Homework 9/Solutions

Section	Exercises
6.2	15
6.3	1,3,8,9bc,11
6.6	7,9
6.7	2,6

(**Section 6.2 Exercise 15**). *Prove Theorem 6.7:*

Let $f : \mathbf{V} \to \mathbf{W}$ and $g : \mathbf{W} \to \mathbf{X}$ be linear. Then $g \circ f$ is linear.

Let $u, v \in V$ and $a, b \in \mathbb{R}$. Then

$$\begin{aligned}
& (g \circ f)(au + bv) \\
= \;& g\Big(f(au + bv)\Big) && \text{– definition of composition} \\
= \;& g\Big(a\big(f(u)\big) + b\big(f(v)\big)\Big) && \text{– since } f \text{ is linear and N6.1.4} \\
= \;& a\Big(g\big(f(u)\big)\Big) + b\Big(g\big(f(v)\big)\Big) && \text{– since } g \text{ is linear and N6.1.4} \\
= \;& a\Big((g \circ f)(u)\Big) + b\Big((g \circ f)(v)\Big) && \text{– definition of composition,twice}
\end{aligned}$$

and so $g \circ f$ is linear by Lemma N6.1.4.

(**Section 6.3 Exercise 8**). *Consider the linear map $D : \mathbb{P}_3 \to \mathbb{P}_3$ defined by $D(p) = p'$.*

(a) Find the matrix of D relative to the basis $B = (1, x, x^2, x^3)$ used for both the domain and codomain.

(b) Find the matrix of D relative to the basis $B^* = \Big(1, x-1, (x+1)^2, (x-1)^3\Big)$ used for both the domain and the codomain.

(a)

$$\begin{array}{c|cccc}
v_j & 1 & x & x^2 & x^3 \\
\hline
D(v_j) & 0 & 1 & 2x & 3x^2 \\
[D(v_j)]_B & \begin{pmatrix}0\\0\\0\\0\end{pmatrix} & \begin{pmatrix}1\\0\\0\\0\end{pmatrix} & \begin{pmatrix}0\\2\\0\\0\end{pmatrix} & \begin{pmatrix}0\\0\\3\\0\end{pmatrix}
\end{array}
\quad \text{and so} \quad A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(b)

v_j^*	1	$x-1$	$(x+1)^2$	$(x-1)^3$
$D(v_j^*)$	0	1	$2(x+1)$	$3(x-1)^2$
$[D(v_j^*)]_{B^*}$	$\begin{pmatrix}0\\0\\0\\0\end{pmatrix}$	$\begin{pmatrix}1\\0\\0\\0\end{pmatrix}$	$\begin{pmatrix}4\\2\\0\\0\end{pmatrix}$	$\begin{pmatrix}-12\\-12\\3\\0\end{pmatrix}$

(Note here that $x+1=(x-1)+2$ and $(x-1)^2=\big((x+1)-2\big)^2=(x+1)^2-4(x+1)+4=(x+1)^2-4(x-1)-4$)

Thus

$$A^*=\begin{bmatrix}0&1&4&-12\\0&0&2&-12\\0&0&0&3\\0&0&0&0\end{bmatrix}$$

(**Section 6.6 Exercise 6**). *Suppose $T:\mathbf{V}\to\mathbf{W}$ is linear. Let $(v_1,\ldots,v_n)$ be list in V such that $T(v_1,\ldots,v_n)$ is a linearly independent in $\mathbf{W}$. Show that $(v_1,\ldots,v_n)$ is linearly independent in $\mathbf{V}$.*

Let $r_1,\ldots,r_n\in\mathbb{R}$ such that

$$r_1v_1+r_2v_2+\ldots+r_nv_n=\mathbf{0}.$$

Applying T to both sides of this equations we obtain:

$$T(r_1v_1+r_2v_2+\ldots+r_nv_n)=T(\mathbf{0}),$$

and so using Theorem 6.2

$$r_1T(v_1)+r_2T(v_2)+\ldots+r_nT(v_n)=\mathbf{0}.$$

Since $T(v_1,\ldots,v_n)$ is a linearly independent, we conclude that

$$r_1=0,r_2=0,\ldots,r_n=0.$$

Thus $(v_1,\ldots,v_n)$ is linearly independent in $\mathbf{V}$.

A. *Let $T:\mathbf{V}\to\mathbf{W}$ be linear. Prove that $T(u-v)=T(u)-T(v)$ for all $u,v\in V$.*

$$\begin{aligned} T(u-v) &= T\big(u+(-v)\big) && \text{– definition of '-'} \\ &= T(u)+T(-v) && \text{– since } T \text{ is linear} \\ &= T(u)+\big(-T(v)\big) && \text{– Theorem 6.2} \\ &= T(u)-T(v) && \text{– definition of '-'} \end{aligned}$$

B. *Let $f : I \to J$ be a function. Prove that $f \circ \mathrm{id}_I = f$ and $\mathrm{id}_J \circ f = f$.*

Let $i \in I$. Then

$$\begin{aligned} (f \circ \mathrm{id}_I)(i) &= f\big(\mathrm{id}_I(i)\big) && \text{– definition of composition} \\ &= f(i) && \text{– definition of } \mathrm{id}_I \end{aligned}$$

Thus $f \circ \mathrm{id}_I = f$ be A.2.2.
Also

$$\begin{aligned} (\mathrm{id}_J \circ f)(i) &= \mathrm{id}_J\big(f(i)\big) && \text{– definition of composition} \\ &= f(i) && \text{– definition of } \mathrm{id}_J \end{aligned}$$

Thus $\mathrm{id}_J \circ f = f$ by A.2.2.