MTH 309-4 **Linear Algebra I** **F11**

Recitation 12

(A) Let A be an $n \times n$ matrix and $(v_1, \dots, v_n)$ a linearly independent list of eigenvectors of A, so $(v_1, \dots, v_n)$ is a basis for $\mathbb{R}^n$ and for all $1 \leq i \leq n$ there exists $\lambda_i \in \mathbb{R}$ with $Av_i = \lambda_i v_i$. Set

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & \ddots & 0 & 0 & 0 \\ 0 & 0 & \lambda_3 & \ddots & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & \lambda_{n-2} & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 & \lambda_{n-1} & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & \lambda_n \end{bmatrix}$$

and

$$P = [v_1, \dots, v_n]$$

that is, the ith column of P is the vector v_i . Prove:
(A1) P is an invertible matrix.
(A2) $PD = AP$
(A3) A and D are similar matrices.

(B) Let

$$A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}.$$

(B1) Find the characteristic polynomial of A.
(B2) Find the eigenvalues of A.
(B3) Find bases of the eigenspaces of A.
(B4) Find a diagonal matrix D and an invertible matrix P such that $D = P^{-1}AP$.

(C) Let $T : \mathbf{V} \to \mathbf{V}$ be a linear function and $f = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ a polynomial. For k a non-negative integer define T^k inductively by $T^0 = \mathrm{id}_V$ and $T^{k+1} = T^k \circ T$. Let $f(T) : V \to V$ be the function $a_n T^n + a_{n-1}T^{n-1} + \ldots + a_1 T + \mathrm{id}_V$. Prove that:

(C1) $f(T) : \mathbf{V} \to \mathbf{V}$ is linear.

(C2) If v is an eigenvector of T associated to $\lambda \in \mathbb{R}$, then v is an eigenvector of $f(T)$ associated to $f(\lambda)$.

(D) Let $T : \mathbf{V} \to \mathbf{V}$ be a linear function with $\dim \mathrm{Im}T = k < \infty$, Prove that T has at most $k+1$ distinct eigenvalues.