MTH 309-4 **Linear Algebra I** **F11**

Recitation 10

Remark: You are allowed to use the following Theorem:

Theorem 6.27 *Let $T : V \to W$ a linear. If V is finite dimensional, then also* $\mathrm{Im}T$ *is finite dimensional and*

$$\dim \mathrm{Im}T + \dim \ker T = \dim V$$

(This is a fairly easy consequence of $\dim \mathrm{Col}A + \dim \mathrm{Nul}A = n$ for all $m \times n$-matrices A and will be proved in class sometime this week)

(A) Let V and W be finite dimensional vector spaces and $T : V \to W$ a linear function. If $S : W \to W$ is an isomorphism prove:

(a) $\ker(S \circ T) = \ker(T)$.

(b) $\dim(\mathrm{Im}(S \circ T)) = \dim(\mathrm{Im}(T))$.

(c) Give an example of a linear function $T : V \to W$ and an isomorphism $S : W \to W$ with $\mathrm{Im}(S \circ T)) \neq \mathrm{Im}(T)$.

(B) Let V and W be finite dimensional vector spaces and $T : V \to W$ a linear function. If $L : V \to V$ is an isomorphism prove:

(a) $\mathrm{Im}(T \circ L) = \mathrm{Im}(T)$.

(b) $\dim(\ker(T \circ L)) = \dim(\ker(T))$.

(c) Give an example of a linear function $T : V \to W$ and an isomorphism $L : V \to V$ with $\ker(T \circ L) \neq \ker(T)$.

(C) Let V and W be finite-dimensional vector spaces with ordered bases $B = (v_1, \dots, v_n)$ of V and $B' = (u_1, \dots, u_m)$ of W. Let $T : V \to W$ be a linear function. Consider the diagram

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, "C_B"'] & W \arrow[d, "C_{B'}"] \\ S & T \end{tikzcd}$$

and prove

(a) There exists a unique $m \times n$ matrix A such that

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, "C_B"'] & W \arrow[d, "C_{B'}"] \\ S \arrow[r, "L_A"'] & T \end{tikzcd}$$

commutes.

(b) The ith column of A is the coordinate vector $[T(v_i)]_{B'}$, that is, A is the matrix of T relative to bases B and B'.

(c) T is 1-1 if and only if L_A is 1-1.

(d) T is onto if and only if L_A is onto.

(e) T has an inverse function if and only if the matrix A is invertible.

(D) Let V and W be finite-dimensional vector spaces. Show that there exists an onto linear function $T : V \to W$ if and only if $\dim W \leq \dim V$.