MTH 309-4 **Linear Algebra I** **F11**

Recitation 9

(A) Consider the linear function $L_A : \mathbb{R}^4 \to \mathbb{R}^3, x \to Ax$, where

$$A = \begin{bmatrix} 1 & 3 & 1 & 1 \\ 2 & -2 & 1 & 2 \\ -1 & -5 & 0 & 1 \end{bmatrix}$$

(A1) Recall that the kernel of L_A is defined to be

$$\ker L_A = \{v \in \mathbb{R}^4 | L_A(v) = \mathbf{0}\}.$$

and that $\ker L_A$ is a subspace of $\mathbb{R}^4$. Verify that the kernel of L_A is exactly the solution set of the homogeneous linear system of equations $Ax = \mathbf{0}$. Find a basis of $\ker L_A$.

(A2) Say your basis of L_A is the list $(u_1, \ldots, u_i)$. Extend this list to a basis (u_1, u_2, u_3, u_4) of $\mathbb{R}^4$.

(A3) Recall that the image of L_A is defined to be

$$\mathrm{Im}L_A = \{L_A(v) \mid v \in \mathbb{R}^4\}.$$

and that $\mathrm{Im}L_A$ is a subspace of $\mathbb{R}^3$
(i) Prove that $\mathrm{Im}L_A$ is the subspace of $\mathbb{R}^3$ spanned by the columns of A.
(ii) Prove that the list $\big(L_A(u_{i+1}), \dots, L_A(u_4)\big)$ is a basis of $\mathrm{Im}L_A$.
(iii) Compute $\dim(\ker L_A) + \dim(\mathrm{Im}L_A)$.

(B) A diagram of sets and maps

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle h}\big\downarrow & & \big\downarrow{\scriptstyle g} \\ S & \xrightarrow[\ell]{} & T \end{array}$$

is called commutative if $g \circ f = \ell \circ h$. Suppose that the given diagram is commutative and that the maps h and g are 1-1 and onto. Show:

(a) f is 1-1 if and only if is ℓ is 1-1.
(b) f is onto if and only if ℓ is onto.
(c) f is 1-1 and onto if and only if ℓ is 1-1 and onto.

(C) Let X, Y, S, and T be sets and $h : X \to S$ and $g : Y \to T$ $1-1$ and onto maps. Let $f : X \to Y$ be a map. Show that there is a unique map $\ell : S \to T$ such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{\;f\;} & Y \\ {\scriptstyle h}\big\downarrow & & \big\downarrow{\scriptstyle g} \\ S & \xrightarrow[\;\ell\;]{} & T \end{array}$$

is commutative.