MTH 309-4 **Linear Algebra I** **F11**

Homework 12/Solutions

Section	Exercises
8.1	5,6,7,8,13,14,15,16
8.2	4,9,15

(**Section 8.1 Exercise 5**). *Suppose λ is an eigenvalue of a linear operator $T : \mathbb{R}^n \to \mathbb{R}^n$. Let A be the matrix of T relative to the standard basis for $\mathbb{R}^n$. Show that the eigenspace $E_T(\lambda)$ of the operator T is equal to the eigenspace $E_A(\lambda)$ of the matrix A.*

By definition

$$E_T(\lambda) = \{x \in \mathbb{R}^n \mid T(x) = \lambda x\} \qquad \text{and} \qquad E_A(\lambda) = \{x \in \mathbb{R}^n \mid Ax = \lambda x\}$$

By Theorem 6.10, $T(x) = Ax$ for all $x \in \mathbb{R}^n$ and so $E_T(\lambda) = E_A(\lambda)$.

(**Section 8.1 Exercise 6**). *Prove that if A is an $n \times n$-matrix, then* $\det(\lambda I - A)$ *defines an nth-degree polynomial in the variable λ.*

We will first prove:

(*) *Let A and B be $n \times n$ matrices. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(\lambda) = \det(\lambda B - A)$. Then f is a polynomial of degree at most n.*

The proof is by induction on n. If $n = 0$, then $f(\lambda) = \det[] = 1$ for all $\lambda \in R$ and so f is a polynomial of degree 0. So (*) holds for $n = 0$. Suppose now that (*) holds for any $n-1 \times n-1$-matrices. Let A and B be $n \times n$-matrices. For $1 \leq j \leq n$ define $f_j : \mathbb{R} \to \mathbb{R}$ by

$$f_j(\lambda) = \det(\lambda B_{1j} - A_{1j})$$

By the induction assumption f_j is a polynomial of degree at most $n - 1$.
Let $\lambda \in R$ and put $C\lambda B - A$. Then $c_{1j} = \lambda b_{1j} - a_{ij}$ and $C_{1j} = \lambda B_{1j} - A_{1j}$. Thus

$$\begin{aligned} f(\lambda) &= \det(\lambda B - A) = \det(C) \\ &= \sum_{j=1}^{n} (-1)^{1+j} c_{1j} \det(C_{ij}) \\ &= \sum_{j=1}^{n} (-1)^{1+j} c_{1j} \det(\lambda B_{1j} - A_{1j}) \\ &= \sum_{j=1}^{n} (-1)^{1+j} (\lambda b_{1j} - a_{1j}) f_j(\lambda) \end{aligned}$$

Thus

$$(**) \qquad f(\lambda) = \sum_{j=1}^{n} (-1)^{1+j}(\lambda b_{1j} - a_{1j}) f_j(\lambda)$$

Since f_j is polynomial of degree at most n, $(-1)^{1+j}(\lambda b_{1j} - a_{1j}) f_j(\lambda)$ defines a polynomial of degree of at most n. The sums of polynomials of degree at most n is a polynomial of degree at most n. So (**) shows that f is a polynomial of degree at most n.

The principal of induction now shows that (*) holds for all non-negative integers n.

Consider now the case $B = I$. By (*) f is polynomial of degree at most n. It remains to show that

(***) If $B = I$, then f has degree n.

For $n = 0$, we have $f = 1$ and so (***) holds for $n = 0$. Suppose now that (***) holds for $n-1$. Note that $(I_n)_{11} = I_{n-1}$ and so $f_1(\lambda) = \det(\lambda I_{n-1} - A_{11})$. Thus by the induction assumption f_1 has degree $n-1$. Since $B = I$, $b_{11} = 1$ and $b_{1j} = 0$ for $1 \leq j \leq n$. So (**) shows

$$(****) \qquad f(\lambda) = (\lambda - a_{11}) f_1(\lambda) + \sum_{j=2}^{n} (-1)^j a_{1j} f_j(\lambda)$$

$a_{1j} f_j(\lambda)$ defines is a polynomial of degree at most $n-1$ and so also $\sum_{j=2}^{n} (-1)^j a_{1j} f_j(\lambda)$ defines a polynomial of degree at most $n-1$. Since f_1 has degree $n-1$, $(\lambda - a_{11}) f_1(\lambda)$ defined a polynomial of degree n. (****) shows that f is a polynomial of degree n.

The principal of induction now shows that (***) holds for all non-negative integers n.

(**Section 8.1 Exercise 8**). *Find the eigenvalues of the following matrices. For each eigenvalue, find a basis for the corresponding eigenspace.*

(a) $\begin{bmatrix} 6 & -24 & -4 \\ 2 & -10 & -2 \\ 1 & 4 & 1 \end{bmatrix}$.

(b) $\begin{bmatrix} 7 & -24 & -6 \\ 2 & -7 & -2 \\ 0 & 0 & 1 \end{bmatrix}$.

(c) $\begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}$.

(d) $\begin{bmatrix} 3 & -7 & -4 \\ -1 & 9 & 4 \\ 2 & -14 & -6 \end{bmatrix}$.

(e) $\begin{bmatrix} -1 & -1 & 10 \\ -1 & -1 & 6 \\ -1 & -1 & 6 \end{bmatrix}$.

(f) $\begin{bmatrix} \frac{1}{2} & -5 & 5 \\ \frac{3}{2} & 0 & -4 \\ \frac{1}{2} & -1 & 0 \end{bmatrix}$.

(a)

$$\det\begin{bmatrix}\lambda-6 & 24 & 4\\ -2 & \lambda+10 & 2\\ -1 & -4 & \lambda-1\end{bmatrix}$$

$$\begin{aligned}
&= (\lambda-6)\Big((\lambda+10)(\lambda-1)-(-4)2\Big)-24\Big(-2(\lambda-1)-(-1)2\Big)+4\Big(-2(-4)-(-1)(\lambda+10)\Big)\\
&= (\lambda-6)(\lambda^2+9\lambda-2)-24(-2\lambda+4)+4(\lambda+18)\\
&= \lambda^3-6\lambda^2+9\lambda^2-54\lambda-2\lambda+12+48\lambda-96+4\lambda+72\\
&= \lambda^3+3\lambda^2-4\lambda-12\\
&= (\lambda^2-4)(\lambda+3)\\
&= (\lambda-2)(\lambda+2)(\lambda+3)
\end{aligned}$$

And so the eigenvalues are

$$\lambda=2,\quad \lambda=-2 \text{ and } \lambda=-3$$

We now use the Gauss Jordan Algorithm to find a basis for $E_A(\lambda)=\text{Nul}(\lambda I-A)$ for each of the three eigenvalues.

For $\lambda=2$:

$$\begin{bmatrix}2-6 & 24 & 4\\ -2 & 2+10 & 2\\ -1 & -4 & 2-1\end{bmatrix}=\begin{bmatrix}-4 & 24 & 4\\ -2 & 12 & 2\\ -1 & -4 & 1\end{bmatrix}\xrightarrow[\substack{-\frac{1}{2}R2\to R2\\ -R3\to R3}]{-\frac{1}{4}R1\to R1}\begin{bmatrix}1 & -6 & -1\\ 1 & -6 & -1\\ 1 & 4 & -11\end{bmatrix}$$

$$\xrightarrow[\substack{R3-R1\to R2\\ R2\leftrightarrow R3}]{R2-R1\to R2}\begin{bmatrix}1 & -6 & -1\\ 0 & -10 & -10\\ 0 & 0 & 0\end{bmatrix}\xrightarrow[R1+6R2\to R1]{-\frac{1}{10}R2\to R2}\begin{bmatrix}1 & 0 & 5\\ 0 & 1 & 1\\ 0 & 0 & 0\end{bmatrix}$$

So $x_1=-5x_3$, $x_2=-x_3$, $x_3=x_3$ and $\Big((-5,-1,1)\Big)$ is a basis for $E_A(2)$.

For $\lambda=-2$:

$$\begin{bmatrix}-2-6 & 24 & 4\\ -2 & -2+10 & 2\\ -1 & -4 & -2-1\end{bmatrix}=\begin{bmatrix}-8 & 24 & 4\\ -2 & 8 & 2\\ -1 & -4 & -3\end{bmatrix}\xrightarrow[\substack{-\frac{1}{2}R2\to R2\\ R1\leftrightarrow R2}]{-\frac{1}{4}R1\to R1}\begin{bmatrix}1 & -4 & -1\\ -2 & 6 & 1\\ -1 & -4 & -3\end{bmatrix}$$

$$\xrightarrow[R3+R1\to R3]{R2+2R1\to R1}\begin{bmatrix}1 & -4 & -1\\ 0 & -2 & -1\\ 0 & -8 & -4\end{bmatrix}\xrightarrow[\substack{R3-4R2\to R3\\ -\frac{1}{2}R2\to R2}]{R1-2R2\to R1}\begin{bmatrix}1 & 0 & 1\\ 0 & 1 & \frac{1}{2}\\ 0 & 0 & 0\end{bmatrix}$$

So $x_1=-x_3$, $x_2=-\frac{1}{2}x_3$, $x_3=x_3$ and (choosing $x_3=-2$) $\Big((2,1,-2)\Big)$ is a basis for $E_A(-2)$.

For $\lambda = -3$:

$$\begin{bmatrix} -3-6 & 24 & 4 \\ -2 & -3+10 & 2 \\ -1 & -4 & -3-1 \end{bmatrix} = \begin{bmatrix} -9 & 24 & 4 \\ -2 & 7 & 2 \\ -1 & -4 & -4 \end{bmatrix} \xrightarrow[R1 \leftrightarrow R3]{-R3 \to R3} \begin{bmatrix} 1 & 4 & 4 \\ -2 & 7 & 2 \\ -9 & 24 & 4 \end{bmatrix}$$

$$\xrightarrow[R3+9R1 \to R3]{R2+2R1 \to R2} \begin{bmatrix} 1 & 4 & 4 \\ 0 & 15 & 10 \\ 0 & 60 & 40 \end{bmatrix} \xrightarrow[\frac{1}{15}R2 \to R2]{R3-3R2 \to R2} \begin{bmatrix} 1 & 4 & 4 \\ 0 & 1 & \frac{2}{3} \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R1-4R2 \to R1} \begin{bmatrix} 1 & 0 & \frac{4}{3} \\ 0 & 1 & \frac{2}{3} \\ 0 & 0 & 0 \end{bmatrix}$$

So $x_1 = -\frac{4}{3}x_3$, $x_2 = -\frac{2}{3}x_3$, $x_3 = x_3$ and (choosing $x_3 = -3$) $\Big((4, 2, -3)\Big)$ is a basis for $E_A(2)$.

(b)

$$\begin{aligned} &\det \begin{bmatrix} \lambda-7 & 24 & 6 \\ -2 & \lambda+7 & 2 \\ 0 & 0 & \lambda-1 \end{bmatrix} \\ = &\ (\lambda-1)\Big((\lambda-7)(\lambda+7) - (-2)24\Big) \\ = &\ (\lambda-1)(\lambda^2-1) \\ = &\ (\lambda-1)^2(\lambda+1) \end{aligned}$$

And so the eigenvalues are

$$\lambda = 1, \quad \lambda = -1$$

We now use the Gauss Jordan Algorithm to find a basis for $E_A(\lambda) = \text{Nul}(\lambda I - A)$ for each of the two eigenvalues.

$\lambda = 1$

$$\begin{bmatrix} 1-7 & 24 & 6 \\ -2 & 1+7 & 2 \\ 0 & 0 & 1-1 \end{bmatrix} = \begin{bmatrix} -6 & 24 & 6 \\ -2 & 8 & 2 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow[R2+2R1 \to R2]{-\frac{1}{6}R1 \to R1} \begin{bmatrix} 1 & -4 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

So $x_1 = 4x_2 + x_3$, $x_2 = x_2$, $x_3 = x_3$ and $\Big((4, 1, 0), (1, 0, 1)\Big)$ is a basis for $E_A(1)$

$\lambda = -1$

$$\begin{bmatrix} -1-7 & 24 & 6 \\ -2 & -1+7 & 2 \\ 0 & 0 & -1-1 \end{bmatrix} = \begin{bmatrix} -8 & 24 & 6 \\ -2 & 6 & 2 \\ 0 & 0 & -2 \end{bmatrix} \xrightarrow{\substack{\frac{1}{2}R1 \to R1 \\ -\frac{1}{2}R2 \to R2 \\ -\frac{1}{2}R3 \to R3 \\ R1 \leftrightarrow R2}} \begin{bmatrix} 1 & -3 & -1 \\ -4 & 12 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R2+4R1 \to R2} \begin{bmatrix} 1 & -3 & -1 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{\substack{R1-R2 \to R1 \\ R3+R2 \to R3 \\ -R2 \to R2}} \begin{bmatrix} 1 & -3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

So $x_1 = 3x_2$, $x_2 = x_2$, $x_3 = 0$ and $\Big((3,1,0)\Big)$ is a basis for $E_A(-1)$

(c)

$$\det \begin{bmatrix} \lambda-3 & -1 & 0 \\ 0 & \lambda-3 & -1 \\ 0 & 0 & \lambda-3 \end{bmatrix} = (\lambda-3)\Big((\lambda-3)(\lambda-3)-0(-1)\Big) = (\lambda-3)^3$$

and so $\lambda = 3$ is the only eigenvalue. We now use the Gauss Jordan Algorithm to find a basis for $E_A(3) = \mathrm{Nul}(3I - A)$.

$$\begin{bmatrix} 3-3 & -1 & 0 \\ 0 & 3-3 & -1 \\ 0 & 0 & 3-3 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow[-R2 \to R2]{-R1 \to R1} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

So $x_1 = x_1$, $x_2 = 0$, $x_3 = 0$ and $\Big((1,0,0)\Big)$ is a basis for $E_A(3)$

(d)

$$\det \begin{bmatrix} \lambda-3 & 7 & 4 \\ 1 & \lambda-9 & -4 \\ -2 & 14 & \lambda+6 \end{bmatrix}$$

$$\begin{aligned}
&= (\lambda-3)\Big((\lambda-9)(\lambda+6)-14(-4)\Big) - 7\Big(1(\lambda+6)-(-2)(-4)\Big) + 4\Big(1(14)-(-2)(\lambda-9)\Big) \\
&= (\lambda-3)(\lambda^2-3\lambda+2) - 7(\lambda-2) + 4(2\lambda-4) \\
& (\lambda-3)(\lambda-1)(\lambda-2) - 7(\lambda-2) + 8(\lambda-2) \\
&= (\lambda^2-4\lambda+3-7+8)(\lambda-2) \\
&= (\lambda^2-4\lambda+4)(\lambda-2) \\
&= (\lambda-2)^3
\end{aligned}$$

and so $\lambda = 2$ is the only eigenvalue. We now use the Gauss Jordan Algorithm to find a basis for $E_A(2) = \mathrm{Nul}(2I - A)$.

$$\begin{bmatrix} 2-3 & 7 & 4 \\ 1 & 2-9 & -4 \\ -2 & 14 & 2+6 \end{bmatrix} = \begin{bmatrix} -1 & 7 & 4 \\ 1 & -7 & -4 \\ -2 & 14 & 8 \end{bmatrix} \xrightarrow[\substack{R1+R2 \to R2 \\ -R1 \to R1}]{R3+2R2 \to R2} \begin{bmatrix} 1 & -7 & -4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

So $x_1 = 7x_2 + 4x_3$, $x_2 = x_2$, $x_3 = x_3$ and $\Big((7,1,0),(4,0,1)\Big)$ is a basis for $E_A(2)$

(e)

$$\begin{bmatrix} \lambda+1 & 1 & -10 \\ 1 & \lambda+1 & -6 \\ 1 & 1 & \lambda-6 \end{bmatrix} \xrightarrow[R1-R2 \to R1]{R3-R2 \to R3} \begin{bmatrix} \lambda & -\lambda & -4 \\ 1 & \lambda+1 & -6 \\ 0 & -\lambda & \lambda \end{bmatrix} \xrightarrow[\substack{R1 \leftrightarrow R2 \\ -R3 \to R3}]{R1-R3 \to R1}$$

$$\begin{bmatrix} 1 & \lambda+1 & -6 \\ \lambda & 0 & -\lambda-4 \\ 0 & \lambda & -\lambda \end{bmatrix} \xrightarrow[R2 \leftrightarrow R3]{R2 - \lambda R1 \to R2} \begin{bmatrix} 1 & \lambda+1 & -6 \\ 0 & \lambda & -\lambda \\ 0 & -\lambda^2-\lambda & 5\lambda-4 \end{bmatrix}$$

$$\xrightarrow{R3 + (\lambda+1)R2 \to R3} \begin{bmatrix} 1 & \lambda+1 & -6 \\ 0 & \lambda & -\lambda \\ 0 & 0 & -(\lambda-2)^2 \end{bmatrix}$$

So the eigenvalues are $\lambda = 0$ and $\lambda = 2$. We now continue to use the Gauss Jordan Algorithm to find a basis for $E_A(\lambda) = \mathrm{Nul}(\lambda I - A)$ for each of the two eigenvalues.

For $\lambda = 0$.

$$\begin{bmatrix} 0+1 & 0+1 & -6 \\ 0 & 0 & -0 \\ 0 & 0 & -(0-2)^2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -6 \\ 0 & 0 & 0 \\ 0 & 0 & -4 \end{bmatrix} \xrightarrow[\substack{R1 + 6R3 \to R1 \\ R2 \leftrightarrow R3}]{R3 + (\lambda+1) - \frac{14}{R}3 \to R3} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

So $x_1 = -x_2$, $x_2 = x_2$, $x_3 = 0$ and $\Big((-1,1,0)\Big)$ is a basis for $E_A(0)$

For $\lambda = 2$

$$\begin{bmatrix} 1 & 2+1 & -6 \\ 0 & 2 & -2 \\ 0 & 0 & -(2-2)^2 \end{bmatrix} = \begin{bmatrix} 1 & 3 & -6 \\ 0 & 2 & -2 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow[R1 - 3R2 \to R1]{\frac{1}{2}R2 \to R2} \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

So $x_1 = 3x_3$, $x_2 = x_3$, $x_3 = x_3$ and $\Big((3,1,1)\Big)$ is a basis for $E_A(2)$

(f)

$$\det \begin{bmatrix} \lambda - \frac{1}{2} & 5 & -5 \\ -\frac{3}{2} & \lambda & 4 \\ -\frac{1}{2} & 1 & \lambda \end{bmatrix}$$

$$\begin{aligned}
&= \frac{1}{2} \det \begin{bmatrix} 2\lambda-1 & 5 & -5 \\ -3 & \lambda & 4 \\ -1 & 1 & \lambda \end{bmatrix} \\
&= \frac{1}{2}\Bigg((2\lambda-1)\Big(\lambda^2-4\Big) - 5\Big((-3)\lambda - (-1)4\Big) + (-5)\Big((-3)1 - (-1)\lambda\Big)\Bigg) \\
&= \frac{1}{2}\Big((2\lambda^3 - \lambda^2 - 8\lambda + 4) - 5(4 - 3\lambda) - 5(\lambda - 3)\Big) \\
&= \frac{1}{2}\Big(2\lambda^3 - \lambda^2 + 2\lambda - 1\Big) \\
&= \frac{1}{2}(2\lambda - 1)(\lambda^2 + 1)
\end{aligned}$$

and so $\lambda = \frac{1}{2}$ is the only eigenvalue. We now use the Gauss Jordan Algorithm to find a basis for $E_A(\frac{1}{2}) = \mathrm{Nul}(\frac{1}{2}I - A)$.

$$\begin{bmatrix} 2\frac{1}{2}-1 & 5 & -5 \\ -3 & \frac{1}{2} & 4 \\ -1 & 1 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 0 & 5 & -5 \\ -3 & \frac{1}{2} & 4 \\ -1 & 1 & \frac{1}{2} \end{bmatrix} \begin{matrix} \frac{1}{5}R1 \to R1 \\ -R3 \to R3 \\ R1 \leftrightarrow R3 \\ R2 \leftrightarrow R3 \end{matrix} \begin{bmatrix} 1 & -1 & -\frac{1}{2} \\ 0 & 1 & -1 \\ -3 & \frac{1}{2} & 4 \end{bmatrix}$$

$$\begin{matrix} R3+3R1 \to R3 \\ R1+R2 \to R2 \end{matrix} \begin{bmatrix} 1 & 0 & -\frac{3}{2} \\ 0 & 1 & -1 \\ 0 & -\frac{5}{2} & \frac{5}{2} \end{bmatrix} \quad R3+\tfrac{5}{2}R2 \to R3 \quad \begin{bmatrix} 1 & 0 & -\frac{3}{2} \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

So $x_1 = \frac{3}{2}x_3$, $x_2 = x_3$, $x_3 = x_3$ and (choosing $x_3 = 2$) $\Big((3,2,2)\Big)$ is a basis for $E_A(3)$

(**Section 8.1 Exercise 8**). *Compute the eigenvalues of the matrices*

(a) $\begin{bmatrix} -5 & 9 \\ 0 & 3 \end{bmatrix}$

(b) $\begin{bmatrix} 7 & 4 & -3 \\ 0 & -3 & 9 \\ 0 & 0 & 2 \end{bmatrix}$.

(c) $\begin{bmatrix} -2 & 8 & 7 & 4 \\ 0 & 3 & 5 & 9 \\ 0 & 0 & 1 & 9 \\ 0 & 0 & 0 & 5 \end{bmatrix}$

(d) $\begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}$

(e) What feature of these matrices makes it relatively easy to compute their eigenvalues?

(f) Formulate a general result suggested by this observation.

(g) Prove your conjecture.

(e) They are upper triangular:

Definition II *An $n \times n$ matrix is called upper triangular if $a_{ij} = 0$ for all $1 \le j < i \le n$.*

(f): **Conjecture III** *Let A be an upper triangular $n \times n$-matrix. Then*

$$\det A = a_{11}a_{22}\ldots a_{nn}$$

Conjecture IV *Let A be an upper triangular $n \times n$-matrix. Then*

$$\chi_A(\lambda) = (\lambda - a_{11})(\lambda - a_{22})\ldots(\lambda - a_{nn}).$$

and so the eigenvalues of A are

$$\lambda = a_{11}, \quad \lambda = a_{22}, \quad \ldots, \quad \lambda = a_{nn}$$

.

(g): *Proof of Conjecture III*: If $n = 1$ then $\det A = a_{11}$ and Conjecture III holds. (Conjecture III also holds for $n = 0$ as long as one defines the product of an empty list of elements to be 1)

Suppose now that Conjecture III holds for all upper triangular $(n-1)\times(n-1)$-matrices. Let A be an upper triangular $n \times n$-matrix. Expanding the determinant along row n we get

$$\det A = \sum_{j=1}^{n}(-1)^{n+j} a_{nj} \det A_{nj}.$$

Since A is upper triangular, $a_{nj} = 0$ for all $1 \leq j < n$. Also $(-1)^{n+n} = 1$ and so

$$\det A = a_{nn} \det A_{nn} = (\det A_{nn}) a_{nn}$$

Since A is upper triangular, $a_{ij} = 0$ for all $1 \leq j < i \leq n-1$. So A_{nn} is upper triangular, and the $\det A_{nn} = a_{11}a_{22}\ldots a_{(n-1)(n-1)}$ by the induction assumption. Hence

$$\det A = (\det A_{nn})a_{nn} = a_{11}a_{22}\ldots a_{(n-1)(n-1)}a_{nn}$$

So if Conjecture III holds for $n-1$ it also holds for n. Thus by the principal of induction, the Conjecture holds for all positive integers n.

Proof of Conjecture IV: Let A be an upper triangular matrix and $\lambda \in \mathbb{R}$. Put $B = \lambda I - A$ Note that λI is upper triangular and so also B is upper triangular. Thus by Conjecture III

$$\det B = b_{11}\ldots b_{nn}$$

We have $b_{ii} = \lambda - a_{ii}$ and so

$$\chi_A(\lambda) = \det(\lambda I - A) = \det B = b_{11}\ldots b_{nn} = (\lambda - a_{11})(\lambda - a_{22})\ldots(\lambda - a_{nn})$$

Since the eigenvalues of A are the roots of χ_A, we conclude that the eigenvalues of A are $\lambda = a_{11}, \lambda = a_{22}, \ldots, \lambda = a_{nn}$ and Conjecture IV is proved.

The proven conjecture 4 now allows us to solve (a)-(d)

(a) The eigenvalues are -5 and 3.

(b) The eigenvalues are 7, -3 and 2.

(c) The eigenvalues are -2, 3, 1 and 5.
(d) The eigenvalues are a, d and f.

(**Section 8.1 Exercise 13**). *Suppose v is an eigenvector of an $n \times n$ n matrix A associated with the eigenvalue λ. Suppose P is a invertible $n \times n$-matrix. Show that $P^{-1}v$ is an eigenvector of $P^{-1}AP$ associated to λ.*

Since v is an eigenvector of A associated to λ, $v \neq \mathbf{0}$ and $Av = \lambda v$.

Since P^{-1} is invertible, $\text{Nul}P^{-1} = \{z\}$ and since $v \neq \mathbf{0}$ we conclude that $P^{-1}v \neq \mathbf{0}$. Also

$$(P^{-1}AP)(P^{-1}v) = P^{-1}\Big(A\big((PP^{-1})v\big)\Big) = P^{-1}\big(A(Iv)\big) = P^{-1}(Av) = P^{-1}(\lambda v) = \lambda(P^{-1}v)$$

and so $P^{-1}v$ is an eigenvector of $P^{-1}AP$ associated to λ.

(**Section 8.1 Exercise 14**). *Suppose that A and A' are $n \times n$- matrices. Suppose v is an eigenvector of A associated with the eigenvalue λ. Suppose v is also an eigenvector of A' associated with the eigenvalue λ'. Show that v is an eigenvector of $A + A'$ associated with $\lambda + \lambda'$.*

Since v is an eigenvector of A associated with the eigenvalue λ we have $v \neq \mathbf{0}$ and $Av = \lambda v$. And since v is also an eigenvector of A' associated with the eigenvalue λ', $A'v = \lambda' v$. Thus

$$(A + A')v = Av + A'v = \lambda v + \lambda' v = (\lambda + \lambda')v$$

Hence v is an eigenvector of $A + A'$ associated with $\lambda + \lambda'$.

(**Section 8.1 Exercise 15**). *Suppose v is an eigenvector of an $n \times n$ matrix A associated with the eigenvalue λ*

(a) Show that v is an eigenvector of A^2. With what eigenvalue is it associated?

(b) State and prove a generalization of your result in part a to higher powers of A

(c) What can you say about eigenvalues and eigenvectors of A^{-1} and other negative powers of A.

(b) We will prove:

(*) Let n be a non-negative integer. Then v is an eigenvector of A^n associated to λ^n.

We have $A^0 v = Iv = v = 1v = \lambda^0 v$ and so (*) holds for $n = 0$. Suppose (*) holds for n. Then $A^n v = \lambda^n v$. Also since v is an eigenvector of A associated to λ, $v \neq \mathbf{0}$ and $Av = \lambda v$. Thus

$$A^{n+1}v = (A^n A)v = A^n(Av) = A^n(\lambda v) = \lambda(A^n v) = \lambda(\lambda^n v) = (\lambda\lambda^n)(v) = \lambda^{n+1}v$$

Hence v is an eigenvector of A^{n+1} associated to λ^{n+1}. Thus (*) holds for $n + 1$ and by the Principal of Induction, (*) holds for all non-negative integers n.

(a) By (*) applied with $n = 2$, v is an eigenvector of A^2 associated to λ^2.

(c) Suppose now that A is invertible. We will show

(**) $\lambda \neq 0$ and v is an eigenvector of A^{-1} associated to λ^{-1}.

We have

$$\begin{array}{rrcll} & Av & = & \lambda v & \quad - \, v \text{ is an eigenvector of } A \text{ associated to } \lambda \\ \Longrightarrow & A^{-1}(Av) & = & A^{-1}(\lambda v) & \\ \Longrightarrow & (A^{-1}A)v & = & \lambda(A^{-1}v) & \\ \Longrightarrow & Iv & = & \lambda(A^{-1}v) & \\ \Longrightarrow & v & = & \lambda(A^{-1}v) & \end{array}$$

Since $v \neq \mathbf{0}$ and $0x = \mathbf{0}$ for all $x \in R^n$ we conclude that $\lambda \neq 0$. Then multiplying the previous equation by λ^{-1} gives $A^{-1}v = \lambda^{-1}v$. So (**) holds. Next we prove

(***) Let n be an integer. Then v is an eigenvector of A^n associated to λ^n.

From (*) we know that (***) holds if $n \geq 0$. So suppose $n < 0$ and put $m = -n$. By (**) we can apply (*) to m, A^{-1} and λ^{-1} in place of n, A and λ. Thus

$$v \text{ is an eigenvector of } (A^{-1})^m \text{ associated to } (\lambda^{-1})^m.$$

Since $A^n = A^{-m} = (A^{-1})^m$ and $\lambda^n = \lambda^{-m} = (\lambda^{-1})^m$ we conclude that (***) holds.

(**Section 8.1 Exercise 16**). *Suppose $T : V \to V$ and $T' : V \to V$ are linear operators on a vector space V. Suppose v is an eigenvector of T associated with the eigenvalue λ. Suppose v is also an eigenvector of T' associated with the eigenvalue λ'. Show that v is an eigenvector of the composition $T' \circ T$. What is the eigenvalue?*

Since v is an eigenvector of T associated with the eigenvalue λ, $v \neq \mathbf{0}$ and $T(v) = \lambda v$. Since v is an eigenvector of T' associated with the eigenvalue λ', $T'(v) = \lambda' v$. Thus

$$(T' \circ T)(v) = T'\big(T(v)\big) = T'(\lambda v) = \lambda\big(T'(v)\big) = \lambda(\lambda' v) = (\lambda\lambda')v$$

So v is an eigenvector of $T' \circ T$ associated to $\lambda\lambda'$.

(**Section 8.2 Exercise 4**). *(a) Show that* $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ *is similar to* $\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$.

(b) Show that $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 7 \end{bmatrix}$ *is similar to* $\begin{bmatrix} 7 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

(a) Put $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ and $A' = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$. Consider the basis $E' = (e_2, e_1)$ of $\mathbb{R}^2$. Since $L_A(e_2) = Ae_2 = a_2 = 3e_2 = 3e_2 + 0e_1$ and $L_A(e_1) = Ae_1 = a_1 = 2e_1 = 0e_2 + 2e_1$, the matrix of L_A with respect to E' is

$$\Big[[L_A(e_2)]_{E'}, [L_A(e_1)]_{E'}\Big] = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} = A'.$$

Note also that A is the matrix of L_A with respect to the standard basis, and so by Lemma N8.2.2, A' is similar to A.

(b) Put $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 7 \end{bmatrix}$ and $A' = \begin{bmatrix} 7 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$. Consider basis $E' = (e_3, e_1, e_2)$ of $\mathbb{R}^2$. Since $L_A(e_3) = Ae_3 = a_3 = 7e_3 = 7e_3 + 0e_1 + 0e_2$, $L_A(e_1) = Ae_1 = a_1 = 2e_1 = 0e_3 + 2e_1 + 0e_2$, and $L_A(e_2) = Ae_2 = 3a_2 = 3e_2 = 0e_3 + 0e_1 + 3e_2$ the matrix of L_A with respect to E' is

$$\Big[[L_A(e_3)]_{E'}, [L_A(e_1)]_{E'}, [L_A(e_2)]_{E'}\Big] = \begin{bmatrix} 7 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} = A'.$$

Note also that A is the matrix of L_A with respect to the standard basis, and so by Lemma N8.2.2, A' is similar to A

(**Section 8.2 Exercise 9**). *Find two matrices that have the same characteristic polynomial but are not similar.*

Let $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and $A' = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Note that both A and A' are upper triangular and so by Conjecture IV, $\chi_A(\lambda) = (\lambda - 0)(\lambda - 0) = \chi_{A'}(\lambda)$. Thus A and A' have the same characteristic polynomial. Let P be an invertible 2×2-matrix, then $P^{-1}AP = AP = A$ and so $A' \neq P^{-1}AP$. Thus A' is not similar A.

(**Section 8.2 Exercise 15**). *Suppose A is the matrix of a linear map $T : \mathbf{V} \to \mathbf{V}$ relative to the basis B of the n-dimensional vector space V.*

(a) Prove that if v is an eigenvector of T associated to λ then $[v]_B$ is an eigenvector for A associated to λ.

(b) Prove that if $x \in \mathbb{R}^n$ is an eigenvector of A associated to λ then $L_B(x)$ is an eigenvector for T associated to λ.

Let $v \in V$ and put $x \in \mathbb{R}^n$. Since C_B is the inverse of L_B, $x = [v]_B$ if and only of $L_B(x)$. Suppose that $x = [v]_B$ Then

$$\begin{array}{rcll} & Tv & = & \lambda v & \\ \iff & [Tv]_B & = & [\lambda v]_B & \text{– since } C_B \text{ is 1-1} \\ \iff & A[v]_B & = & \lambda[v]_B & \text{– Theorem 6.11 },C_B \text{ is linear} \\ \iff & Ax & = & \lambda x & \text{– since } x = [v]_B \end{array}$$

The forward direction gives (a) and since $v = L_B(x)$, the backward direction gives (b).