)!} && = \frac{(-1)^k 2^{1+2k}}{(k+1)!} \prod_{i=0}^{k} \left(\frac{1}{2} - i\right) \\
&= \frac{(-1)^k 2^{1+2k}}{(k+1)!} \prod_{i=0}^{k} \frac{1-2i}{2} && = \frac{(-1)^k 2^k}{(k+1)!} \prod_{i=0}^{k} (1-2i) && = \frac{2^k}{(k+1)!} \prod_{i=1}^{k} (2i-1) \\
&= \frac{2^k}{(k+1)!} \frac{(2k)!}{\prod_{i=1}^{k} 2k} && = \frac{1}{(k+1)k!} \frac{(2k)!}{k!} && = \frac{1}{k+1} \frac{(2k)!}{k!(2k-k)!} \\
&= \frac{1}{k+1} \binom{2k}{k}
\end{aligned}$$

Thus

$$C_k = \frac{1}{k+1} \binom{2k}{k}.$$

So for example

$$C_3 = \frac{1}{3+1} \binom{2 \cdot 3}{3} = \frac{1}{\not{4}} \frac{\not{6} \cdot 5 \cdot \not{4}}{\not{3} \cdot \not{2}} = 5$$

and

$$C_4 = \frac{1}{4+1} \binom{2 \cdot 4}{4} = \frac{1}{\not{5}} \frac{\not{8} \cdot 7 \cdot \not{6}^{2} \cdot \not{5}}{\not{4} \cdot \not{3} \cdot \not{2}} = 14$$

just as we have seen above.

1.6.5 Changing Money

How many ways are there to make change for one dollar bill with coins?

penny	nickel	dime	quarter	half dollar coin	one dollar coin
1	5	10	25	50	100

So we need to count the number of sequence $(m_1, m_2, m_3, m_4, m_5, m_6)$ of natural numbers with

$$100 = m_1 + 5m_2 + 10m_2 + 25m_4 + 50m_5 + 100m_6$$

Consider the general problem:

Definition 1.6.19. *Let $A = (a_1, \ldots, a_k)$ be a sequence of positive integers. Then*

$$d_n^A = |\{(m_1, m_2, \ldots, m_k) \mid m_i \in \mathbb{N}, m_1 a_1 + m_2 a_2 + \ldots m_k a_k = n\}|$$

(So d_n^A is the number of ways to make change for n cents with k-distinct coins of values $a_1, a_2, \ldots, a_k$ (in cents). For $n \in \mathbb{N}$

$D^A(x)$ denotes the corresponding generating function, that is

$$D^A(x) = \sum_{n \in \mathbb{N}} d_n^A x^n$$

Example 1.6.20. Compute d_n^A and $D^A(x)$ if

(a) $A = (1)$.

We only have one coin, a penny. For each n there is only one way to make change with only pennies, take n pennies. so

$$D^A(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

(b) $A = (5)$.

We only have one coin, a nickel. If $n = 5k$ there is a unique way to make change: take k nickels. So $d_n^A = 1$. If $5 \nmid n$, there is no way to make change, so $d_n^A = 0$. Thus

$$D^A(x) = \sum_{k=0}^{\infty} x^{5k} = \frac{1}{1-x^5}$$

(c) $A = (a)$ for some $a \in \mathbb{Z}^+$. Then

$$d_n^A = \begin{cases} 1 & \text{if } a \mid n \\ 0 & \text{if } a \nmid n \end{cases} \quad \text{and} \quad D^A(x) = \sum_{k=0}^{\infty} x^{ak} = \frac{1}{1-x^a}$$

Theorem 1.6.21. *Let A and B be sequences of positive integers. Then*

$$D^{AB}(x) = D^A(x)D^B(x)$$

Proof. Let $n \in \mathbb{N}$ and let k and l be the length of A and B. Suppose we are making change for n-cents with the $k + l$ coins with values AB. Let i be the contribution from the first k coins and j the contribution from the last coins. Then $n = i + j$. There are d_i^A ways to make change for i-cents with the fist k coins and D_j^B ways to make change j-cents with the last l coins. Thus

$$d_n^{AB} = \sum_{\substack{i,j \in \mathbb{N} \\ i+j=n}} d_i^A D_j^B$$

and so d_n^{AB} is exactly the coefficient of x^n in $D^A(x)D^B(x)$. Thus $D^{AB}(x) = D^A(x)D^B(x)$. □

It follows that

Corollary 1.6.22. *Let $A = (a_1, \ldots, a_k)$ be a sequence of positive integers. Then*

$$D^A(x) = \prod_{i=1}^{k} D^{(a_i)}(x) = \prod_{i=1}^{k} \sum_{k_i=0}^{\infty} x^{k_i a_i} = \prod_{i=1}^{k} \frac{1}{1 - x^{a_i}}$$

Example 1.6.23. The generating function to makes change with pennies, nickels, dimes, quarters, and half dollars

$$\frac{1}{(1-x)(1-x^5)(1-x^{10})(1-x^{25})(1-x^{50})(1-x^{100})}$$

We will now develop a recursive formula to compute $D^A(x)$. :

Theorem 1.6.24. *Let $k \in \mathbb{Z}^+$, let $n \in \mathbb{N}$ and let $A = (a_1, \ldots, a_k)$ be a sequence of positive integers. Put $B = (a_1, \ldots, a_{k-1})$ and $a = a_k$. Then*

$$d_n^A = \begin{cases} d_n^B & \text{if } n < a \\ d_n^B + d_{n-a}^A & \text{if } n \geq a \end{cases}$$

Proof. Note that $A = Ba$. So by 1.6.21

$$D^A(x) = D^{Ba}(x) = D^B(x)D^{(a)}(x) = D^B(x)\frac{1}{1 - x^a}$$

Hence

$$(1 - x^a)D^A(x) = D^B(x)$$

and thus

$$D^A(x) = D^B(x) + x^a D^A(x)$$

it follow thats

$$d_n^A = \begin{cases} d_n^B & \text{if } n < a \\ d_n^B + d_{n-a}^A & \text{if } n \ge a \end{cases}$$

□

We now compute the numbers of ways to make change for a half-dollar with pennies, nickels and half dollars.

n			0	5	10	15	20	25	30	35	40	45	50
$d_n^{(1)}$			1	1	1	1	1	1	1	1	1	1	1
$d_{n-5}^{(1,5)}$				1	2	3	4	5	6	7	8	9	10
$d_n^{(1,5)}$	$= d_n^{(1)}$	$+d_{n-5}^{(1,5)}$	1	2	3	4	5	6	7	8	9	10	11
$d_{n-10}^{(1,5,10)}$					1	2	4	6	9	12	16	20	25
$d_n^{(1,5,10)}$	$= d_n^{(1,5)}$	$+d_{n-10}^{(1,5,10)}$	1	2	4	6	9	12	16	20	25	30	36
$d_{n-25}^{(1,5,10,25)}$								1					13
$d_n^{(1,5,10,25)}$	$= d_{n-10}^{(1,5,10)}$	$+d_{n-25}^{(1,5,10,25)}$	1					13					49
$d_{n-50}^{(1,5,10,25,50)}$													1
$d_n^{(1,5,10,25,50)}$	$= d_n^{(1,5,10,25)}$	$+d_{n-50}^{(1,5,10,25,50)}$	1										$\boxed{50}$

Next we will derive a general formula for $A(x) := D^{(1,5,10)}(x)$. The same method also works for $D^{(1,5,10,25,50,100)}$ but the details of the computations are more complex (see the text book).

We have

$$A(x) = \frac{1}{(1-x)(1-x^5)(1-x^{10})} = \frac{\frac{1-x^5}{1-x}}{(1-x^5)(1-x^5)(1-x^{10})} = \frac{1+x+x^2+x^3+x^4}{(1-x^5)^2(1-x^{10})}$$

and so

$$A(x) = (1+x+x^2+x^3+x^4))B(x^5)$$

where

$$B(x) = \frac{1}{(1-x)^2(1-x^2)}$$

Let a_n and b_n be the coefficients of x^n in $A(x)$ and $B(x)$ respectively. Then

Then

$$A(x) = (1+x+x^2+x^3+x^4))B(x^5) = \sum_{r=0}^{4} x^r \cdot \sum_{q=0}^{\infty} b_q x^{5q} = \sum_{q=0}^{\infty}\sum_{r=0}^{4} b_q x^{5q+r}$$

Thus

$$a_n = b_q \quad \text{where } q, r \in \mathbb{N} \text{ with } n = 5q + r \text{ and } r \leq 4$$

For example $a_{23} = a_{5\cdot4+3} = b_4$ and $a_{1001} = \alpha_{5\cdot200+1} = b_{200}$.
So it remains to compute $B(x)$.

$$\begin{aligned}
B(x) &= \frac{1}{(1-x)^2(1-x^2)} \\
&= \frac{\left(\frac{1-x^2}{1-x}\right)}{(1-x^2)^2(1-x^2)} \\
&= \frac{(1+x)^2}{(1-x^2)^3} \\
&= \frac{1+2x+x^2}{(1-x^2)^3} \\
&= (1+2x+x^2)C(x)
\end{aligned}$$

where $C(x) = \frac{1}{(1-x^2)^3)}$. Recall from 1.6.13 that

$$\frac{1}{(1-x)^m} = \sum_{k\in\mathbb{Z}} \binom{k+m-1}{k} x^k = \sum_{k\in\mathbb{Z}} \binom{k+m-1}{m-1} x^k$$

For $m = 3$ and with x^2 on place of x:

$$C(x) = \frac{1}{(1-x^2)^3} = \sum_{k\in\mathbb{Z}} \binom{k+3-1}{3-1} (x^2)^k = \sum_{k\in\mathbb{Z}} \binom{k+2}{2} x^{2k}$$

Thus

$$\begin{aligned}
B(x) &= (1+2x+x^2)C(x) \\
&= C(x) + xC(x) + x^2C(x) \\
&= \sum_{k\in\mathbb{Z}} \binom{k+2}{2} x^{2k} + \sum_{k\in\mathbb{Z}} 2\binom{k+2}{2} x^{2k+1} + \sum_{k\in\mathbb{Z}} \binom{k+2}{2} x^{2k+2} \\
&= \sum_{k\in\mathbb{Z}} \left[\binom{k+2}{2} x^{2k} - \binom{k+1}{2}\right] x^{2k} + \sum_{k\in\mathbb{Z}} 2\binom{k+2}{2} x^{2k+1} \qquad k \to k-1 \text{ in the last summand}
\end{aligned}$$

We compute

$$\binom{k+2}{2} + \binom{k+1}{2} = \frac{(k+2)(k+1)}{2} + \frac{k+1)k}{2} = \frac{(k+1)(k+2+k)}{2} = (k+1)(k+1) = (k+1)^2$$

and

$$2\binom{k+2}{2} = 2\frac{(k+2)(k+1)}{2} = (k+2)(k+1)$$

Thus

$$B(x) = \sum_{k\in\mathbb{Z}} (k+1)^{2k} x^{2k} + \sum_{k\in\mathbb{Z}} (k+2)(k+1) x^{2k+1}$$

Hence

$$b_q = \begin{cases} (k+1)^2 & \text{if } q = 2k \\ (k+2)(k+1) & \text{if } q = 2k+1 \end{cases}$$

Let $q = 2k + s$ with $s = 0, 1$ and $n = 5q + r$ with $0 \le r \le 4$. Put $t = 5s + r$. Then

$$n = 5q + r = 5(2k + s) + r = 10k + 5s + r = 10k + t$$

Note that $0 \le t \le 4$ if $s = 0$ and $5 \le t \le 9$ if $s = 1$. Hence

$$a_{10k+t} = a_n = b_q = \begin{cases} (k+1)^2 & \text{if } 0 \le t \le 4 \\ (k+2)(k+1) & \text{if } 5 \le t \le 9 \end{cases}$$

For example the number of ways to make change for 20 bill using pennies, nickels and dimes is

$$a_{2000} = a_{200\cdot 10+0} = (201)^2 = 40401$$

1.6.6 Recurrence Relations

Let $b, c \in \mathbb{R}$ and $A = (a_n)_{n\in\mathbb{N}}$ a sequence of real numbers such that

$$(*) \qquad a_n = ba_{n-1} + c$$

for all $n \in \mathbb{Z}^+$.

Let $G(x) = G_A(x) = \sum_{n=0}^{\infty} a_n x^n$ be the generating function of A. We compute

$$\begin{aligned}
G(x) &= a_0 + \sum_{n=1}^{\infty} a_n x^n \\
&= a_0 + \sum_{n=1}^{\infty} (ba_{n-1} + c)x^n && \mid (*) \\
&= a_0 + x \sum_{n=1}^{\infty} (ba_{n-1} + c)x^{n-1} \\
&= a_0 + x \sum_{n=0}^{\infty} (ba_n + c)x^n && \mid \text{substitution:} n \to n+1 \\
&= a_0 + bx \sum_{n=0}^{\infty} a_n x^n + cx \sum_{n=0}^{\infty} x^n \\
&= a_0 + bxG(x) + \frac{cx}{1-x} && \mid \text{Definition of } G(x), \text{geometric series}
\end{aligned}$$

Hence

$$(1-bx)G(x) = a_0 + \frac{cx}{1-x}$$

and so

$$G(x) = \frac{a_0}{1-bx} + \frac{cx}{(1-x)(1-bx)}$$

To determine the power series for $G(x)$ we first find $A, B \in \mathbb{R}$ with

$$\frac{x}{(1-x)(1-bx)} = \frac{A}{1-x} + \frac{B}{1-bx}$$

Multiplying with $(1-x)(1-bx)$ gives

$$x = A(1-bx) + B(1-x) = (A+B) - (Ab+B)x$$

and so

$$\begin{aligned}
A + B = 0 &\quad \text{and} \quad Ab + B = -1 \\
B = -A &\quad \text{and} \quad -1 = Ab - A = A(b-1)
\end{aligned}$$

and if $b \neq 1$

$$A = \frac{1}{1-b} \quad \text{and} \quad B = -A = -\frac{1}{1-b}$$

So

$$\frac{x}{(1-x)(1-bx)} = \frac{1}{1-b}\left(\frac{1}{1-x} - \frac{1}{1-bx}\right)$$

and

$$\begin{aligned}G(x) &= \frac{a_0}{1-bx} + \frac{cx}{(1-x)(1-bx)}\\ &= \frac{a_0}{1-bx} + \frac{c}{1-b}\frac{1}{1-x} - \frac{c}{1-b}\frac{1}{1-bx}\\ &= \left(a_0 - \frac{c}{1-b}\right)\frac{1}{1-bx} + \frac{c}{1-b}\frac{1}{1-x}\\ &= \sum_{n=0}^{\infty}\left[\left((a_0 - \frac{c}{1-b}\right)b^n + \frac{c}{1-b}\right]x^n\end{aligned}$$

We conclude that

$$a_n = \left(a_0 - \frac{c}{1-b}\right)b^n + \frac{c}{1-b}$$

1.7 Polya's Theory of Counting

1.7.1 Groups

Definition 1.7.1. *A group is a pair $(G, \circ)$ such that G is a set, $\circ$ is a function and the following five conditions hold:*

(i) *$a \circ b$ is defined for all $a, b \in G$. (More precisely, (a, b) is in the domain of $\circ$ for all $a, b \in G$ and we denote the image of (a, b) under $\circ$ by $a \circ b$.)*

(ii) *$a \circ b \in G$ for all $a, b \in G$.*

(iii) *$(a \circ b) \circ c = a \circ (b \circ c)$ for all $a, b, c \in G$.*

(iv) *There exist an element in G, denoted by e and called the identity of G, such that $a \circ e = a = e \circ a$ for all a in G.*

(v) *For each $a \in G$, there exists an element in G, denoted by a^{-1} and called'a inverse', such $a \circ a^{-1} = e = a^{-1} \circ a$.*

Example 1.7.2. (a) $(\mathbb{Z}, +)$, $(\mathbb{Q}, +)$ and $(\mathbb{R}, +)$ all a groups.

(b) $(\mathbb{Z}, \cdot)$ is not a group.

(c) Let $\mathbb{Q}^\sharp = \mathbb{Q} \smallsetminus \{0\}$. Then $(\mathbb{Q}^\sharp, \cdot)$ is a group.

(d) Let I be a set and define $\mathrm{Sym}(I)$ to be the set of bijections from I to I. Then $(\mathrm{Sym}(I), \circ)$ is a group, where $\circ$ denotes composition of functions.

(e) Let $n \in \mathbb{Z}^+$ and $\mathrm{GL}_n(\mathbb{R})$ the set of $n \times n$-matrices with real coefficients and non-zero determinant. Then $(\mathrm{GL}_n(\mathbb{R}), \cdot)$ is a group, where $\cdot$ denotes matrix multiplication.

If $n \in \mathbb{Z}^+$ we write $\mathrm{Sym}(n)$ for $\mathrm{Sym}(\{1,2,\dots,n\}$. Let $\pi \in \mathrm{Sym}(n)$. We will denote π by the two rows :

$$\begin{pmatrix} 1 & 2 & \dots & n \\ \pi(1) & \pi(2) & \dots & \pi(n) \end{pmatrix}$$

For example

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 6 & 2 & 5 & 1 \end{pmatrix}$$

is the permutation $\pi \in \mathrm{Sym}(6)$ with

$$\pi(1) = 3, \quad \pi(2) = 4, \quad ,\pi(3) = 6, \quad \pi(4) = 2, \quad \pi(5) = 5, \quad \pi(6) = 1$$

We also write $\pi : i \mapsto j$ to indicate that $\pi(i) = j$. For example with the above π

$$\pi : \quad 1 \mapsto 3 \mapsto 6 \mapsto 1, \quad 2 \mapsto 4 \mapsto 2, \quad 5 \mapsto 5$$

Most often we will use cycle notation: The above π is denoted by

$$(1,3,6)(2,4)(5)$$

If all the number only have one digit we will leave out the commas. We also usually don't list the cycles of length 1:

$$(136)(24)$$

Note that the cycle notation is not unique. For example

$$(136)(24) \quad \text{and} \quad (42)(631)$$

denote the same element of $\mathrm{Sym}(6)$. To make it unique we usually use the 'standard cycle notation': Each cycle starts with the smallest element of that cycle, and the first elements of each cycle appear in increasing order. So $(136)(24)$ is in standard cycle notation, but neither $(42)(631)$ nor $(24)(136)$ are.

Definition 1.7.3. *Let G be a group, $a \in G$ and $n \in \mathbb{Z}$.*

(a) *Define a^n recursively as follows:*

$$a^n = \begin{cases} e & \text{if } n = 0 \\ aa^{n-1} & \text{if} n > 0 \end{cases}$$

(b) *If there exists $n \in \mathbb{Z}^+$ with $a^n = e$, then the order of a is the smallest such n. Otherwise, a has infinite order. We denote the order of a by $|a|$.*

Example 1.7.4. Let $\pi = (136)(24)$. Compute π^n for all $n \in \mathbb{Z}$. What is $|\pi|$?

$$\begin{aligned}
\pi^0 &= (1)(136)(24) \\
\pi^1 &= (136)(24) \\
\pi^2&(136)(24 \circ (136)(24) = (163)(2)(4) = (163) \\
\pi^3 &= (136)(24) \circ (163) = (1)(24)(3)(6) = (24) \\
\pi^4 &= 136)(24) \circ (24) = (136) \\
\pi^5 &= (136)(24) \circ (136) = (163)(24) \\
\pi^6 &= (136)(24) \circ (163)(24) = (1) \\
\pi^7 &= (136)(24) = \pi \\
\pi^{-1} &= (631)(42) = \pi^5 \\
\pi^{-2} &= (\pi^5)^2 = \pi^{10} = \pi^4
\end{aligned}$$

So $|\pi| = 6$ and $\pi^n = \pi^r$ if r is the remainder of n when divided by 6.

Definition 1.7.5. *Let $(G, \circ)$. Then H is called a subgroup of G, and we write $H \leq G$, provided that $H \subseteq G$ and $(H, \circ)$ is group.*

Definition 1.7.6. *Let G be a group and H a subset of G. Then*

$$\langle H \rangle = \bigcap_{\substack{K \leq G \\ H \subseteq K}} K$$

So $\langle H \rangle$ is intersection of all the subgroups of G which contain H. $\langle H \rangle$ is called the subgroup of G generated by H.

Remark 1.7.7. *Let G be a group and $H \subseteq G$. Then $\langle H \rangle$ is the smallest subgroup of G containing H, that is*

(a) *$H \subseteq \langle H \rangle$,*

(b) *$\langle H \rangle \leq G$, and*

(c) *If K is a subgroup of G with $H \subseteq K$, then $\langle H \rangle \subseteq K$.*

Example 1.7.8. Find the subgroup of Sym(6) generated by (136)(24)

Put $\pi = ((136)(24)$ and put $H = \langle \pi \rangle$. Since H is a subgroup of G, Thus $e = (1) \in H$ and H is closed under multiplication and inverses. It follows that $\pi^n \in H$ for all $n \in \mathbb{Z}$.

Thus

$$\{(1), (136)(24), (163), (24), (136), (163)(24)\} \subseteq H$$

One can verify that the set on left side is subgroup of Sym(6) and so is equal to H.

Definition 1.7.9. *Let G be a group and $a \in G$.*

(a) *The order of the group G is the cardinality of the set G.*

(b) *G is called cyclic if $G = \langle g \rangle$ for some $g \in G$.*

(c) *Let $n \in \mathbb{Z}^+$. Then C_n denotes the subgroup $\langle (123\ldots n) \rangle$ of $\mathrm{Sym}(n)$.*

Theorem 1.7.10. *Let G be a group, $a \in G$ and $i, j \in \mathbb{Z}$. Put $n := |a|$.*

(a) $\langle a \rangle = \{a^m \mid m \in \mathbb{Z}\}$.

(b) *Suppose n is infinite. Then $a^i = a^j$ if and only if $i = j$. In particular, $a^i = e$ if and only if $i = 0$.*

(c) *Suppose n is finite. Then $a^i = a^j$ if and only if n divides $j - i$. In particular, $a^i = e$ if and only if n divides i. Moreover, $\langle a \rangle = \{e, a, a^2, \ldots, a^{n-1}\}$.*

(d) $|a| = |\langle a \rangle|$.

Example 1.7.11. Let $n \in \mathbb{Z}^+$.

(1) $\mathrm{Sym}(n)$ as order $n!$.

(2) $(\mathbb{Z}, +)$ is a cyclic group of infinite order.

(3) $(123\ldots n)$ has order n in $\mathrm{Sym}(n)$ and C_n is a cyclic group of order n.

Let $\pi = (1234\ldots n)$. Then

$$1 \overset{\pi}{\mapsto} 2 \overset{\pi}{\mapsto} 3 \overset{\pi}{\mapsto} \ldots \overset{\pi}{\mapsto} n-1 \overset{\pi}{\mapsto} n \overset{\pi}{\mapsto} 1$$

Thus $\pi^k(1) = 1 + k \neq 1$ for all $1 \leq k \leq n-1$ and $\pi^n(1) = 1$. Inn particular, $\pi^k \neq (1)$ for all $1 \leq k \leq n$. Similarly we have $\pi^n(i) = i$ for all $1 \leq i \leq n$ and so $\pi^n = (1)$. Thus $|\pi| = n$.

Remark 1.7.12. (a) *C_n can be viewed as the group of rotations of a regular n-gon.*

(b) *D_n is the group of symmetries of a regular n-gon, consisting of n-rotations and n reflections. For example*

$$\mathrm{D}_4 = \{(1), (1234), (13)(24), (1432), (12)(34), (14)(23), (13), (24)$$

and

$$\mathrm{D}_5 = \{(1), (12345), (13524), (14253), (15432), (25)(34), (13)(45), (24)(15), (12)(35), (14)(23)\}$$

1.7.2 Burnside's Lemma

Definition 1.7.13. *A function $*$ is an action of the group $(G, \circ)$ on the set S if*

(i) *$a * s$ is defined for all $a \in G$ and $s \in S$. (More precisely, (a, s) is in the domain of $*$ for all $a \in G$ and $s \in S$ and we denote the image of (a, s) under $*$ by $a * s$.)*

(ii) *$a * s \in S$ for all $a \in G$, $s \in S$.*

(iii) *$(a \circ b) * s = a * (b * s)$ for all $a, b \in G$ and $s \in S$.*

(iv) *$e * s = s$ for all $s \in S$.*

Example 1.7.14. (1) $\mathrm{Sym}(I)$ acts on I via $\pi * s = \pi(s)$ for all $\pi \in \mathrm{Sym}(I)$, $s \in S$.

(2) $\mathrm{GL}_n(\mathbb{R})$ acts on $\mathbb{R}^n$ via $A * v = Av$ for all $A \in \mathrm{GL}_n(\mathbb{R})$ and $v \in \mathbb{R}^n$ For example for $n = 2$,

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} * \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$$

(3) Let $(G, \circ)$ be a group. Then $\circ$ is an action of G on G, called the action by left multiplication.

(4) Let V be the set of vertices of a regular n-gon and E the set of edges. (Here we view an edge as set of size two consisting of two adjacent vertices. Then $\overline{D}_n$ acts on V via $\pi * v = \pi(v)$ and on E via $\pi * \{a, b\} = \{\pi(a), \pi(n)\}$.

For example $(13)(24) * \{1, 4\} = \{2, 3\}$.

Theorem 1.7.15. *Let $*$ be a action of the group G on the set S. For $a \in G$ let π_a be th function from S to S defined by $\pi_a(s) = a * s$. Then*

(a) *Let $a, b \in G$ then $\pi_{a \circ b} = \pi_a \circ \pi_b$.*

(b) $\pi_e = \mathrm{id}_S$.

(c) *Let $a \in G$. Then π_a is a bijection with inverse $\pi_{a^{-1}}$. In particular, $\pi_a \in \mathrm{Sym}(S)$.*

Proof. Let $s \in S$.

(a):

$$\begin{aligned} \pi_{a \circ b}(s) &= (a \circ b) * s && \text{– Definition of } \pi_{a \circ b} \\ &= a * (b * s) && \text{– Definition of an action} \\ &= \pi_a(\pi_b(s)) && \text{– Definition of } \pi_a \text{ and } \pi_b \\ &= (\pi_a \circ \pi_b)(s) && \text{– Definition of composition of function} \end{aligned}$$

Since this holds for all $s \in S$ we get $\pi_{a \circ b} = \pi_a \circ \pi_b$.

(b)

$$\begin{aligned}\pi_e(s) &= e * s && \text{–Definition of } \pi_e \\ &= s && \text{–Definition of an action} \\ &= \mathrm{id}_S(s) && \text{–Definition of } \mathrm{id}_S\end{aligned}$$

Since this holds for all $s \in S$ we get $\pi_e = \mathrm{id}_S$.

(c) Using (a) and (b) we compute

$$\begin{aligned}\pi_a \circ \pi_{a^{-1}} &= \pi_{a \circ a^{-1}} && \text{– (a)} \\ &= \pi_e && \text{– Definition of a group} \\ &= \mathrm{id}_S && \text{– (a)}\end{aligned}$$

$$\begin{aligned}\pi_{a^{-1}} \circ \pi_a &= \pi_{a^{-1} \circ a} && \text{– (a)} \\ &= \pi_e && \text{– Definition of a group} \\ &= \mathrm{id}_S && \text{– (a)}\end{aligned}$$

Hence $\pi_{a^{-1}}$ is an indeed an inverse of the function π_a. This π_a is a bijection and (c) holds. □

Remark 1.7.16. (a) *If $(G, \circ)$ is a group, then we will often just say that G is a group and write ab for $a \circ b$.*

(b) *If $*$ is an action of the group G on the set S, we will often just say that G is a group acting on the set S and write as for $a * s$.*

Theorem 1.7.17. *Let G be a group acting on the set S and let $g \in G$ and $s, t \in G$.*

(a) *$g^{-1}(gs) = s$.*

(b) *If $gs = gt$, then s, t.*

Proof. (a)

$$g^{-1}(gs) = (g^{-1}g)s = es = s$$

(b) Since $gs = gt$ we get $g^{-1}(gs) = g^{-1}(gt)$ and so (a) implies $s = t$. □

Definition 1.7.18. *Let S be a set, $R \subseteq S \times S$. Put $\sim = (S, R)$.*

(a) *$\sim$ is called a relation on the set S.*

(b) *We write $a \sim b$ if $a, b \in S$ and $(a, b) \in R$.*

(c) $\sim$ *is called reflexive if* $a \sim a$ *for all* $a \in S$.

(d) $\sim$ *is called symmetric if* $b \sim a$ *for all* a, b *with* $a \sim b$,

(e) $\sim$ *is called transitive if* $a \sim c$ *for all* a, b, c *with* $a \sim b$ *and* $b \sim c$.

(f) $\sim$ *is called a equivalence relation of* $\sim$ *is reflexive, symmetric and transitive.*

(g) *Let* $a \in S$. *Then* $[a]_\sim = \{b \in S \mid a \sim b\}$. $[a]_\sim$ *is called the class of a with respect to* $\sim$.

(h) $S/\sim = \{[a]_\sim \mid a \in S\}$.

Theorem 1.7.19. *Let* $\sim$ *be an equivalence relation on the set* S *and* $a \in S$. *there exists a unique* $B \in S/\sim$ *with* $a \in B$, *namely* $B = [a]_\sim$. *In particular,* $S/\sim$ *is a partition of* S.

Proof. Let $B \in S/\sim$. We need to show that $a \in B$ if and only if $B = [a]_\sim$.

$\Longleftarrow$: Suppose $B = [a]_\sim$. Since $\sim$ is reflexive, $a \sim a$ and so $a \in [a]_\sim = B$.

$\Longrightarrow$: Suppose that $a \in B$. By definition of $S/\sim$, $B = [b]_\sim$ for some $b \in S$. Since $a \in B = [b]_\sim$ we get

$$(*) \qquad b \sim a$$

.

Next we show

$$(**) \qquad [a]_\sim \subseteq B$$

For this let $d \in [a]_\sim$. Then $a \sim d$. By $(*)$ $b \sim a$. Thus $b \sim a$ and $a \sim d$, and since $\sim$ is transitive we get $b \sim d$. Thus $d \in [b]_\sim = B$. We proved that $d \in B$ for all $d \in [a]_\sim$ and so $[a]_\sim \subseteq B$. Thus $(**)$ is proved.

$$(***) \qquad B \subseteq [a]_\sim.$$

By $(*)$ $b \sim a$. Since $\sim$ is symmetric this gives $a \sim b$ and so $b \in [a]_\sim$. Applying $(**)$ with a and b interchanged gives $[b] \sim \subseteq [a]_\sim$ and so $B \subseteq [a]_\sim$.

Observe that $(**)$ and $(***)$ gives $[a]_\sim = B$. □

Theorem 1.7.20. *Let* G *be a group acting on a set* S. *Define the relation* $\equiv_G$ *on* S *by* $s \equiv_G t$ *if there exists* $g \in G$ *with* $t = gs$. *Then* $\equiv_G$ *is equivalence relation on* S.

Proof. Let $r, s, t \in S$. We write $s \sim t$ for $s \equiv_G t$.

Note that $s = es$ and so $s \sim s$.

Suppose that $s \sim t$. Then there exists $g \in G$ with $t = gs$. Then $g^{-1}t = g^{-1}(gs) = s$ and so $t \sim s$.

Suppose that $r \sim s$ and $s \sim t$. Then there exists $g, h \in G$ with $s = gr$ and $t = hs$. Thus

$$t = hs = h(gr) = (hg)r$$

and so $r \sim s$. □

Definition 1.7.21. *Let $*$ be an action of the group G be a group acting in a set S, $s \in S$ and $g \in G$.*

(a) $G * s = \{g * s \mid s \in G\}$. *Note that $G * a$ is the class of a in S with respect to the relation '$\equiv_G$'. G is called the orbit of s in S under G.*

(b) $G_s = \{g \in G \mid g * s = s\}$. *$G_s$ is called the stabilizer of s in G.*

(c) $S_g = \{s \in S \mid g * s = s\}$. *$S_g$ is called the set of fixed-points of g in S.*

(d) $S/G = \{G * s \mid s \in S\}$. *Note that $S/G = S/\equiv_G$ is the set of classes of '$\equiv_G$' in S.*

Example 1.7.22. Let $S = \{1, 2, 3, 4\}$ and $G = \mathrm{Sym}(4) = \mathrm{Sym}(S)$.

Then

$$G_4 = \{\pi \in \mathrm{Sym}(4) \mid \pi(4) = 4\} = \mathrm{Sym}(3).$$

Put $D = \langle(124) = \{(1), (124), (142)\}$. Then

$$D1 = \{(1)(1), (124)(1), (142)(1)\} = \{1, 2, 4\}, \quad D2 == \{(1)(2), (124)(2), (142)(2)\} = \{2, 4, 1\} = D1 = D4$$

and

$$D3 = \{3\}$$

So the set of orbits D on S is

$$S/D = \big\{\{1, 2, 4\}, \{3\}\big\}.$$

Let $g = (1, 3, 4)$. The $S_g = \{2\}$.

Note that

$$|G| = 4!2 = 24, \quad |G1| = |\{1, 2, 3, 4\} = 4 \quad |G_1| = |\mathrm{Sym}(3)| = 3!6.$$

So

$$|G| = |G1| \cdot |G_1|$$

Theorem 1.7.23. *Let G be a group acting on the set S, $s \in S$ and A an orbit of G on S. Then*

$$|G| = |Gs| \cdot |G_s| = \sum_{t \in A} |G_t|$$

Proof. Recall that $Gs = \{gs \mid g \in G\}$. So for each $a \in G_s$ choose $g_a \in G$ with

$$(*) \qquad t = g_t s$$

Define

$$F: \quad Gs \times G_s \to G, \quad (a,h) \mapsto g_a h$$

We will show that F is a bijection. To show that F is injective let $a, b \in Gs$ and $h, k \in G_s$ with $F(a,h) = F(b,k)$. Then

$$\begin{aligned}
& g_a h = g_b k \\
\Longrightarrow \quad & (g_a h)s = (g_b k)s \\
\Longrightarrow \quad & g_a(hs) = g_b(ks) && \text{– Definition of an action} \\
\Longrightarrow \quad & g_a s = g_b s && \text{– } h, k \in G_s \\
\Longrightarrow \quad & a = b && \text{– } (*)
\end{aligned}$$

Hence also $g_a = g_b$ and so $g_a h = g_b k = g_a k$. Now 1.7.17 implies $h = k$. We proved $a = b$ and $h = k$, so $(a,h) = (b,k)$ and so F is injective. To show that F is surjective, let $h \in G$. Put $a := hs$. Then $(*)$ shows that

$$g_a s = a = hs$$

and so

$$(g_a^{-1} h)s = g_a^{-1}(hs) = g_a^{-1}(g_a s) = s$$

Thus $g_a^{-1} h \in G_s$ and

$$F(a, g_a^{-1} h) = g_a(g_a^{-1} h) = h$$

Thus F is surjective. We proved that F is a bijection and so

$$|G| = |Gs \times G_s| = |Gs| \cdot |G_s|$$

So the first equality holds. For the second we may assume that $s \in A$. Then $A = Gs$. Then

$$|G| = |A||G_s|$$

In particular, $|G_t| = |G_s|$ for all $t \in A$ and so

$$|G| = |A||G_s| = \sum_{t \in A} |G_s| = \sum_{t \in A} |G_t|$$

□

Theorem 1.7.24 (Burnside)**.** *Let G be a group acting on a set S. Then*

$$\sum_{g \in G} |S_g| = \sum_{s \in S} |G_s| = |S/G| \cdot |G|$$

Proof. Let

$$D = \{(g, s) \mid g \in G, s \in S, gs = s\}$$

We will compute $|D|$ in two ways:
For a fixed $g \in G$, $(g, s) \in D$ if and only of $s \in S_g$. So there are $|S_g|$ choices for s. Hence

$$|D| = \sum_{g \in G} |S_g|$$

For a fixed s, $(g, s) \in D$ if and only of $g \in G_s$. So there are $|G_s|$ choices for s. Hence

$$|D| = \sum_{s \in S} |G_s|$$

So the first equality in Burnside's Theorem holds.

Let A be an orbit of G on S. Then by 1.7.23 $|G| = \sum_{s \in S} |G_s|$. Also G/S is a partition of S and so

$$\sum_{s \in S} |G_s| = \sum_{A \in S/G} \sum_{s \in A} |G_s| = \sum_{A \in S/G} |G| = |S/G| \cdot |G|$$

so also the second equality holds. □

Definition 1.7.25. *Let S and T be sets. A coloring of S with T is a function c from S to T. We denote the set of coloring of S with T by $\mathrm{Co}(T, S)$. Note that if $T = \{1, 2, 3, \ldots n\}$, when a coloring of S with T is a sequence $\big(c(i)\big)_{i=1}^{n}$ from T.*

Example 1.7.26. Let $S = \{1, 2, 3, 4\}$ and $T = \{g, w\}$. Then

$$\begin{aligned}\mathrm{Co}(S, T) = \{&gggg, gggw, ggwg, ggww, gwgg, gwgw, gwwg, gwww,\\ &wggg, wggw, wgwg, wgww, wwgg, wwgw, wwwg, wwww\}\end{aligned}$$

Now view S has the set of vertices of a square and define two coloring to be equivalent, if the second is rotation of the first. So for example $ggww, wggw, wwgg, gwwg$ are equivalent to each other. What are the equivalence class?

$$\{gggg\}$$
$$\{wggg, gwgg, ggwg, gggw\}$$
$$\{ggww, wggw, wwgg, gwwg\}$$
$$\{gwgw, wgwg\}$$
$$\{gwww, wgww, wwgw, wwwg\}$$
$$\{wwww\}$$

So there are six equivalence class.

Theorem 1.7.27. *Let G be a group acting on a set S, and let T be a set. Then G acts on $\mathrm{Co}(S,T)$ via*

$$(g * c)(s) = c(g^{-1}s)$$

for all $g \in G$, $c \in \mathrm{Co}(T,S)$ and $s \in S$

Proof. Observe that $g * c = g \circ \pi_{g^{-1}}$, where $\pi_g : S \to S, g \mapsto gs$ is the function defined in 1.7.15. As $\pi_{g^{-1}} = \pi_g^{-1}$ we get

$$g * c = c \circ \pi_g^{-1}$$

We compute

$$(g \circ h) * c = c \circ \pi_{gh}^{-1} = c \circ (\pi_g \circ \pi_h)^{-1} = c \circ (\pi_h^{-1} \circ \pi_g^{-1}) = (c \circ \pi_h^{-1}) \circ \pi_g^{-1} = g * (h * c)$$

and

$$e * c = c \circ \pi_e^{-1} = c \circ \mathrm{id}_S^{-1} = c \circ \mathrm{id}_S = c$$

So $*$ is indeed an action of G in $\mathrm{Co}(S,T)$. □

Theorem 1.7.28. *S, T be sets.*

(a) $|\mathrm{Co}(S,T)| = |T|^{|S|}$.

(b) *Let G be group acting on S. Then $|\mathrm{Co}(S,T)_G| = |T|^{|S/G|}$.*

(c) *Let $g \in G$ and let k_g be the number of cycles of g on S, that is the number of cycles of the permutation π_g of S, including the cycles of length 1. Then $|\mathrm{Co}(S,T)_g| = |T|^{k_g}$.*

Proof. (a) Let $c \in \mathrm{Co}(S,T)$. For each $s \in S$, $c(s)$ is one of the $|T|$ elements of T. So the product rule shows $|\mathrm{Co}(S,T)| = |T|^{|S|}$.

(b) Let $c \in \mathrm{Co}(S,T)$. Then

$$\begin{aligned}
& c \in \mathrm{Co}(S,T)_G \\
\iff\quad & c = g * c \text{ for all } g \in G \\
\iff\quad & c(s) = c(g^{-1}s) \text{ for all } g \in G, s \in S \\
\iff\quad & c(s) = c(gs) \text{ for all } g \in G, s \in S \\
\iff\quad & c(s) = c(t) \text{for all } A \in S/G, s,t \in A
\end{aligned}$$

So $c \in \mathrm{Co}(S,T)$, then for each $A \in S/G$ we have $|T|$ choices for the common color of the elements of A. Thus $|\mathrm{Co}(S,T)_G = |T|^{|S/G|}$.

(c) Put $H = \langle g \rangle$. Each orbits of H on S consist of the elements of a cycle of g on S. So k_g is the number of orbits of H on S, that is $|S/H|$. Hence (c) is a special case of (b). □

Corollary 1.7.29. *Let S and T be sets and let G be group acting on S. For $g \in G$ let k_g be the number of cycles of g on S Then the number of orbits of G on $\mathrm{Co}(S,T)$ is*

$$|\mathrm{Co}(S,T)/G| = \frac{1}{|G|} \sum_{\pi \in G} |T|^{k_\pi}$$

Proof. By Burnside's Theorem

$$|G| \cdot |\mathrm{Co}(S,T)/G| = \sum_{g \in G} |\mathrm{Co}(S,T)_g|$$

and by (1.7.28)(c) $|\mathrm{Co}(S,T)_g| = |T|^{k_g}$. Thus

$$|\mathrm{Co}(S,T)/G| = \frac{1}{|G|} \sum_{\pi \in G} |\mathrm{Co}(S,T)|_g = \frac{1}{|G|} \sum_{\pi \in G} |T|^{k_\pi}.$$

□

Example 1.7.30. Use 1.7.29 to determine the number of orbits of C_4 and D_4 on $\mathrm{Co}(\{1,2,3,4\},\{g,w\})$ and on $\mathrm{Co}(\{1,2,3,4\},\{r,w,b\})$

We need to compute k_π of for each π in D_4.

elements	m = # elements	k_π	2^{k_π}	$m \cdot 2^{k_\pi}$	3^{k_π}	$m \cdot 3^{k_\pi}$
$(1)(2)(3)(4)$	1	4	16	16	81	81
$(1234), (4321)$	2	1	2	4	3	6
$(13)(14)$	1	2	4	4	9	9
$(12)(34), (14)(23)$	2	2	4	8	9	18
$(13)(2)(4), (14)(2)(3)$	2	3	8	16	27	54

Thus the number of orbits of C_4 are

$$\frac{16+4+4}{|C_4|} = \frac{24}{4} = 6 \quad \text{and} \quad \frac{81+6+9}{|C_4|} = \frac{96}{4} = 24$$

and the number of orbits of D_4 are

$$\frac{16+4+4+8+16}{|D_4|} = \frac{48}{8} \quad \text{and} \quad \frac{81+6+9+18+54}{|D_4|} = \frac{168}{8} = 21$$

1.7.3 The Cycle Index

Definition 1.7.31. *Let G be a group acting on a set S. Put $n = |S|$ and $\underline{x} = (x_i)_{i=1}^n = (x_1, \ldots x_n)$ be a sequence of n indeterminates.*

(a) *Let k_g is the number of cycles of g on S. For $1 \le i \le k_g$ let l_i be the length of i-th cycles of g on S and for $1 \le l \le n$ let m_l be the number of cycles of length l of g on S. Define the multinomial*

$$M_g(\underline{x}) = \prod_{i=1}^{k_g} x_{l_i} = x_{l_1} x_{l_2} \ldots x_{l_k}$$

and note that

$$M_g(\underline{x}) = \prod_{l=1}^{m} x_l^{m_l}$$

Depending on the content we way also write M_g, M_g^S and $M_g^S(\underline{x})$ for M_g.

(b)

$$P_G(\underline{x}) := \frac{1}{|G|} \sum_{g \in G} M_g(\underline{x})$$

$P_G(\underline{x})$ is called the cycle index *of G on S. We also may write $P_G^S(x)$ for $P_G(x)$ to indicated the dependence of $P_G(x)$ on the set S.*

Example 1.7.32. Compute $P_{C_4}(\underline{x})$ and $P_{D_4}(\underline{x})$.

The elements of C_4 are $(1)(2)(3)(4)$, $(1234), (13)(24)$ and $(14)(23)$. We compute

$$\begin{aligned}
M_{(1)(2)(3)(4)} &= x_1 x_1 x_1 x_1 = x_1^4 \\
M_{(1234)} &= x_4 \\
M_{(13)(24)} &= x_2 x_2 = x_2^2 \\
M_{(1432)} &= x_4
\end{aligned}$$

and so

$$P_{\mathrm{C}_4} = \frac{1}{|C_4|}\sum_{g\in \mathrm{C}_4} M_g = \frac{1}{4}(x_1^4 + x_4 + x_2^2 + x_4) = \frac{1}{4}(x_1^4 + 2x_4 + x_2^2)$$

The elements of D_4 which are not in C_4 are the four reflections $(13)(2)(4), (24)(1)(3)$, $(12)(34)$ and $(14)(23)$. We compute

$$\begin{aligned}
M_{(13)(2)(4)} &= x_2x_1x_1 = x_1^2x_2\\
M_{(24)(1)(3)} &= x_2x_1x_1 = x_1^2x_2\\
M_{(12)(34)} &= x_2x_2 \quad = x_2^2\\
M_{(14)(23)} &= x_2x_2 \quad = x_2^2
\end{aligned}$$

Thus

$$P_{\mathrm{D}_4} = \frac{1}{|D_4|}\sum_{g\in \mathrm{D}_4} M_g = \frac{1}{8}(x_1^4 + 2x_4 + x_2^2 + x_1 2x_2 + x_1^2x_2 + x_2^2 + x_2^2) = \frac{1}{8}(x_1^4 + 2x_4 + 3x_2^2 + 2x_1^2x_2)$$

Theorem 1.7.33. *Let S and T be sets and G a group acting on S. Then number of orbits of G on $\mathrm{Co}(S,T)$ is*

$$|\mathrm{Co}(S,T)/G| = P_G^S(|T|, |T|, \dots, |T|)$$

Proof. Let $n = |S|$ and $g \in G$. Then

$$M_g(x_1, \dots, x_n) = \prod_{i=1}^{k_g} x_{l_i}$$

and so

$$M_g(|T|, \dots, |T|) = \prod_{i=1}^{k_g} |T| = |T|^{k_g}$$

As $P_G(\underline{x}) = \frac{1}{|G|}\sum_{g\in G} M_g(\underline{x})$ we get

$$P_G(|T|, \dots |T|) = \frac{1}{|G|}\sum_{g\in G} M_g(|T|, \dots |T|) = \frac{1}{|G|}\sum_{g\in G} |T|^{k_g}$$

By 1.7.29 the latter sum in equal to $|\mathrm{Co}(S,T)/G|$ and so indeed

$$|\mathrm{Co}(S,T)/G| = P_G(|T|, |T|, \dots, |T|)$$

□

Example 1.7.34. Use the cycle index to compute the number of orbits of D_4 on $\mathrm{Co}(\{1,2,3,4\}, \{g, w\})$.

By 1.7.32

$$P_{D_4}(\underline{x}) = \frac{1}{8}(x_1^4 + 2x_4 + 3x_2^2 + 2x_1^2x_2.$$

Also $|\{g, w\}| = 2$ and so by 1.7.33 the number of orbits is

$$P_{D_4}(2,2,2,2) == \frac{1}{8}(2^4 + 2\cdot 2 + 3\cdot 2^2 + 2\cdot 2^2\cdot 2) = \frac{1}{8}(16+4+12+16) = \frac{48}{8} = 6$$

Note that this is the same answer as in 1.7.30.

Our next goal is to compute $P_{D_n}(\underline{x})$ for arbitrary $n \in \mathbb{Z}^+$.

For this we need to determine the lengths of the cycles of the rotation and the reflection. We start with an example

Example 1.7.35. Let $\pi = (1,2,3,4,\dots,20)$ be the rotation by 18°. For $i = 1,2,3,4,5,12$ compute the order of π^i and the number and lengths of cycles of π^i.

$\pi^1 = \pi$ has order 20 and has 1 cycle of length 20.

$\pi^2 = (1,3,5,\dots,19)(2,4,6,\dots,20)$ has order 10 and as 2 cycles of length 10.

$\pi^3 = (1,4,7,10,13,16,19,2,5,8,11,14,17,20,3,6,9,12,15,18)$ has order 20 and as 1 cycle of length 20.

$\pi^4 = (1,5,9,13,17)(2,6,10,14)(3,7,11,15,19)(4,8,12,16,20)$ has order 5 and has 4 cycles of length 5.

$\pi^5 = (1,6,11,16)(2,7,12,17)(3,8,13,18)(4,9,14,19)(5,10,15,20)$ has order 4 and has 5-cycles of length 4.

$\pi^{12} = (1,13,5,17,9)(2,14,6,18,10)(3,15,7,19,11)(4,16,8,20,12)$ has order 5 has 4-cycles of length 5.

The general pattern:

Let $l \in \mathbb{Z}$. Then π^l order k and has m cycles of length k, where $m = \gcd(l,20)$ and $km = 20$.

How many rotation of order 4 are in C_{20}. Let $0 \le l < 20$. The π^l as order 4 if and only if $4\cdot\gcd(l,20) = 20$, that is $\gcd(l,20) = 5$. Hence l is one of 5, 15. So there are two such elements. Note that any such l can be written as $5d$ where $0 \le d < 4$ and $\gcd(d,4)$. So there are $\phi(4)$ such elements.

In general, if $k \in \mathbb{Z}^+$ with $k|20$, then any rotation of order k in C_{20} can be written as π^{md}, where $m = \frac{n}{k}$ and $0 \le d < k$ with $\gcd(d,k) = 1$.

Theorem 1.7.36. *Let G be a cyclic group of finite order n and $g \in G$ with $G = \langle g\rangle$.*

(a) *Let $l \in \mathbb{Z}$. Then $|g^l| = \frac{n}{\gcd(l,n)}$. In particular, $|g^l|$ divides n.*

(b) *Let $k \in \mathbb{Z}^+$ with $k|n$ and let $h \in G$. Put $m = \frac{n}{k}$. Then $|h| = k$ if and only if $h = g^{md}$ for some $d \in \mathbb{Z}$ with $0 \le d < k$ and $\gcd(d,k) = 1$. Moreover, this d is uniquely determined by h. In particular, then number of elements of order k in G is $\phi(k)$.*

(c) *Let $h \in C_n$ and put $k = |h|$. Then h has exactly $\frac{n}{k}$ cycles, each of length k.*

Proof. (a): Let $m \in \mathbb{Z}$ then $(g^l)^m = e$ if and only of $g^{lm} = e$ and, since $|g| = n$, if and only of n divides lm. The latter holds if and only of $\frac{n}{\gcd(l,n)}$ divides m. Hence the smallest positive integer with $(g^l)^m = e$ is $k = \frac{n}{\gcd(l,n)}$

(b): Since $h \in G = \langle g \rangle$ and $|G| = n$ there exists a unique $l \in \mathbb{Z}$ with $0 \le l < n$ and $h = g^l$.

Suppose that $|h| = k$. Then (a) shows that $k = \frac{n}{\gcd(n,l)} = \frac{n}{e} m = \frac{n}{k} = e$ and so m divides l. If $h = g^{md}$ with $0 \le d < k = \frac{n}{m}$, then $0 \le md < n$ and so $l = md$ and again m divides l.

So we may assume that m divides l. Put $d = \frac{l}{m}$. Then d is the unique integer such that $l = md$. Since l is the unique integer with $h = g^l$ and $0 \le l < n = mk$, d is the unique integer with $0 \le d < k$ and $h = g^{kd}$. It remains to show that $|h| = k$ if and only if $\gcd(k, d) = 1$.

$$
\begin{aligned}
& |h| = k \\
\Longleftrightarrow\quad & k = \frac{n}{\gcd(l,n)} && -\text{(a)} \\
\Longleftrightarrow\quad & \gcd(l,n) = \frac{n}{k} \\
\Longleftrightarrow\quad & \gcd(md, mk) = m && -\, l = md, n = mk \\
\Longleftrightarrow\quad & m \cdot \gcd(d,k) = m \\
\Longleftrightarrow\quad & \gcd(d,k) = 1
\end{aligned}
$$

Hence (b) holds.

(c) Since $\pi \in \mathrm{C}_n$, π is a rotation of order k of a regular n-gon. Let a be a vertex of the n-gon. For any $1 \le i < k$, π^i is a non-trivial rotation and so $\pi^i(a) \neq a$ On the other hand $\pi^k = e$ and so $\pi^k(a) = a$. So the cycle containing a is

$$\big(a, \pi(a), \pi^2(a), \ldots, \pi^{k-1}(a)\big)$$

We proved that each cycle of π has length k. Since there are n-vertices, π must have $\frac{n}{k}$ cycles. □

Example 1.7.37. Let $n \in \mathbb{Z}^+$. Compute $P_{\mathrm{C}_n}(\underline{x})$ and $P_{\mathrm{D}_n}(\underline{x})$.

We start with $P_{\mathrm{C}_n}(\underline{x})$. Let $k \in \mathbb{Z}^+$ with $k|n$. From 1.7.36 each element of C_n has order dividing n, and C_n has $\phi(k)$ elements of order k. Moreover, if $g \in \mathrm{C}_n$ has order k, then g has $\frac{n}{k}$ cycles of length k. So $M_g = x_k^{\frac{n}{k}}$. Hence for each $k \in \mathbb{Z}^+$ with $k|n$ there are $\phi(k)$ elements with $M_g = x_k^{\frac{n}{k}}$. Thus

$$P_{\mathrm{C}_n}(\underline{x}) = \frac{1}{|\mathrm{C}_n|} \sum_{g \in \mathrm{C}_n} M_g = \frac{1}{n} \sum_{\substack{k \in \mathbb{Z}^+ \\ k|n}} \phi(k) x_k^{\frac{n}{k}}$$

For example, using the formula

$$\phi\left(\prod_{i=1}^m p_i^{l_i}\right) = \prod_{i=1}^l p_i^{l_i - 1}(p_i - 1)$$

we compute

$$\begin{aligned} P_{C_{20}}(\underline{x}) &= \frac{1}{20}\left(\phi(1)x_1^{20} + \phi(2)x_2^{10} + \phi(4)x_4^5 + \phi(5)x_5^4 + \phi(10)x_{10}^2 + \phi(20)x_{20}^1\right) \\ &= \frac{1}{20}\left(x_1^{20} + x_2^{10} + 2x_4^5 + 4x_5^4 + 4x_{10}^2 + 8x_{20}\right) \end{aligned}$$

Next we compute $P_{D_n}(\underline{x})$. For this we still need to compute the cycles for the n-reflections.

Let $g \in D_n$ be a reflection. Then $g^2 = (1)$ and so the cycles all of length 1 or 2. Note that there are two kinds of reflection: Reflection at a line through a vertex, and reflections through a mid-point of an edge. These reflections behave differently when n is even vs. when n is odd.

Suppose first that $n = 2m + 1$ is odd. The point opposite to a vertex is the mid-point of an edge and vice versa. It follows g fixes exactly one of the vertices. Hence π has one cycle of length 1 and m cycles of length 2. Thus

$$M_g = x_1 x_2^m$$

Since there are n reflections we conclude:

$$P_{D_n}(\underline{x}) = \frac{1}{|D_n|}\sum_{g \in D_n} M_g = \frac{1}{2n}\left(\sum_{\substack{k \in \mathbb{Z}^+ \\ k \mid n}} \phi(k) x_k^{\frac{n}{k}} \; + \; n x_1 x_2^m\right)$$

Suppose next that $n = 2m$ is even. Then the point opposite to a vertex is vertex and the point opposite to the mid-point of edge is the mid-point of edge. So if g is reflection at a line through a vertex, then g has two fixed points and otherwise none. In the first case g has two cycles of length 1 and $m - 1$ cycles of length 2, so $M_g = x_1^2 x_2^{m-1}$. In the second case g has no cycles of length 1 and m cycles of length 2, so $M_g = x_2^m$. There are m reflections of each of the two kinds, thus

$$P_{D_n}(\underline{x}) = \frac{1}{|D_n|}\sum_{g \in D_n} M_g = \frac{1}{2n}\left(\sum_{\substack{k \in \mathbb{Z}^+ \\ k \mid n}} \phi(k) x_k^{\frac{n}{k}} \; + \; m x_1^2 x_2^{m-1} + m x_2^m\right)$$

For example if $n = 20 = 2 \cdot 10$, then

$$\begin{aligned} P_{D_{20}}(\underline{x}) &= \frac{1}{40}\left(x_1^{20} + x_2^{10} + 2x_4^5 + 4x_5^4 + 4x_{10}^2 + 8x_{20} + 10x_1^2 x_2^9 + 10x_2^{10}\right) \\ &= \frac{1}{40}\left(x_1^{20} + 11x_2^{10} + 2x_4^5 + 4x_5^4 + 4x_{10}^2 + 8x_{20} + 10x_1^2 x_2^9\right) \end{aligned}$$

Chapter 2

Graph Theory

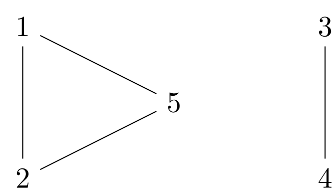
2.1 Introductory Concepts

2.1.1 The Basic

Definition 2.1.1. *A graph G is a pair (V, E) such that V is a set and E is a subset of $\mathcal{P}_2(V)$. The elements of V are called the vertices of G and the elements of E are called the edges of G. A graph G is called finite if V is finite. All graphs in these lecture notes are assumed to be finite.*

Note that every edge of G is a set $\{a, b\}$ with $a, b \in V$ and $a \neq b$.

Example 2.1.2. Let $V = \{1, 2, 3, 4, 5\}$, $E = \{\{1, 2\}, \{3, 4\}, \{1, 5\}, \{2, 5\}\}$. Then $G = (V, E)$ is a graph with 5 vertices and 4 edges. We can visualize this graph by the diagram



Definition 2.1.3. *Let $G = (V, E)$ be graph, $v, w \in V$, $e \in E$ and $S \subseteq V$.*

(a) *The order of G is the cardinality $|V|$ of V.*

(b) *The size of G is the cardinality $|E|$*

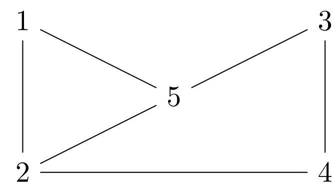
(c) *We will write vw for $\{v, w\}$. If a formula $\Phi(S)$ is defined for S being a set of vertices and also for S being an edge, then in $\Phi(\{v, w\})$ we view $\{v, w\}$ has set of two vertices and in $\Phi(vw)$ we view vw as an edge.*

(d) *We say that v is adjacent to w or that v and w are adjacent, if vw is an edge of G, that is if $\{v, w\} \in E$.*

(e) *v and w are nonadjacent in G if vw is not an edge of G.*

(f) *If $v \in e$ we say that v is a (end) vertex of v, that e is incident with v and that v is incident to e.*

Example 2.1.4. Consider the following graph G



G has order 5 and size 6.
5 is adjacent to $1, 2, 3$ and nonadjacent to 4.
4 is incident with 24 and not incident with 15.

Definition 2.1.5. (a) *The (open) neighborhood of v in G, denoted by $\mathrm{N}(v)$ is the set of vertices adjacent to v, that is*

$$\mathrm{N}(v) = \{w \in V \mid vw \in E\}$$

(b) *The closed neighborhood of v in G, denoted by $\mathrm{N}[v]$ is the set consisting of v and all vertices adjacent to v, that is*

$$\mathrm{N}[v] = \{v\} \cup \mathrm{N}(v)$$

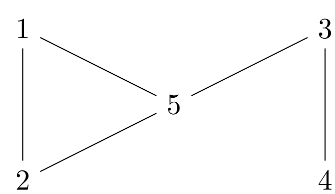
(c) *The (open) neighborhood of S in G, denoted by $\mathrm{N}(S)$ is the set of vertices adjacent to some vertex in S, that is*

$$\mathrm{N}(S) = \bigcup_{s \in S} \mathrm{N}(s).$$

(d) *The closed neighborhood of S in G, denoted by $\mathrm{N}(S)$ is the vertices which are in S or are adjacent to some vertex in S, that is*

$$\mathrm{N}(S) = \bigcup_{s \in S} \mathrm{N}[s].$$

Example 2.1.6. Consider the following graph G



$$\mathrm{N}(1) = \{2,5\}, \quad \mathrm{N}[1] = \{1,2,5\}, \quad \mathrm{N}\left(\{2,4\}\right) = \{1,3,5\} \quad \text{and} \quad \mathrm{N}\left[\{2,4\}\right] = \{1,2,3,4,5\}$$

Definition 2.1.7. *Let $G = (V, E)$ be a graph.*

(a) *The degree of a vertex v, is the cardinality of* $\mathrm{N}(v)$*, that is the number of vertices adjacent to* v*.*

(b) *The maximum degree of G, denoted by $\Delta(G)$, is the largest degree of a vertex of G, that is*

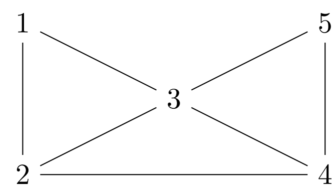
$$\Delta(G) = \max_{v \in V} \deg(x)$$

(c) *The minimum degree of G, denoted by $\delta(G)$, is the smallest degree of a vertex of G, that is*

$$\delta(G) = \min_{v \in V} \deg(x)$$

(d) *The degree sequence of G is the decreasing sequence obtained from the multiset* $[\deg(v) \mid v \in V]$.

Example 2.1.8. Let G be the graph



Then

$$\deg(1) = 2, \quad \deg(2) = 3, \quad \deg(3) = 4, \quad \deg(4) = 3, \quad \text{and} \quad \deg(5) = 2$$

So

$$\Delta(G) = 4 \quad \text{and} \quad \delta(G) = 2$$

and the degree sequence is

$$43322$$

Theorem 2.1.9. *Let (V, E) be a graph. Then*

$$\sum_{v \in V} \deg(v) = 2|E|$$

In particular, the number of vertices of odd degree is even.

Proof. Let $I = \{(v, w) \in V \times V \mid vw \in E\}$. We count the elements of I in two ways.

First note that any edge vw of G gives rise to two elements of I, namely (v, w) and (w, v). Thus $|I| = 2|E|$. Also

$$|I| = \sum_{v \in V} |\{w \in V \mid vw \in E\}| = \sum_{v \in V} |\mathrm{N}(v)| = \sum_{v \in V} \deg(v)$$

□

Definition 2.1.10. *Let $G = (V, E)$ be a graph, let $k \in \mathbb{Z}^+$, let $u, v \in V$ and and let $W = w_0 w_2 \dots w_k$ be a sequence of vertices.*

(a) *W is called a $u - v$ walk if $u = w_0, v = w_k$ and w_{i-1} is adjacent to w_i for each $1 \le i \le k$.*

(b) *Suppose W is a $u - v$ walk. Let $1 \le i \le k$ and put $e_i = e_{i-1}e_i$. Then e_i is called an edge of W. Then $(e_i)_{i=1}^n$ is called the sequence of edges of W. The length of the walk W is defined to be k (so the length of sequences of edges of W). u and v are called the end vertices of W.*

(c) *A trail is a walk with no repeated edges.*

(d) *A path is a walk with no repeated vertices.*

(e) *A $u - v$ walk is called closed if $u = v$.*

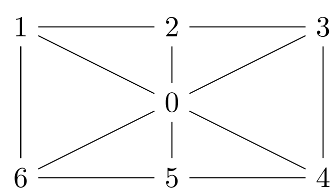
(f) *A circuit is a closed trail of length at least 1.*

(g) *W is a cycle if W is a circuit and $w_1 \dots w_k$ is a path.*

Remark 2.1.11. (a) *A path with no repeated vertex cannot have a repeated edge. So every path is a trail.*

(b) *If vw is an edge in a graph, then $v \neq w$ and vwv has a repeated edge. Thus all circuits and all cycles have length at least three.*

Example 2.1.12. Consider the graph



01235 is not a walk.

102302 is a walk but neither closed nor a trail.

012056 is a trail, but neither closed nor a path.

123456 is a path, but is not closed.

0120320 is a closed walk, but not a circuit.

0120650 is a circuit, but a not cycle.

1234561 is a cycle.

Definition 2.1.13. *A subwalk of a walk W is a subsequence of W which is a walk.*

Theorem 2.1.14. *Let W be a $u - v$ walk in the graph G. Then every shortest $u - v$ subwalk of W is a path in G.*

Proof. $S = s_1 \ldots s_n$ be a $u-v$ subwalk of W of smallest length. Suppose S has a repeated vertex. Then there exist $1 \le i < j \le n$ with $s_i = s_j$.

Suppose $j = n$. Then $s_1 = u$ and $s_i = s_j = s_n = v$ and so

$$s_1 s_2 \ldots s_i$$

is a $u-v$-subwalk of W of shorter length than S, a contradiction.

Suppose that $j < n$. Since $s_j s_{j+1}$ is an edge, we see that $s_i s_{j+1}$ is an edge. Also $s_1 = u$ and $s_n = v$, so

$$s_1 s_2 \ldots s_i s_{j+1} \ldots s_n$$

is a $u-v$-subwalk of W of shorter length than S, again a contradiction. □

Definition 2.1.15. *Let $G = (V, E)$ be graph.*

(a) *Let $v \in V$. Then $G - v$ is the graph obtained from G by removing v and all edges incident with v. That is*

$$V - v = V \setminus \{v\}, \quad E - v = E \setminus \{e \in E \mid v \in e\}, \quad \textit{and} \quad G - v = (V - v, E - v).$$

(b) *Let $S \subseteq V$. Then $G - S$ is the graph obtained from G by removing each vertex of S and all edges incident with a vertex of S. That is*

$$V - S = V \setminus S, \quad E - S = E \setminus \{e \in E \mid s \in e \text{ for some } s \in S\}, \quad \textit{and} \quad G - S = (V - S, E - S).$$

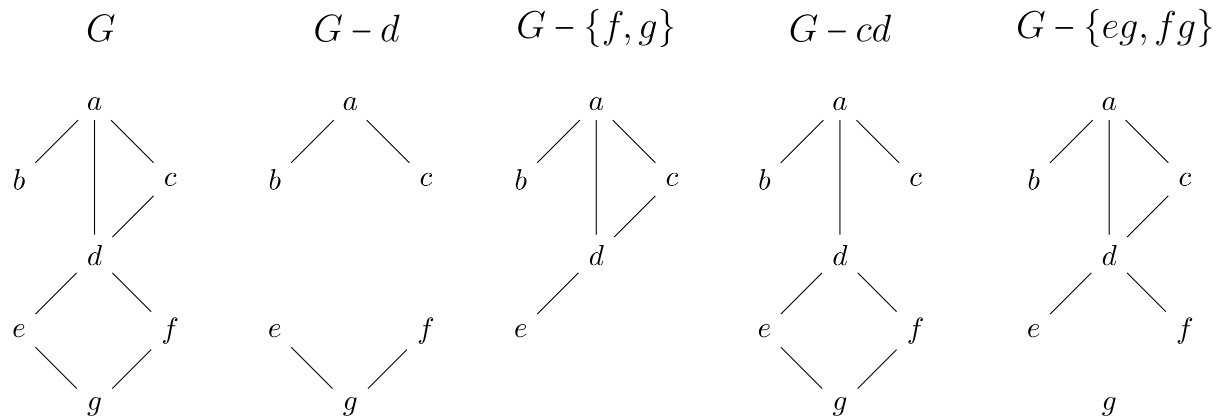
(c) *Let $e \in E$. Then $G - e$ is the graph obtained from G by removing e. That is*

$$V - e = V, \quad E - e = E \setminus \{e\}, \quad \textit{and} \quad G - e = (V - e, E - e).$$

(d) *Let $T \subseteq E$. Then $G - T$ is the graph obtained from G by removing each edge of T. That is*

$$V - T = V \setminus T, \quad E - T = E \setminus T, \quad \textit{and} \quad G - T = (V - T, E - T).$$

Example 2.1.16.

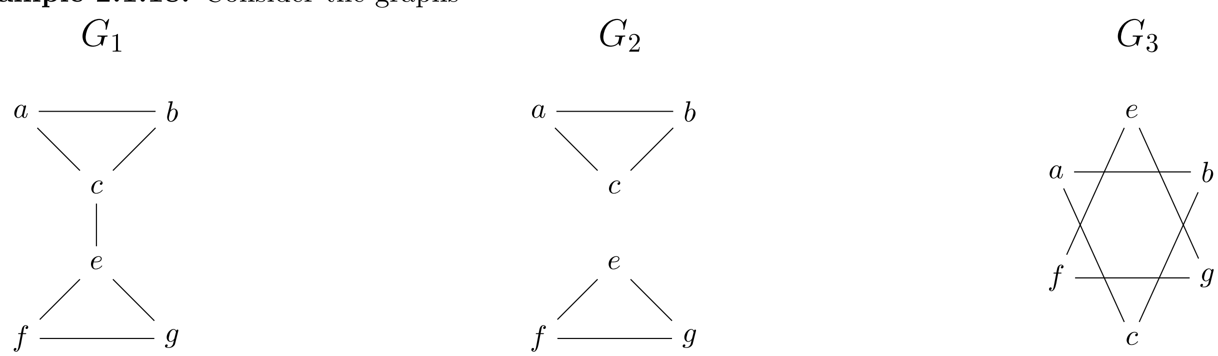


Definition 2.1.17. *Let G be graph.*

(a) *Let u, v be vertices of G. We say that u is* connected to *v in G if there exists a $u - v$ walk in G.*

(b) *We say that G is* connected *if G has order at least 1, and u is connected to v in G whenever u and v are vertices of G. G is* disconnected *if G has order at least 1 and is not connected.*

Example 2.1.18. Consider the graphs



G_1 is connected.

a is not connected to e in G_2. So G_2 is not connected.

G_2 and G_3 are the exact same graphs. So G_3 is not connected.

Theorem 2.1.19. *The relation 'is connected to' on the vertices of a graph is an equivalence relation.*

Proof. Let u, v, w be vertices of the graph.
v is a $v - v$ walk, so the relation is connected.
If $w_1 \dots w_n$ is a $v - w$ walk, then $w_n \dots w_1$ is a $w - v$ walk, so the relation is symmetric.
If $v_1 \dots v_n$ is a $u - v$ walk, and $w_1, \dots w_m$ is a vw walk, then $v_n = v = w_1$ and so

$$v_1 \dots v_n w_2 \dots w_m$$

is a $u - w$ walk, so the relation is transitive. □

Definition 2.1.20. *The equivalence classes of the relation 'connected to' on the vertices of graph are called the connected components of the graph.*

Example 2.1.21. The graph G_1 in 2.1.18 has one connected component, and the graphs G_2 and G_3 each have two connected components.

Definition 2.1.22. *Let G be a graph of order n.*

(a) *A vertex v of G is called a cut vertex if $G - v$ has more connected components than G.*

(b) *An edge e of G is called a bridge if $G - e$ has more connected components than G.*

(c) *A set of vertices S of G is called a cut set if $G - S$ is disconnected.*

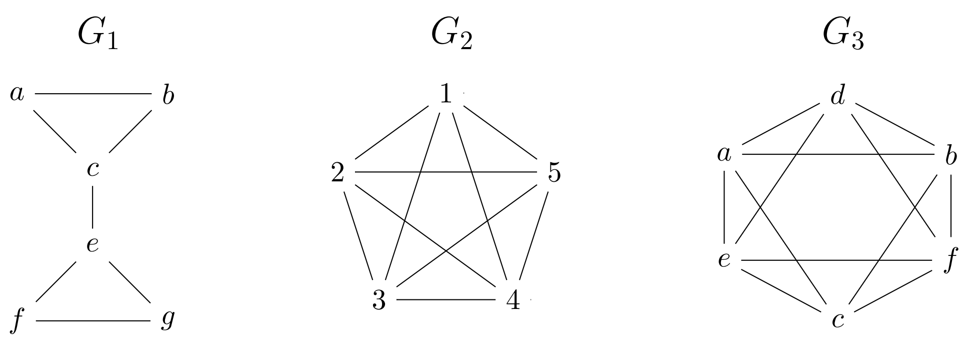
(d) *$\kappa(G)$ is the minimum size of a cut set of G, with $\kappa(G) = n - 1$ if G is a graph of order n with no cut set. $\kappa(G)$ is called the connectivity of G.*

(e) *A graph is called complete if vw is an edge whenever v and w are distinct vertices of V.*

(f) *Let $k \in \mathbb{N}$. We say that G is k-connected if $G - S$ is connected of order at least 2 for all sets of vertices S with $|S| < k$.*

Remark 2.1.23. *A graph G of order n is k-connected if and only if $k < n$ and any subgraph obtained by removing less than k vertices from G is connected.*

Example 2.1.24. Consider the graphs



c is a cut vertex of G_1, ec is bridge of G_1, $\{c\}$ is a cut set of G_1 and $\kappa(G_1) = 1$. G_1 is 1-connected but not 2-connected.

G_2 is complete, does not have a cut vertex, does not have a cut set and $\kappa(G_2) = 5 - 1 = 4$.

G_3 has no cut vertex, $\{a, b, e, f\}$ is a cut set and $\kappa(G_3) = 4$.

Theorem 2.1.25. *Let $n \in \mathbb{Z}^+, k \in \mathbb{N}$ and let G be a graph of order n.*

(a) *G is 0-connected.*

(b) *G is 1-connected if and only if G is connected and $n \geq 2$.*

(c) *G is complete if and only G has no cut set and if and only if G is $n-1$-connected.*

(d) *If G is k-connected, then $k \leq n-1$.*

(e) *$\kappa(G)$ is the largest integer m such that G is m-connected. In particular, G is k-connected if and only if $k \leq \kappa(G)$.*

Proof. (a): There are no sets of size less than 0, so G is 0-connected.

(b): $\varnothing$ is the only set of size less than 1 and $G - \varnothing = G$. Hence G is 1-connected if and only if G is connected and $n \geq 2$.

(c): Suppose G is complete and S is a subset of V. Since G is complete also $G - S$ is complete and thus $G - S$ is connected. So S is not a cut set.

Suppose G does not have a cut set and let $S \subseteq V$ with $|S| < n-1$. Then $G - S$ has order at least two and is connected. Thus G is $n-1$-connected.

Suppose G is $n-1$-connected and let v and w be distinct vertices of G. Put $S = V \setminus \{v, w\}$. Then $|S| = n-2 < n-1$ and since G is $n-1$ connected we conclude that $G - S$ is connected. The only vertices of $G - S$ are v and w and so vw must be an edge. Thus G is complete.

(d) Let $v \in V$ and put $S = V \setminus \{v\}$. Then $G - S$ has order 1. If $|S| < k$ then since G is k-connected we conclude that $G - S$ has order at least 2, a contradiction. Hence $k \leq |S| = n-1$.

(e) By (d) $k \leq n-1$, whenever G is k-connected. Thus there exists a largest integer m such that G is m-connected. Moreover $m \leq n-1$.

Suppose first that there does no exist a cut set of G. Then (c) shows that G is $n-1$ connected and so $m \geq n-1$. Thus $m = n-1$. Moreover, the definition of $\kappa(G)$ implies $\kappa(G) = n-1$. Hence $m = \kappa(G)$ in this case.

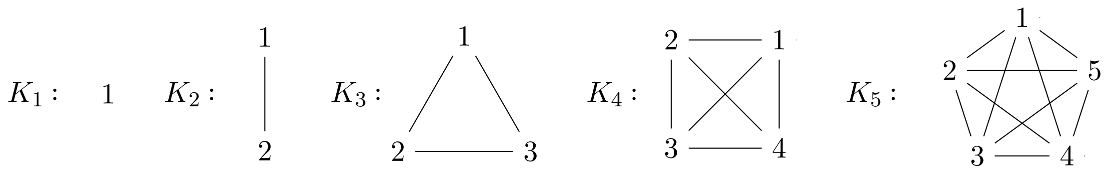
Suppose next that there does exist a cut set and let T be a cut set of minimal size. So $|T| = \kappa(G)$ by definition of $\kappa(G)$. As $G - T$ is not connected and G is m-connected we cannot have $|T| < m$. So $m \leq |T|$. Now let $S \subseteq V$ with $|S| < |T|$. By the minimal choice of $|T|$ we see that $G - S$ is connected. Also $G - S$ has order larger than $G - T$. But $G - T$ is disconnected and so has order at least 2. Hence also $G - S$ has size at least 2 and so G is $|T|$-connect. It follows that $m \geq |\mathrm{T}|$ and so $m = |T| = \kappa(G)$. □

2.1.2 Special Graphes

In the section we name a few classes of graphs.

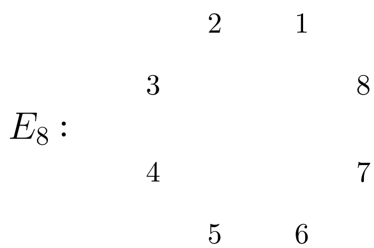
Complete Graphs

K_n denotes a complete graph of order n.



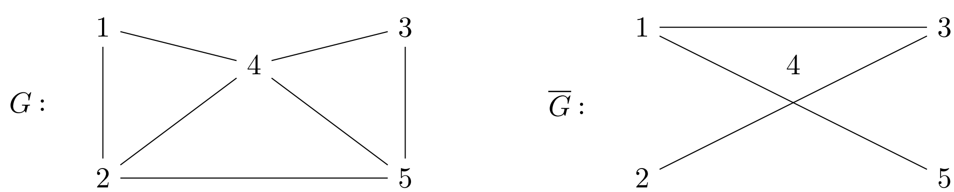
Empty Graphs

A graph $G = (V, E)$ is called *empty* if G has no edges, that is $E = \varnothing$. E_n denotes an empty graph of order n.



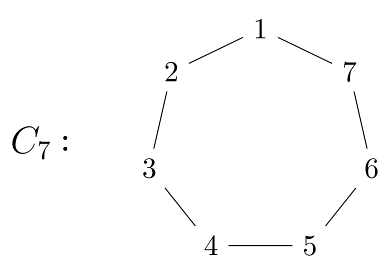
Complements

Let $G = (V, E)$ be graph. Then the *complement* $\overline{G}$ of G is the graph $(V, \mathcal{P}_2(V) \setminus E)$, so G and $\overline{G}$ have the same vertices and vw is an edge of $\overline{G}$ if and only if it is not an edge of G.



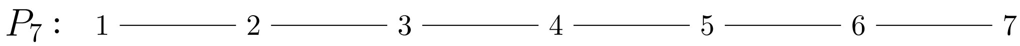
Cycles

C_n denotes graph consisting of the vertices and edges of a cycle of length n:



Paths

P_n denotes any graph consisting of the vertices and edges of a path of length $n-1$:

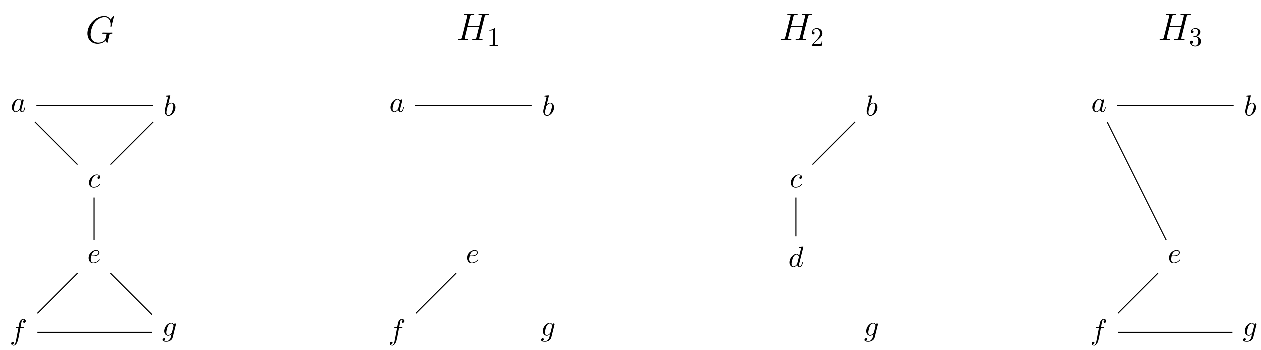


Subgraphs

Definition 2.1.26. (a) *Let $G = (V, E)$ be a graph. Then $\mathrm{V}(G) = V$ and $\mathrm{E}(G) = E$.*

(b) *Let H and G be graphs. We say that H is a* subgraph *of G if $\mathrm{V}(H) \subseteq \mathrm{V}(G)$ and $\mathrm{E}(H) \subseteq \mathrm{E}(G)$. In this case we also say that G contains H and write $H \subseteq G$.*

Is H_i a subgraph of G?

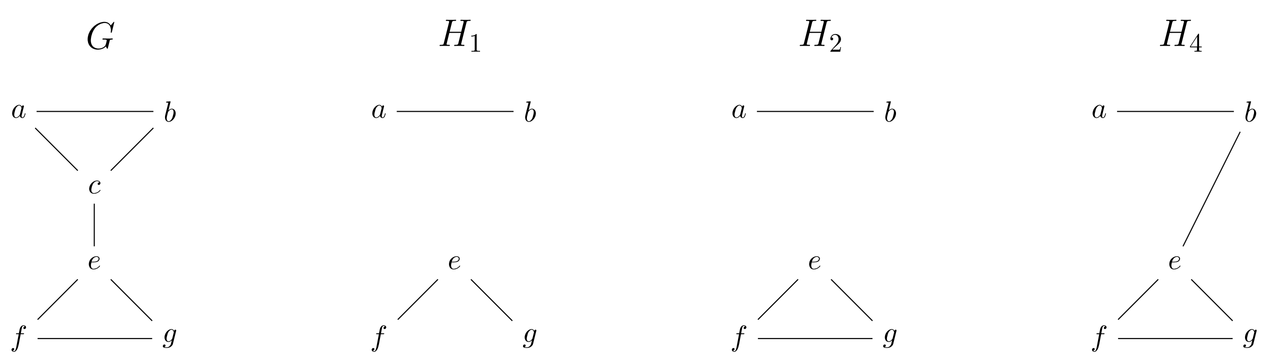


H_1 is a subgraph.
H_2 is not a subgraph, since d is a vertex of H_2 but not of G.
H_3 is not a subgraph, since ae is an edge of H_3 but not of G.

Induced subgraphs

Definition 2.1.27. *Let $G = (V, E)$ be a graph and $W \subseteq V$. The graph $(W, \mathcal{P}_2(W) \cap E)$ is called the subgraph of G induced by W and denoted by $\langle W \rangle_G$ or by $\langle W \rangle$. Note that if $v, w \in W$, then vw is an edge of $\langle W \rangle$ if and only if vw is an edge of G.*

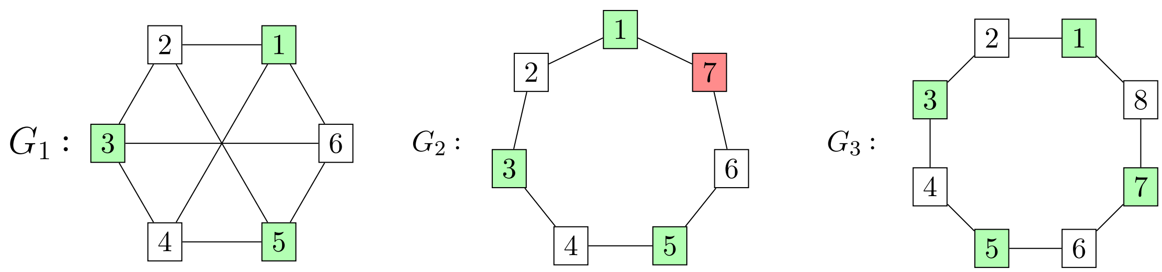
Is H_i an induced subgraph of G?



H_1 is not an induced subgraph of G, since fg is an edge in G but not in H_1.
H_2 is an induced subgraph.
H_3 is not an induced subgraph, since be is an edge of H_3 but not of G.

Bipartite Graphs

Definition 2.1.28. *A bipartition of the graph $G = (V, E)$ is a partition (V_1, V_2) of V such that each edge has one vertex in V_1 and the other in V_2. A graph is called bipartite if it has a bipartition.*



G_1 and G_3 are bipartite and G_2 is not.

Theorem 2.1.29. *A graph is bipartite if and only if all closed walks have even length.*

Proof. Let $G = (V, E)$ be a graph.

Suppose first that (V_1, V_2) is a bipartition of G. Let $W = v_1 v_2 \dots v_n$ be a walk in G say with $v_1 \in V_1$. Then $v_2 \in V_2$, $v_3 \in V_1$, and in general $v_i \in V_1$ if i is odd and $v_i \in V_2$ of i is even. If W is a walk then $v_n = v_1 \in V_1$ and so n is odd. Hence the length of W, namely $n - 1$, is even.

Suppose next that all closed walks in G have even length. Let $G_1, \dots G_m$ be the connect component of G. Assume that (U_i, W_i) is a bipartition of G_i. Put $U = \bigcup_{i=1}^m U_i$ and $W = \bigcup_{i=1}^m W_i$. Then (U, W) is a bipartition of G.

So we may assume that G is connected. If G has no edges, then $(V, \varnothing)$ is a bipartition of G. So we may assume that V has an edge xy.

Let v, w in V. We say that v is evenly connected to w in G if there exists a $v - w$ walk of even length in G. Just as in 2.1.19 we see that 'is evenly connected' is an equivalence relation. Let V_1 and V_2 be the equivalence class of x and y, respectively. We will show that (V_1, V_2) is a bipartition of G.

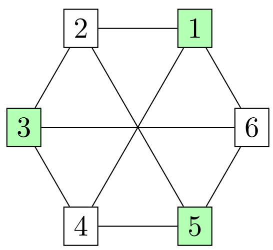
Let $v \in V$. Since G is connected there exists an $x - v$ walk W. If W has even length, then $v \in V_1$. If W has odd length, then yW is a $y - v$ walk of even length and so $v \in V_2$. Thus $V = V_1 \cup V_2$.

Let vw be any edge of G. Since $V = V_1 \cup V_2$ we can choose $i, j \in \{1, 2\}$ with $v \in V_i$ and $w \in V_j$. Let U a $v - w$-walk. Then Uv is a $v - v$ walk and so closed. Thus Uv has even length. It follows that U has odd length for all such U and so w is not contained in the equivalence class of v. Thus $W \notin V_j$. Hence $V_i \neq V_j$ and $i \neq j$. It follows that exactly one of v and w is in V_1 and the other in V_2. This applied to the edge xy shows that $V_1 \neq V_2$ and so $V_1 \cap V_2 = \varnothing$. Thus (V_1, V_2) is a partition of V and a bipartition of G. So G is indeed bipartite. □

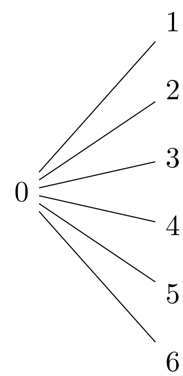
Definition 2.1.30. *A bipartition* (V_1, V_2) *of a graph is called complete if* $v_1 v_2$ *is an edge of* G *for all* $v_1 \in V_1$ *and* v_2 *in* V_2. *A graph is called complete bipartite if it has a complete bipartition* (V_1, V_2). *Such a graph is denoted by* $K_{|V_1|,|V_2|}$.

Example 2.1.31.

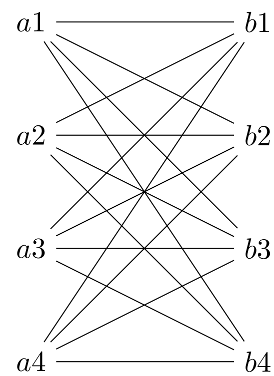
$K_{3,3}$:



$K_{1,6}$:



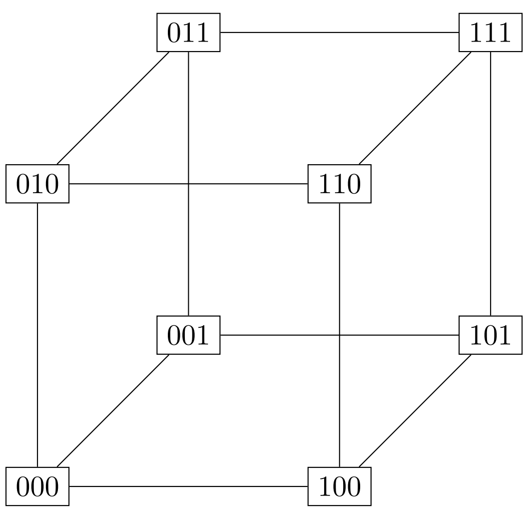
$K_{4,4}$:

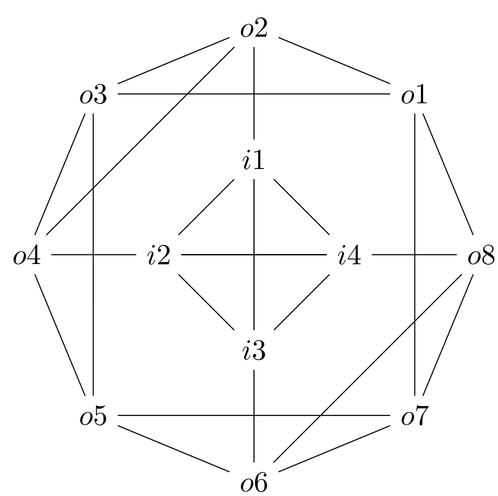


Regular Graphs

Let G be graph and $r \in \mathbb{N}$. G is called regular of degree r (or r-regular) if $\deg(v) = r$ for all vertices v of G.

G is 0-regular graph if and only if its empty If G has order n, then G is $n-1$-regular if and only if G is complete. Below are examples of a 3-regular and a 4-regular graph:





Embeddings and Isomorphisms

Definition 2.1.32. *Let* H *and* G *be graphs.*

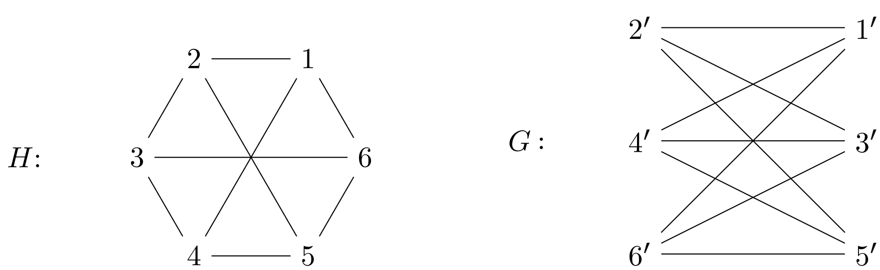
(a) *Let I and J be sets, K a subset of I and $f : I \to J$ a function. Then $f(K) := \{f(k) \mid k \in K\}$ Note that $f(K) \subseteq J$.*

(b) *A homomorphism from H to G is a function $f : \mathrm{V}(H) \to \mathrm{V}(G)$ such that $f(e) \in \mathrm{E}(G)$ for $e \in \mathrm{E}(H)$. (So f sends the edges of H to edges of G.)*

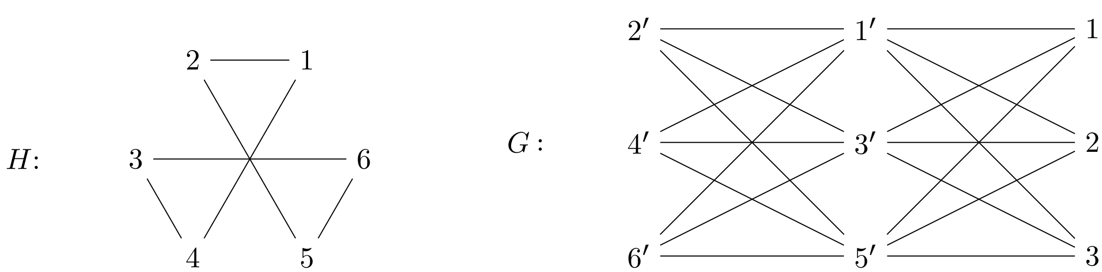
(c) *An embedding of H into G is a homomorphism f from G to H such that $f : \mathrm{V}(H) \to \mathrm{V}(G)$ is injective. We say that H is embedded in G, if there exists an embedding of H into G.*

(d) *An isomorphism from H to G is a homomorphism f from G to H such that the function $f : \mathrm{V}(H) \to \mathrm{V}(G), v \to f(v)$ and the function $\mathrm{E}(H) \to \mathrm{E}(G)$, $e \mapsto f(e)$ both are bijections. We say that H is isomorphic to G if there exists an isomorphism from H to G.*

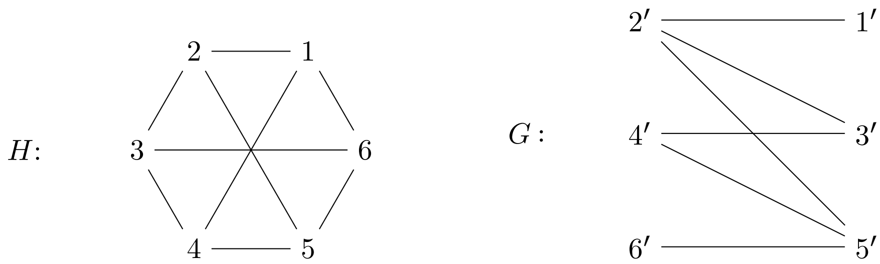
Example 2.1.33. Let H and G be the graphs and $f : \mathrm{V}(H) \to \mathrm{V}(G)$ a function. If $v \in \mathrm{V}(H)$ we write v' for $f(v)$. In the examples below, decide whether f is a homomorphism, an embedding, an isomorphism or neither.



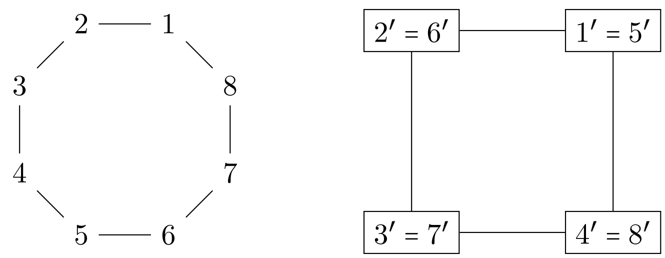
f is an isomorphism (both graphs are $K_{3,3}$).



f is an embedding, but not an isomorphism.

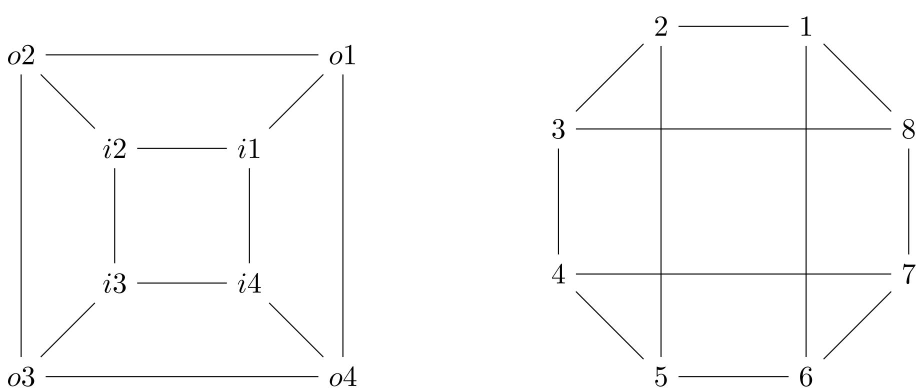


f is not a homomorphism.

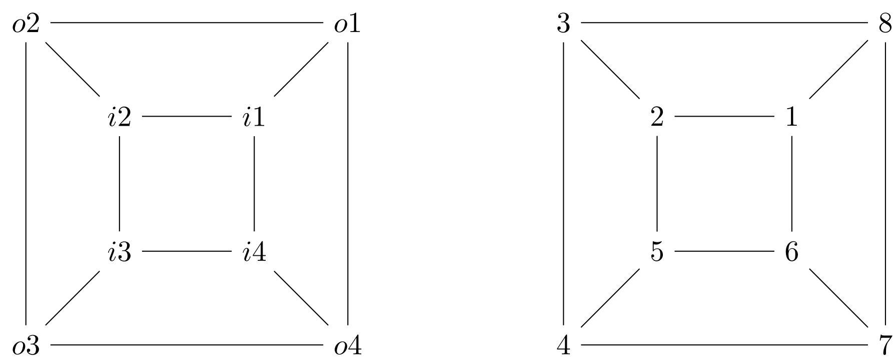


f is a homomorphism, but not an embedding.

Example 2.1.34. Are the following two graphs isomorphic:



Yes, redraw the second graph:

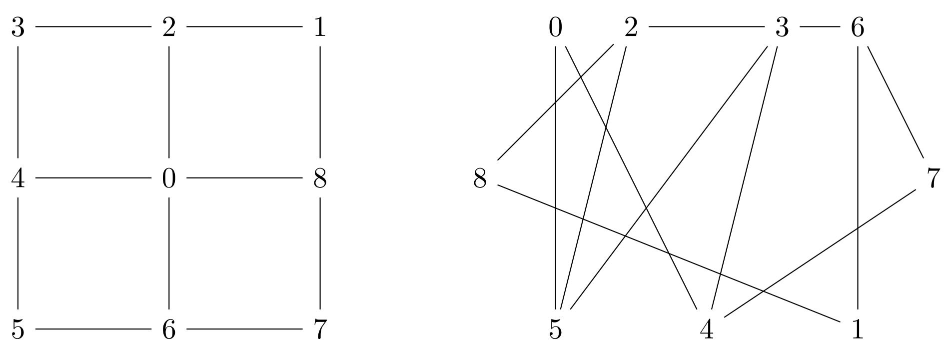


And so the function

$$f: \quad o1 \mapsto 8, \quad o2 \mapsto 3, \quad o3 \mapsto 4, \quad o4 \mapsto 7, \quad i1 \mapsto 1, \quad i2 \mapsto 2, \quad i3 \mapsto 5, \quad i4 \mapsto 6$$

is an isomorphism.

Example 2.1.35. Are the following two graphs isomorphic?



One can often distinguish graphs by their degree sequence:

The degrees of the first graphs are: 423232323. So the degree sequence is 433332222.

The degrees of the second graphs are: 223433322. So the degree sequence is 433332222.

But since the degrees sequences are the same, we cannot conclude that the graphs are not isomorphic. But we can also not conclude that they are isomorphic:

The first graph has no cycles of length 3 (indeed $\{\{0,1,3,5,7\},\{0,2,4,6\}\}$ is a bipartition), but the second has the cycle 2352 of length 3. Thus the graphs are not isomorphic.

Theorem 2.1.36. *Let H and G be graphs and $f : \mathrm{V}(H) \to \mathrm{V}(G)$ a function. Put*

$$H' := \Big(f\big(\mathrm{V}(H)\big), f\big(\mathrm{E}(H)\big) \Big)$$

and define

$$f' : \quad \mathrm{V}(H) \to f\big(\mathrm{V}(H)\big), \quad v \mapsto f(v).$$

(a) *f is homomorphism if and only if H' is a subgraph of G.*

(b) *Suppose f is a homomorphism. Then also f' is a homomorphism of graphs. Moreover, f is an embedding if and only if f' is an isomorphism.*

(c) *H is embedded in G if and only if H is isomorphic to a subgraph of G.*

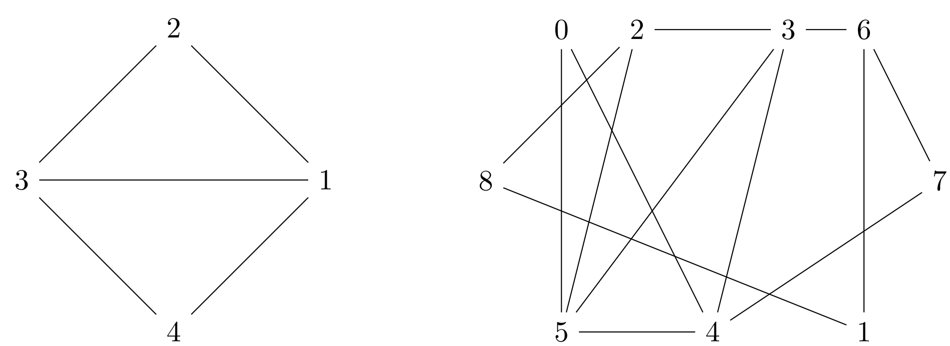
Proof. (a) Both statement just say that $f(e)$ is an edge of G for all $e \in \mathrm{E}(H)$.

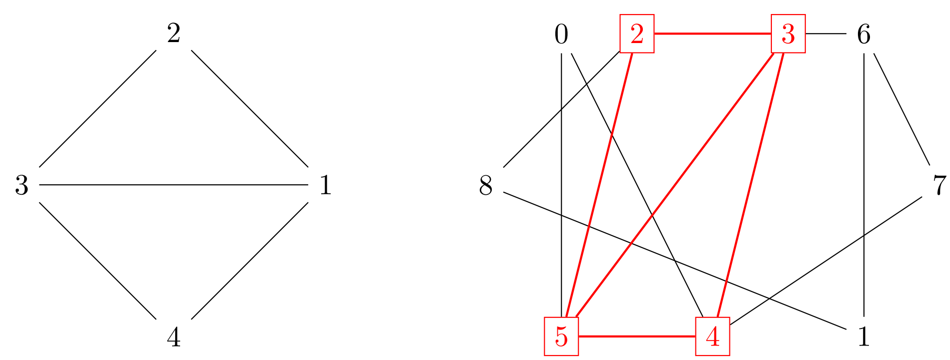
(b) By definition of H', each $f(e)$, $e \in \mathrm{E}(H)$ is an an edge of H'. So f' is a homomorphism. Again by definition of H', f' is surjective. As $f(v) = f'(v)$ for all $v \in \mathrm{V}(H)$, f is injective if and only of f' is injective. Thus f is an embedding if an only of f' is an isomorphism.

(c) $\Longrightarrow$: If f is an embedding, then (a) and (b) show that H' is a subgraph of G and $f' : H \to H'$ is an isomorphism. Thus H' is a subgraph of G isomorphic to H.

$\Longleftarrow$: Suppose H^* is a subgraph of G isomorphic to H. Let $g^* : H \to H^*$ be an isomorphism and define $g : \mathrm{V}(H) \to \mathrm{V}(G), v \to g^*(v)$. Then g is an embedding of H in G. □

Example 2.1.37. Is the first graph embedded in the second?





So

$$f: \quad 1 \mapsto 3, \quad 2 \mapsto 2 \quad 3 \mapsto 5, \quad 4 \mapsto 4$$

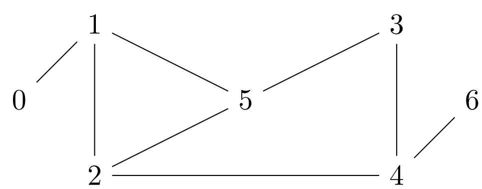
is an embedding of the first graph in the second.

2.2 Distances in Graphs

2.2.1 Basic Properties of Distance

Definition 2.2.1. *Let G be connected graph and x, y vertices of G. The distance from x to y, denoted by $\mathrm{d}_G(x,y)$, is the length of a shortest $x - y$ walk in G. We will usually just write $\mathrm{d}(x,y)$ for the more precise $\mathrm{d}_G(x,y)$.*

Example 2.2.2. What is $\mathrm{d}(0,6)$ in the following graph:



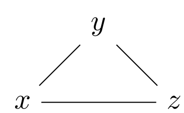
01246 is shortest walk from 0 to 6. So $\mathrm{d}(0,6) = 4$.

Definition 2.2.3. *A metric on the set S is a function $M : S \times S \to \mathbb{R}$ such that for all $x, y, z \in S$.*

(i) *$M(x,y) \geq 0$ with equality if and only of $x = y$.*

(ii) *$M(x,y) = M(y,x)$.*

(iii) *$M(x,z) \leq \mathrm{M}(x,y) + M(y,z)$ [Triangular Inequality].*



Theorem 2.2.4. *Let $G = (V, E)$ be connected graph.*

(a) *$\mathrm{d}(\cdot,\cdot)$ is a metric on V.*

(b) *Let $x, y \in V$. Then $\mathrm{d}(x,y)$ is the length of a shortest path from x to y.*

Proof. (a) Let $x, y, z \in V$. Let $x = v_0, v_1 \dots, v_n = y$ be a shortest $x - y$-walk and let $y = w_0, w_1, \dots, w_m = z$ be a shortest $y - z$-walk. So $n = \mathrm{d}(x,y)$ and $m = \mathrm{d}(y,z)$. We will now verify the three conditions on a metric:

(i) The length of any path is in $\mathbb{N}$. So $\mathrm{d}(x,y) \geq 0$. If $\mathrm{d}(x,y) = 0$, then $n = 0$ and so $x = v_0 = v_n = y$.

(ii) $y = v_n, \dots, v_0 = x$ is a $y - x$ walk of length n, so $\mathrm{d}(y,x) \leq n = \mathrm{d}(x,y)$. For the same reason $\mathrm{d}(x,y) \leq \mathrm{d}(y,x)$, so $\mathrm{d}(x,y) = \mathrm{d}(y,x)$.

(iii) $x = v_0, \dots, v_n = y = w_0, w_1, \dots w_m = z$ is a $x - z$ walk of length $n + m$. So $\mathrm{d}(x,z) \leq n + m = \mathrm{d}(x,y) + \mathrm{d}(y,z)$.

(b) By 2.1.14 any shortest $x - y$-walk is a path. □

Definition 2.2.5. *Let $G = (V, E)$ be a connected graph and $v \in V$.*

(a) *The eccentricity of v in G is defined as*

$$\operatorname{ecc}_G(v) = \operatorname{ecc}(v) = \max_{w \in V} \operatorname{d}(v, w)$$

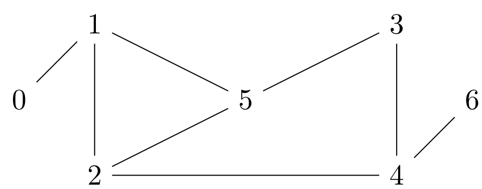
(b) *The radius of G is defined as*

$$\operatorname{rad}(G) = \min_{v \in V} \operatorname{ecc}_G(v).$$

(c) *The diameter of G is defined as*

$$\operatorname{diam}(G) = \max_{v \in V} \operatorname{ecc}_G(v) = \max_{v,w \in V} \operatorname{d}(v, w)$$

Example 2.2.6. Let G be the graph



ecc(0) =?

4

ecc(2) =?

2

rad(G) =?.

2

diam(G) =?.

4

Example 2.2.7. What is the radius and diameter of a complete graph of order n?

$$\operatorname{rad}(K_n) = \operatorname{diam}(K_n) = \begin{cases} 1 & \text{if } n \geq 2 \\ 0 & \text{if } n = 1 \end{cases}$$

Theorem 2.2.8. *Let G be a connected graph. Then*

$$\mathrm{rad}(G) \leq \mathrm{diam}(G) \leq 2\,\mathrm{rad}(G)$$

Proof. Recall that $\mathrm{rad}(G)$ and $\mathrm{diam}(G)$ are the minimum and maximum of the eccentricities, respectively. Thus $\mathrm{rad}(G) \leq \mathrm{diam}(G)$.

Let $x, z \in V$ with $\mathrm{diam}(G) = \mathrm{d}(x,z)$ and $y \in V$ with $\mathrm{ecc}(y) = \mathrm{rad}(G)$. Then

$$\mathrm{diam}(G) = \mathrm{d}(x,z) \leq \mathrm{d}(x,y) + \mathrm{d}(y,z) = \mathrm{d}(y,x) + \mathrm{d}(y,z) \leq \mathrm{ecc}(y) + \mathrm{ecc}(y) = 2\mathrm{ecc}(y) = 2\,\mathrm{rad}(G).$$

□

2.2.2 Graphs and Matrices

Definition 2.2.9. *Let I, J and R be sets.*

(a) $I \times J := \{(i,j) \mid i \in I, j \in J\}$.

(b) *An $I \times J$-matrix with coefficients in R is a function $M : I \times J \to R$. We will write m_{ij} for the image of (i,j) under M and denote M by $[m_{ij}]_{\substack{i\in I\\ j\in J}}$. m_{ij} is called the ij-coefficients of M. We will also write $[M]_{ij}$ for m_{ij}.*

(c) *Let $n, m \in \mathbb{N}$. An $n \times m$-matrix is an $\{1,2,\ldots,n\} \times \{1,2,\ldots,m\}$-matrix.*

Notation 2.2.10. *Notations for matrices*

(1) *We will often write an $I \times J$-matrix as an array. For example*

M	x	y	z
a	0	1	2
b	1	2	3
c	2	3	4
d	3	4	5

stands for the $\{a,b,c,d\} \times \{x,y,z\}$ matrix M with coefficients in $\mathbb{Z}$ such that $m_{ax} = 0$, $m_{ay} = 1$, $m_{bx} = 1$, $m_{cz} = 4$, ...

(2) *$n \times m$-matrices are denoted by an $n \times m$-array in square brackets. For example*

$$M = \begin{bmatrix} 0 & 1 & 2 \\ 4 & 5 & 6 \end{bmatrix}$$

denotes the 2×3 matrix M with $m_{11} = 0, m_{12} = 1$, $m_{21} = 4$, $m_{23} = 6, \ldots$.

(3) *Suppose* $I = \{a_1, \dots, a_n\}$ *and* $J = \{b_1, \dots b_m\}$. *Then we can view an* $I \times J$ *matrix* M *as the* $n \times m$ *matrix* $[M_{a_i b_j}]_{\substack{1 \le i \le n \\ 1 \le j \le m}}$. *For example we can view the matrix in (1) as the* 4×3 *matrix*

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}$$

Note that the $n \times m$ *matrix depends an the order of the elements* $a_1, \dots a_n$ *and of the elements* $b_1, \dots b_m$.

Definition 2.2.11. *Let* $G = (V, E)$ *be a graph. The adjacency matrix* $A = [a_{vw}]_{\substack{v \in V \\ w \in V}}$ *of* G *is the* $V \times V$ *matrix define by*

$$a_{vw} = \begin{cases} 1 & \text{if } vw \in E \\ 0 & \text{if } vw \notin E \end{cases}$$

Example 2.2.12. Compute the adjacency matrix of the cycle $\begin{array}{ccc} a & — & b \\ | & & | \\ d & — & c \end{array}$

A	a	b	c	d
a	0	1	0	1
b	1	0	1	0
c	0	1	0	1
d	1	0	1	0

Definition 2.2.13. *Let* I, J, K *be sets with* J *finite.*

(a) A *be an* $I \times J$ *-matrix and* B *a* $J \times K$ *matrix with coefficients in* $\mathbb{R}$. *Then* AB *is the* $I \times K$ *matrix defined by*

$$[AB]_{ik} = \sum_{j \in J} [A]_{ij} [B]_{jk}$$

for all $i \in I, k \in K$.

(b) *The identity matrix on* I, *denoted by* Id_I, *is the* $I \times I$*-matrix defined by*

$$[\mathrm{Id}_I]_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

(c) *Let $n \in \mathbb{N}$ and A a $J \times J$-matrix with coefficients in $\mathbb{R}$. Then A^n is recursively defined by*

$$A^0 = \mathrm{Id}_J \qquad \text{and} \qquad A^{n+1} = AA^n$$

Example 2.2.14. Compute A^2 and A^3 for the adjacency matrix A of the cycle $\begin{array}{ccc} a & \text{---} & b \\ | & & | \\ d & \text{---} & c \end{array}$

$$A^2 = \begin{array}{c|cccc} A & a & b & c & d \\ \hline a & 0 & 1 & 0 & 1 \\ b & 1 & 0 & 1 & 0 \\ c & 0 & 1 & 0 & 1 \\ d & 1 & 0 & 1 & 0 \end{array} \cdot \begin{array}{c|cccc} A & a & b & c & d \\ \hline a & 0 & 1 & 0 & 1 \\ b & 1 & 0 & 1 & 0 \\ c & 0 & 1 & 0 & 1 \\ d & 1 & 0 & 1 & 0 \end{array} = \begin{array}{c|cccc} A^2 & a & b & c & d \\ \hline a & 2 & 0 & 2 & 0 \\ b & 0 & 2 & 0 & 2 \\ c & 2 & 0 & 2 & 0 \\ d & 0 & 2 & 0 & 2 \end{array}$$

and

$$A^3 = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 \\ 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 4 & 0 & 4 \\ 4 & 0 & 4 & 0 \\ 0 & 4 & 0 & 4 \\ 4 & 0 & 4 & 0 \end{bmatrix}$$

For each vertex x count the number of bx of walks of length 2 and of length

Length 2: $b-a$: 0 $b-b$: 2 $b-c$: 0 $b-d$: 2

Length 3: $b-a$: 4 $b-b$: 0 $b-c$: 4 $b-d$: 0

Theorem 2.2.15. *Let A be the adjacency matrix of the graph G. Then for all $n \in \mathbb{N}$ and all vertices v, w, $[A^n]_{vw}$ is the number of $v-w$-walks in G.*

Proof. The proof is by induction on n. Suppose first that $n = 0$. If $v = w$, then v is the unique $v-w$ walk of length 0 and if $v \neq w$, there is no $v-w$ walk of length 0. Thus

$$[A^0]_{vw} = [\mathrm{Id}_V]_{vw} = \begin{cases} 1 & \text{if } v = w \\ 0 & \text{if } v \neq w \end{cases} = \text{number of } v-w \text{ walks of length } 0$$

Assume next that $n \geq 1$ and that the theorem holds for $n-1$. For v, w in V let $k_{vw} = [A^{n-1}]_{vw}$ and $a_{vw} = [A]_{vw}$. By the inductions hypothesis, k_{vw} is the number of $v-w$ walks of length $n-1$. Also $a_{vw} = 1$ if vw is an edge and $a_{vw} = 0$ if vw is not an edge. Consider a $v-w$ walk of length n:

$$v = v_0, u = v_1, v_2, \ldots, v_n = w.$$

Then u is any vertex such that vu is an edge (that is $u \in \mathrm{N}(v)$), and $v_1, \dots, v_n$ is any $u - w$ walk of length $n - 1$. Thus the number of $v - w$ walks of length n is

$$\begin{aligned}
\sum_{u \in \mathrm{N}(v)} k_{uw} &= \sum_{u \in \mathrm{N}(v)} a_{vu} k_{uw} && - \ a_{vu} = 1 \text{ for } u \in \mathrm{N}(v) \\
&= \sum_{u \in V} a_{vu} k_{uw} && - \ a_{vu} = 0 \text{ for } u \notin \mathrm{N}(v) \\
&= \sum_{u \in V} [A]_{vu} [A^{n-1}]_{uw} && - \ \text{definition of } a_{vu} \text{ and } k_{uw} \\
&= [A^n]_{vw} && - \ \text{definition of } A^n = AA^{n-1}
\end{aligned}$$

□

2.3 Forests, Trees and Leaves

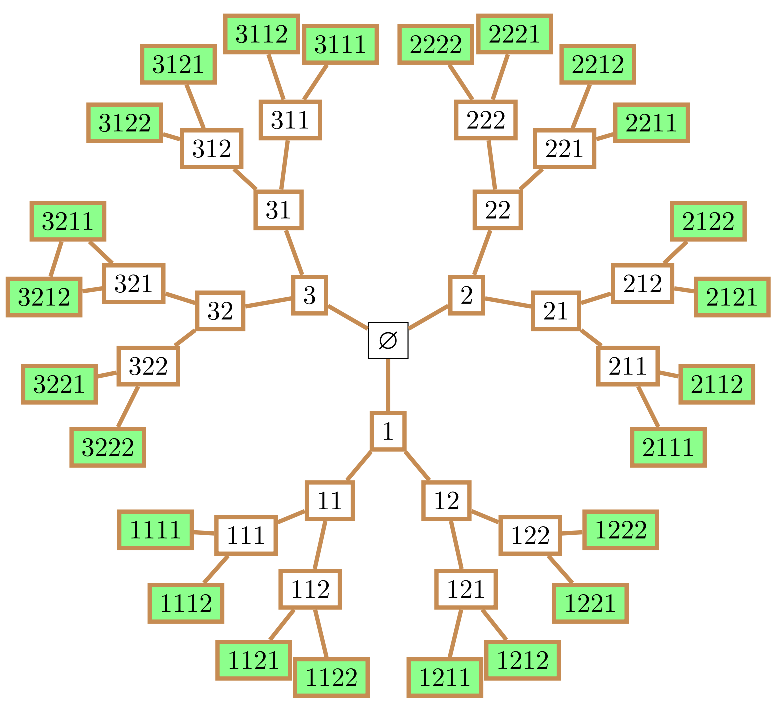
2.3.1 Definition of Forests, Trees and Leaves

Definition 2.3.1. (a) *A graph is called* acyclic *if its has has no cycles. An acyclic graph is called a* forest*.*

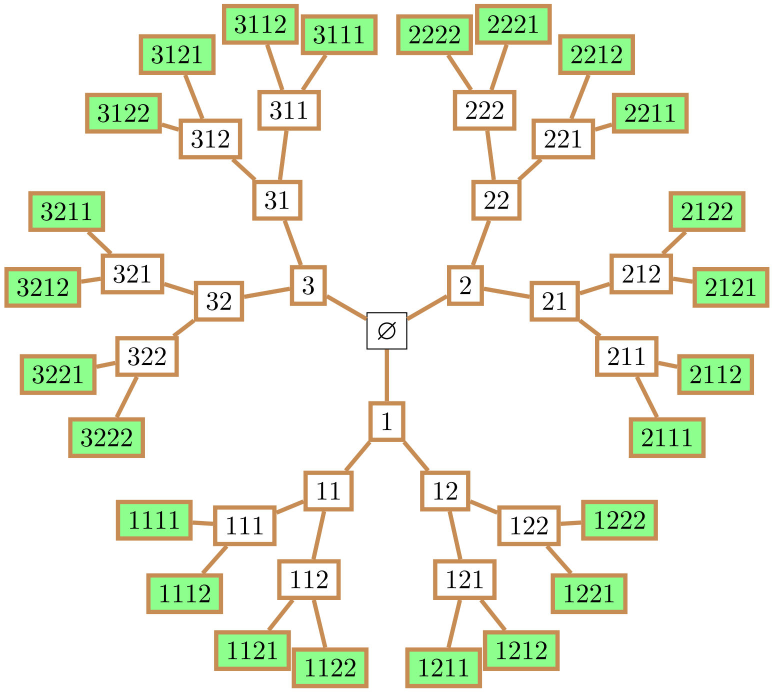
(b) *A* tree *is a connected forest.*

(c) *A leaf of a graph is vertex of degree* 1*.*

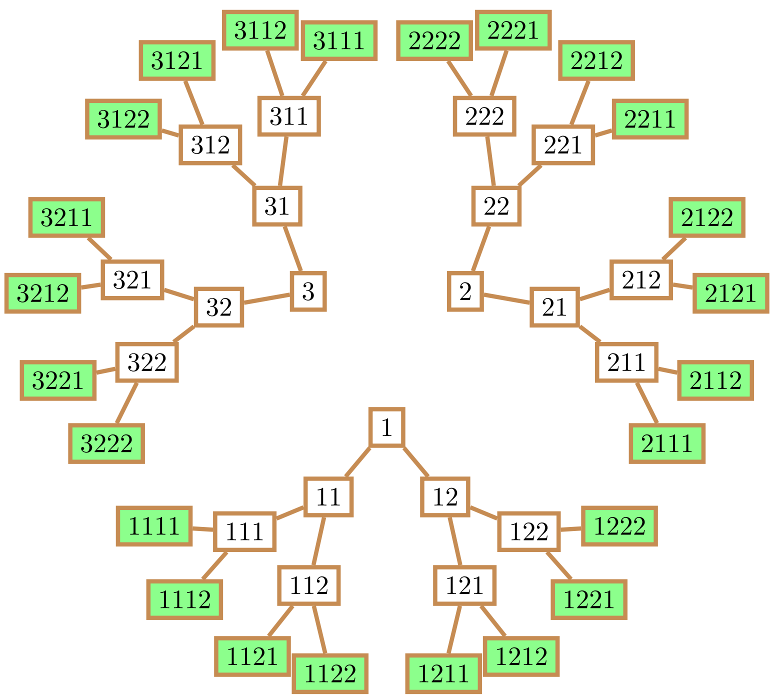
Example 2.3.2. Which of the following graphs are trees? Forest? Find all the leaves.



Not a forest, since $321, 3211, 3212, 321$ is a cycle. The leaves are the green vertices except for 3211 and 3212.



Is a tree. Leaves are the green vertices.



Is a forest, but not a tree since not connected. Leaves are the green vertices.

2.3.2 Basic Properties of Forest, Trees and Leaves

Theorem 2.3.3. *Let G be graph and $v, w \in \mathrm{V}(G)$.*

(a) *Suppose G is forest. Then there exists at most one $v - w$ path in G.*

(b) *Suppose G is a tree. Then there exists a unique $v - w$ path in G.*

Proof. (a) Suppose that G is a forest and let $U = u_0 u_1 \dots u_n$ and $W = w_0 \dots w_m$ be $v - w$ paths in G. We will show by complete induction on n that $U = W$. Suppose that $v = w$. Since U and W are paths this implies $n = m = 0$ and so $U = v = W$. So we may assume the $v \neq w$. In particular, $n \geq 1$ and $m \geq 1$. If $n = m = 1$, then $U = vw = W$. So we may assume that $n + m \geq 3$.

Consider the closed walk

$$v = u_0, u_1, \dots, u_{n-1}, u_n = w = w_m, w_{m-1}, \dots, w_1, w_0 = v$$

of length $n + m$. Since G is acyclic, this is not a cycle. As $n + m \geq 3$, there must be a repeat vertex (not counting the last vertex). As U and W are paths and so do not have repeat vertices this shows that $u_i = w_j$ for some $1 \leq i \leq n - 1$ and $1 \leq j \leq m - 1$. Then

$$v = u_0, u_1, \dots, u_i \quad \text{and} \quad v = w_0, w_1, \dots, w_j = u_i$$

are $v - u_i$ paths. Since $i < n$ the induction hypothesis shows that the two paths are equal.

Similarly,

$$u_i, u_{i+1}, \dots, u_n = w \quad \text{and} \quad u_i = w_j, w_{j+1}, \dots, w_m = w$$

are $u_i - w$ path and since $n - i < n$, they are equal. Hence also $U = W$.

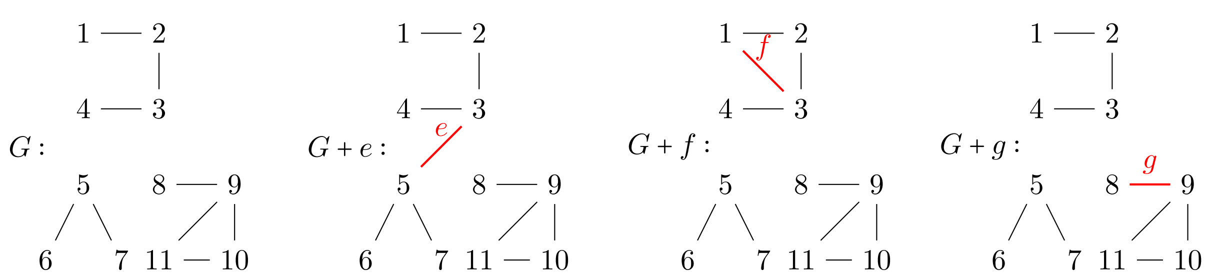
(b) Since trees are connected there exists at least one $v - w$ path and since trees are forests (a) shows there exists at most one $v - w$ path. □

Definition 2.3.4. *Let $G = (V, E)$ be graph and $e = \{v, w\}$ a subset of size 2 of V.*

(a) $G + e = \big(V, E \cup \{e\}\big)$.

(b) *We say that e is internal to G if v is connect to w in G. Otherwise e is called external to G.*

Example 2.3.5. For $x = e$, f and g: Is x internal to G, does there exist a cycle in $G + x$ containing x? Compare the connected components of G and $G + x$.



e is external to G and not contained in cycles. $G + e$ has one connected component less than G.

f is internal to G and contained in the cycle 3123. $G + f$ has the same connected components as G.

g is an edge of G and internal.

Remark 2.3.6. *Any edge of a graph G is internal. The converse is not true unless all connected components of G are complete.*

Theorem 2.3.7. *Let $G = (E, V)$ be a graph and e a subset of size 2 of V with $e \notin E$. Let k be the number of connected components of G.*

(a) *If e is internal to G, then $G + e$ has k-connected components.*

(b) *If e is external to G, then $G + e$ has $k - 1$-connected components*

(c) *e is internal to G if and only if there exists a cycle in $G + e$ containing e.*

(d) *$G + e$ is forest if and only if G is a forest and e is external to G.*

Proof. (a) and (b): Note that any vertices which are connected in G are also connected in $G+e$. Let $e = \{a, b\}$ and let A and B be the connected components of G containing a and b respectively. Note that $A = B$ if and only if a is connected to b in G and so if and only if e is internal to G.

Suppose v and w are vertices of G which are connected in $G + e$, but not in G. We will show that that one of v and w is in A and the other in B. Let $U = u_0 u_1 \dots u_n$ be $v - w$ path in $G + e$ Then U is not a path in G and so e is an edge of U. Since U is a path, U is also trail and so there exists a unique $i \in \mathbb{N}$ with $i < n$ and $e = u_i u_{i+1}$. Without loss $u_i = a$ and $u_{i+1} = b$. Then $u_0 \dots u_i$ is an $v - a$ walk in G and $u_{i+1}, \dots u_n$ is a $b - w$ walk in G. Thus $v \in A$ and $w \in B$. Since v and w are not connected in G we conclude that $A \neq B$, so e is external.

In particular, if e is internal to G then v and w are connected in G if and only if they are connected in $G + e$. So G and $G + e$ have the same connected components and (a) holds.

Suppose that e is external to G. Then $A \neq B$. Since $e = ab$ is an edge of $G + e$, a and b are connected to $G + e$. Let $A = D_1, B = D_2, D_3, \dots D_k$ be the connected components of G. It follows that $A \cup B, D_2, \dots D_k$ are the connected components of G. Thus (b) holds.

(c) Suppose e is external to G. Then a and b are connected in G and so there exist an $a - b$ path $a = v_0, \dots, v_n = b$ in G. Since $a \neq b$, $n \neq 0$ and since $e = ab \notin E$, $n \neq 1$. So $n + 1 \geq 3$ and thus

$$a = v_0, v_1, \dots v_n = b, a$$

is cycle in $G + e$ containing the edge $e = ba$.

Suppose next that $U = v_0 v_1 \dots v_n$ is a cycle in $G + e$ containing e. Without loss $v_i = b$ and $v_{i+1} = a$ for some $0 \leq i < n$. Then

$$a = v_{i+1}, \dots, v_n = v_0, v_1, \dots v_i = b$$

is an $a - b$ path in G. Thus e is external to G.

(d) Observe that each cycle of $G+e$ either is contained in G or contains e. Thus $G+e$ is a forest if and only if G is a forest and there does not exist a cycle of $G+e$ containing e. By (c) this holds if and only if G is a forest and e is external to G. □

Corollary 2.3.8. *Let G be a graph and e an edge of G. Then e is bridge of G if and only if e is external to $G-e$ and if and only if e is not contained in a cycle of G.*

Proof. Put $G' = G - e$. Then e is not an edge of G and $G = G' + e$. By definition e is a bridge of G if and only if G' has more connected components than G. By 2.3.7 this holds if and only if e is external to G' and if and only if e is not contained in a cycle of G. □

Theorem 2.3.9. *Let G be graph of order n and size m, and let k be the number of connected components of G. Then*

$$m \geq n - k$$

with equality if and only if G is a forest.

Proof. The proof is by induction on m. If $m = 0$, then G has no edges. In particular, G is a forest and the connected components of G are the subsets of size 1 of G. Thus $k = n$ and so $m = 0 = n - k$. Thus the theorem holds in this case.

Suppose now that $m \geq 1$. Let e be an edge of G and put $G' = G - e$, so $G = G' + e$ and e is not an edge of G'. Note that G' has order n and size $m-1$. Let k' be the number of connected components of G'. Then by induction

$$m - 1 \geq n - k'$$

with equality if and only if G' is a forest.

Suppose e is an internal edge of G'. Then by 2.3.7 G (that is $G' + e$) is not a forest and $k = k'$. Thus

$$m > m - 1 \geq n - k' = n - k$$

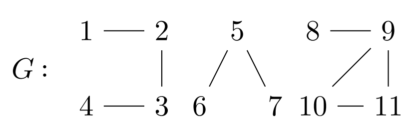
and the theorem holds.

Suppose next that G is an external edge of G'. Then by 2.3.7 $k = k' - 1$ and G (that is $G' + e$) is a forest if and only if G' is a forest. Thus

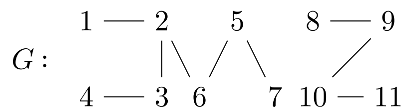
$$m = (m-1) + 1 \geq (n - k') + 1 = n - k$$

with equality if and only if G' is a forest and so if and only if G is a forest. □

Example 2.3.10. Compute the order n, size m and number of connected components. Is $m = n - k$?. Is the graph a forest?



$n = 11$, $m = 9$, $k = 3$. $m = 9 > 8 = 11 - 3 = n - k$. Not a forest: $9, 10, 11, 9$ is a cycle.



$n = 11$, $m = 9$, $k = 2$, $m = 9 = 11 - 2 = n - k$. No cycles so G is a forest.

Corollary 2.3.11. *Let G be of order n and size m. Then the following statements are equivalent.*

(a) *G is a tree.*

(b) *G is a forest and $m = n - 1$.*

(c) *G is connect and $m = n - 1$.*

Proof. Let k be the number of connect components of G. Then 2.3.9 shows that

$(*)$ *$m = n - k$ if and only if G is a forest.*

(a) $\Longrightarrow$ (b): Suppose G is a tree. Then G is a connected forest. So $k = 1$ and $(*)$ shows $m = n - k = n - 1$ and (b) holds.

(b) $\Longrightarrow$ (c): Suppose G is a forest and $m = n - 1$. By $(*)$ $m = n - k$. Hence $k = 1$ and so G is connected.

(c) $\Longrightarrow$ (a): Suppose G is connected and $m = n - 1$. Then $k = 1$ and so $m = n - 1$. Thus $(*)$ shows that G is a forest. As G is connected, this shows that G is a tree. □

Theorem 2.3.12. *Let $G = (V, E)$ be a connect graph of order n.*

(a) *Let W be a proper subset of V, that is $W \subseteq V$ and $\varnothing \neq W \neq V$. Then there exists $w \in W$ and $v \in V \smallsetminus W$ such that vw is an edge.*

(b) *Let $W \subseteq V$ with $W \neq \varnothing$ and $\mathrm{N}(w) \subseteq W$ for all $w \in W$. Then $W = V$.*

(c) *Let H a connect subgraph of order m of G. Then for each $k \in \mathbb{N}$ with $m \leq k \leq n$, there exists a connected subgraph K of order k of G containing H.*

Proof. (a) Since $\varnothing \neq W \neq V$ we can choose $w \in W$ and $v \in V \smallsetminus W$. Since G is connected there exists a $w - v$ walk $v_1 \dots v_n$ in G. Then $v_1 = w \in W$ and we can choose $i \leq n$ maximal with $v_i \in W$. As $v_n = v \notin W$ we have $i \neq n$. Thus $v_i v_{i+1}$ is an edge and by the maximal choice of i we have $v_{i+1} \notin W$.

(b) If $W \neq V$, then (a) shows that there exists $w \in W$ and $v \in V \smallsetminus W$ such that vw is an edge, a contradiction to $\mathrm{N}(w) \subseteq W$.

(c): The proof is by induction on k. If $k = m$, we can choose $K = H$. So suppose $k > m$ and by induction that there exists a connected subgraph L of order $k-1$ of G of containing H. Since $H \subseteq L$, $\mathrm{V}(L) \neq \varnothing$. As $k - 1 < k \leq n$, $V(L) \neq V$. Hence (c) shoes that there exist $w \in \mathrm{V}(L)$ and $v \in V \smallsetminus \mathrm{V}(L)$ such that vw is an edge. Put $K = (\mathrm{V}(L) \cup \{v\}, \mathrm{E}(L) \cup \{vw\})$. Then K is a connect subgraph of G of size k containing L and so also H. □

Theorem 2.3.13. *Let $G = (V, E)$ be a graph.*

(a) *Let $v \in V$ and let H be a connected component of $G - v$. If G is connected, then H contains at least one vertex adjacent to v.*

(b) *Let $S \subseteq V$, $v \in S$ and let H be a connected component of $G - S$. If G is $|S|$-connected, then H contains at least one vertex adjacent to v.*

Proof. (a) Let w a vertex of H. Since G is connected there exists a $v - w$ path $v = v_0, v_1, \dots, v_n = w$ in G. Since $v \neq w$, $n \neq 0$. Then $v_1, \dots, v_n = w$ is $v_1 - w$ path in $G - v$ and so $v_1 \in \mathrm{V}(H)$. Thus v_1 is an vertex of H adjacent to v.

(b) Put $T = S - v$. Then $|T| < |S|$ and since G is $|S|$-connected, $G - T$ is connected. Note that H is connected component of $(G - T) - v$ and so (b) follows from (a) applied to $G - T$ in place of G. □

Theorem 2.3.14. *Let G be a tree of order at least 2 and let v be a vertex of G.*

(a) *Each connect component of $G - v$ contains exactly one element of $\mathrm{N}(v)$. In particular, $G - v$ has exactly $\deg(v)$ connected components.*

(b) *v is a leaf of G if and only if $G - v$ is connected.*

(c) *G has at least two leaves.*

Proof. (a) Let H be a connected component of $G - v$. By (2.3.13)(a) H contains at least one vertex w of $\mathrm{N}(v)$. Suppose now $u \in \mathrm{N}(v)$ with $u \neq w$. Then uvw is a $u - w$ path in G. Since G is a tree 2.3.3 shows that uvw is the unique $u - w$ path in G. Hence there does not exist a $u - w$ path in $G - v$ and so $u \notin \mathrm{V}(H)$. Thus u is vertex of H in $\mathrm{N}(v)$.

(b) Follow from (a).

(c) Let n be the order of G. Let v be any vertex of V. Then by 2.3.12 there exist a connected subgraph H of size $n - 1$ of G containing v Let v' be the unique vertex of G not in H. Then $V(H) = V(G - v')$ and since H is connected also $G - v'$ is connected. Thus (b) shows that v'. is a leaf of G. Since $v \in H$ and $v' \notin H$, we know that $v \neq v'$. This applied to v' shows that v'' is a leaf and $v'' \neq v'$. □

Theorem 2.3.15. *Let T be tree of size m and G a graph of order at least 1 with $\delta(G) \geq m$. Then T is embedded in G.*

Proof. Since T is a tree, T has order $m + 1$. If $m = 0$, then T has order 1 and since G has order at least 1, T is embedded in G.

Suppose now that $m \geq 1$ and that the theorem holds for trees of size $m - 1$. Note that T has order $m + 1 \geq 2$ and so by (2.3.14)(c) T has a leaf v. Put $S = T - v$. Then by the induction assumption there exists an embedding $f : S \to G$. Since $\deg(v) = 1$, v is adjacent to a unique vertex w of T. Put $w' = f(w)$. S is a tree of order m and so

$$|f(\mathrm{V}(S) - w))| = |\mathrm{V}(S)| - 1 = m - 1 < m \leq \delta(G) \leq |\mathrm{N}(w')|$$

So there exists $v' \in \mathrm{N}(w')$ with $v' \notin f(\mathrm{V}(S) - w)$.

Define

$$f^* : \mathrm{V}(T) \to G \quad \text{by} \quad f^*(x) = \begin{cases} f(x) & \text{if } x \in \mathrm{V}(S) \\ v' & \text{if } x = v \end{cases}$$

Note that $v' \neq w'$ and $u \notin f(\mathrm{V}(S) - w)$. Thus $f^*(v) = v' \neq f(x) = f^*(x)$ for all in $\mathrm{V}(S)$. Together with the fact that f is injective this shows that f^* is injective. Let e be an edge of T. If $v \notin e$, then e is an edge of S and so $f^*(e) = f(e)$ is an edge of G. If $v \in e$, then since $\deg(v) = 1$ we have $e = vw$ and so $f^*(e) = f^*(vw) = v'w'$ and so again $f^*(e)$ is an edge of G. Thus f is an injective homomorphism and so an embedding of T in G. □

2.3.3 Spanning Tree

Definition 2.3.16. *Let G be a graph.*

(a) *A tree (forest) of G is a subgraph of G which is a tree (forest).*

(b) *A spanning forest T of G is a maximal forest of G, that is T is a forest of G and if H is forest of G containing T, then $T = H$.*

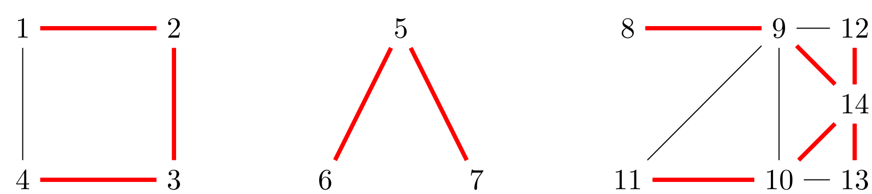
(c) *A spanning forest of G, which is a tree, is called spanning tree of G.*

Remark 2.3.17. *Let T be a spanning forest of the graph G. Then $\mathrm{V}(G) = \mathrm{V}(T)$.*

Proof. Let v be a vertex of G. Then $\big(\mathrm{V}(T) \cup \{v\}, \mathrm{E}(T)\big)$ is forest of G containing T. As T is a maximal forest of G this implies $v \in \mathrm{V}(T)$. So $\mathrm{V}(G) = \mathrm{V}(T)$. □

Example 2.3.18. The red edges are the edges of a spanning forest of the following graph:

G:



Theorem 2.3.19. *Let G be a graph order n. Let k be the number of connected components of G and let T be a forest of G with $\mathrm{V}(T) = \mathrm{V}(G)$. Then the following statements are equivalent.*

(a) *T is spanning forest.*

(b) *All edges of G are internal to T.*

(c) *T and G have the same connected components*

(d) *T has k connected components.*

(e) *T has size $n - k$.*

Proof. (a) $\Longrightarrow$ (b): Suppose T is a spanning forest and let e be an edge of G. Since T is a maximal forest of G, either $e \in \mathrm{E}(T)$ or $T + e$ is not a forest. In the latter case 2.3.7 shows that e is internal to T. The same is true if $e \in \mathrm{E}(T)$ and so all edges of G are internal to T.

(b) $\Longrightarrow$ (a): Suppose all edges of G are internal to T. Let F be a forest of G with $T \subseteq F$. Let e be an edge of F. Suppose e is is not contained in T. Since $T + e \subseteq F$, $T + e$ is a forest. So 2.3.7 shows that e is external to T, a contradiction. Thus $e \in \mathrm{E}(T)$ and so $\mathrm{E}(T) = \mathrm{E}(F)$. Hence T is maximal forest of G, that is a spanning forest.

(b) $\Longrightarrow$ (c): Suppose all edges of G are internal to T. Let v and w be vertices of G which are connected in G. Then there exists a $v - w$ walk $v_0 \dots v_n$ in G. Then each $v_i v_{i+1}$ is internal to T and so there exists a $v_i v_{i+1}$ walk in T. Hence there also exists a $v - w$ walk in T and so v and w are connected in T. Thus the connected components of G and T are the same.

(c) $\Longrightarrow$ (b): Suppose T and G have the same connected components and let $e = vw$ be an edge of G. Then v is connect to w in G and so also connected to w in T. Thus e is internal to G.

(c) $\Longleftrightarrow$ (d): Since T is a subgraph of G, any path in T is also a path in G. Thus any connected of G is a union of connected components of T. Hence G and T have the same connected components if and only if the have the same number of connected components.

(d) $\Longleftrightarrow$ (e): Let m' be the size of T and let k' be the number of connected components of G. Since $\mathrm{V}(G) = \mathrm{V}(T)$, T has order n. As T is a forest 2.3.9 shows that $m' = n - k'$. Thus $m' = n - k$ if and only if $k = k'$. So (d) and (e) are equivalent.

□

Definition 2.3.20. (a) *A* weight *function for the graph G is a function from $W : \mathrm{E}(G) \to \mathbb{R}$.*

(b) *A weighted graph is a pair (G, W), where G is a graph and W is a weight function for G.*

Definition 2.3.21. *Let (G, W) be weighted graph*

(a) *The total weight of a subgraph H of G with respect to W, denoted by $W(H)$, is defined to be*

$$W(H) := \sum_{e \in \mathrm{E}(H)} W(e)$$

(b) *A minimal weight spanning forest of G is a spanning forest of G of minimal total weight.*

Algorithm 2.3.22 (Kruskal)**.** *Let (G, W) be a weighted graph. Define subgraphs T_i of G recursively as follows:*

(0) $T_0 := (\mathrm{V}(G), \varnothing)$

Let $i \in \mathbb{N}$ and suppose recursively that T_i has been defined.

(1) *If there exist an external edge e to T_i in G choose such an edge, say e_i, with $W(e_i)$ minimal and define*

$$T_{i+1} \coloneqq T_i + e_i$$

Repeat Step (1) with $i+1$ in place of i

(2) *If all edges of G are internal to T_i, the algorithm stops. Define $m = i$ and $T = T_m$.*

Then T is a minimal weight spanning forest for G.

Proof. Note that T_0 is a forest. If T_i is a forest, then since e_i is external to T_i also $T_{i+1} = T_i + e_i$ is a forest, see 2.3.7. Thus $T = T_m$ is a forest. By construction all edges of G are internal to T. Thus 2.3.19 shows that T is a spanning forest of G.

It remains to show that T has minimal weight among all the spanning forest. For this let S be any spanning forest of W. Observe that $T_0 \subseteq S$ and so we can n in $\mathbb{N}$ with $n \leq m$ maximal subject to $T_n \subseteq S$. We will use downwards induction on n to show that $W(S) \leq W(T)$.

If $m = n$, then $T = T_m \subseteq S$ and so certainly $W(T) \leq W(S)$.

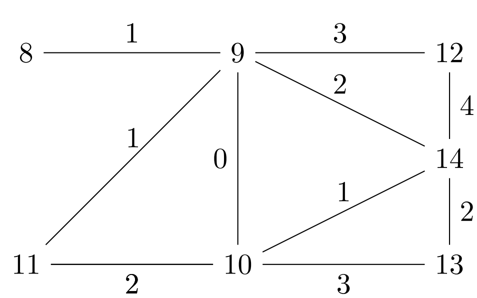
So suppose $n < m$ and that $W(T) \leq W(S')$ for any spanning tree of G with $T_{n+1} \subseteq S'$. Put $e \coloneqq e_n$ and let $v, w \in \mathrm{V}(T)$ with $e = vw$. Then $e \notin \mathrm{V}(S)$. Since S is a maximal forest of G this shows $S + e$ is a not a forest. Hence 2.3.7 shows that e is internal to S, that is there exists a $v - w$ path $P = u_0 u_1 \dots u_l$ in S. Since $e \notin S$, $P \neq vw$ and since $v - w$ is the unique $v - w$ path in T we conclude that P is not contained in T. Thus there exists an edge c of P not contained in T. Since P is the unique $v - w$ path in S and c is not in $S - c$ we conclude that there does not exists a $v - w$ path in $S - c$. Thus e is external to $S - c$. Hence 2.3.7 implies that $S - c + e$ is a forest. Observe that $S - c + e$ has the same size as S, so 2.3.19 shows that $S - c + e$ is a spanning forest of G.

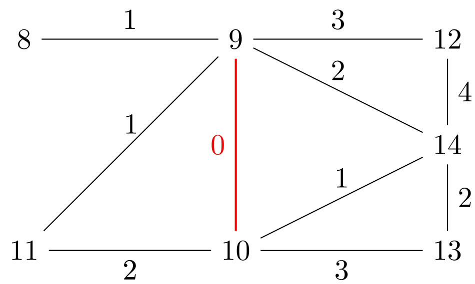
Since $T_n + c \subseteq S$, we see that $T_n + c$ is a forest Also $c \notin \mathrm{V}(T_n)$ and so 2.3.7 shows that c is external to T_n. The minimal choice of $W(e)$ in (1) now implies that $W(e) \leq W(c)$. Thus

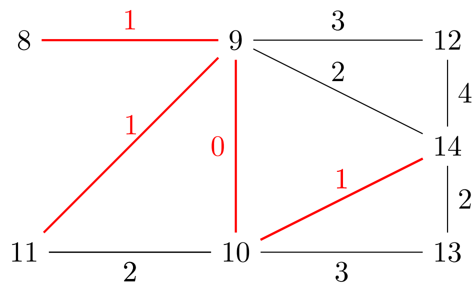
$$W(S + e - c) = W(S) + W(e) - W(c) \leq W(S).$$

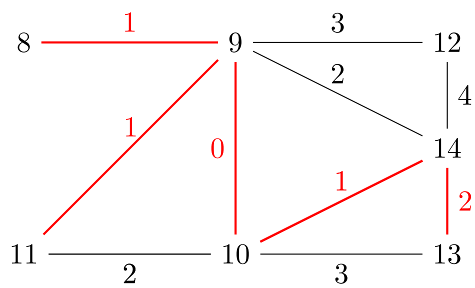
Observe that $T_{n+1} = T_n + e \subseteq S + e - c$. So the induction assumption implies that $W(T) \leq W(S + e - c)$ and so also $W(T) \leq W(S)$. □

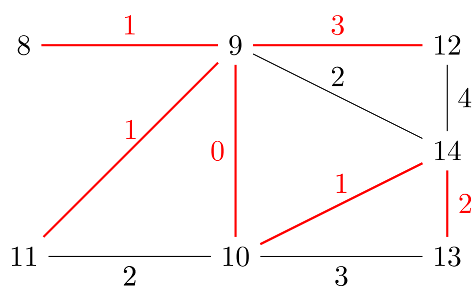
Example 2.3.23. Find a minimal weight spanning tree for the weighted graph





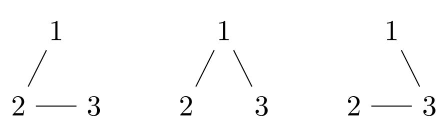






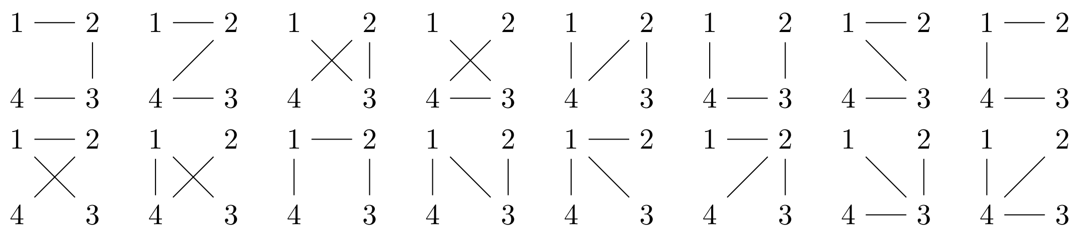
2.3.4 Counting Trees

Example 2.3.24. How many trees with vertex set $\{1, 2, 3\}$?



So there are 3 such trees.

How many trees with vertex set $\{1, 2, 3, 4\}$?



So there are 16 such trees.

Note that $3 = 3^{3-2}$ and $16 = 4^{4-2}$. Let $n \in \mathbb{Z}^+$ and let V be a set of cardinality n, In this subsection we will show that the number of trees with $\mathrm{V}(T) = V$ is n^{n-2} by establishing a bijection between such tress and the sequence of length $n-2$ from V. This bijection depends on a chosen ordering of the vertices:

Definition 2.3.25. *Let S be set. Then an ordering on S is a relation '$<$' on S such that for all $a, b, c \in S$:*

(i) *Exactly one of $a < b, a = b$ and $b < a$ holds; and*

(ii) *if $a < b$ and $b < c$, then $a < c$.*

An ordered set is a pair $(S, <)$, where S is set and $<$ is an ordering on S. An ordered graph is a triple $(V, E, <)$ such that (V, E) is a graph and $(V, <)$ is an ordered set.

Example 2.3.26. (1) How many ordering does a finite set S of cardinality n have?

$n!$:

Any ordering of S can be viewed as a non-repeating sequence $s_1 s_2 \dots s_n$ of length n from S.

Indeed given such a sequence we can define an ordering on S by $s_i < s_j$ if and only if $i < j$.

Conversely, if $<$ is an ordering of S define s_1 to be the smallest element of S and inductively for $1 \le i < n$, define s_{i+1} be the smallest element $S \setminus \{s_1, \dots, s_i\}$. Then $s_1 s_2 \dots s_n$ is a non-repeating sequence of length n from S.

Note that these two functions are inverse to each other.

(2) S be a set of real numbers, and '$<$' the usual 'less than' relation on $\mathbb{R}$, then $(S, <)$ is an ordered set. Henceforth we will view any set of real numbers as an ordered set in this way.

Definition 2.3.27. *We recursively define a function* PS *which assigns to each ordered tree T of order $n \ge 2$ a sequence* $\mathrm{PS}(T)$ *of length $n-2$ from* $\mathrm{V}(T)$*. If $n = 2$, then*

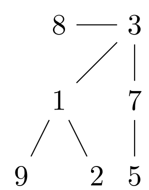
$$\mathrm{PS}(T) = ()$$

the sequence of length 0. *Suppose that* $n \geq 3$. *Let* v *be the smallest leaf of* T *and* w *the unique vertex of* T *adjacent to* v. *Then*

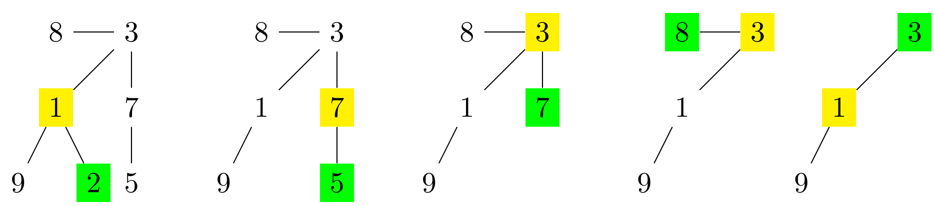
$$\mathrm{PS}(T) = w\mathrm{PS}(T - v)$$

$\mathrm{PS}(T)$ *is called the Prüfer sequence of* T.

Example 2.3.28. Compute the Prüfer sequence of the ordered tree



The leaves are $8, 9, 2$ and 5. So the smallest leaf is 2. The vertex adjacent to 2 is 1. Thus 1 is the first element of the Prüfer sequence. Remove 2 from the tree and continue. (Here the green vertex is the smallest leaf and the yellow vertex the unique neighbor of the green vertex). Stop then only two vertices will be left.



Hence the Prüfer sequence is

$$17331$$

Given the sequence 17331 of the tree T in the previous example. Can we recover T? Suppose we can determine from the sequence 17331 that 2 was smallest leaf of T. Then 7331 is the sequence for the tree $T - 2$ and 1 is the unique element of $T - 2$ adjacent to 1. Recursively we can recover $T - 2$ from 7331 and then we get T by adding the vertex 2 and the edge 12. Note that in the example 2 is the smallest element which does not appear in the sequence 17331. The next theorem shows that this is always the case.

Theorem 2.3.29. *Let* T *be an ordered tree of order* $n \geq 2$. *Let* v *be smallest leaf of* T *and let* w *the unique vertex of* T *adjacent to* v.

(a) *Suppose* $n \geq 3$. *Then* w *is the first element of* $\mathrm{PS}(T)$.

(b) *Let* u *be a vertex of* T. *Then* u *appears with the multiplicity* $\deg(v) - 1$ *in* $\mathrm{PS}(T)$.

(c) *v is the smallest vertex of T which does not appear in* $\mathrm{PS}(T)$*.*

Proof. (a) By definition $\mathrm{PS}(T) = w\mathrm{PS}(T-v)$, so (a) holds.

(b) The proof is by induction on n.

Suppose $n = 2$. Then $\mathrm{PS}(T) = ()$. Thus both vertices of T appear with multiplicity 0 in $\mathrm{PS}(T)$. Also both vertices have degree 1 and so (b) holds in this case.

Suppose next that $n \geq 3$. Then $\mathrm{PS}(T) = w\mathrm{PS}(T-v)$. By definition, $\mathrm{PS}(T-v)$ is a sequence from $T-v$. Also $v \neq w$ and so v does not appear in $\mathrm{PS}(T)$. Since v is leaf, $\deg(v) - 1 = 1 - 1 = 0$. So (b) hold for $u = v$.

$u \neq v$.

By induction u appears with multiplicity $\deg_{T-v}(u) - 1$ in $\mathrm{PS}(T-v)$. Since w is adjacent to v, $\deg_T(w) = \deg_{T-v}(w) + 1$. And since $\mathrm{PS}(T) = w\mathrm{PS}(T-v)$, w appears once more in $\mathrm{PS}(T)$ than in $\mathrm{PS}(T-v)$. So (b) also holds for $u = w$.

Suppose $u \neq v$ and $u \neq w$. Since w is the unique vertex of T adjacent to v, u is not adjacent to v. Thus $\deg_T(u) = \deg_{T-v}(u)$. As $\mathrm{PS}(T) = w\mathrm{PS}(T-v)$, u appears with the same multiplicity in $\mathrm{PS}(T)$ as in $\mathrm{PS}(T-v)$. So (b) also holds in this last case.

(c) By (b) the leaves are exactly the elements of T which do not appear in $\mathrm{PS}(T)$. Thus (c) holds. □

Definition 2.3.30. *We will recursively define a function* PT *which assigns to each pair (V, σ) such that V is a finite ordered set of size $n \geq 2$ and σ is a sequence of length $n-2$ from V, a tree* $\mathrm{PT}(V, \sigma)$ *with set of vertices V.*

Suppose $n = 2$. Then $\mathrm{PS}(V, \sigma)$ *is defined to be the complete graph on V.*

Suppose that $n \geq 3$. Since σ has length less than n there exist at least one element of V which does not appear in σ. Let v be the smallest such element and let w be the first element of σ. Let σ' be the sequence of length $n-3$ with $\sigma = w\sigma'$. Since v does not appear in σ, σ' is sequence of length $n-3$ from $V - v$. Let E' be the set of edges of $\mathrm{PT}(V - v, \sigma')$*. Define*

$$\mathrm{PT}(V, \sigma) := \big(V, E' \cup \{vw\}\big)$$

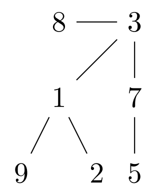
$\mathrm{PT}(V, \sigma)$ *is called the Prüfer tree of σ with ordered vertex set V.*

Example 2.3.31. Compute the Prüfer tree of 17331 with ordered vertex set $\{1, 2, 3, 5, 7, 8, 9\}$.

We compute

V	σ
1 2 35789	1 7331
13 5 789	7 331
13 7 89	3 31
13 8 9	3 1
1 3 9	1
19	

and so the tree is



Theorem 2.3.32. *Let V be a finite ordered set of size $n \geq 2$ and let σ be sequence of length $n-2$ from V. Let v be the smallest element of V which does not appear in σ.*

(a) *Suppose that $n \geq 3$ and let w be the first element of σ. Then w is the unique vertex of $\mathrm{PT}(V,\sigma)$ adjacent to v.*

(b) *Let $u \in V$. The degree of u in $\mathrm{PT}(V,\sigma)$ is $\mathrm{mult}_u(\sigma)+1$.*

(c) *v is the smallest leaf of $\mathrm{PT}(V,\sigma)$.*

Proof. Put $T := \mathrm{PT}(V,\sigma)$

(a) Let $\sigma = w\sigma'$ where σ' is a sequence of length $n-3$ from V. Put $T' := \mathrm{PT}(V-v,\sigma')$. By definition of T, $\mathrm{E}(T) = \{vw\} \cup \mathrm{E}(T')$. Since $\mathrm{V}(T') = V - v$, v not an vertex of T' and so also not contained in any edge of T'. Thus vw is the unique edge of T containing v and so w is the unique vertex of T adjacent to v.

(b) Suppose first that $n=2$. Then σ has length 0 and so both vertices of V appear with multiplicity 0 in σ. Since $n=2$, the definition of T shows that T is the complete graph on V and so both vertices have degree 1 in T. Thus (b) holds in this case.

Suppose now that $n \geq 3$. By (a) w is the unique vertex of T adjacent to v, so $\deg(v)=1$. Also v does not appear in σ. Thus (b) holds for $u=v$.

Suppose next that $u \neq v$. By induction

$$\deg_{T'}(u) = \mathrm{mult}_u(\sigma')+1.$$

Since $\sigma = w\sigma'$,

$$\mathrm{mult}_u(\sigma) = \begin{cases} \mathrm{mult}_u(\sigma')+1 & \text{if } u = w \\ \mathrm{mult}_u(\sigma') & \text{if } u \neq w \end{cases}$$

Since $\mathrm{E}(T) = \mathrm{E}(T' \cup \{vw\})$ and $u \neq v$,

$$\deg_T(u) = \begin{cases} \deg_{T'}(u)+1 & \text{if } u = w \\ \deg_{T'}(u) & \text{if } u \neq w \end{cases} = \begin{cases} (\mathrm{mult}_u(\sigma')+1)+1 & \text{if } u = w \\ \mathrm{mult}_u(\sigma')+1 & \text{if } u \neq w \end{cases} = \mathrm{mult}_u(\sigma)+1$$

(c): By (b) the leaves of T are exactly the vertices of T which do not appear in σ. So (c) holds. □

Theorem 2.3.33. *Let $n \in \mathbb{Z}^+$ and let V be a set of cardinality n. Then there exist exactly n^{n-2} trees with vertex set V.*

Proof. Suppose $n = 1$. Then there exists exactly one graph with vertex set V, namely the empty graph. Also $1^{1-2} = 1^{-1} = 1$ and so the theorem holds in this case.

So we may assume $n \geq 2$. Choose some ordering on V.

The function PS assigns to each tree with vertex set V a sequence of length $n-2$ from V. The function $\mathrm{PT}(V, \cdot)$ assigns to each sequences of length $n-2$ from V a tree with vertex set V. Note that 2.3.29 and 2.3.32 imply that these two assignments are inverse to each other. Hence the number of trees with vertex set V is equal to number of sequences of length $n-2$ from the set V. As $|V| = n$, the latter number is n^{n-2}. □

2.4 Eulerian Circuits and Hamiltonian Paths

2.4.1 Eulerian Circuits

Definition 2.4.1. *Let G be a graph and W a walk in G.*

(a) $\mathrm{V}(W)$ *is the set of vertices and* $\mathrm{E}(W)$ *is the set of edges of* W.

(b) W *is called* Eulerian *in* G *if* $\mathrm{E}(W) = \mathrm{E}(G)$.

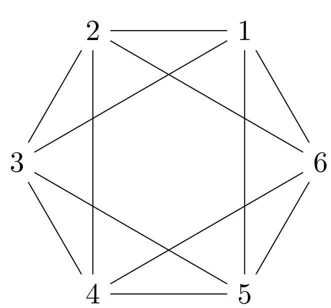
(c) G *is a* Eulerian graph *if G has a closed Eulerian trail.*

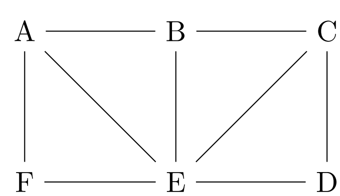
(d) *A trail T is called extendable in G if there exists a vertex w of G such that Tw is trail. Otherwise T is called non-extendable.*

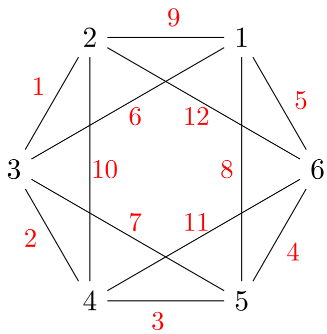
(e) *Let $W = w_0w_1\dots w_n$ and let W' be a walk in G. Then W' is called a rotation of W if either $W' = W$ or W is closed and there exists $i \in \mathbb{Z}^+$ with $i \leq n$ and*

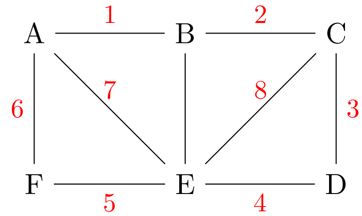
$$W' = w_iw_{i+1}\dots w_nw_1\dots w_{i-1}w_i$$

Example 2.4.2. If possible, find a Eulerian circuit.









So the first graph is Eulerian while the second does not seem to be Eulerian.

Theorem 2.4.3. *Let T be a trail in the graph G.*

(a) *Let w be the last vertex of T. Then T is closed if and only if w is incident with an even number of edges of T.*

(b) *Let v be any vertex of T. Then v is incident with an odd number of edges of T if and only if T is not closed and v is an end vertex of T.*

Proof. (a) Let $T = t_0 \dots t_n$,so $w = t_n$. If $n = 0$, then T is closed and w is not incident with 0 edges of T, so the theorem holds in this case.

So suppose $m \geq 1$. Put $T' = t_1 \dots t_n$. Let m and m' be the number of edges of T and T', respectively, incident with w. By induction m' is even if and only if T' is closed.

If w is incident with t_0t_1 (that is $w = t_0$ or $w = t_1$), then $m = m' + 1$. Otherwise $m = m'$.

Suppose first that $w = t_1$. Then $m = m' + 1$, $t_1 = t_n$, T' is closed and m' is even. So m is odd. Since t_0t_1 is an edge, $t_0 \neq t_1 = w$ and so T is not closed.

Suppose next that $w \neq t_1$. Then T' is not closed and so m' is odd. If $w = t_0$ then T is closed and $m = m' + 1$ is even. If $w \neq t_0$, then T is not closed and $m = m'$ is odd. Thus (a) is proved.

(b) If v is not a vertex of T, then v is incident with 0 edges of T and (b) holds.

If v is the last vertex of T, (b) follows from (a). If v is the first vertex of T, then (b) follows from (a) applied to the inverse path of T.

It remains to consider the case there $v \neq t_0, w \neq t_n$ and $v = t_i$ for some $1 \leq i \leq n-1$. Then v is an end vertex of the two trails $t_0t_1 \dots t_i$ and $t_it_{i+1} \dots t_n$. Neither of the two trails is closed, so v is incident with an odd number of edges of each trail, and hence with an even number of edges of T. □

Theorem 2.4.4. *Let W be a non-extendable trail in the connected graph G. Let w be last vertex of W.*

(a) *Any edge of G which is incident with w is an edge of W.*

(b) *W is closed if and only if* $\deg(w)$ *is even.*

(c) *Suppose that W is closed and no rotation of W is extendable. Then W is closed Eulerian trail in G.*

Proof. Let $W = w_0, w_1, \dots, w_n$.

(a) Suppose wv is an edge of G which is not edge of W. Then

$$w_0, w_1, \dots, w_n = w, v$$

is a trail in G, contrary to the assumption that W is not extendable.

Hence all edge incident with w are edges of W.

(b) Observe that $\deg(w)$ is the number of edges of G incident with w. By (a) this is the number of edges of W incident with w. By 2.4.3 the latter number is even if and only if W is closed.

(c) Let u be any vertex of W. Then there exists a rotation W' of W with last vertex u. By the hypothesis of (c) W' is not extendable. So we can apply (a) and (b) to W' and conclude that all edges incident with u edges of W.

If $\mathrm{V}(W) \neq \mathrm{V}(G)$, then (2.3.12)(a) shows that there exists $v \in \mathrm{V}(W)$ and $w \in \mathrm{V}(G) \setminus \mathrm{V}(W)$ such that vw is an edge. But then vw is not an edge of W, a contradiction. Thus $\mathrm{V}(W) = \mathrm{V}(G)$. Hence every edge of V is incident with a vertex of W, and so $\mathrm{E}(W) = \mathrm{E}(G)$. Thus W is a closed Eulerian trail in G. □

Algorithm 2.4.5. *Let G be graph. Define $m \in \mathbb{N}$ and trails T_i, $0 \leq i \leq m$ recursively as follows.*

(0) *Define $T_0 = ()$, the unique trail without vertices.*

(1) *Suppose recursively that $i \in \mathbb{N}$ and that T_i has been defined.*

  (i) *If some rotation of T_i is extendable, choose an extendable rotation T_i' of T_i and a vertex v_i such that $T_i'v_i$ is a trail and define $T_{i+1} = T_i'v_i$. Repeat Step 1 with $i+1$ in place of i.*

  (ii) *If no rotation of T_i is extendable, define $m = i$ and the algorithm stops.*

If G is connected and $\deg(v)$ is even for all vertices v of G, then T_m is a closed Eulerian trail in G.

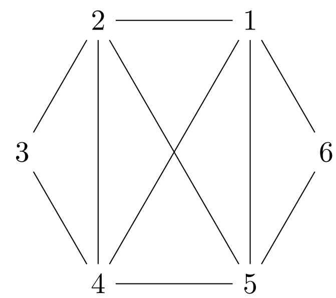
Proof. Put $T = T_m$. By construction, T is a trail of G and no rotation of T is extendable in G. Suppose now G is connected and $\deg(v)$ is even for all vertices v of G. Let w be the last vertex of T. Then $\deg(w)$ is even and (2.4.4)(b) shows that T is closed. Now (2.4.4)(c) implies that T is closed Eulerian trail in G. □

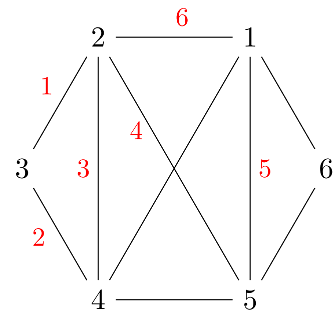
Corollary 2.4.6. *Let G be a connected graph. Then G is Eulerian if and only if all vertices of G have even degree.*

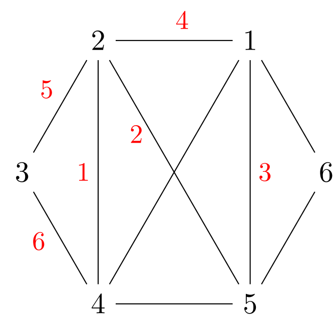
Proof. Suppose G is Eulerian. Then there exists a closed Eulerian trail W in G. Let $v \in V$. Then $\deg(v)$ is the number of edges of G (and so also of W) which are incident with v and so is even by 2.4.3.

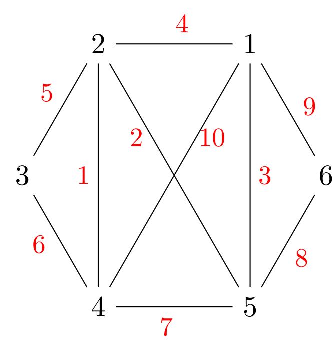
Suppose all vertex of G have even degree. Then 2.4.5 shows that G has closed Euclidean trail. □

Example 2.4.7. Use 2.4.5 to find a Eulerian circuit in the graph









Theorem 2.4.8. *Let G be a connected graph. Then G has a Eulerian trail if and only if G has at most two vertices of odd degree.*

Proof. If G has an Eulerian trail T, then 2.4.3 shows that all vertices except possible the end vertices of T have even degree.

Suppose next that at most two vertices of G have even degree.

If G has no vertices of odd degree, then 2.4.5 shows that G has closed Euclidean trail and we are done.

So suppose that G has one or two vertices of odd degree. By 2.1.9 G has an even number of vertices of odd degree. Thus G has excatly two vertices, say v and w of odd degree.

Assume that vw is an edge of G. Then $G - vw$ is a graph all of those vertices have even degree and so has closed Eulerian trail T. Replacing T be a rotation we may assume that v is the last vertex of T. Then Tw is a Eulerian trail of G.

Assume that vw is not an edge of G. Then $G + vw$ is a graph all of those vertices have even degree and so has closed Eulerian trail T. Replacing T be T be a rotation we may assume that vw is the last edge of T. Removing the last vertex of T, we obtain a Eulerian trail of G. □

2.4.2 Hamiltonian Paths

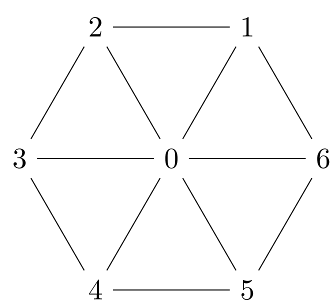
Definition 2.4.9. *Let G be a graph.*

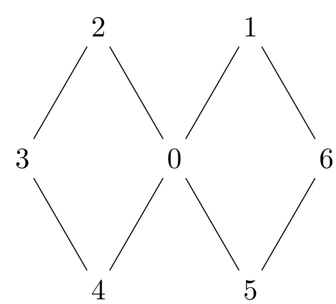
(a) *A walk in G is called Hamiltonian if* $\mathrm{V}(H) = \mathrm{V}(G)$.

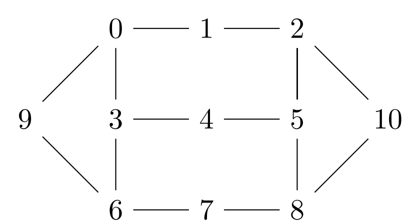
(b) *G is called traceable if G has a Hamiltonian path.*

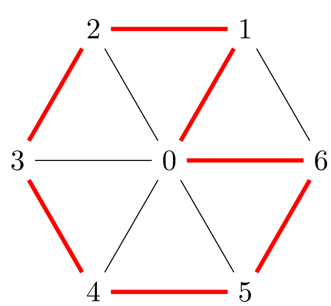
(c) *G is called Hamiltonian if G has a Hamiltonian cycle.*

Example 2.4.10. Is G traceable? Is G Hamiltonian?

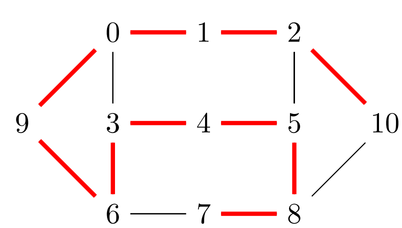








Not Traceable since 0 is a cut vertex



Theorem 2.4.11. *Let P be a longest path in graph G and v a end vertex of P. Then $\mathrm{N}(v) \subseteq \mathrm{V}(P)$.*

Proof. Replacing P be its inverse if necessary we may assume that v is the last vertex of P. Let $w \in \mathrm{N}(v)$. Then vw is an edge and so Pw is a walk. Since P is a longest walk we conclude that Pw is not a path, thus $w \in \mathrm{V}(P)$. □

Theorem 2.4.12. *Let A and B be subsets of the finite set D. If $|A| + |B| > |D|$, then $A \cap B \neq \varnothing$.*

Proof.
$$|A| + |B| - |A \cap B| = |A \cup B| \leq |D| < |A| + |B|$$
and so $|A \cap B| > 0$. □

Theorem 2.4.13. *Let $G = (V, E)$ be a graph of order $n \geq 3$. If $\delta(G) \geq \frac{n}{2}$, then G is Hamiltonian.*

Proof. Observe that

$(*)$ $\deg(v) \geq \delta(G) \geq \frac{n}{2}$ *for all $v \in V$.*

Next we show that

$(**)$ *G is connected.*

Let $v, w \in V$. Then
$$|\mathrm{N}[v]| + |\mathrm{N}(w)| \geq \left(\frac{n}{2} + 1)\right) + \frac{n}{2} > n = |V|$$
and so there exists $u \in \mathrm{N}[v] \cap \mathrm{N}(w)$. If $u = v$, then vw is a $v - w$ walk and if $u \neq v$ then vuw is a $v - w$ walk. Thus v is connected to w in G.

Let $P = v_1 \dots v_m$ be a longest path in G.

$(* * *)$ *There exists a cycle C of length m in G.*

For $1 \leq i \leq n - 1$, define $v_i^+ = v_{i+1}$. By 2.4.11 both $\mathrm{N}(v_1)$ and $\mathrm{N}(v_m)$ are contained in $\mathrm{V}(P)$. As $v_m \notin \mathrm{N}(v_m)$ we can define $\mathrm{N}^+(v_m) = \{u^+ \mid u \in \mathrm{N}(v_m)\}$. Then both $\mathrm{N}(v_1)$ and $\mathrm{N}^+(v_m)$ are subsets of $\{v_2, v_3, \dots, v_m\}$ of size at least $\frac{n}{2}$. Note that $\frac{n}{2} + \frac{n}{2} = n \geq m > m - 1$ and so $\mathrm{N}(v_1) \cap \mathrm{N}^+(v_m) \neq \varnothing$. Thus there exists $1 \leq i \leq n - 1$ with $v_i \in \mathrm{N}(v_m)$ and $v_{i+1} = v_i^+ \in \mathrm{N}(V_1)$. It follows that
$$v_1, v_2, \dots v_i, v_m, v_{m-1}, \dots, v_{i+1}, v_1$$
is a cycle of length m in G.

$(+)$ $\mathrm{N}(v) \subseteq \mathrm{V}(C)$ *for all $v \in \mathrm{V}(C)$.*

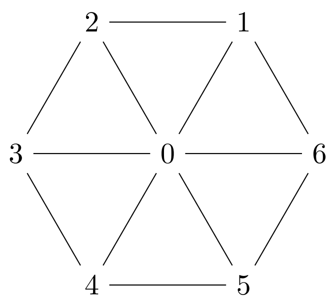
Note that there exists a rotation $C' = c_0 c_1 \dots c_m$ of C with v as last vertex As m is the length of a longest path, $c_1 \dots c_m$ is a longest path of G and 2.4.11 shows that $\mathrm{N}(v) \subseteq \mathrm{V}(C') = \mathrm{V}(C)$.

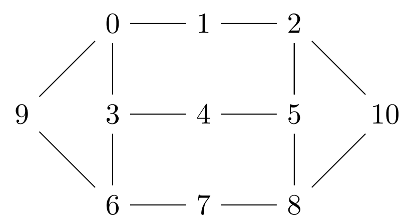
By $(**)$ G is connected and by $(+)$ $\mathrm{V}(v) \subseteq \mathrm{V}(C)$. It follows that $\mathrm{V}(C) = V$, see (2.3.12)(b). Thus C is an Hamiltonian circle. □

Definition 2.4.14. *Let G be graph and $W \subseteq \mathrm{V}(G)$. W is called* independent *if no two vertices from W are adjacent in G. $\alpha(G)$ is the largest size of an independent subset of $\mathrm{V}(G)$. $\alpha(G)$ is called the independence number of G.*

Remark 2.4.15. *Let G be graph and $W \subseteq \mathrm{V}(G)$. Then W is independent if and only of the subgraph of G induced by W is empty. In particular, $\alpha(G)$ is the largest order of an empty induced subgraph of G.*

Example 2.4.16. Compute the independence number of G.





The independent subsets of maximal size in the first graph are:

$$\{1,3,5\} \qquad \text{and} \qquad \{2,4,6\}$$

Thus $\alpha(G) = 3$
In the second graph there is a unique independent subsets of maximal size namely

$$\{1,3,5,7,9,10\}$$

Thus $\alpha(G) = 6$.

Theorem 2.4.17. *Let G be graph and $W \subseteq \mathrm{V}(G)$. Let H be a connected component of $G - W$ and let $U = W \cap \mathrm{N}(\mathrm{V}(H))$. If $|U| < \kappa(G)$, then $U = W$ and $H = G - W$.*

Proof. Suppose $|U| < \kappa(G)$. By definition of $\kappa(G)$, G has no cut set of size less than $\kappa(G)$ and so $G - U$ is connected. Suppose that $\mathrm{V}(G - U) \neq \mathrm{V}(H)$. Then by (2.3.12)(b) there exists an edge ab of $G - U$ with $a \in \mathrm{V}(H)$ and $b \in \mathrm{V}(G - U) \setminus \mathrm{V}(H)$. Then $b \notin U$ and $b \in \mathrm{N}(\mathrm{V}(H))$, so $b \notin W$. Hence ab is an edge of $G - W$. But then a and b are in the same connected component of $G - W$, a contradiction since $a \in \mathrm{V}(H)$ and $b \notin \mathrm{V}(H)$.

Thus $\mathrm{V}(G - U) = \mathrm{V}(H)$. As $U \subseteq W$ and $W \cap \mathrm{V}(H) = \varnothing$, this implies $U = W$ and $\mathrm{V}(G - W) = \mathrm{V}(H)$. So also $H = G - W$. □

Theorem 2.4.18. *Let $G = (V, E)$ be a graph of order $n \geq 3$ with $\alpha(G) \leq \kappa(G)$. Then G is Hamiltonian.*

Proof. We will first show that

(∗) $\kappa(G) \geq 2$.

Otherwise $\alpha(G) \leq \kappa(G) \leq 1$. Let $v, w \in \mathrm{V}$ with $v \neq w$. Since $\alpha(G) \leq 1$, $\{v, w\}$ is not independent and so vw is an edge. Thus G is a complete graph and so $\kappa(G) = n - 1$. Thus $n - 1 \leq 1$ and $n \leq 2$, a contradiction to the assumptions.

Since $\kappa(G) \geq 2$, G is 2-connected and so Exercise 3b on Homework 10 shows that G has a cycle. So we can choose longest cycle $C = c_0c_1 \dots c_m$ in G. If $\mathrm{V}(C) = \mathrm{V}(G)$, then C is a Hamiltonian cycle and we are done. So we may assume that $\mathrm{V}(C) \neq \mathrm{V}(G)$. Thus there exists connected component H of $G - \mathrm{V}(C)$. Define

$$W = \{w \in \mathrm{V}(C) \mid w \text{ is adjacent to some } w' \in \mathrm{V}(H)\}$$

For $0 \leq i \leq m - 1$ define $c_i^+ = c_{i+1}$ and put $W^+ = \{w^+ \mid w \in W\}$. For $w \in W$ choose $w' \in \mathrm{V}(H)$ such that w is adjacent to w'.

(∗∗) *Let $w \in W$. Then $w^+ \notin W$.*

Otherwise there exists $0 \leq i \leq n - 1$ such that $c_i \in W$ and $c_{i+1} = c_i^+ \in W$. Since H is connected there exists a $c_i' - c_{i+1}'$ walk $h_0 \dots h_l$ in H. As $\mathrm{V}(H) \cap \mathrm{V}(C) = \varnothing$ we conclude that

$$c_0, c_1 \dots, c_i, c_0' = h_0, \dots h_l = c_{i+1}', c_{i+1}, c_{i+2}, \dots, c_m$$

is a cycle in G is length $m + l > m$, a contradiction to the maximal choice of C.

(∗ ∗ ∗) $|W| \geq \kappa(G)$.

Suppose that $|W| < \kappa(G)$. Then 2.4.17 shows that $W = \mathrm{V}(C)$. Let $w \in \mathrm{V}(C)$. It follows that $w \in W$ and $w^+ \in W$, a contradiction to (∗∗).

(+) *W^+ is independent.*

Otherwise there exists $c_i, c_j \in W$ and $c_{i+1} = c_i^+$ is adjacent to $c_j^+ = c_{j+1}$. Since H is connected there exists a $c_i' - c_j'$ path $h_0 \dots h_l$ in H. Then

$$c_0, c_1, \dots c_i, c_i' = h_0, h_1 \dots, h_l = c_j', c_j, c_{j-1}, \dots c_{i+2}, c_{i+1}, c_{j+1}, c_{j+2}, \dots c_0$$

is a cycle of length $m + l$, contradiction the maximal choice of C. Thus (+) holds.

We are now able to derive a contradiction.

Let $v \in \mathrm{V}(H)$. Let $w \in W$, then (∗∗) shows that $w^+ \notin W$ and so v is not adjacent to w^+. As W^+ is independent we conclude that also $W^+ \cup \{v\}$ is independent. Hence $|W^+ \cup \{v\}| \leq \alpha(G)$ and so $|W^| < \alpha(G)$. Thus

$$\kappa(G) \leq |W| = |W^+| < \alpha(G) \leq \kappa(G).$$

a contradiction. □

2.5 Colorings of graphs

2.5.1 Definitions

Definition 2.5.1. *Let $G = (V, E)$ be a graph. .*

(a) *Let S be a set. A coloring of G with S is a function $K : V \to S$ (that is a coloring of V with S) such that $K(v) \neq K(w)$ whenever vw is an edge of G.*

(b) *Let $k \in \mathbb{N}$. Then G is called k-colorable if there exists a coloring of G with a set of size k.*

Remark 2.5.2. *Let $G = (V, E)$ be a graph of order n. Then $\mathrm{id}_V : V \to V, v \to v$ is a coloring of G with V. In particular, G is n-colorable.*

Remark 2.5.3. *Let G be a graph. Let $k \in \mathbb{N}$. Then G is k-colorable if and only if there exists a partition $(V_1, \dots, V_k)$ of $\mathrm{V}(G)$ such that each V_i is independent in G.*

Definition 2.5.4. *Let G be a graph. The chromatic number of G, denote by $\chi(G)$ is the smallest k in $\mathbb{N}$ such that G is k-colorable.*

Example 2.5.5. Let G be a graph.

(1) $\chi(G) = 1$ if and only if

G is empty.

(2) $\chi(G) \leq 2$ if and only if

G is bipartite and if and only if G is has no odd cycles.

Example 2.5.6. Let $n \in \mathbb{N}$ with $n \geq 2$

(1) Compute $\chi(P_n)$

$\chi(P_n) = 2$.

(2) Compute $\chi(C_n)$

$$\chi(C_n) = \begin{cases} 2 & \text{if } n \text{ is even} \\ 3 & \text{if } n \text{ is odd} \end{cases}$$

(3) Compute $\chi(K_n)$

$\chi(K_n) = n$.

Algorithm 2.5.7 (Greedy Algorithm)**.** *Let G be an ordered graph and S an ordered set. Define a coloring K of G with S recursively as follows:*

Let v be vertex of G and suppose $K(w)$ has been defined for all vertices $w < v$.

(a) *Compute $S' = \{K(w) \mid w \in \mathrm{N}(v), w < v\}$.*

(b) (1) *If $S = S'$, the algorithm stops and has failed.*

(2) *If $S \neq S'$, let s be the smallest element of $S \setminus S'$ and define $K(v) = s$.*

(c) (1) *If v is the largest element of V, the algorithm stops successfully.*

(2) *Otherwise go to Step (a) with v replaced by the smallest vertex larger than v.*

Theorem 2.5.8. *Let G be graph. Then $\chi(G) \leq \Delta(G) + 1$.*

Proof. Order G and let S be any ordered set of size $\Delta(G)+1$. Apply 2.5.7 Note that $|S'| \leq |\deg(v) < |\Delta(G)| + 1 = |S|$. Thus $S \neq S'$ and the algorithm does not fail. □

Theorem 2.5.9. *Let G be a connected graph of order n. Then either $G = K_n$, or n is even and $G = C_{n-1}$, or $\chi(G) \leq \Delta(G)$.*

Proof. Put $k := \Delta(G)$ and let S be any ordered set of size k.

$(*)$ *If G is not regular, then $\chi(G) \leq k$.*

Suppose G is not regular. Then there exists w of $\mathrm{V}(G)$ with $\deg(w) \neq k$ and so $\deg(w) < k = |S|$.
Choose an ordering of V such that $u < v$ whenever $\mathrm{d}(w,u) > \mathrm{d}(w,v)$. (So the smallest element of V are the elements furthest from w, and w (the closest element to w) is the largest element of V. We will show that the Greedy algorithm does not fail. Let $v \in \mathrm{V}(G)$ and suppose $K(w)$ has been defined for all vertices less than v. If $v = w$, then $\deg(v) < |S|$ and so $|S'| \leq \deg v < |S|$. If $w \neq v$, then there exists a vertex u adjacent to v with $\mathrm{d}(w,u) = \mathrm{d}(w,v) - 1 < \mathrm{d}(w,v)$. Then $u > v$ and so $|S'| < \deg(v) - 1 < \Delta(G) = |S|$. In either case $S \neq S'$ and the algorithm does not fail.

$(**)$ *Suppose $k \leq 2$. Then the theorem holds.*

Suppose $k \leq 1$, then $n \leq 2$ and so $G = K_n$.
Suppose that $k = 2$. Then G is P_n or C_{n-1}. If $G = C_{n-1}$, n even, the theorem holds. If $G = P_n$ or n is odd and $G = C_{n-1}$, then $\chi(G) = 2$, and again the theorem holds.

Observe that

$(***)$ *If $\mathrm{diam}(G) \leq 1$, then G is complete.*

So we may assume from now on that

$(+)$ *G is regular, $k \geq 3$ and $\mathrm{diam}(G) \geq 2$.*

Case 1. *G is not 2-connected.*

Then G has cut vertex v. Let $H_i, 1 \leq i \leq l$ be the connected components of $G - v$. Let G_i be subgraph of H induced by $\mathrm{V}(H) \cup \{v\}$. By (2.3.13)(a) there exists a vertex h_i in H_i adjacent to v. Hence $\deg_{G_i}(v) < \deg_G(v) = k$. Since G is k-regular we have $\deg_{G_i}(w) = k$ for $w \in \mathrm{V}(H_i)$. Thus G_i is not regular, and so by $(*)$ implies that there exists coloring K_i of G_i with S. Pick $s \in S$ and choose permutation π_i of S with $\pi_i(K_i(v)) = s$. Define $L_i = \pi_i \circ K_i$. Then L_i is a coloring of G_i and

$L_i(v) = s$ for all $1 \le i \le l$. Hence we can define $L : V \to S$ by $L(u) = L_i(u)$ if $u \in \mathrm{V}(G_i)$. If uw is an edge of G then uw is an edge of some G_i and so $L(u) = L_i(u) \neq L_i(w) = L(u)$. Thus L is a coloring of G with S

Since $\operatorname{diam}(G) \ge 2$ we can choose vertices a and b with $\mathrm{d}((,a),b) = 2$.

Case 2. *G is 2-connected, but $G - a - b$ is not connected.*

Let $H_i 1 \le i \le l$ be the connected components of $G - a - b$. Let G_i be subgraph of H induced by $\mathrm{V}(H) \cup \{a, b\}$. By (2.3.13)(a) there exists a vertex a_i in H_i adjacent to a. Hence $\deg_{G_i}(v) < k$ Since G is k- regular we have $\deg_{G_i}(w) = k$ for $w \in \mathrm{V}(H_i)$. Thus G_i is not regular, and so by $(*)$ implies that there exists coloring K_i of G_i with S.

Suppose first for each $1 \le i \le l$, $\deg c \in \{a, \beta\}$ with $\deg_{G_i}(c) \le k-2$. Let $\{a, b\} = \{c, d\}$. Then there exists $s \in S$ with $s \neq K_i(d)$ and $s \neq K_i(w)$ for all w in $\mathrm{V}(H_i) \cap \mathrm{N}(c)$. Hence we may assume that $K_i(a) \neq K_i(b)$. Hence we can choose a coloring L_i of G_i such that $L_i(x) = K_j(x)$ for all $x \in \{a, b\}$ and all for all $1 \le i, j \le l$.

Hence we can define $L : V \to S$ by $L(u) = L_i(u)$ if $u \in \mathrm{V}(G_i)$. If uw is an edge of G then uw is an edge of some G_i and so $L(u) = L_i(u) \neq L_i(w) = L(u)$. Thus L is a coloring of G with S

Case 3. *$G - a - b$ is connected.*

Put $v_1 = a$ and $v_2 = b$. We will now choose vertices $v_3, \ldots v_n$ in $G - a - b$ recursively as follows Since $\mathrm{d}(a, b) = 2$ there exists a path $a - b$ path acb of length two in G. Put $v_n = c$. Let $3 \le i \le n$ and suppose we already choose $v_{i+1}, \ldots v_n$. Put $U = \{v_{i+1}, \ldots v_n\}$. Since $G - a - b$ is connected there exists a vertex w in $G - a - b$ with $w \notin U$ such that w is adjacent to some v_j in U. Choose $v_i = w$.

Order G such that $v_1 < v_2 < \ldots < v_n$. It remains to show that the Greedy algorithm does not fails.

Suppose $K(v_l)$ has been defined for all $1 \le l < i$.

Suppose $i = 1$. Then $S' = \varnothing$ and so $K(v_1)$ is the smallest element of S.

Suppose $i = 2$. Since $\mathrm{d}(a, b) = 2$, v_2 is not adjacent to v_1. Thus $S' = \varnothing$ and so $K(v_2)$ is the smallest element of S. In particular, $K(v_1) = K(v_2)$.

Suppose $3 \le i < n$. Then v_i is adjacent to some v_t with $t > i$ and so $|S'| < |S|$.

Suppose $i = n$. Then $v_n = c$ and v_n is adjacent to v_1 and v_2. As $K(v_1) = K(v_2)$ this shows that $|S'| < |S|$.

Thus $S \neq S'$ in all cases and so the Greedy algorithm succeeds.

□

2.6 Chromatic Polynomials

Definition 2.6.1. *Let G be a graph and S a set. Then $\mathrm{Co}(G, S)$ is the set of colorings of G with S and $c_G(S)$ denotes number of colorings of G with S. If $k \in \mathbb{N}$, then $c_G(k) = C_G(S)$, where S is any set of size k.*

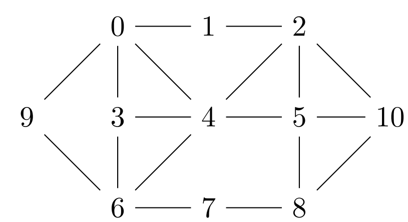
Example 2.6.2. Compute $c_{K_n}(k)$ and $c_{E_n}(k)$.
$c_{K_n}(k) = k^{\underline{n}}$
$c_{E_n}(k) = k^n$

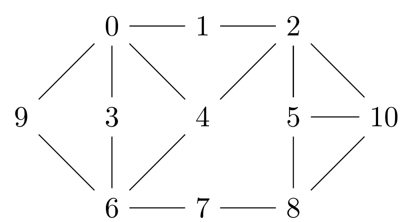
Definition 2.6.3. *Let $G = (V, E)$ be a graph and $e = vw$ an edge of G. Then G/e is the graph obtained from G by removing e and identifying v and w.*

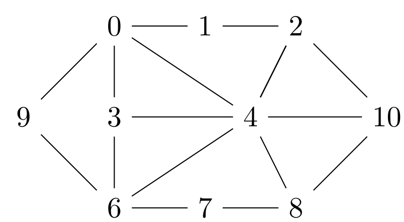
More formally for $x \in V$ define $\overline{x} = x$ if $x \neq w$ and $\overline{x} = v$ if $x = w$. For an edge $f = \{y, z\}$ of $G - e$ define $\overline{f} = \{\overline{y}, \overline{z}\}$. Define $V/e = \{\overline{x} \mid x \in V\} = V - w$, $E/e = \{\overline{f} \mid f \in E - e\}$ and $G/e = (V/e, E/e)$. [1]

Remark 2.6.4. *Let $G = (V, E)$ be a graph, $e = vw$ an edge of G and $x, y \in V \setminus \{v, w\}$. Then x is adjacent to y in G/e if and only if x is adjacent to y in G. And x is adjacent to v in G/e if and only if x is adjacent to v or to w in G.*

Example 2.6.5. Compute $G - 45$ and $G/45$.







Theorem 2.6.6. *Let $G = (V, E)$ be a graph, S a set and $e = vw$ an edge of G.*

(a) *The colorings of G are exactly the colorings K of $G - e$ with $K(v) \neq K(w)$.*

(b) *If K is a coloring of V, restrict K to a coloring $K\!\downarrow$ of $V - w$ via $K\!\downarrow(u) = K(u)$ for all $u = V - u$.*

If L is a coloring of $V - w$, extend L to a coloring $L\!\uparrow$ of V by

$$L\!\uparrow(u) = \begin{cases} L(u) & \text{if } u \in V - w \\ L(v) & \text{if } u = w \end{cases}$$

Then the function $K \mapsto K\!\downarrow$ is a bijection between the coloring of $G - e$ with $K(v) = K(w)$ and the colorings of G/e. The inverse is given by $L \to L\!\uparrow$.

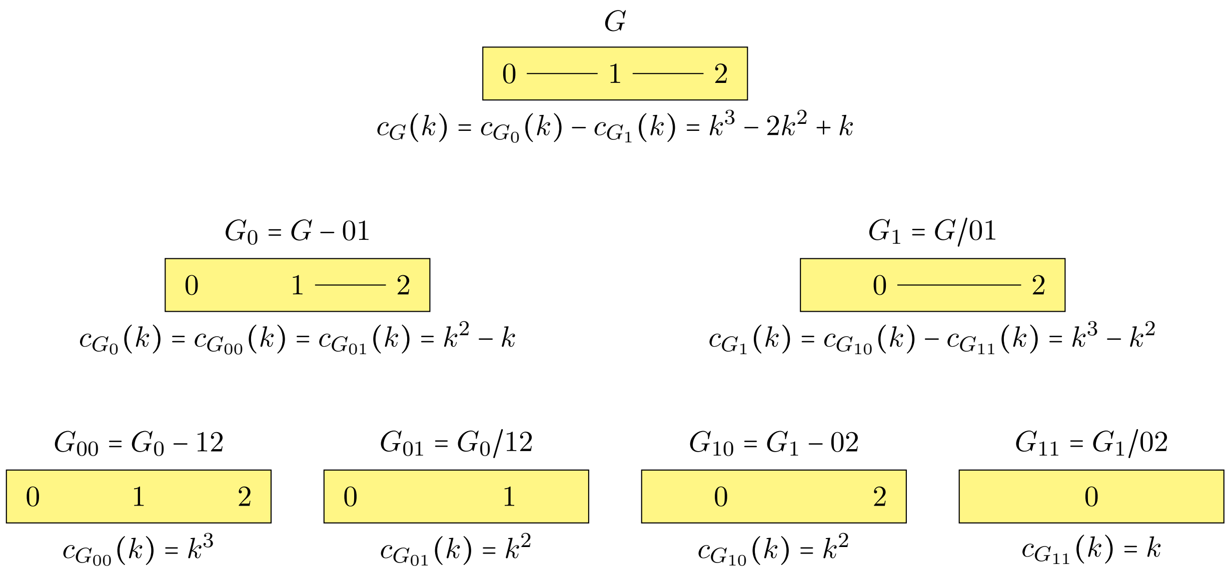
[1] Note that the definition of E/e depends on the pair (v, w) and not only the edge e, but since the two graphs G/vw and G/wv are isomorphic, we still use the ambiguous notation G/e.

(c) $c_G(S) = c_{G-e}(S) - c_{G/e}(S)$.

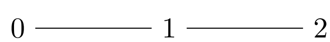
Remark 2.6.7. *Let G be graph with connected components $G_1, G_2, \ldots G_l$ and $k \in \mathbb{N}$. Then*

$$c_G(k) = c_{G_1}(k)c_{G_2}(k)\ldots C_{G_l}(k)$$

Example 2.6.8. Use Theorem 2.6.6 to compute $c_{P_3}(k)$.



Example 2.6.9. Use a counting argument to compute $c_{P_3}(k)$.



We can choose $K(0)$ to be any of the k-colors. Then $K(1)$ can be any color other than $K(0)$, so there are $k-1$ choices for $K(1)$. Now $K(2)$ can be any color other than $K(1)$, so there are $k-1$ choices for $K(2)$. Altogether we see that

$$c_{P_n}(k) = k(k-1)(k-1) = k(k^2 - 2k + 1) = k^3 - 2k^2 - k$$

Theorem 2.6.10. *Let G be a graph of order $n \geq 1$.*

(a) *$c_G(k)$ is a polynomial of degree n.*

(b) *The leading coefficient of $c_G(k)$ is 1.*

(c) *The constant coefficient of $c_G(k)$ is 0.*

(d) *The coefficients of $c_G(k)$ alternate in sign.*

(e) *The absolute value of the coefficients of k^{n-1} is the size of G.*

2.7 Matchings

2.7.1 Definitions and Examples

Remark 2.7.1. *Let G be a graph. Then the following statements are equivalent:*

(a) *G is* 1*-regular.*

(b) *Each connected component of G has order* 2*.*

Definition 2.7.2. *Let G be a graph.*

(a) *A matching in G is a* 1*-regular subgraph of G.*

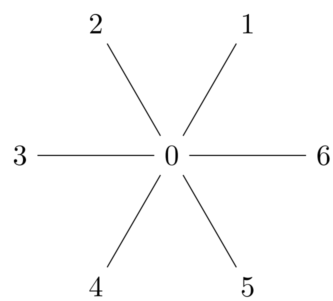
(b) *A maximum matching of G is matching in G of largest order.*

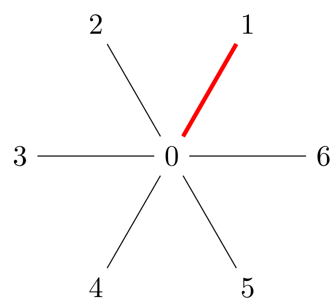
(c) *A perfect matching of G is a matching in G with vertex set* $\mathrm{V}(G)$*.*

(d) *Let M be matching of G and v, w vertices of G. We say that x matches y in M if xy is an edge of M. In this case we also say that that y is the match of x in M.*

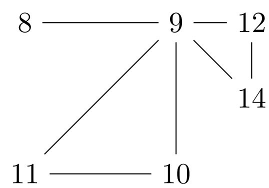
Example 2.7.3. Find a maximum matchings:

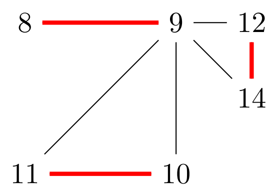
(1)



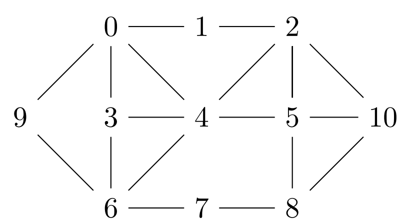


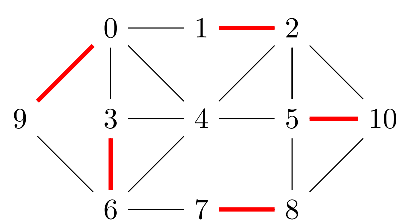
(2)





(3)





2.7.2 Hall's Marriage Theorem and SDRs

Definition 2.7.4. *Let M be matching in the graph G and P a walk in G*

(a) *P is called M-alternating if the edges of P alternate between M-edges and non M-edges, that is, if $e_1 \dots e_k$ is the edge sequence of P and $1 \leq i < k$, then $e_i \in \mathrm{E}(M)$ if and only of $e_{i+1} \notin \mathrm{E}(M)$.*

(b) *P is called M-augmenting if P is M-alternating of length at least 1 and none of the end vertices of P are in M*

Theorem 2.7.5. *Let M and M' be matchings of the graph G. Suppose there neither exists a M-augmented nor an M'-augmented path in G. Then M and M' have the same size and order.*

Proof. Let H be the union of the two graphs M and M', so $\mathrm{V}(H) = \mathrm{V}(M) \cup \mathrm{V}(M')$ and $\mathrm{E}(H) = \mathrm{E}(M) \cup \mathrm{E}(M')$.

Let v a vertex of H. If $v \in M$ then x is incident with exactly one edge from M and at most one edge from M', thus

$$(*) \qquad 1 \le \deg_H(v) \le 2.$$

Let K be a connected component of H. We will now show that

$(**)$ *The number M-edges in K is equal to the number of M' edges in K.*

Since $1 \le \deg_K(v) \le 2$ for all vertices v of K, K is either path graph or a cycle graph of size at least 1 , In particular, there exists a path or cycle $P = v_0 \dots v_n$ of length at least 1 in K with $\mathrm{E}(P) = \mathrm{E}(K)$. Put $e_i = e_{i-1}e_i$. If e_i is in M, then e_{i+1} is not in M and so e_i is in M'. Thus P is an alternating M and an alternating M' walk.

Suppose that n is even. Then P (and so also K) has exactly $\frac{n}{2}$ edges from M and $\frac{n}{2}$ edges from M'. So $(**)$ holds.

Suppose next that n is odd.

Asssume e_1 is in M and in M'. If $n \ge 2$, e_2 would be neither in M nor in M', a contradiction. Thus $n = 1$ and e_1 is the only edge of K, so $(**)$ holds in this case.

Note that the setup is symmetric in M and M'. So we may assume now that e_1 is in M' but not in M. Since n is odd and P is M'-alternating, also e_n is in M'. Note that e_1 is the only M-edge incident with v_0 and that $v_0 \ne v_n$. Thus e_1 is the only edge of P and so also of K) incident with v_0. Thus v_0 is not incident to any M-edge of K and so $v_0 \notin \mathrm{V}(M)$. The same holds for v_n. Thus M is an M-augmented path, a contradiction to the hypothesis of the theorem. Thus $(**)$ is proved.

By $(**)$ each connected component of H has same number of M-edges as M' edges. Thus the same holds for H. As H contains all M and all M' edges, this show that M and M' have the same size. For any 1-regular graph the order is twice the size, so M and M' also have the same order. □

Theorem 2.7.6. *A matching M in the graph G is maximum if and only there does not exist a M-augmenting path in G.*

Proof. Suppose that there does exist am M-augmenting path $P = v_0v_1 \dots v_n$ in G. Put $e_i = v_{i-1}v_i$. Then v_0 and v_n are not in M and so also e_1 and e_n are not in M. Since P is alternating we conclude that n is odd, that

$$e_1, e_3 \dots, e_n \notin \mathrm{E}(M), \quad \text{and} \quad e_2, e_4, \dots, e_{n-1} \in \mathrm{E}(M)$$

Put $F = \{e_2, e_4, \dots, e_{n-1}\}$ and $F' = \{e_1, e_3, \dots, e_n\}$. So F and F' u is the set of M- and non M-edges of P, respectively Consider the graph M' with vertices $\mathrm{V}(M) \cup \{v_0, v_n\}$ and edges $(\mathrm{E}(M) \cup F') \setminus F$.

Next we show

$(*)$ *Let v be a vertex of M'. Then* $\deg_{M'}(v) = 1$.

We need to show that v is incident with a unique edge of M'.

Suppose $v = v_0$ or v_n then $v \notin \mathrm{E}(M)$ and so v is not incident with any edge of M. Thus e_1 (if $v = v_0$) and e_n (if $v = v_n$) is the unique edge of M' incident with v.

Suppose that $v = v_i$ for some $1 \leq i < n$. Let $i = 2j - k$ with $j \in \mathbb{Z}^+$ and $k \in \{0, 1\}$. Then e_{2j} is the unique M-edge incident with v. Thus v is not incident with any edge in $\mathrm{E}(M) \smallsetminus F$. It follows that e_{2j-1} is the unique edge in M' incident v.

Suppose finally with $v \notin \mathrm{V}(P)$. Then $v \in \mathrm{E}(M)$ and so v is incident with a unique edge e of M. As $v \notin \mathrm{V}(P)$, v is not incident with any edge of P. Thus $e \notin F$ and so $e \in \mathrm{E}(M')$. Moreover, e is incident with any edge in F', so e is the unique edge of M' incident with v.

Thus $(*)$ is proved.

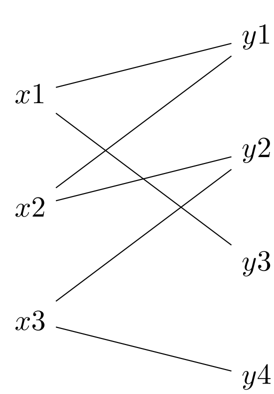
By $(*)$ M' is 1-regular, and so a matching in G. But M' has two more vertices (namely v_0 and v_n) than the maximum matching M, a contradiction.

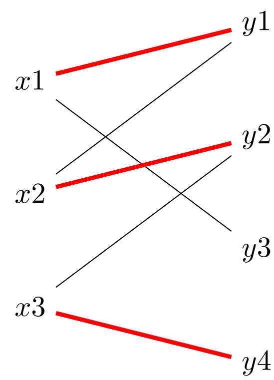
Thus completes the proof of the forward direction. Suppose now that there does not exist an M-augmented path in G. Let M' be any maximum matching of G. By the forward direction there also does not exist an M'-augmented path, so 2.7.5 shows that M and M' have the same order. Hence, as M' is maximum, so is M. □

Definition 2.7.7. *Let (X, Y) be a bipartition of the graph G. We say that X matches into Y if there exists a matching M of G with $X \subseteq \mathrm{V}(M)$.*

Example 2.7.8. Does X match into Y?

(1)





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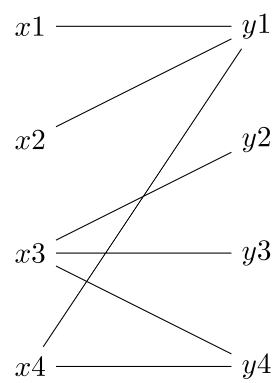


No, since both $x1$ and $x2$ would have to match $y1$.

Theorem 2.7.9. *Let (X,Y) be a bipartition of the graph G and let M be maximum matching of G. Let $W = X \setminus \mathrm{V}(M)$. Let A be the set of all vertices v of G such that there exist $w \in W$ and an M-alternating $w - v$ walk. Put $S = A \cap X$, $T = A \cap Y$ and $T^+ = S \cap \mathrm{V}(M)$. Then*

(a) $S = W \cup T^+$, $S \cap T^+ = \varnothing$ *and* $T \cup T^+ \subseteq \mathrm{V}(M)$.

(b) *The M-matches of the elements of T are the elements of T^+ and vice versa. In particular,* $|T| = |T^+|$.

(c) $\mathrm{N}(S) = T$.

Proof. Let $v \in A$. Then there exist $w \in W$ and an M-alternating path

$$P: \quad v_0 = w, v_1, \dots, v_{m-1}, v_m = v.$$

Suppose that $m \geq 1$., Note that $w \notin \mathrm{V}(M)$ and by 2.7.6 there does not exist an M-augmenting path in G. It follows that

(∗) $$v \in \mathrm{V}(M).$$

As $w \notin \mathrm{V}(M)$, v_0v_1 is non M-edge. As P is M-alternating, this shows that $v_{l-1}v_l$ is an M-edge if and only if l is even. Since G is bipartite and $w \in X$, $v_l \in X$ if and only of l is even. For $l = m$ (and so $v_l = v$) we conclude

(∗∗) *$v_{m-1}v_m$ is an M-edge if and only if $v \in X$.*

(a) If $w \in W$, then w is an alternating path of length 0 and so $w \in A \cap X = S$. Thus $W \subseteq S$. As $W = X \smallsetminus \mathrm{M}(V)$ we also get $W = S \smallsetminus (S \cap \mathrm{V}(M) = S \smallsetminus T^+$. So $S = W \cup T^+$ and $W \cap T^+ = \varnothing$. If $v \in T = A \cap Y$, then $v \neq w$ and so (∗) shows that $v \in \mathrm{V}(M)$. Thus $T \subseteq \mathrm{V}(M)$.

(b) Suppose that $v \in T^+$. Then (∗∗) shows that $v_{m-1}v$ is an M-edge. Thus v_{m-1} is the M match of v, note also that $v_{m-1} \in T$ and so the M-matches of the elements of T^+ are all contained on T.

Suppose that $v \in T$. By (a) $v \in \mathrm{M}(V)$ and so v has an M-match u. Then $u \in X \cap \mathrm{N}(V)$. From (∗∗) we know that $v_{m-1}v$ is non-M-edge. If $u = v_l$ for some $0 \leq l \leq m$ then $v_0, \ldots v_l$ is M-alternating $w - u$-path. Otherwise $w = v_0, \ldots, v_{m-1}, v_m = v, u$ is an M-alternating path. In either case, $u \in A \cap X \cap \mathrm{V}(M) = T^+$. So the M-matches of the elements of T are all contained on T.

(c) Suppose $v \in S$ and let $u \in \mathrm{N}(v)$. We need to show that $u \in T$. Since G is bipartite and $v \in S \subseteq X$ we get $u \in Y$.

If $v \in W$, then $v - u$ is an M-alternating path and so $u \in A \cap Y = T$.

So suppose $v \notin W$. Then $v \in T^+$ and $m \geq 1$. If $vu \in \mathrm{E}(M)$, then u is the M-match of v and (b) shows that $u \in T$. So assume that $vu \notin \mathrm{E}(M)$. By (∗∗) $v_{m-1}v$ is an M-edge. If $u = v_l$ for some $0 \leq l \leq m$ then $v_0, \ldots v_l$ is M-alternating $w - u$-path. Otherwise $w = v_0, \ldots, v_{m-1}, v_m = v, u$ is an M-alternating path. In either case, $u \in A \cap Y = T$. □

Theorem 2.7.10 (Hall's Marriage Theorem)**.** *Let (X, Y) be a bipartiton of the graph G. Then X matches into Y if and only if $|S| \leq |\mathrm{N}(S)|$ for all $S \subseteq X$.*

Proof. Suppose first that X matches into Y. Then there exists a matching M of G with $X \subseteq \mathrm{V}(M)$. Let $S \subseteq X$. Then $S \subseteq \mathrm{V}(M)$. As M is 1-regular we get $\mathrm{N}_M(S)| = |S|$. But $\mathrm{N}_M(S)|leq\mathrm{N}_G(S)$ and so $|S|leq\mathrm{N}_G(S)$.

Suppose next that $|S| \leq |\mathrm{N}(S)|$ for all $S \subseteq X$ and let M be a maximum matching of G. We use the notation from 2.7.9. Then $W = X \smallsetminus \mathrm{V}(M)$, $\mathrm{N}(S) = T$, $S = W \cup T^+$, $W \cap T^+ \varnothing$ and $|T| = |T^+$. Hence

$$|W| + |T^+| = |W \cup T^+| = |S| \leq |\mathrm{N}(S)| = |T| = |T^+|$$

Thus $|W| = 0$ and so $W = \varnothing$ and $X \subseteq \mathrm{V}(M)$. Hence X matches into Y. □

Definition 2.7.11. *Let X and Y be a sets and $A = (A_x)_{x \in X}$ a family of subsets of Y (so for each $x \in X$, A_x is a subsets of Y).*

(a) *A system of representatives for A is a family $(a_x)_{x \in X}$ of elements of Y such that $a_x \in A_x$ for all $x \in X$.*

(b) *A system of distinct representatives (SDR) for A is a system of representatives $(a_x)_{x \in X}$ for A such that $a_x \neq a_y$ for all $x, y \in X$ with $x \neq y$.*

Notation 2.7.12. (a) *Let $X = \{1, 2, \dots n\}$. Note that a family $(a_x)_{x \in X}$ is just a sequence of length n. So we also use the notations $(a_i)_{i=1}^n$ and $(a_1, \dots, a_n)$*

(b) *Let $X = \{x_1, \dots, x_n\}$ be a set. Then we denote the family $A = (A_x)_{x \in X}$ by*

x_1	x_2	$\dots$	x_n
A_{x_1}	A_{x_2}	$\dots$	A_{x_n}

Example 2.7.13. (a) Does $\big(\{2,8\}, \{8\}, \{5,7\}, \{2,4,8\}, \{7\}\big)$ have an SDR?

Yes, exactly one, namely $(2, 8, 5, 4, 7)$.

(b) Does $\big(\{1,2\}, \{1,3\}, \{2,3\}, \{1,4\}, \{2,4\}\big)$ have an SDR?

No, any SDR would be a non-repeating sequence of length five from a set of size four, which is impossible for example by the Pigeonhole principal.

Definition 2.7.14. *Let X and Y be disjoint sets.*

(a) *Let $A = (A_x)_{x \in X}$ be family of subsets of Y. Then G_A is the graph with vertex set $X \cup Y$ and edges $\{x, y\}$, $x \in X, y \in A_x$.*

(b) *Let $a = (a_x)_{x \in X}$ be family of elements of Y. Then M_a is the the graph with vertices x and a_x, $x \in X$, and edges $\{x, a_x\}$, $x \in X$.*

Example 2.7.15. Let $X = \{1, 2, 3\}$, $Y = \{a, b, c, d\}$. Compute G_A and M_a for

A:

1	2	3
$\{a, b\}$	$\{b, d\}$	$\{a, c, d\}$

and a:

1	2	3
a	b	d

Theorem 2.7.16. *Let X and Y be disjoint sets. Let $A = (A_x)_{x \in X}$ be family of subsets of Y and let $a = (a_x)_{x \in X}$ be family of elements of Y.*

(a) *(X, Y) is a bipartition of G_A.*

(b) $\mathrm{N}(x) = A_x$ *for all $x \in X$.*

(c) *The function $A \to G_A$ is a bijection between the families $(A_x)_{x \in X}$ of subsets of Y and the graphs with bipartition (X, Y).*

(d) *a is an SDR of A if and only if M_a is matching of G_A.*

(e) *The function $a \mapsto M_a$ is a bijection between the SDRs of A and the matchings M of G with $X \subseteq \mathrm{V}(M)$.*

Corollary 2.7.17. *Let X and Y be disjoints sets and $A = (A_x)_{x\in X}$ a family of subsets of Y. Then A has an SDR if and only if*

$$|S| \le \left|\bigcup_{s\in S} A_s\right|$$

for all $S \subseteq X$.

Proof. Consider the graph G_A defined in 2.7.14. Let $S \subseteq X$ and $s \in S$. By (2.7.16)(b) $\mathrm{N}(s) = A_s$ and so

$$\mathrm{N}(S) = \bigcup_{s\in S} \mathrm{N}(s) = \bigcup_{s\in S} A_s$$

Thus

$$|S| \le \left|\bigcup_{s\in S} A_s\right|$$

for all $S \subseteq X$ if and only f $\mathrm{N}(S) \le |S|$ for all $S \subseteq X$. By 2.7.10 the latter holds if and only if X matches Y, that is if and only if there exists a matching M of G_A with $X \subseteq \mathrm{V}(M)$, and so (2.7.16)(e) if and only if A has an SDR. □

Example 2.7.18. Does

$$\Big(\{a,c,d,g\},\{a,b,e,f,g\},\{a,c,g\},\{c,d,g\},\{b,d,e,g\},\{a,d\},\{a,d,g\}\Big)$$

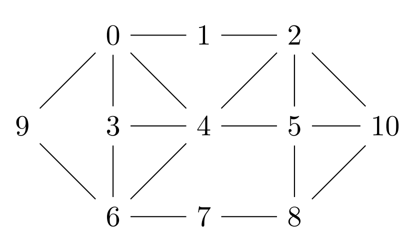
have a system of distinct representatives?

No, since the union of five of the sets, namely the first, third, fourth, sixth seventh, is equal to $\{a,c,d,g,\}$ and so has size less than 5.

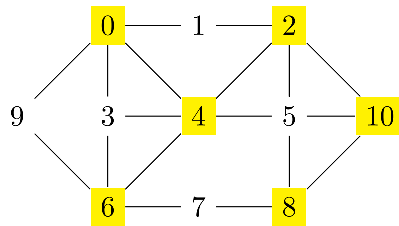
2.7.3 Edge Covers

Definition 2.7.19. *A edge cover for the graph G is a set of vertices C of G such that each edge of G is incident with at least one vertex of C. A minimum edge cover is an edge cover of smallest cardinality.*

Example 2.7.20. Find a minimum edge cover.



The claim that the yellow vertices in the following diagram form an edge covering.



This is the edge covering with six vertices. Any edge covering needs three vertices from the cycle $0, 3, 4, 6, 9, 0$ and three vertices from the path $1, 2, 5, 10, 8, 7$. So six is the minimum size of an edge covering.

Theorem 2.7.21. *Let G be a bipartite graph. Then number of edges in a maximum matching of G is equal to the number of vertices in minimum edge cover.*

Proof. Let M be a maximum matching for G and C a minimum edge cover of G. Note that C must be contain a vertex from each edge of M and since the edge of M are disjoint $|C| \geq \mathrm{E}(M)$. So to complete the proof we just need to find an some edge covering D of G with $|D| = |\mathrm{E}(M))$.

Let (X, Y) be an bipartition of G. Define W, S, T and T^+ as in 2.7.9. Then $W = X \setminus \mathrm{V}(M)$, $S = W \cup T^+$, $T^+ = \mathrm{V}(M) \cap S$ and $|T| = |T^+|$. Put $D = (X \setminus S) \cup T$. Then $X \setminus S = (X \cap \mathrm{V}(M)) \setminus T^+$ and $|D| = |X \cap \mathrm{V}(M)| = |\mathrm{E}(M)$.

It remains to show that D is an edge cover of G. For this let $e = xy$ be an edge of G with $x \in X$ and $y \in Y$. By 2.7.9 $\mathrm{N}(S) = T$. So if $x \in S$, then $y \in T \subseteq D$ and if $x \notin S$, then $x \in X \setminus S \subseteq D$. In either case e is incident with a vertex in D and so D is indeed an edge cover. □

Bibliography

[1] J. Harris, J. Hirst, M. Mossinghoff, *Combinatorics and Graph Theory* second edition, Springer Undergraduate Texts in Mathematics **2008**.

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