MTH 309-4 **Linear Algebra I** **F11**

Recitation 7

(A) Let V be a finite dimensional vector space, $U \subseteq V$ a subspace of V, and $S : U \to W$ a linear function from U to a vector space W. Show that there is a linear function $T : V \to W$ with $T(u) = S(u)$ for all $u \in U$.

(B) Let $T : \mathbb{P}_3 \to \mathbb{R}^2$ be the function defined by

$$T(p) = \begin{bmatrix} p(0) \\ p(1) \end{bmatrix}$$

for all p in $\mathbb{P}_3$.
(a) Show that T is a linear function.
(b) Show that the set

$$\ker(T) = \{p \in \mathbb{P}_3 \mid T(p) = \mathbf{0}\}$$

is a subspace of $\mathbb{P}_3$ and find a basis of $\ker T$.

(C) Let V and W be vector spaces with bases $B_V = (v_1, \ldots, v_n)$ and $B_W = (w_1, \ldots, w_m)$
Consider the set of linear functions from V to W:

$$\mathbb{L}(V, W) = \{T : V \to W \mid T \text{ a linear function}\}.$$

$\mathbb{L}(V, W)$ is a vector space if we define addition and scalar multiplication as follows:
For all $T, S \in \mathbb{L}(V, W)$, $r \in \mathbb{R}$, and $v \in V$:

$$(T + S)(v) = T(v) + S(v) \text{ and } (rT)(v) = r(T(v))$$

(a) Prove that for all $1 \leq i \leq n$ and all $1 \leq j \leq m$ there is a unique linear function

$$S_{ij} : V \to W \text{ with } S_{ij}(v_k) = \begin{cases} w_j & \text{if } k = i \\ 0 & \text{if } k \neq i \end{cases} \qquad \text{for all } 1 \leq k \leq n$$

(b) Prove that the list

$$B = (S_{ij})_{1 \le i \le n; 1 \le j \le m} = (S_{11}, \dots, S_{1m}, S_{21}, \dots S_{2m}, \dots, S_{n1}, \dots, S_{nm})$$

is linearly independent in $\mathbb{L}(V, W)$.

(c) Let B be as in (b). Prove that B spans $\mathbb{L}(V, W)$.

(d) Prove that $\dim \mathbb{L}(V, W) = nm$.