MTH 309-4 **Linear Algebra I** **F11**

Homework 10/Solutions

Section	Exercises
6.4	1,2
6.5	1,2,3,4,6

(**Section 6.5 Exercise 1**). *Consider the linear map $T : \mathbb{P}_2 \to \mathbb{P}_3$ defined by $T(p) = xp$. Consider the basis $B = (1, x, x^2)$ for $\mathbb{P}_2$ and the basis $B^* = (1, x, x^2, x^3)$ for $\mathbb{P}_3$.*

(a) Find the matrix A of T relative to B and B^*

(b) Find the matrix A^* of the differentiation operator $D : \mathbb{P}_3 \to \mathbb{P}_2$ relative to B^* and B. (So $D(p) = p'$ where p' is the derivative of p).

(c) Use Theorem 6.11 to compute the matrix of the composition $D \circ T : \mathbb{P}_2 \to \mathbb{P}_2$ relative to B and B.

(d) Confirm that this is the product A^*A.

(e) Use Theorem 6.11 to compute the matrix of the composition $T \circ D : \mathbb{P}_3 \to \mathbb{P}_3$ relative to B^* and B^*.

(f) Confirm that this is the product AA^*.

(a):

$$\begin{array}{c|ccc} v_j & 1 & x & x^2 \\ \hline T(v_j) & x & x^2 & x^3 \\ [T(v_j)]_{B^*} & \begin{pmatrix}0\\1\\0\\0\end{pmatrix} & \begin{pmatrix}0\\0\\1\\0\end{pmatrix} & \begin{pmatrix}0\\0\\0\\1\end{pmatrix} \end{array} \quad \text{and so} \quad A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(b)

$$\begin{array}{c|cccc} v_j^* & 1 & x & x^2 & x^3 \\ \hline D(v_j^*) & 0 & 1 & 2x & 3x^2 \\ [D(v_j^*)]_B & \begin{pmatrix}0\\0\\0\end{pmatrix} & \begin{pmatrix}1\\0\\0\end{pmatrix} & \begin{pmatrix}0\\2\\0\end{pmatrix} & \begin{pmatrix}0\\0\\3\end{pmatrix} \end{array} \quad \text{and so} \quad A^* = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

(c)

$$\begin{array}{c|ccc} v_j & 1 & x & x^2 \\ \hline T(v_j) & x & x^2 & x^3 \\ D(T(v_j) & 1 & 2x & 3x^2 \\ [(D\circ T)(v_j)]_B & \begin{pmatrix}1\\0\\0\end{pmatrix} & \begin{pmatrix}0\\2\\0\end{pmatrix} & \begin{pmatrix}0\\0\\3\end{pmatrix} \end{array} \quad \text{and so} \quad A^{**} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

(c)

$$A^*A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} = A^{**}$$

(e)

$$\begin{array}{c|cccc} v_j^* & 1 & x & x^2 & x^3 \\ \hline D(v_j^*) & 0 & 1 & 2x & 3x^2 \\ T(D(v_j^*)) & 0 & x & 2x^2 & 3x^3 \\ [(T\circ D)(v_j^*)]_B & \begin{pmatrix}0\\0\\0\\0\end{pmatrix} & \begin{pmatrix}0\\1\\0\\0\end{pmatrix} & \begin{pmatrix}0\\0\\2\\0\end{pmatrix} & \begin{pmatrix}0\\0\\0\\3\end{pmatrix} \end{array} \quad \text{and so} \quad A^{***} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

(f)

$$AA^* = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} = A^{***}$$

(**Section 6.5 Exercise 2**). *Consider the basis* $B^* = (1+x, x, 1-x^2)$ *and* $B = (x, 1-x, 1+x^2)$ *of* $\mathbb{P}_2$. *Find the change-of-basis matrix for changing from* B^* *to* B.

$$\begin{aligned}
1+x &= 2x + 1(1-x) + 0(1+x^2)\\
x &= 1x + 0(1-x) + 0(1+x^2)\\
1-x^2 &= 2x + 2(1-x) + (-1)(1+x^2)
\end{aligned}$$

So the change-of-basis matrix is

$$\begin{bmatrix} 2 & 1 & 2 \\ 1 & 0 & 2 \\ 0 & 0 & -1 \end{bmatrix}$$

(**Section 6.5 Exercise 3**). *Show that the matrix for changing from an ordered basis $(u_1, \dots, u_n)$ for $\mathbb{R}^n$ to the standard basis for $\mathbb{R}^n$ consists of the columns $u_1, \dots, u_n$ in that order.*

Let $B = (u_1, \dots, u_n)$ and let $E = (e_1, \dots, e_n)$. Let P be the change-of-matrix basis from B to E. Then by Theorem 6.15

$$(*) \qquad P = \Big[[u_1]_E, [u_2]_E, \dots, [u_n]_E \Big]$$

By an example in class (see N3.4.6 in the notes) $[x]_E = x$ for all $x \in E$. In particular, $[u_j]_E = u_j$ for all $1 \leq j \leq n$ and so (*) gives

$$P = \Big[u_1, u_2, \dots, u_n \Big]$$

A. *Let* $\mathbf{V}$ *and* $\mathbf{W}$ *be vector spaces, let $B = (v_1, \dots, v_n)$ be basis for* $\mathbf{V}$ *and let $E = (u_1, \dots, u_n)$ be a list in W.*

(a) Show that $L_E \circ C_B$ is a linear and $(L_E \circ C_B)(v_j) = u_j$ for all $1 \leq j \leq n$.

(b) Let $T : \mathbf{V} \to \mathbf{W}$ be a linear with $T(v_j) = u_j$ for all $1 \leq j \leq n$. Show that $T = L_E \circ C_B$.

(a) By N6.2.8(c) C_B is an isomorphism, in particular C_B is linear. By N6.1.5 L_E is linear. By Theorem 6.7 a composition of linear function is linear and so $L_E \circ C_B$ is linear. We compute

$$\begin{aligned}
(L_E \circ C_B)(v_j) &= L_E\big(C_B(v_j)\big) && \text{– definition of composition}\\
&= L_E(e_j) && \text{– N6.2.8d}\\
&= v_j && \text{– N6.1.5d}
\end{aligned}$$

(b) Using (a) and the assumptions in (b) we have $(L_E \circ C_B)(v_j) = u_j = T(v_j)$. Since B is a basis for V, B spans V. Also both $L_E \circ C_B$ and T are linear. Thus Theorem 6.3 implies $T = L_E \circ C_B$.

B. *Let* $\mathbf{V}$ *and* $\mathbf{W}$ *be vector spaces with basis* B *and* D *respectively.*

Let $n = \dim V$ *and* $m = \dim W$.

For a linear function T *from* $\mathbf{V}$ *to* $\mathbf{W}$ *let* A_T *be the matrix of* T *with respect to* B *and* D.

Define the function $\alpha : \mathrm{L}(\mathbf{V}, \mathbf{W}) \to \mathbb{M}(m, n)$ *by* $\alpha(T) = A_T$ *for all* $T \in \mathrm{L}(\mathbf{V}, \mathbf{W})$.

Define the function $\beta : \mathbb{M}(m, n) \to \mathrm{L}(\mathbf{V}, \mathbf{W})$ *by* $\beta(A) = L_D \circ L_A \circ C_B$ *for all* $A \in \mathbb{M}(m, n)$.

(a) Show that α *is linear.*

(b) Show that β *is linear.*

(c) Show that β *is an inverse of* α.

(d) Show that the vector space $\mathbf{L}(\mathbf{V}, \mathbf{W})$ *is isomorphic to the vector space* $\mathbb{M}(m, n)$.

(a) Let $T, S \in \mathrm{L}(\mathbf{V}, \mathbf{W})$. Then by Lemma N6.4.2(a), the matrix for $T + S$ with respect to B and D is $A_T + A_S$. So

$$\alpha(T + S) = A_{T+S} = A_T + A_S = \alpha(T) + \alpha(S).$$

Let $r \in \mathbb{R}$. Then by Lemma N6.4.2(b) the matrix for rT (with respect to B and D is rA_T. So

$$\alpha(rT) = rA_T = rA_T = r\alpha(T)$$

Thus α is linear.

(c) Let $A \in \mathbb{M}(m, n)$ and $T \in \mathrm{L}(\mathbf{V}, \mathbf{W})$. Then Theorem 6.11(a,e) in the notes:

$$A = A_T \iff T = L_D \circ L_A \circ C_B$$

So by definition of α and β:

$$(*) \qquad A = \alpha(T) \iff T = \beta(A)$$

Thus by Lemma A.5.6(a,g) in the appendix of the notes, β is an inverse of α.

(b) By (a) α is linear and by (c), β is the inverse of α. So by Theorem 6.8, β is linear.

(d) By (a) α is linear and by (c) α is invertible. Thus by definition, α is an isomorphism. Since α is a function from $\mathbf{L}(\mathbf{V}, \mathbf{W})$ to $\mathbb{M}(m, n)$, this implies that $\mathbf{L}(\mathbf{V}, \mathbf{W})$ is isomorphic to $\mathbb{M}(m, n)$.

C. *Retain the notation from Exercise B. Suppose* $B = (v_1, \ldots, v_n)$ *and* $D = (w_1, \ldots, w_m)$. *For* $1 \leq i \leq m$ *and* $1 \leq j \leq n$ *let* T_{ij} *be unique linear function from* $\mathbf{V}$ *to* $\mathbf{W}$ *with*

$$T_{ij}(v_k) = \begin{cases} w_i & \textit{if } k = j \\ \mathbf{0_W} & \textit{if } k \neq j \end{cases}$$

for all $1 \leq k \leq n$*. Also let* A_{ij} *be* $m \times n$ *matrix whose* (i,j) *entry is* 1 *and all other entries zero. Show that*

$$\alpha(T_{ij}) = A_{ij} \text{ and } T_{ij} = \beta(A_{ij})$$

Fix $1 \leq i \leq m$ and $1 \leq j \leq n$. Let $A = \alpha(T_{ij})$. So $A = A_{T_{ij}}$ is the matrix of T_{ij} with respect to B and D. Let $1 \leq k \leq n$. By definition of A, column k of A is

$$a_k = [T_{ij}(v_k)]_D.$$

Suppose that $k \neq j$. Then by definition of T_{ij}, $T_{ij}(v_k) = \mathbf{0}$ and so

$$a_k = [T_{ij}(v_k)]_D = [\mathbf{0}]_D = \mathbf{0}$$

Thus

$$a_{lk} = 0 \text{ for all } 1 \leq l \leq m \text{ and } 1 \leq k \leq n \text{ with } k \neq j$$

Suppose that $k = j$. By definition of T_{ij}, $T_{ij}(v_j) = w_i$. Thus

$$a_j = [T_{ij}(v_j)]_D = [w_i]_D = C_D(w_i) = e_i$$

where the last equality follows for example from N6.2.8d.

Hence

$$a_{ij} = 1 \text{ and } a_{lj} = 0 \text{ for all } 1 \leq l \leq m \text{ with } l \neq i$$

So the (i,j)-entry of A is 1, while all other entries are 0. Thus $A = A_{ij}$.

We proved that $\alpha(T_{ij}) = A_{ij}$. Thus statement (*) in Exercise B shows that $\beta(A_{ij}) = T_{ij}$.