MTH 309-4 **Linear Algebra I** **F11**

Recitation 6

Remark: You are allowed to use the fact that the length of any basis in a finite dimensional vector space V is equal to the dimension of V

Consider the following subspaces of $\mathbb{R}^5$:

$$U = \operatorname{span}\big((-1,-3,0,1,1),(3,5,8,-3,1),(-2,-6,4,2,1)\big).$$

and

$$W = \operatorname{span}\big((0,-1,2,1,1),(0,-4,12,0,3)\big).$$

(A) Recall that the intersection of two subspaces of a vector space is again a subspace. Find a basis B_0 of $U \cap W$.

(B) Extend the basis $B_0 = (v_1, \ldots, v_n)$ of $U \cap W$ to a basis $B_1 = (v_1, \ldots, v_n, u_1, \ldots, u_k)$ of U and a basis $B_2 = (v_1, \ldots, v_n, w_1, \ldots, w_l)$ of W. Show that

$$(v_1, \ldots, v_n, u_1, \ldots, u_k, w_1, \ldots, w_l)$$

is basis for $U + W$. (Recall that the set $U + W$ is defined by

$$U + W = \{u + w | u \in U \text{ and } w \in W\}$$

It is known that $U + W$ is a subspace.)

(C) Verify the formula:

$$\dim(U + W) = \dim(U) + \dim(W) - \dim(U \cap W).$$

(D) The formula of (C) holds true in general, that is, the following theorem can be proved:

Theorem. *Let V be a finite dimensional vector space and U and W subspaces of V. Then*

$$\dim(U + W) = \dim(U) + \dim(W) - \dim(U \cap W).$$

Discuss strategies how one could prove this theorem.

(E) Let $U, W \subseteq \mathbb{R}^8$ be subspaces. Prove:
(a) If $\dim(U) = 3$, $\dim(W) = 5$, and $U + W = \mathbb{R}^8$, then $U \cap W = \{\mathbf{0}\}$.
(b) If $\dim(U) = \dim(W) = 5$, then $U \cap W \neq \{\mathbf{0}\}$.

(F) True or False? Justify your answer:
Let V be a finite dimensional vector space.

(F1) If $B = (v_1, \ldots, v_n)$ is a linearly independent list in V, then every spanning list of V has at length at least n.
True - False?
REASON:

(F2) Every vector space is finite dimensional.
True - False?
REASON:

(F3) If a finite list of nonzero vectors spans V then some sublist of B is a basis of V.
True - False?
REASON:

(F4) If dim $V \geq 1$ then V has infinitely many different bases.
True - False?
REASON:

(F5) Let $S, T \subseteq V$ be subspaces of V with $V = S + T$. Let $(v_1, \ldots, v_n, u_1, \ldots, u_k)$ be basis of S and $(v_1, \ldots, v_n, w_1, \ldots, w_l)$ a basis of T. Suppose that $u_i \neq w_j$ for all $1 \leq i \leq k$ and $1 \leq j \leq l$. Then

$$(v_1, \ldots, v_n, u_1, \ldots, u_k, w_1, \ldots, w_l)$$

is a basis of V.
True - False?
REASON:

(F6) If the list $(v_1, \ldots, v_p)$ spans V and if T is a list of length larger than p, then T is linearly dependent.
True - False?
REASON: