MTH 309-4 **Linear Algebra I** **F11**

Recitation 4

(A) Consider the vectors $v_1 = (1,2,1)$, $v_2 = (0,1,2)$, $v_3 = (2,1,0)$, $v_4 = (1,0,2)$ and $w = (1,2,3)$ in $\mathbb{R}^3$.
(a) Write w in two different ways as a linear combination of (v_1, v_2, v_3, v_4).
(b) Prove that there are infinitely many different ways to write w as a linear combination of (v_1, v_2, v_3, v_4).
(Hint to (b): Can you write the zero vector in different ways as a linear combination (v_1, v_2, v_3, v_4)?)

(B) Generalize the statement of (A) by showing:
Let V be a vector space and $v_1, v_2, \ldots, v_n \in V$. If a vector $w \in \operatorname{span}(v_1, \ldots, v_n)$ can be written as a linear combination of $(v_1, \ldots, v_n)$ in two different ways, then there are infinitely many different ways to write w as a linear combination of $(v_1, \ldots, v_n)$.

(C) Let V be a vector space and $v_1, v_2, \ldots, v_n \in V$. Prove that if $v_i \in \text{span}(v_1, \ldots v_{i-1}, v_{i+1}, \ldots, v_n)$ for some $1 \leq i \leq n$, then

$$\text{span}(v_1, \ldots, v_n) = \text{span}(v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n)$$

In particular, if $v_i = 0$ for some $1 \leq i \leq n$, then

$$\text{span}(v_1, \ldots, v_n) = \text{span}(v_1, \ldots v_{i-1}, v_{i+1}, \ldots, v_n)$$

(D) Let V be a vector space and $u_1, u_2, \ldots, u_m$, $v_1, v_2, \ldots, v_n$ vectors in V. Show:
(a) If $\{u_1, \ldots, u_m\} \subseteq \text{span}(v_1, \ldots, v_n)$, then

$$\text{span}(u_1, \ldots, u_m) \subseteq \text{span}(v_1, \ldots, v_n)$$

(b)

$$\text{span}(v_1, \ldots, v_n) = \text{span}(v_1 - v_2, v_2 - v_3, \ldots, v_{n-1} - v_n, v_n)$$

(E) Let V be a vector space and $u_1, u_2, \ldots, u_m$, $v_1, v_2, \ldots, v_n \in V$. Prove:

$$\operatorname{span}(u_1, \ldots, u_m, v_1, \ldots, v_n) = \operatorname{span}(u_1, \ldots, u_m) + \operatorname{span}(v_1, \ldots, v_n)$$