Homework 11/Solutions

Section	Exercises
6.8	3,4,5
7.2	1,2b
7.3	10,11,12

(**Section 6.8 Exercise 3**). *Which pairs of the following vector spaces are isomorphic?*

$$\mathbb{R}^7, \mathbb{R}^{12}, \mathbb{M}(3,3), \mathbb{M}(3,4), \mathbb{M}(4,3), \mathbb{P}_6, \mathbb{P}_8, \mathbb{P}_{11}, \mathbb{P}$$

Since $\dim \mathbb{R}^n = n$, $\dim \mathbb{M}(m,n) = mn$ and $\dim \mathbb{P}_n = n+1$ we obtain the following chart:

V	$\mathbb{R}^7$	$\mathbb{R}^{12}$	$\mathbb{M}(3,3)$	$\mathbb{M}(3,4)$	$\mathbb{M}(4,3)$	$\mathbb{P}_6$	$\mathbb{P}_8$	$\mathbb{P}_{11}$	$\mathbb{P}$
$\dim V$	7	12	9	12	12	7	9	12	∞

By Theorem 6.30 two finite dimensional vector spaces are isomorphic if and only if they have the same dimension. Also a finite dimensional vectors space is not isomorphic to an infinite dimensional vector space. So we obtain the following (unordered) pairs of isomorphic vector spaces

$\dim V = 7$: $\{\mathbb{R}^7, \mathbb{P}_6\}$

$\dim V = 9$: $\{\mathbb{R}^9, \mathbb{M}(3,3)\}$

$\dim V = 12$: $\{\mathbb{R}^{12}, \mathbb{M}(3,4)\}$, $\{\mathbb{R}^{12}, \mathbb{P}_{11}\}$, $\{\mathbb{P}_{11}, \mathbb{M}(3,4)\}$,
$\{\mathbb{R}^{12}, \mathbb{M}(4,3)\}$, $\{\mathbb{P}_{11}, \mathbb{M}(4,3)\}$, $\{\mathbb{M}(3,4), \mathbb{M}(4,3)\}$

(**Section 6.8 Exercise 4**). *(a) . For any vector space V, show that $\mathrm{id}_V : V \to V$ is an isomorphism.*

(b) Suppose $T : V \to V'$ is an isomorphism from the vector space V to the vector space V'. Prove that T is invertible and that T^{-1} is an isomorphism from V' to V.

(c) Suppose $T : V \to V'$ and $T' : V \to V''$ are isomorphisms. Prove that $T' \circ T : V \to V''$ is an isomorphism.

(a) By Lemma A.5.2 in the appendix of the notes, $\mathrm{id}_V \circ \mathrm{id}_V = \mathrm{id}_V$. So id_V is an inverse of id_V. Also by **Section 6.1 Exercise 6**, id_V is linear and so id_V is an isomorphism.

(b) This is obvious with the definition of an isomorphism in the notes. (But observe that according to Theorem 6.8 in the notes, the definition of an isomorphism in the notes is equivalent to the definition in the book).

(c) By (b) T and T' are invertible and so by A.5.6 in the appendix of the notes, $T' \circ T$ is invertible. By Theorem 6.7, $T' \circ T$ is invertible and so by Theorem 6.8, $T' \circ T$ is an isomorphism.

(**Section 6.8 Exercise 5**). *(a) Show that any vector space V is isomorphic to itself.*

(b) Show that if a vector space V is isomorphic to a vector space V' , then V' is isomorphic to V.

(c) Show that if the vector space V is isomorphic to the vector space V' and V' is isomorphic to the vector space V'' , then V is isomorphic to V''.

Recall call first that by definition a vector space V is isomorphic a vector space W if and only if there exists an isomorphism $T : V \to W$.

(a) By **Section 6.8 Exercise 4**a, $\mathrm{id}_V : V \to V$ is an isomorphism. So V is isomorphic to V.

(b) Suppose that the vector space V is isomorphic to the vector space V'. Then there there exists an isomorphism $T : V \to V'$. By **Section 6.8 Exercise 4**b, $T^{-1} : V' \to V$ is an isomorphism and so V' is isomorphic to V.

(c) Suppose the vector space V is isomorphic to the vector space V' and V' is isomorphic to the vector space V''. Then there exist an isomorphism $T : V \to V'$ and an isomorphism $T' : V' \to V''$. By **Section 6.8 Exercise 4**c, $T' \circ T : V \to V''$ is an isomorphism and so V is isomorphic to V''.

(**Section 7.2 Exercise 2**b). *Use Theorem 7.4 to compute* $\det \begin{bmatrix} -2 & 2 & 7 & 0 \\ 0 & 9 & 0 & 3 \\ 0 & 5 & 0 & 0 \\ 4 & 1 & 8 & 2 \end{bmatrix}$ *by keeping track of the changes that occur as you apply row operations to put the matrix in reduced row-echelon form.*

$$\begin{bmatrix} -2 & 2 & 7 & 0 \\ 0 & 9 & 0 & 3 \\ 0 & 5 & 0 & 0 \\ 4 & 1 & 8 & 2 \end{bmatrix} \xrightarrow[-30]{\begin{array}{lr} R4 + 2R1 \to R4 & 1 \\ -\frac{1}{2}R1 \to R1 & -2 \\ \frac{1}{3}R2 \to R2 & 3 \\ \frac{1}{5}R3 \to R3 & 5 \end{array}} \begin{bmatrix} 1 & -2 & -\frac{7}{2} & 0 \\ 0 & 3 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 5 & 22 & 2 \end{bmatrix} \xrightarrow[-30]{\begin{array}{lr} R1 + 2R3 \to R1 & 1 \\ R2 - 3R3 \to R2 & 1 \\ R4 - 5R3 \to R4 & 1 \end{array}} \begin{bmatrix} 1 & 0 & -\frac{7}{2} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 22 & 2 \end{bmatrix}$$

$$\xrightarrow[-660]{\begin{array}{lr} \frac{1}{22}R4 \to R4 & 22 \\ R2 \leftrightarrow R3 & -1 \\ R3 \leftrightarrow R4 & -1 \end{array}} \begin{bmatrix} 1 & 0 & -\frac{7}{2} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{1}{11} \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow[-660]{\begin{array}{lr} R3 - \frac{1}{11}R4 \to R4 & 1 \\ R1 + \frac{7}{2}R3 \to R3 & 1 \end{array}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

So the determinant is -660.

(**Section 7.3 Exercise 10**). *Suppose A is a square matrix. Use induction to prove for any integer $n \geq 0$, that* $\det A^n = (\det A)^n$.

Recall first that A^n is inductively defined by

$$A^0 = I, \quad \text{and} \quad A^{n+1} = A^n A$$

By definition of a regular determinant function $\det I = 1$. Thus $\det A^0 = \det I = 1 = (\det A)^0$ and so $\det A^n = (\det A)^n$ holds for $n = 0$.

Suppose now that

$$(*) \qquad \det A^n = (\det A)^n$$

for some non-negative integer n.

Then

$$\begin{aligned} \det A^{n+1} &= \det A^n A && \text{– definition of } A^{n+1} \\ &= \det A^n \det A && \text{– Theorem 7.7} \\ &= (\det A)^n \det A && \text{– by the induction assumption } (*) \\ &= (\det A)^{n+1} && \text{– Property of } \mathbb{R} \end{aligned}$$

So $\det A^{n+1} = (\det A)^{n+1}$. Thus by the principal of induction, $\det A^n = (\det A)^n$ holds for all non-negative integers n.

(**Section 7.3 Exercise 11**). *Prove that if the square matrix A is invertible, then*

$$\det A^{-1} = (\det A)^{-1}.$$

Since A is invertible, $AA^{-1} = I$. Thus by Theorem 7.7, $\det(A)\det(A^{-1}) = \det(AA^{-1}) = \det I$. By definition of a regular determinant function $\det I = 1$ and so $\det(A)\det(A^{-1}) = 1$. Thus $\det A^{-1} = (\det A)^{-1}$.

(**Section 7.3 Exercise 11**). *Prove that if A and P are $n \times n$- matrices and P is invertible, then*

$$\det(P^{-1}AP) = \det A$$

Using Theorem 7.7 twice we compute

$$\begin{aligned} \det(P^{-1}AP) = \det\Big(P^{-1}(AP)\Big) \qquad &= \det P^{-1} \det(AP) \\ = \det P^{-1} \det A \det P &= \det P^{-1} \det P \det A \end{aligned}$$

By **Section 7.3 Exercise 10**, $\det(P^{-1}) = (\det P)^{-1}$ and so $\det P^{-1} \det P = 1$. Thus $\det(P^{-1}AP) = \det A$.

A. *Fill in all the ? in the proof of the following Theorem:*

Theorem A. *Let A be a $m \times n$ matrix and B its reduced row echelon form. Let $x_{f_1}, \dots, x_{f_t}$ be the free variables of B and let s be number of non-zero rows of B . Let $(e_1, \dots, e_n)$ be the standard basis for $\mathbb{R}^n$ and let b^k be row k of B. Then $(b^1, \dots, b^s)$ is a basis for* Row$A$ *and $(b^1, \dots, b^s, e_{f_1}, \dots, e_{f_t})$ is basis for $\mathbb{R}^n$.*

Proof. Put

$$D = (b^1, \dots, b^s, e_{f_1}, \dots, e_{f_t})$$

Note that $(b^1, \dots, b^s)$ is the list of non-zero rows of B. By Theorem $\boxed{\text{N3.7.5}}$ $(b^1, \dots, b^s)$ is a basis for Row$A$. So we just need to show that D is a basis for $\mathbb{R}^n$. Note that s is the number lead variables and so $n = s + t$. Thus D is a list of length n in the $\boxed{n}$-dimensional vector space $\mathbb{R}^n$. So by Theorem $\boxed{\text{N3.5.5}}$

(∗) D is basis of $\mathbb{R}^n$ if and only if D is linearly independent.

To show that D is linearly independent, let $r_1, \dots, r_s, u_1, \dots, u_t \in \boxed{\mathbb{R}}$ such that

$$(**) \qquad r_1 b^1 + \dots + r_s b^s + u_1 e_{f_1} + \dots + u_t e_{f_t} = \boxed{\mathbf{0}}$$

Let $1 \le k \le s$ and let b_{kl_k} be the leading 1 in b^k. Then b_{kl_k} is the only non-zero entry in Column l_k of B and so the l_k entry of b^j is $\boxed{0}$ for all $1 \le j \le s$ with $j \ne k$. Since x_{l_k} is a leading variable, $l_k \ne f_j$ for all $1 \le j \le t$ and so also the l_k entry of e_{f_j} is $\boxed{0}$. Thus the l_k entry of the linear combination on the left side of the equation (∗∗) is $\boxed{r_k}$. Hence $\boxed{r_k} = 0$ for all $1 \le k \le s$. Thus (∗∗) implies

$$u_1 e_{f_1} + \dots + u_t e_{f_t} = \mathbf{0}$$

Since $(e_1, \dots, e_n)$ is $\boxed{\text{a basis and so linearly independent}}$ this gives $u_j = \boxed{0}$ for all $1 \le j \le t$. Thus D is $\boxed{\text{linearly independent}}$, and so by (∗) D is a basis for $\mathbb{R}^n$. □

B. *Let* $V = \text{span}\Big((1,0,1,1,1), (3,3,0,3,3), (1,1,0,1,1)\Big)$. *Find a basis for* V *and extend it to a basis of* $\mathbb{R}^5$. Hint: *Use Theorem A to find both bases simultaneously.*

We use the Gauss-Jordan Algorithm to compute the reduced row echelon form of the matrix A formed by the the above list of vectors as rows.

$$\begin{bmatrix} 1 & 0 & 1 & 1 & 1 \\ 3 & 3 & 0 & 3 & 3 \\ 1 & 1 & 0 & 1 & 1 \end{bmatrix} \xrightarrow[R3 - R1 \to R3]{R2 - 3R1 \to R2} \begin{bmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 3 & -3 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \end{bmatrix} \xrightarrow[\frac{1}{3}R2 \to R2]{R3 - \frac{1}{3}R2 \to R3} \begin{bmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus by Theorem I,

$$\left(\begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} \right)$$

is a basis for Col$A = V$ and since x_3, x_4, x_5 are the free variables

$$\left(\begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right)$$

is a basis for $\mathbb{R}^5$.